Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 16–24

CHARACTERIZATIONS OF CLASSICAL ORTHOGONAL POLYNOMIALS ON QUADRATIC LATTICES

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Abstract: This paper is devoted to characterizations of classical orthogonal polynomials on quadratic lattices by using a matrix approach. In this form we recover the Hahn, Geronimus, Tricomi and Bochner type characterizations of classical orthogonal polynomials on quadratic lattices. Moreover a new characterization is also presented. From the Bochner type characterization we derive the three-term recurrence relation coefficients for these polynomials.

Keywords: Classical orthogonal polynomials, quadratic lattices, characterization theorems, Divided-difference operators.
AMS Subject Classification (2010): 33C45.

1. Introduction

Classical continuous orthogonal polynomial sequences can be characterized by different properties, using different approaches. Probably the first results in this direction go back to Bochner [3], Favard [5] and Hahn [10]. Moreover, some recent characterizations can be found in [2, 6, 7], by using either differential operators as Bochner or linear functionals as introduced by Maroni [17, 16]. Recently a new characterization of classical continuous, discrete and their q-analogues was given by Verde-Star [22, 23] by using a matrix approach.

A general presentation of classical continuous orthogonal polynomials in terms of solutions of certain differential equations have been done by Nikiforov et al. [18, 19, 20]. In this direction, classical orthogonal polynomials

Received May 25, 2016.

The first author acknowledges hospitality from the AIMS-Cameroon during her visits in 2015 and 2016. The second author was partially supported by the Centre for Mathematics of the University of Coimbra – UID/MAT/00324/2013, funded by the Portuguese Government through FCT/MEC and co-funded by the European Regional Development Fund through the Partnership Agreement PT2020. The third author acknowledges support from the AIMS-Cameroon 2015–2016 research grant and the hospitality and financial support during his visit to Universidade de Vigo in July 2015. The last author thanks the hospitality of the African Institute for Mathematical Sciences (AIMS-Cameroon), where the this research was started during his visits in November 2014, and May and June 2015.

are solution of the second order linear differential equation

$$\sigma(x)y''(x) + \tau(x)y'(x) - \lambda y(x) = 0$$

where σ and τ are polynomials of at most second and first degree, respectively.

The above differential equation can be replaced by a difference equation, giving rise to classical orthogonal polynomials of a discrete variable [18, Chapter 2], if we consider a discretization with constant mesh, or classical orthogonal polynomials on nonuniform lattices [18, Chapter 3] if we consider a class of lattices with variable mesh $\mu(t)$. We would like to notice that divided-difference operators associated with the special non-uniform lattices have appeared in many studies of orthogonal polynomials of a discrete variable. For example see the early studies by Hahn [9, 10, 11, 12], the foundational work by Askey and Wilson [1] and the monograph of Nikiforov, Suslov and Uvarov [18].

As indicated in [18, Theorem 1, page 59] some restrictions must be imposed on the lattice $\mu(t)$ giving rise to the following classification of the lattices:

(1) Linear lattices if $\mu(t) = c_2 t + c_3$ with $c_2 \neq 0$.
(2) Quadratic lattices if $\mu(t) = c_1 t^2 + c_2 t + c_3$, with $c_1 \neq 0$.
(3) q-linear lattices if $\mu(t) = c_5 q^t + c_6$, with $c_5 \neq 0$.
(4) q-quadratic lattices if $\mu(t) = c_4 q^t + c_5 q^{-t} + c_6$ with $c_5 c_6 \neq 0$.

The characterization theorems of classical orthogonal polynomials in the cases of linear and q-linear lattices by using matrix approach have been obtained in [23]. We would like to emphasize that this approach has not yet been used in the case of quadratic or q-quadratic lattices, despite the importance in many applications of the families belonging to these classes (e.g. Racah or Wilson orthogonal polynomials).

In a recent paper [6] the authors gave a characterization theorem for classical orthogonal polynomials on a lattice as described above by using the Pearson-type equation. Moreover, in [7] and by using the functional approach, the authors stated and proved a characterization theorem for classical orthogonal polynomials on non-uniform lattices including the Askey-Wilson polynomials.

The main aim of this paper is to present a new characterization of classical orthogonal polynomials on quadratic lattices, by using a matrix approach. In doing so, we reinterpret in matrix form previous characterizations classical orthogonal polynomials on quadratic lattices, showing that previous results of [22, 23] on classical continuous orthogonal polynomials, discrete and their

q-analogues, can be generalized to nonuniform lattices. In this way, we obtain the Hahn, Geronimus, Tricomi, and Bochner type characterizations. Moreover, by using the method presented by Vicente Gonçalves, we explicitly obtain the coefficients in the three-term recurrence relation satisfied by classical orthogonal polynomials on nonuniform lattices from the second order linear divided-difference equation they satisfy.

This work is organized as follows: in section 2 we introduce the basic definitions and notations. In section 3 we reinterpret the Hahn, Geronimus, Tricomi, and Bochner characterizations of classical orthogonal polynomials on quadratic lattices by using a matrix approach and derive a new characterization of these polynomials. Finally, in section 4 we extend the method of Vicente Gonçalves to obtain the coefficients of the three-term recurrence relation of classical orthogonal polynomials on quadratic lattices from the second-order linear divided-difference equation they satisfy.

2. Basic definitions and notations

Let us consider the quadratic lattice

$$\mu(t) = c_1\, t^2 + c_2\, t + c_3, \tag{1}$$

where c_1, c_2, and c_3 are constants and in what follows we shall assume that $c_1 = 1$, i.e. a pure quadratic lattice. Notice that the particular case $c_1 = 0$, i.e. linear lattices, have been considered in [23], and as mentioned before our intention is to show that that matrix approach can be followed in the case of nonuniform lattices. Let $P_n(\mu(t))$ be a monic polynomial of degree n in the lattice $\mu(t)$,

$$P_n(\mu(t)) = P_n = p_{n,n} + p_{n-1,n}\vartheta_1(t) + p_{n-2,n}\vartheta_2(t) + \cdots + p_{1,n}\vartheta_{n-1}(t) + \vartheta_n(t), \tag{2}$$

where the basis $\left\{\vartheta_n(t)\right\}_{n\geq 0}$ is defined by [8, Equation (54), p. 416]

$$\vartheta_n(t) = (-4)^{-n}\, (2t + 1/2 + c_2)_n\, (-2t + 1/2 - c_2)_n, \tag{3}$$

and $(A)_n = A(A+1)\cdots(A+n-1)$ with $(A)_0 = 1$ denotes the Pochhammer symbol. Let us further define

$$\begin{aligned}
\mathcal{P} &= \begin{bmatrix} P_0 & P_1 & P_2 & \cdots \end{bmatrix}^{\mathsf{T}} = \begin{bmatrix} 1 & p_{1,1} + \vartheta_1(t) & p_{2,2} + p_{1,2}\,\vartheta_1(t) + \vartheta_2(t) & \cdots \end{bmatrix}^{\mathsf{T}} \\
&= A \begin{bmatrix} 1 & \vartheta_1(t) & \vartheta_2(t) & \cdots \end{bmatrix}^{\mathsf{T}},
\end{aligned}$$

where

$$A = \begin{bmatrix} 1 & 0 & & \\ p_{1,1} & 1 & 0 & \\ p_{2,2} & p_{1,2} & 1 & \ddots \\ & \ddots & \ddots & \ddots \end{bmatrix}. \tag{4}$$

The difference operators $\mathbb{D}$ and $\mathbb{S}$ [14, 15] are defined by

$$\mathbb{D}f(t) = \frac{f(t+1/2) - f(t-1/2)}{\mu(t+1/2) - \mu(t-1/2)} \quad \text{and} \quad \mathbb{S}f(t) = \frac{f(t+1/2) + f(t-1/2)}{2}.$$

Notice that the above divided-difference operators transform polynomials of degree n in the lattice $\mu(t)$ defined in (1) into polynomials of respectively degree $n-1$ and n in the same variable $\mu(t)$. Since [8]

$$\mathbb{D}\,\vartheta_n(t) = n\,\vartheta_{n-1}(t), \tag{5}$$

we have

$$\begin{aligned} \mathcal{P}' &= \begin{bmatrix}\mathbb{D}\,P_1 & \frac{1}{2}\mathbb{D}\,P_2 & \frac{1}{3}\mathbb{D}\,P_3 & \cdots\end{bmatrix}^{\mathsf{T}} = \begin{bmatrix}1 & p_{1,2} + \vartheta_1(t) & \cdots\end{bmatrix}^{\mathsf{T}} \\ &= \tilde{A}\begin{bmatrix}1 & \vartheta_1(t) & \vartheta_2(t) & \cdots\end{bmatrix}^{\mathsf{T}}, \end{aligned}$$

where

$$\tilde{A} = \tilde{D}\,A\,D, \tag{6}$$

with

$$\tilde{D} = \begin{bmatrix} 0 & 1 & & & \\ & 0 & \frac{1}{2} & & \\ & & 0 & \frac{1}{3} & \\ & & & \ddots & \ddots \end{bmatrix}, \quad \text{and} \quad D = \begin{bmatrix} 0 & & & \\ 1 & 0 & & \\ & 2 & 0 & \\ & & \ddots & \ddots \end{bmatrix}. \tag{7}$$

Let us assume that $\{P_n\}_{n\geq 0}$ is a sequence of monic orthogonal polynomials on a quadratic lattice $\mu(t)$. Then, the three-term recurrence relation satisfied by $\{P_n\}_{n\geq 0}$ reads as

$$\mu(t)P_n = P_{n+1} + \beta_n P_n + \gamma_n P_{n-1}, \tag{8}$$

with initial conditions $P_0 = 1$, $P_1 = \mu(t) - \beta_0$.

Lemma 1. *The three-term recurrence relation* (8) *can be written in matrix form as*

$$L\,A = A\,\boldsymbol{X^1},$$

where

$$\boldsymbol{X^1} := X + \operatorname{diag}\{f_0, f_1, \ldots\}, \tag{9}$$

with

$$L = \begin{bmatrix} \beta_0 & 1 & & & \\ \gamma_1 & \beta_1 & 1 & & \\ & \gamma_2 & \beta_2 & \ddots & \\ & & \ddots & \ddots & \end{bmatrix}, \qquad X = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & \ddots & \ddots \end{bmatrix}, \tag{10}$$

and the coefficients f_n *are defined as* [8, Equation (34), p. 411]

$$\mu(t)\,\vartheta_n(t) = \vartheta_{n+1}(t) + f_n\,\vartheta_n(t), \quad n = 0, 1, \ldots, \tag{11}$$

and explicitly given by

$$f_n = -\frac{c_2^2}{4} + \frac{1}{16}(2n+1)^2 + c_3. \tag{12}$$

Proof: Using (11) and the linear independence of $\{\vartheta_n(t)\}_{n\geq 0}$ we get

$$\begin{aligned} \mu(t)\,\begin{bmatrix}1 & \vartheta_1(t) & \vartheta_2(t) & \cdots\end{bmatrix}^{\mathsf{T}} &= \big(\operatorname{diag}\{f_0, f_1, \ldots\} + X\big)\,\begin{bmatrix}1 & \vartheta_1(t) & \vartheta_2(t) & \cdots\end{bmatrix}^{\mathsf{T}} \\ &= \boldsymbol{X^1}\,\begin{bmatrix}1 & \vartheta_1(t) & \vartheta_2(t) & \cdots\end{bmatrix}^{\mathsf{T}}; \end{aligned}$$

and from $L\,\mathcal{P} = \mu(t)\,\mathcal{P}$, the result follows. ■

Lemma 2. *The following matrix relation holds true*

$$AD\tilde{A}^{-1} = AD\tilde{D}A^{-1}D = AJA^{-1}D = D,$$

where

$$J = \begin{bmatrix} 0 & 0 & 0 & 0 & \\ 0 & 1 & 0 & 0 & \ddots \\ 0 & 0 & 1 & 0 & \ddots \\ & \ddots & \ddots & \ddots & \ddots \end{bmatrix}. \tag{13}$$

Proof: The result follows from the definitions of the matrices A, D, $\tilde{A}$, $\tilde{D}$ and J. ■

3. Characterizations of classical orthogonal polynomials on quadratic lattices

3.1. Hahn's characterization. Let us assume that the sequence $\{\frac{1}{n}\mathbb{D}P_n = P_n'\}_{n\geq 1}$ is also orthogonal (Hahn's characterization). Then, the three-term recurrence relation satisfied by $\{P_n'\}_{n\geq 1}$

$$P_1' = 1, \quad P_2' = \mu(t) - \beta_0', \quad \mu(t)P_n' = P_{n+1}' + \beta_n' P_n' + \gamma_n' P_{n-1}', \; n \in \mathbb{N},$$

can be written in matrix form as

$$M\,\mathcal{P}' = \mu\,\mathcal{P}', \tag{14}$$

with

$$M = \begin{bmatrix} \beta_0' & 1 & & \\ \gamma_1' & \beta_1' & 1 & \\ & \gamma_2' & \beta_2' & \ddots \\ & & \ddots & \ddots \end{bmatrix}. \tag{15}$$

Thus, by using the definition of $\mathcal{P}'$ it yields

$$M\,\tilde{A} = \tilde{A}\,\boldsymbol{X^1}. \tag{16}$$

where $\boldsymbol{X^1}$ has been defined in (9).

As conclusion, we have the Hahn-type characterization of classical orthogonal polynomials on quadratic lattices by using the matrix approach

Theorem 1. *The sequence $\{P_n\}_{n\geq 0}$ is of classical orthogonal polynomials on the quadratic lattice $\mu(t)$ defined in (1) if and only if (16) holds true, where the matrices M, $\tilde{A}$ and $\boldsymbol{X^1}$ are defined in (15), (6), and (9), respectively.*

3.2. Geronimus' characterization. Classical orthogonal polynomials on quadratic lattices can be also characterized from the following algebraic relation (Geronimus' characterization)

$$\mathbb{S}\,P_n = P_{n+1}' + \ell_n^1\,P_n' + \ell_n^2\,P_{n-1}',$$

i.e. each element of the sequence $\{\mathbb{S}P_n\}_{n\geq 0}$ can be expressed as a linear combination of three consecutive elements of the sequence $\{P_n'\}_{n\geq 1}$. By using that [8, Equation (31), p. 410]

$$\mathbb{S}\,\vartheta_n(t) = \vartheta_n(t) + g_n\vartheta_{n-1}(t), \qquad g_n = \frac{n\,(2\,n-1)}{4}, \tag{17}$$

the Geronimus characterization can be written in matrix form as

$$\mathbb{S}\,\mathcal{P} = U\,\mathcal{P}', \tag{18}$$

where

$$U = \begin{bmatrix} 1 & 0 & & \\ \ell_1^1 & 1/2 & 0 & \\ \ell_2^2 & \ell_2^1/2 & 1/3 & \ddots \\ & \ddots & \ddots & \ddots \end{bmatrix}, \tag{19}$$

and

$$\mathbb{S}\,\mathcal{P} = A \begin{bmatrix} 1 \\ \mathbb{S}\,\vartheta_1(t) \\ \mathbb{S}\,\vartheta_2(t) \\ \vdots \end{bmatrix} = A \begin{bmatrix} 1 & 0 & & \\ g_1 & 1 & 0 & \\ 0 & g_2 & 1 & \ddots \\ & \ddots & \ddots & \ddots \end{bmatrix} \begin{bmatrix} 1 \\ \vartheta_1 \\ \vartheta_2 \\ \vdots \end{bmatrix},$$

or

$$A\,G = U\,\tilde{A}\,, \quad \text{i.e.} \quad A\,G\,\tilde{A}^{-1} = U\,, \tag{20}$$

where

$$G = \begin{bmatrix} 1 & 0 & & \\ g_1 & 1 & 0 & \\ 0 & g_2 & 1 & \ddots \\ & \ddots & \ddots & \ddots \end{bmatrix}.$$

By applying the divided-difference operator $\mathbb{D}$ to the three-term recurrence relation (8) satisfied by the sequence $\{P_n\}_{n\geq 0}$, by using product rule of the operator $\mathbb{D}$, we obtain

$$\mathbb{S}\,P_n = \mathbb{D}\,P_{n+1} + \beta_n\,\mathbb{D}\,P_n + \gamma_n\,\mathbb{D}\,P_{n-1} - \mathbb{S}\,\mu(t)\,\mathbb{D}\,P_n,$$

which in matrix form can be expressed as

$$\mathbb{S}\,\mathcal{P} = L\,D\,\mathcal{P}' - D\,\mathbb{S}\,\mu(t)\,\mathcal{P}',$$

with

$$\mathbb{S}\,\mu(t) = \mu(t) + \frac{1}{4}.$$

From the recurrence relation (14) for the sequence of divided-differences $\{P'_n\}_{n\geq 1}$ we have $\mathbb{S}\,\mathcal{P} = L\,D\,\mathcal{P}' - D\,\big(M + \frac{1}{4}\,I\big)\,\mathcal{P}'$, or, by (18) $U\,\mathcal{P}' = L\,D\,\mathcal{P}' - D\,\big(M + \frac{1}{4}\,I\big)\,\mathcal{P}'$, i.e.

$$U = L\,D - D\,\Big(M + \frac{1}{4}\,I\Big). \tag{21}$$

Therefore, we obtain the Geronimus-type characterization of classical orthogonal polynomials on quadratic lattices in matrix form as

Theorem 2. *$\{P_n\}_{n\geq 0}$ is a sequence of classical orthogonal polynomials on the quadratic lattice $\mu(t)$ defined in (1) if and only if (21) holds true, where U, L, D, and M are defined in (19), (10), (7), and (15), respectively and I denotes the identity matrix.*

3.3. A new characterization of classical orthogonal polynomials on quadratic lattices. Let us recall Lemma 2 as well as the following properties

$$LU = L^2D - LDM - \frac{1}{4}LD,$$
$$UM = LDM - DM^2 - \frac{1}{4}DM,$$
$$AJA^{-1}D = D, \quad \tilde{D}D = I, \quad \text{and} \quad D\tilde{D} = J,$$

where J has been defined in (13) and

$$G = \begin{bmatrix} 1 & 0 & & \\ g_1 & 1 & 0 & \\ 0 & g_2 & 1 & \ddots \\ & \ddots & \ddots & \ddots \end{bmatrix} = I + E, \qquad E = \begin{bmatrix} 0 & 0 & & \\ g_1 & 0 & 0 & \\ 0 & g_2 & 0 & \ddots \\ & \ddots & \ddots & \ddots \end{bmatrix}. \tag{22}$$

We have

$$L^2\,D - 2L\,D\,M + D\,M^2 + \frac{1}{4}(D\,M - L\,D) = A\Big(\boldsymbol{X^1}\,E - E\,\boldsymbol{X^1}\Big)\tilde{A}^{-1}\,,$$

with

$$\boldsymbol{X^1}E - E\boldsymbol{X^1} = \begin{bmatrix} g_1 & 0 & 0 & 0 & \\ g_1(f_1 - f_0) & g_2 - g_1 & 0 & 0 & \ddots \\ 0 & g_2(f_2 - f_1) & g_3 - g_2 & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \end{bmatrix}.$$

Since

$$f_{n+1} - f_n = \frac{1}{2}(n+1), \quad f_{n+1} + f_n = \frac{1}{2}(n+1)^2 + 2(c_3 + \frac{1}{16} - \frac{c_2^2}{4}),$$

where f_n are given in (12), as well as

$$g_{n+1} - g_n = \frac{1}{4}(4n+1) = \frac{1}{2}(2n+1) - \frac{1}{4},$$

we have

$$g_{n+1}(f_{n+1}-f_n) = \frac{1}{2}(n+1)(f_{n+1}+f_n) - \frac{1}{4}(n+1)(f_{n+1}-f_n) - \big(c_3 + \frac{1}{16} - \frac{c_2^2}{4}\big)(n+1)\,.$$

Thus,

$$\boldsymbol{X^1}E - E\boldsymbol{X^1} = \frac{1}{2}(\boldsymbol{X^1}D + D\boldsymbol{X^1}) - \frac{1}{4}(\boldsymbol{X^1}D - D\boldsymbol{X^1}) - \big(c_3 + \frac{1}{16} - \frac{c_2^2}{4}\big)D, \tag{23}$$

and from (2) we obtain

$$
\begin{aligned}
A\boldsymbol{X}^{\mathbf{1}} D\tilde{A}^{-1} &= A\boldsymbol{X}^{\mathbf{1}} A^{-1}(AD\tilde{A}^{-1}) = LD, \\
AD\boldsymbol{X}^{\mathbf{1}}\tilde{A}^{-1} &= (AD\tilde{A}^{-1})\tilde{A}\boldsymbol{X}^{\mathbf{1}}\tilde{A}^{-1} = DM.
\end{aligned}
$$

By using (23) we obtain

$$
A(\boldsymbol{X}^{\mathbf{1}} E - E\boldsymbol{X}^{\mathbf{1}})\tilde{A}^{-1} = \frac{1}{2}(LD + DM) - \frac{1}{4}(LD - DM) - (c_3 + \frac{1}{16} - \frac{c_2^2}{4})D.
$$

Therefore,

$$
\begin{aligned}
L^2 D - 2LDM + DM^2 - \frac{1}{4}(LD - DM) \\
= \frac{1}{2}(LD + DM) - \frac{1}{4}(LD - DM) - \big(c_3 + \frac{1}{16} - \frac{c_2^2}{4}\big)D.
\end{aligned}
$$

We are now in conditions to state a new characterization of classical orthogonal polynomials on quadratic lattices, which is the extension of previous works [22, 23]:

Theorem 3. *$\{P_n\}_{n\geq 0}$ is a sequence of classical orthogonal polynomials on the quadratic lattice $\mu(t)$ defined in (1) if and only if*

$$
L^2 D - 2LDM + DM^2 - \frac{1}{2}(LD + DM) + \big(c_3 + \frac{1}{16} - \frac{c_2^2}{4}\big)D = 0,
$$

holds true, where the matrices L, D, and M are defined in (10), (7), and (15), respectively, and the quadratic lattice $\mu(t)$ depends on the constants c_2 and c_3.

3.4. Tricomi's characterization. Classical orthogonal polynomials on quadratic lattices can be also characterized in terms of a structure relation of the form (Tricomi's characterization) [21]

$$
\phi\, \mathbb{D}\, P_n = g_n^0\, \mathbb{S}\, P_{n+1} + g_n^1\, \mathbb{S}\, P_n + g_n^2\, \mathbb{S}\, P_{n-1},
$$

where ϕ is a polynomial of at most degree 2 in the lattice $\mu(t)$, which can be written in matrix form as

$$
\phi\, \mathcal{P}' = W\, \mathbb{S}\, \mathcal{P},
$$

with

$$
W = \begin{bmatrix} g_1^2 & g_1^1 & g_1^0 & 0 & \\ 0 & g_2^2 & g_2^2 & g_2^0 & \ddots \\ & \ddots & \ddots & \ddots & \ddots \end{bmatrix}. \tag{24}
$$

Thus, $\tilde{A}\,\phi(X+\mathrm{diag}) := \tilde{A}\,\phi(\boldsymbol{X^1}) = W\,A\,G$. Notice that

$$\left(\boldsymbol{X^1}\right)^2 = \begin{bmatrix} f_0^2 & f_0+f_1 & 1 & 0 & \\ 0 & f_1^2 & f_1+f_2 & 1 & \ddots \\ & \ddots & \ddots & \ddots & \ddots \end{bmatrix}.$$

Therefore $\tilde{A}\,\phi(\boldsymbol{X^1}) = W\,A\,G$. As $A\,G = U\,\tilde{A}$, multiplying the first equation by U (left) and using the second identity in (20) we have $U\,W\,A\,G = A\,G\,\phi(\boldsymbol{X^1})$ or equivalently $U\,W = (A\,G)\,\phi(\boldsymbol{X^1})\,(A\,G)^{-1}$, i.e.

$$U\,W = \phi\big((A\,G)\,\boldsymbol{X^1}\,(A\,G)^{-1}\big). \tag{25}$$

Multiplying now first by W (left) the second equation and applying the first equation we obtain

$$W\,U\,\tilde{A} = \tilde{A}\,\phi\big(\boldsymbol{X^1}\big) \quad \text{or} \quad W\,U = \tilde{A}\,\phi\big(\boldsymbol{X^1}\big)\,\tilde{A}^{-1}, \quad \text{and}$$

$$W\,U = \phi\big(\tilde{A}\,\boldsymbol{X^1}\,\tilde{A}^{-1}\big). \tag{26}$$

Thus, we can rewrite the Tricomi-type characterization of classical orthogonal polynomials on quadratic lattices by using the matrix approach as

Theorem 4. *$\{P_n\}_{n\geq 0}$ is a sequence of classical orthogonal polynomials on the quadratic lattice $\mu(t)$ defined in (1) if and only if (25) and (26) hold true, where the matrices W, U, $\tilde{A}$, and $\boldsymbol{X}^1$ are defined in (24), (19), (6), and (9), respectively.*

3.5. Bochner's characterization. Classical orthogonal polynomials on quadratic lattices are solution of a second-order divided-difference equation (Bochner's characterization) [7]

$$\phi\,\mathbb{D}^2\,P_n + \psi\,\mathbb{S}\,\mathbb{D}\,P_n = \lambda_n\,P_n \tag{27}$$

where

$$\phi \equiv \phi(\mu(t)) = a_0(\mu(t))^2 + a_1\mu(t) + a_2, \,, \quad \psi \equiv \psi(\mu(t)) = b_0\mu(t) + b_1\,, \tag{28}$$

are polynomials of at most degree 2 and 1 in the lattice $\mu(t)$. We can express the above characterization in matrix form as

$$A\,D^2\,\phi(\boldsymbol{X^1}) + A\,D\,G\,\psi(\boldsymbol{X^1}) = \Lambda\,A\,,$$

with

$$\lambda_n = n\left((n-1)a_0 + b_0\right), \quad \Lambda = \mathrm{diag}\{\lambda_0, \lambda_1, \lambda_3, \ldots\}. \tag{29}$$

The Bochner equation can be written as an algebraic Sylvester equation in A, namely

$$A\,D\big(D\,\phi(\boldsymbol{X}^1) + G\,\psi(\boldsymbol{X}^1)\big) = \Lambda\,A\,. \tag{30}$$

Therefore, given ϕ and ψ the matrix A is determined if and only if the point spectra of Λ and $D^2\,\phi(\boldsymbol{X^1}) + D\,G\,\psi(\boldsymbol{X^1})$ be disjoint i.e. the given matrices do not have common eigenvalues.

As a conclusion, we have the following Bochner-type characterization of classical orthogonal polynomials on quadratic lattices as

Theorem 5. *$\{P_n\}_{n\geq 0}$ is a sequence of classical orthogonal polynomials on the quadratic lattice $\mu(t)$ defined in (1) if and only if (30) holds true, assuming that $\lambda_n \neq \lambda_m$ for any $n, m = 0, 1, 2, \ldots,\ n \neq m$, where the matrices A, D, $\boldsymbol{X}^1$, G, and Λ are defined in (4), (7), (9), (22), and (29), respectively.*

4. Solving the Bochner-type equation

In 1942 and 1943 [4] Vicente Gonçalves published two papers [24, 25] about classical orthogonal polynomials (Hermite, Jacobi, Laguerre and Bessel), proving the following result. Let $\sigma(x) = a_0x^2 + a_1x + a_2$, $\tau(x) = b_0x + b_1$, and $\lambda_n = n((n-1)a_0 + b_0)$. Assuming that for each n there exists a unique monic polynomial solution of the the equation, each element of the monic polynomial sequence $\{y_n\}_{n\geq 0}$ satisfies

$$\sigma(x)y''(x) + \tau(x)y'(x) - \lambda_n y = 0, \qquad (n = 0, 1, 2, \ldots)$$

then the monic polynomial sequence $\{y_n\}_{n\geq 0}$ satisfies the above equation if and only if $\{y_n\}_{n\geq 0}$ satisfies a three-term recurrence relation

$$xy_n = y_{n+1} + \beta_n y_n + \gamma_n y_{n-1}, \quad n \geq 1,$$

where the two sequences of real numbers $\{\beta_n\}_{n\geq 0}$ and $\{\gamma_n\}_{n\geq 1}$ are fully determined by the constants a_0, a_1, a_2, b_0, and b_1.

Next we reinterpret the above result for quadratic lattices. From (27) let us introduce the Bochner-type operator

$$\mathrm{L}_n = \phi\,\mathbb{D}^2 + \psi\,\mathbb{S}\,\mathbb{D} - \lambda_n\,I, \tag{31}$$

where ϕ and ψ are polynomials in the lattice $\mu(t)$ defined in (28). We shall assume that L_n has for each nonnegative integer n a unique monic polynomial solution of degree exactly n in the quadratic lattice $\mu(t)$, denoted by $P_n \equiv P_n(\mu(t))$, i.e. $P_n = \vartheta_n(t) + p_{1,n}\vartheta_{n-1}(t) + p_{2,n}\vartheta_{n-2}(t) +$ terms of lower degree and

$$\mathrm{L}_n\left(P_n\right) = 0, \quad n = 0, 1, \ldots .$$

Notice that L_n acting on a polynomial g of degree n in the lattice $\mu(t)$ gives a new polynomial of degree at most n in the lattice $\mu(t)$. Let us recall the expression (2) of the polynomial P_n in terms of the basis $\{\vartheta_n(t)\}$. First, we state a result for the unicity of monic polynomial solution of the Bochner-type equation (27).

Lemma 3. *For each n, the unicity of monic polynomial solution of the Bochner-type equation (27) is equivalent to*

(1) $\lambda_j = \lambda_n$ *has* $j = n$ *as unique solution in* $\mathbb{N}$*;*
(2) $\lambda_k \neq 0$*,* $k = 0, 1, \ldots, n-1$*.*

Proof: The result can be deduced as in the classical continuous case [18, Chapter 1] by considering the polynomial solution given in terms of the basis $\vartheta_n(t)$ defined in (3). ■

Lemma 4. *There exists a sequence $\{\beta_n\}_{n\in\mathbb{N}}$ such that the polynomial*

$$U_n(\mu(t)) = \mathrm{L}_{n+1}\big((\mu(t) - \beta_n)P_n\big), \tag{32}$$

has degree $n-1$ in the lattice $\mu(t)$, for each $n \in \mathbb{N}$. Moreover

$$\beta_n = p_{1,n} + f_n + \frac{k_{1,n+1}}{\lambda_n - \lambda_{n+1}}, \tag{33}$$

and $U_n(\mu(t)) = t_n\,\vartheta_{n-1} + \cdots$ where

$$t_n = k_{2,n+1} + (f_n + p_{1,n} - \beta_n)k_{1,n} + (p_{1,n}f_{n-1} + p_{2,n} - \beta_n p_{1,n})(\lambda_{n-1} - \lambda_{n+1}), \tag{34}$$

and

$$\begin{cases} k_{0,j} = a_0 j(j-1) + b_0 j - \lambda_n, \\ k_{1,j} = a_0 j(j-1)(f_{j-1} + f_{j-2}) + b_0 j f_{j-1} + a_1 j(j-1) + b_0 j g_{j-1} + b_1 j, \\ k_{2,j} = a_0 j(j-1) f_{j-2}^2 + a_1 j(j-1) f_{j-2} + b_0 j g_{j-1} f_{j-2} \\ \qquad\qquad + a_2 j(j-1) + b_1 j g_{j-1}. \end{cases} \tag{35}$$

Proof: From (2) we have

$$\begin{aligned} &\big(\mu(t) - \beta_n\big)\, P_n(\mu(t)) \\ &= \vartheta_{n+1}(t) + (f_n + p_{1,n} - \beta_n)\,\vartheta_n(t) + (p_{1,n}f_{n-1} + p_{2,n} - \beta_n p_{1,n})\,\vartheta_{n-1}(t) + \cdots . \end{aligned}$$

By using (5) and (11),

$$\mathrm{L}_n\left(\vartheta_j(t)\right) = k_{0,j}\,\vartheta_j(t) + k_{1,j}\,\vartheta_{j-1}(t) + k_{2,j}\,\vartheta_{j-2}(t), \tag{36}$$

where $k_{i,j}$ are defined in (35). Therefore,

$$\begin{aligned}
\mathrm{L}_{n+1}\big(\vartheta_{n+1}(t)\big) &= [a_0 n(n+1) + b_0(n+1) - \lambda_{n+1}]\,\vartheta_{n+1}(t) \\
&\quad + k_{1,n+1}\,\vartheta_n(t) + k_{2,n+1}\,\vartheta_{n-1}(t), \\
\mathrm{L}_{n+1}\big(\vartheta_n(t)\big) &= [a_0 n(n-1) + b_0 n - \lambda_{n+1}]\,\vartheta_n(t) + k_{1,n}\,\vartheta_{n-1}(t) + k_{2,n}\,\vartheta_{n-2}(t), \\
\mathrm{L}_{n+1}\big(\vartheta_{n-1}(t)\big) &= [a_0 (n-1)(n-2) + b_0(n-1) - \lambda_{n+1}]\,\vartheta_{n-1}(t) \\
&\quad + k_{1,n-1}\,\vartheta_{n-2}(t) + k_{2,n-1}\,\vartheta_{n-3}(t).
\end{aligned}$$

As a consequence,

$$\begin{aligned}
U_n = &\big[a_0 n(n+1) + b_0(n+1) - \lambda_{n+1}\big]\,\vartheta_{n+1}(t) \\
&+ \big[k_{1,n+1} + (f_n + p_{1,n} - \beta_n)(a_0 n(n-1) + b_0 n - \lambda_{n+1})\big]\,\vartheta_n(t) \\
&+ \big[k_{2,n+1} + (f_n + p_{1,n} - \beta_n)k_{1,n} + (p_{1,n} f_{n-1} + p_{2,n} - \beta_n p_{1,n}) \\
&\qquad \times (a_0(n-1)(n-2) + b_0(n-1) - \lambda_{n+1})\big]\,\vartheta_{n-1}(t) + \cdots .
\end{aligned}$$

Thus, the coefficient in ϑ_{n+1} is zero since $\lambda_{n+1} = a_0 n(n+1) + b_0(n+1)$. Moreover, in order that $U_n(\mu(t))$ in (32) be a polynomial of degree $n-1$ in $\mu(t)$ we get (33) as well as $\lambda_{n+1} \neq \lambda_n$. Finally, we also obtain that the coefficient in ϑ_{n-1} in (32) is given by (34). ∎

In order to continue with the method of Vicente Gonçalves for quadratic lattices, and since the proofs are rather technical, we shall first state the results, while the complete proofs are detailed later.

Lemma 5. *For each natural number n we have $\mathrm{L}_{n-1}\big(U_n(\mu(t)\big) = 0$, where $U_n(\mu(t))$ is defined in (32).*

From the unicity of solution of Bochner's equation, there exists a constant t_n such that

$$U_n = t_n\, P_{n-1}\,. \tag{37}$$

Lemma 6. *Let P_n be the unique monic polynomial solution of degree n in the quadratic lattice $\mu(t)$ of the Bochner equation (27). Then, there exist sequences $\{\beta_n\}_{n\geq 0}$ and $\{\gamma_n\}_{n\geq 1}$ such that the following three-term recurrence relation holds*

$$P_{n+1} = (\mu(t) - \beta_n)\, P_n - \gamma_n\, P_{n-1}. \tag{38}$$

More precisely, β_n is given in (33) and

$$\gamma_n = \frac{t_n}{\lambda_{n-1} - \lambda_{n+1}}. \tag{39}$$

As a summary of the previous results we have

Theorem 6. *Let P_n be the monic polynomial solution of degree n in the quadratic lattice $\mu(t)$ of the second-order linear divided-difference equation (27), where the polynomials ϕ and ψ are given in (28), respectively, and the eigenvalue λ_n is given in (29). Then, the coefficients β_n and γ_n of the three-term recurrence relation (38) satisfied by the sequence $\{P_n\}_{n\geq 0}$ are given by*

$$\beta_n = p_{1,n} - p_{1,n+1} + f_n, \tag{40}$$

$$\gamma_n = p_{1,n}\left(f_{n-1} - \beta_n\right) + p_{2,n} - p_{2,n+1}, \tag{41}$$

where

$$p_{1,n} = -\frac{n\left(a(n-1)\left(f_{n-2}+f_{n-1}\right)+b(n-1)+r\left(f_{n-1}+g_{n-1}\right)+s\right)}{\lambda_{n-1}-\lambda_n}, \tag{42}$$

$$\begin{aligned} p_{2,n} = {} & -\frac{1}{\lambda_{n-2}-\lambda_n}\left\{(n-1)\left(p_{1,n}\left(a(n-2)\left(f_{n-3}+f_{n-2}\right)+b(n-2)\right.\right.\right. \\ & \left.\left.\left. +r\left(f_{n-2}+g_{n-2}\right)+s\right)+n\left(f_{n-2}\left(af_{n-2}+b\right)+c\right)\right)+ng_{n-1}\left(rf_{n-2}+s\right)\right\}, \end{aligned} \tag{43}$$

$$\lambda_n = n(a(n-1)+r), \tag{44}$$

and the coefficients f_n and g_n are given in (12) and (17), respectively.

Example 1. As an example of application of the previous results, let us recall that monic Racah polynomials can be defined in terms of hypergeometric series as [13, page 190]

$$\begin{aligned} r_n(\alpha,\beta,\gamma,\delta;t) = r_n(t) = {} & \frac{(\alpha+1)_n\,(\beta+\delta+1)_n\,(\gamma+1)_n}{(n+\alpha+\beta+1)_n} \\ & \times\ {}_4\phi_3\left(\left.\begin{matrix}-n, n+\alpha+\beta+1, -t, t+\gamma+\delta+1\\ \alpha+1, \beta+\delta+1, \gamma+1\end{matrix}\,\right|\,1;,\right) \qquad n=0,1,\ldots,N, \end{aligned}$$

where $r_n(\alpha,\beta,\gamma,\delta;t)$ is a polynomial of degree n in the quadratic lattice $\mu(t) = t(t+\gamma+\delta+1)$. Racah polynomials satisfy a second-order linear divided-difference equation which can be written as a Bochner-type equation of the form (27) where ϕ is the polynomial of degree two in the lattice $\mu(t)$ given by

$$\begin{aligned} \phi(\mu(t)) = {} & -(\mu(t))^2 + \frac{1}{2}(-\alpha(2\beta+\delta+\gamma+3)+\beta(\delta-\gamma-3) \\ & -2(\delta\gamma+\delta+\gamma+2))\mu(t) - \frac{1}{2}(\alpha+1)(\gamma+1)(\beta+\delta+1)(\delta+\gamma+1), \end{aligned}$$

τ is the polynomial of degree one in the lattice $\mu(t)$ given by

$$\tau(\mu(t)) = -(\alpha+\beta+2)\mu(t) - (\alpha+1)(\gamma+1)(\beta+\delta+1),$$

and the eigenvalues λ_n are given by $\lambda_n = -n(\alpha+\beta+n+1)$. If we apply Theorem 6 we obtain exactly the coefficients of the three-term recurrence relation [13, Eq. (9.2.4)]. In a similar way, Theorem 6 can be applied to obtain the coefficients of the three-term recurrence relation satisfied by any sequence of monic orthogonal polynomials solution of a Bochner-type equation on a quadratic lattice (27), assuming that the equation has a unique monic polynomial solution for each positive integer n.

Proof of Lemma 5: We shall need the following relations

a) $\mathbb{D}[f\,g] = \mathbb{S}f\,\mathbb{D}g + \mathbb{D}f\,\mathbb{S}g$,
b) $\mathbb{S}[f\,g] = m_2(t)\mathbb{D}f\,\mathbb{D}g + \mathbb{S}f\,\mathbb{S}g$, with $m_2(t) = \mu(t) + \delta_x$,
c) $\mathbb{S}[\mu(t)] = \mu(t) + 1/4$,
d) $\mathbb{D}\mathbb{S}f = \mathbb{S}\mathbb{D}f + m_1\,\mathbb{D}^2 f$, with $m_1 = 1/2$,
e) $\mathbb{S}^2 f = m_1\,\mathbb{S}\mathbb{D}f + m_2(t)\,\mathbb{D}^2 f + f$.

From the definition of the linear operator L_n and the polynomial $U_n(\mu(t))$ we have

$$\begin{aligned}
U_n &= \mathrm{L}_{n+1}\big((\mu(t)-\beta_n)P_n\big)\\
&= \phi\,\mathbb{D}^2\big((\mu(t)-\beta_n)P_n\big) + \psi\,\mathbb{S}\,\mathbb{D}\big((\mu(t)-\beta_n)P_n\big) - \lambda_{n+1}\big((\mu(t)-\beta_n)P_n\big)\\
&= \phi\,\mathbb{D}\Big((\mu(t){+}\frac{1}{4}-\beta_n)\mathbb{D}P_n + \mathbb{S}P_n\Big) + \psi\,\mathbb{S}\Big(\mu(t){+}\frac{1}{4}-\beta_n)\mathbb{D}P_n + \mathbb{S}P_n\Big)\\
&\qquad\qquad - \lambda_{n+1}\big((\mu(t)-\beta_n)P_n\big)\\
&= \phi\big((\mu(t)-\beta_n+1)\mathbb{D}^2P_n + 2\mathbb{S}\,\mathbb{D}P_n\big) + \psi\big(2(\mu(t)+\delta_x)\mathbb{D}^2P_n\\
&\qquad\qquad + (\mu(t)-\beta_n+1)\mathbb{S}\,\mathbb{D}P_n + P_n\big) - \lambda_{n+1}\big((\mu(t)-\beta_n)P_n\big)\\
&= (\mu(t)-\beta_n+1)\big(\phi\mathbb{D}^2P_n + \psi\mathbb{S}\,\mathbb{D}P_n\big) + 2\phi\mathbb{S}\,\mathbb{D}P_n\\
&\qquad\qquad + \psi\big(2(\mu(t)+\delta_x)\mathbb{D}^2P_n + P_n\big) - \lambda_{n+1}\big((\mu(t)-\beta_n)P_n\big)\\
&= (\mu(t)-\beta_n+1)\lambda_nP_n + 2\phi\mathbb{S}\,\mathbb{D}P_n + \psi\big(2(\mu(t)+\delta_x)\mathbb{D}^2P_n + P_n\big)\\
&\qquad\qquad - \lambda_{n+1}\big((\mu(t)-\beta_n)P_n\big)\\
&= 2\phi\mathbb{S}\,\mathbb{D}P_n + \psi P_n + (\mu(t)-\beta_n)\lambda_nP_n - \lambda_{n+1}(\mu(t)-\beta_n)P_n\\
&\qquad\qquad + \lambda_nP_n + 2\psi(\mu(t)+\delta_x)\mathbb{D}^2P_n\\
&= 2\phi\mathbb{S}\,\mathbb{D}P_n + \psi P_n + (\lambda_n-\lambda_{n+1})(\mu(t)-\beta_n)P_n + \lambda_nP_n + 2\psi(\mu(t)+\delta_x)\mathbb{D}^2P_n.
\end{aligned}$$

As a consequence,

$$
\begin{aligned}
&\mathrm{L}_{n-1}\left(U_n(\mu(t)\right)\\
&=\mathrm{L}_{n-1}\left[2\phi\mathbb{S}\,\mathbb{D}P_n+\psi P_n+(\lambda_n-\lambda_{n+1})(\mu(t)-\beta_n)P_n+\lambda_nP_n+2m_2(t)\,\psi\mathbb{D}^2P_n\right].
\end{aligned}
$$

We shall now obtain a number of properties which shall be used in the proof. First,

$$
\begin{aligned}
\mathrm{L}_{n+1}&\left[(\lambda_n-\lambda_{n+1})(\mu(t)-\beta_n)P_n\right]\\
&=2(\lambda_n-\lambda_{n+1})\phi\mathbb{S}\,\mathbb{D}P_n+(\lambda_n-\lambda_{n+1})(\lambda_n-\lambda_{n-1})(\mu(t)-\beta_n)P_n\\
&+(\lambda_n-\lambda_{n+1})\psi P_n+\lambda_n(\lambda_n-\lambda_{n+1})P_n+2(\lambda_n-\lambda_{n+1})m_2(t)\,\psi\,\mathbb{D}^2P_n. \quad (45)
\end{aligned}
$$

Moreover,

$$
\mathrm{L}_{n-1}\left[\lambda_nP_n\right]=\lambda_n(\lambda_n-\lambda_{n-1})P_n. \quad (46)
$$

We shall also need the following relations:

$$
\begin{aligned}
\mathbb{S}^2\mathbb{D}^2P_n&=m_2(t)\mathbb{D}^4P_n+m_1\,\mathbb{S}\mathbb{D}^3P_n+\mathbb{D}^2P_n,\\
\mathbb{D}\mathbb{S}\mathbb{D}^2P_n&=\mathbb{S}\mathbb{D}^3P_n+m_1\,\mathbb{D}^4P_n,\\
\mathbb{D}^2\mathbb{S}\mathbb{D}P_n&=\mathbb{D}\mathbb{S}\mathbb{D}^2P_n+m_1\,\mathbb{D}^4P_n=\mathbb{S}\mathbb{D}^3P_n+2m_1\,\mathbb{D}^4P_n,\\
\mathbb{S}\mathbb{D}\mathbb{S}\mathbb{D}P_n&=\mathbb{S}^2\mathbb{D}^2P_n+m_1\,\mathbb{S}\mathbb{D}^3P_n=m_2(t)\mathbb{D}^4P_n+2m_1\,\mathbb{S}\mathbb{D}^3P_n+\mathbb{D}^2P_n,\\
\mathbb{D}\mathbb{S}^2\mathbb{D}P_n&=\mathbb{S}\mathbb{D}\mathbb{S}\mathbb{D}P_n+m_1\,\mathbb{D}^2\mathbb{S}\mathbb{D}P_n=(m_2(t)+2m_1^2)\,\mathbb{D}^4P_n+3m_1\,\mathbb{S}\mathbb{D}^3P_n+\mathbb{D}^2P_n,\\
\mathbb{S}^3\mathbb{D}P_n&=m_2(t)\,\mathbb{D}^2\mathbb{S}\mathbb{D}P_n+m_1\,\mathbb{S}\mathbb{D}\mathbb{S}\mathbb{D}P_n+\mathbb{S}\mathbb{D}P_n\\
&=(m_2(t)+2m_1^2)\,\mathbb{S}\mathbb{D}^3P_n+3m_1m_2(t)\,\mathbb{D}^4P_n+m_1\mathbb{D}^2P_n+\mathbb{S}\mathbb{D}P_n.
\end{aligned}
$$

Also,

$$
\begin{aligned}
\mathbb{D}^2\left[\phi\mathbb{S}\,\mathbb{D}P_n\right]&=\mathbb{D}^2(\phi)\,\mathbb{D}P_n+\mathbb{S}\mathbb{D}(\phi)\,\mathbb{D}\mathbb{S}^2\mathbb{D}P_n+\mathbb{D}\mathbb{S}(\phi)\,\mathbb{S}\mathbb{D}\mathbb{S}\mathbb{D}P_n+\mathbb{S}^2(\phi)\,\mathbb{D}^2\mathbb{S}\mathbb{D}P_n\\
&=\mathbb{D}^2(\phi)\left[(m_2(t)+2m_1^2)\,\mathbb{S}\mathbb{D}^3P_n+3m_1m_2(t)\,\mathbb{D}^4P_n+m_1\mathbb{D}^2P_n+\mathbb{S}\mathbb{D}P_n\right]\\
&+\mathbb{S}\mathbb{D}(\phi)\left[(m_2(t)+2m_1^2)\,\mathbb{D}^4P_n+3m_1\,\mathbb{S}\mathbb{D}^3P_n+\mathbb{D}^2P_n\right]\\
&+\mathbb{D}\mathbb{S}(\phi)\left[m_2(t)\mathbb{D}^4P_n+2m_1\,\mathbb{S}\mathbb{D}^3P_n+\mathbb{D}^2P_n\right]+\mathbb{S}^2(\phi)\left[\mathbb{S}\mathbb{D}^3P_n+2m_1\,\mathbb{D}^4P_n\right]\\
&=\left[3m_1m_2(t)\,\mathbb{D}^2(\phi)+m_2(t)\,\mathbb{S}\mathbb{D}(\phi)+2m_1^2\mathbb{S}\mathbb{D}(\phi)+m_2(t)\mathbb{D}\mathbb{S}(\phi)+2m_1\mathbb{S}^2(\phi)\right]\mathbb{D}^4P_n\\
&+\left[m_2(t)\,\mathbb{D}^2(\phi)+2m_1^2\,\mathbb{D}^2(\phi)+3m_1\mathbb{S}\mathbb{D}(\phi)+2m_1\mathbb{D}\mathbb{S}(\phi)+\mathbb{S}^2(\phi)\right]\mathbb{S}\mathbb{D}^3P_n\\
&+\left[m_1\,\mathbb{D}^2(\phi)+\mathbb{S}\mathbb{D}(\phi)+\mathbb{D}\mathbb{S}(\phi)\right]\mathbb{D}^2P_n+\mathbb{D}^2(\phi)\,\mathbb{S}\mathbb{D}P_n.
\end{aligned}
$$

Note that

$$\begin{aligned}\mathbb{D}^2\big[\phi\,\mathbb{D}^2P_n\big] &= \mathbb{D}^2(\phi)\,\mathbb{S}^2\mathbb{D}^2P_n + \mathbb{SD}(\phi)\,\mathbb{DSD}^2P_n + \mathbb{DS}(\phi)\,\mathbb{SD}^3P_n + \mathbb{S}^2(\phi)\,\mathbb{D}^4P_n\\ &= \big[m_2(t)\,\mathbb{D}^2(\phi) + m_1\,\mathbb{SD}(\phi) + \mathbb{S}^2(\phi)\big]\,\mathbb{D}^4P_n\\ &\quad + \big[m_1\,\mathbb{D}^2(\phi) + \mathbb{SD}(\phi) + \mathbb{DS}(\phi)\big]\,\mathbb{SD}^3P_n + \mathbb{D}^2(\phi)\,\mathbb{D}^2P_n,\end{aligned}$$

as well as

$$\begin{aligned}\mathbb{SD}\big[\phi\,\mathbb{D}^2P_n\big] &= \mathbb{SD}(\phi)\,\mathbb{S}^2\mathbb{D}^2P_n + \mathbb{S}^2(\phi)\,\mathbb{SD}^3P_n + m_2(t)\,\big[\mathbb{D}^2(\phi)\,\mathbb{DSD}^2P_n\\ &\quad + \mathbb{DS}(\phi)\,\mathbb{D}^4P_n\big] = \big[m_2(t)\,\mathbb{SD}(\phi) + m_1m_2(t)\,\mathbb{D}^2(\phi) + m_2(t)\,\mathbb{DS}(\phi)\big]\,\mathbb{D}^4P_n\\ &\quad + \big[m_1\,\mathbb{SD}(\phi) + \mathbb{S}^2(\phi) + m_2(t)\,\mathbb{D}^2(\phi)\big]\,\mathbb{SD}^3P_n + \mathbb{SD}(\phi)\,\mathbb{D}^2P_n.\end{aligned}$$

As a consequence,

$$\begin{aligned}\mathbb{D}^2\big[\phi\mathbb{S}\,\mathbb{D}P_n\big] &= 2m_1\,\mathbb{D}^2\big[\phi\,\mathbb{D}^2P_n\big] + \mathbb{SD}\big[\phi\,\mathbb{D}^2P_n\big] + \big[m_1\,\mathbb{D}^2(\phi)\\ &\quad + \mathbb{SD}(\phi) + \mathbb{DS}(\phi)\big]\,\mathbb{D}^2P_n + \mathbb{D}^2(\phi)\,\mathbb{SD}P_n - 2m_1\,\mathbb{D}^2(\phi)\,\mathbb{D}^2P_n - \mathbb{SD}(\phi)\,\mathbb{D}^2P_n\\ &= 2m_1\,\mathbb{D}^2\big[\phi\,\mathbb{D}^2P_n\big] + \mathbb{SD}\big[\phi\,\mathbb{D}^2P_n\big] + \mathbb{SD}(\phi)\,\mathbb{D}^2P_n + \mathbb{D}^2(\phi)\,\mathbb{SD}P_n.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathbb{D}^2\big[2\phi\mathbb{S}\,\mathbb{D}P_n\big] &= 4m_1\,\mathbb{D}^2\big[\phi\,\mathbb{D}^2P_n\big]\\ &\quad + 2\mathbb{SD}\big[\phi\,\mathbb{D}^2P_n\big] + 2\mathbb{SD}(\phi)\,\mathbb{D}^2P_n + 2\mathbb{D}^2(\phi)\,\mathbb{SD}P_n.\end{aligned} \tag{47}$$

Furthermore,

$$\begin{aligned}\mathbb{D}^2\big[\psi\,m_2(t)\,\mathbb{D}^2P_n\big] &= \mathbb{D}^2\big[\psi\,m_2(t)\big]\,\mathbb{S}^2\mathbb{D}^2P_n + \mathbb{SD}\big[\psi\,m_2(t)\big]\,\mathbb{DSD}^2P_n\\ &\quad + \mathbb{DS}\big[\psi\,m_2(t)\big]\,\mathbb{SD}^3P_n + \mathbb{S}^2\big[\psi\,m_2(t)\big]\,\mathbb{D}^4P_n.\end{aligned}$$

We have

$$\begin{aligned}\mathbb{D}^2\big[\psi\,m_2(t)\big] &= 4m_1\,\mathbb{SD}(\psi),\\ \mathbb{SD}\big[\psi\,m_2(t)\big] &= m_2(t)\,\mathbb{SD}(\psi) + 2m_1^2\,\mathbb{SD}(\psi) + 2m_1\,\mathbb{S}^2(\psi),\\ \mathbb{DS}\big[\psi\,m_2(t)\big] &= m_2(t)\,\mathbb{SD}(\psi) + 6m_1^2\,\mathbb{SD}(\psi) + 2m_1\,\mathbb{S}^2(\psi),\\ \mathbb{S}^2\big[\psi\,m_2(t)\big] &= 4m_1m_2(t)\,\mathbb{SD}(\psi) + 2m_1^3\,\mathbb{SD}(\psi) + (2m_1^2 + m_2(t)\,\mathbb{S}^2(\psi).\end{aligned}$$

Then,

$$
\begin{aligned}
\mathbb{D}^2\big[\psi\, m_2(t)\,\mathbb{D}^2 P_n\big] &= 4m_1\,\mathbb{S}\mathbb{D}(\psi)\,\big[m_2(t)\mathbb{D}^4 P_n + m_1\,\mathbb{S}\mathbb{D}^3 P_n + \mathbb{D}^2 P_n\big] \\
&\quad + \big[m_2(t)\,\mathbb{S}\mathbb{D}(\psi) + 2m_1^2\,\mathbb{S}\mathbb{D}(\psi) + 2m_1\,\mathbb{S}^2(\psi)\big]\big[\mathbb{S}\mathbb{D}^3 P_n + m_1\,\mathbb{D}^4 P_n\big] \\
&\quad + \big[m_2(t)\,\mathbb{S}\mathbb{D}(\psi) + 6m_1^2\,\mathbb{S}\mathbb{D}(\psi) + 2m_1\,\mathbb{S}^2(\psi)\big]\,\mathbb{S}\mathbb{D}^3 P_n \\
&\quad + \big[4m_1 m_2(t)\,\mathbb{S}\mathbb{D}(\psi) + 2m_1^3\,\mathbb{S}\mathbb{D}(\psi) + (2m_1^2 + m_2(t))\,\mathbb{S}^2(\psi)\big]\,\mathbb{D}^4 P_n \\
&= \big[9m_1 m_2(t)\,\mathbb{S}\mathbb{D}(\psi) + 4m_1^3\,\mathbb{S}\mathbb{D}(\psi) + 4m_1^2\,\mathbb{S}^2(\psi) + m_2(t)\,\mathbb{S}^2(\psi)\big]\,\mathbb{D}^4 P_n \\
&\quad + \big[12m_1^2\,\mathbb{S}\mathbb{D}(\psi) + 2m_2(t)\,\mathbb{S}\mathbb{D}(\psi) + 4m_1\,\mathbb{S}\mathbb{D}(\psi) + 4m_1\,\mathbb{S}^2(\psi)\big]\,\mathbb{S}\mathbb{D}^3 P_n \\
&\quad + 4m_1\,\mathbb{S}\mathbb{D}(\psi)\,\mathbb{D}^2 P_n.
\end{aligned}
$$

Notice that

$$
\begin{aligned}
\mathbb{D}^2\big[\psi\,\mathbb{S}\mathbb{D}P_n\big] &= \mathbb{D}^2(\psi)\,\mathbb{S}^2\mathbb{D}^2 P_n + \mathbb{S}\mathbb{D}(\psi)\,\mathbb{D}\mathbb{S}^2\mathbb{D}P_n + \mathbb{D}\mathbb{S}(\psi)\,\mathbb{S}\mathbb{D}\mathbb{S}\mathbb{D}P_n \\
&\quad + \mathbb{S}^2(\psi)\,\mathbb{D}^2\mathbb{S}\mathbb{D}P_n = \mathbb{S}\mathbb{D}(\psi)\,\big[(m_2(t) + 2m_1^2)\,\mathbb{D}^4 P_n + 3m_1\,\mathbb{S}\mathbb{D}^3 P_n + \mathbb{D}^2 P_n\big] \\
&\quad + \mathbb{D}\mathbb{S}(\psi)\,\big[m_2(t)\mathbb{D}^4 P_n + 2m_1\,\mathbb{S}\mathbb{D}^3 P_n + \mathbb{D}^2 P_n\big] + \mathbb{S}^2(\psi)\,\big[\mathbb{S}\mathbb{D}^3 P_n + 2m_1\,\mathbb{D}^4 P_n\big] \\
&= \big[2m_2(t)\,\mathbb{S}\mathbb{D}(\psi) + 2m_1^2\,\mathbb{S}\mathbb{D}(\psi) + 2m_1\,\mathbb{S}^2(\psi)\big]\,\mathbb{D}^4 P_n \\
&\quad + \big[5m_1\,\mathbb{S}\mathbb{D}(\psi) + \mathbb{S}^2(\psi)\big]\,\mathbb{S}\mathbb{D}^3 P_n + 2\mathbb{S}\mathbb{D}(\psi)\,\mathbb{D}^2 P_n,
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbb{S}\mathbb{D}\big[\psi\,\mathbb{S}\mathbb{D}P_n\big] &= \mathbb{S}\mathbb{D}(\psi)\,\mathbb{S}^3\mathbb{D}P_n + \mathbb{S}^2(\psi)\,\mathbb{S}\mathbb{D}\mathbb{S}\mathbb{D}P_n \\
&\quad + m_2(t)\big[\mathbb{D}^2(\psi)\,\mathbb{D}\mathbb{S}^2\mathbb{D}P_n + \mathbb{D}\mathbb{S}(\psi)\,\mathbb{D}^2\mathbb{S}\mathbb{D}P_n\big] \\
&= \mathbb{S}\mathbb{D}(\psi)\,\big[(m_2(t) + 2m_1^2)\,\mathbb{S}\mathbb{D}^3 P_n + 3m_1 m_2(t)\,\mathbb{D}^4 P_n + m_1\mathbb{D}^2 P_n + \mathbb{S}\mathbb{D}P_n\big] \\
&\quad + \mathbb{S}^2(\psi)\,\big[m_2(t)\,\mathbb{D}^4 P_n + 2m_1\,\mathbb{S}\mathbb{D}^3 P_n + \mathbb{D}^2 P_n\big] + m_2(t)\,\mathbb{D}\mathbb{S}(\psi)\,\big[\mathbb{S}\mathbb{D}^3 P_n + 2m_1\,\mathbb{D}^4 P_n\big] \\
&= \big[5m_1 m_2(t)\,\mathbb{S}\mathbb{D}(\psi) + m_2(t)\,\mathbb{S}^2(\psi)\big]\,\mathbb{D}^4 P_n \\
&\quad + \big[2m_2(t)\,\mathbb{S}\mathbb{D}(\psi) + 2m_1^2\,\mathbb{S}\mathbb{D}(\psi) + 2m_1\,\mathbb{S}^2(\psi)\big]\,\mathbb{S}\mathbb{D}^3 P_n \\
&\quad + \big[m_1\,\mathbb{S}\mathbb{D}(\psi) + \mathbb{S}^2(\psi)\big]\,\mathbb{D}^2 P_n + \mathbb{S}\mathbb{D}(\psi)\,\mathbb{S}\mathbb{D}P_n.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\mathbb{D}^2\big[\psi\, m_2(t)\,\mathbb{D}^2 P_n\big] &= 2m_1\,\mathbb{D}^2\big[\psi\,\mathbb{S}\mathbb{D}P_n\big] + \mathbb{S}\mathbb{D}\big[\psi\,\mathbb{S}\mathbb{D}P_n\big] \\
&\quad - \big[m_1\,\mathbb{S}\mathbb{D}(\psi)\,\mathbb{D}^2 P_n + \mathbb{S}^2(\psi)\,\mathbb{D}^2 P_n\big] - \mathbb{S}\mathbb{D}(\psi)\,\mathbb{S}\mathbb{D}P_n.
\end{aligned}
$$

We also have $\mathbb{D}^2\big[\psi\, P_n\big] = 2\mathbb{S}\mathbb{D}(\psi)\,\mathbb{S}\mathbb{D}P_n + \big[m_1\,\mathbb{S}\mathbb{D}(\psi) + \mathbb{S}^2(\psi)\big]\,\mathbb{D}^2 P_n$, and so

$$\begin{aligned}&2\mathbb{D}^2\big[\psi\, m_2(t)\,\mathbb{D}^2 P_n\big] + \mathbb{D}^2\big[\psi\, P_n\big]\\ &\quad = 4m_1\,\mathbb{D}^2\big[\psi\,\mathbb{S}\mathbb{D}P_n\big] + 2\mathbb{S}\mathbb{D}\big[\psi\,\mathbb{S}\mathbb{D}P_n\big] - m_1\,\mathbb{S}\mathbb{D}(\psi)\,\mathbb{D}^2 P_n - \mathbb{S}^2(\psi)\,\mathbb{D}^2 P_n. \quad (48)\end{aligned}$$

From (47) and (48) it yields

$$\begin{aligned}\mathbb{D}^2\big[2\phi\mathbb{S}\,\mathbb{D}P_n + 2\psi\, m_2(t)\,\mathbb{D}^2 P_n + \psi\, P_n\big] &= 4m_1\,\mathbb{D}^2\big[\phi\,\mathbb{D}^2 P_n\big] + 2\mathbb{S}\mathbb{D}\big[\phi\,\mathbb{D}^2 P_n\big]\\ + 2\mathbb{S}\mathbb{D}(\phi)\,\mathbb{D}^2 P_n + 2\mathbb{D}^2(\phi)\,\mathbb{S}\mathbb{D}P_n &+ 4m_1\,\mathbb{D}^2\big[\psi\,\mathbb{S}\mathbb{D}P_n\big] + 2\mathbb{S}\mathbb{D}\big[\psi\,\mathbb{S}\mathbb{D}P_n\big]\\ - m_1\,\mathbb{S}\mathbb{D}(\psi)\,\mathbb{D}^2 P_n - \mathbb{S}^2(\psi)\,\mathbb{D}^2 P_n &= 4m_1\,\mathbb{D}^2\big[\lambda_n P_n\big] + 2\mathbb{S}\mathbb{D}\big[\lambda_n P_n\big] + 2\mathbb{S}\mathbb{D}(\phi)\,\mathbb{D}^2 P_n\\ &+ 2\mathbb{D}^2(\phi)\,\mathbb{S}\mathbb{D}P_n - m_1\,\mathbb{S}\mathbb{D}(\psi)\,\mathbb{D}^2 P_n - \mathbb{S}^2(\psi)\,\mathbb{D}^2 P_n.\end{aligned}$$

So

$$\begin{aligned}\mathbb{D}^2\big[2\phi\mathbb{S}\,\mathbb{D}P_n + 2\psi\, m_2(t)\,\mathbb{D}^2 P_n + \psi\, P_n\big] &= 4m_1\lambda_n\,\mathbb{D}^2 P_n + 2\lambda_n\,\mathbb{S}\mathbb{D}P_n\\ + 2\mathbb{S}\mathbb{D}(\phi)\,\mathbb{D}^2 P_n + 2\mathbb{D}^2(\phi)\,\mathbb{S}\mathbb{D}P_n &- m_1\,\mathbb{S}\mathbb{D}(\psi)\,\mathbb{D}^2 P_n - \mathbb{S}^2(\psi)\,\mathbb{D}^2 P_n. \quad (49)\end{aligned}$$

Moreover, we successively get,

$$\begin{aligned}&\mathbb{S}\mathbb{D}\big[\phi\,\mathbb{S}\mathbb{D}P_n\big]\\ &= \mathbb{S}\mathbb{D}(\phi)\,\mathbb{S}^3\mathbb{D}P_n + \mathbb{S}^2(\phi)\,\mathbb{S}\mathbb{D}\mathbb{S}\mathbb{D}P_n + m_2(t)\big[\mathbb{D}^2(\phi)\,\mathbb{D}\mathbb{S}^2\mathbb{D}P_n + \mathbb{D}\mathbb{S}(\phi)\mathbb{D}^2\mathbb{S}\mathbb{D}P_n\big]\\ &= \mathbb{S}\mathbb{D}(\phi)\,\big[(m_2(t) + 2m_1^2)\,\mathbb{S}\mathbb{D}^3 P_n + 3m_1 m_2(t)\,\mathbb{D}^4 P_n + m_1\mathbb{D}^2 P_n + \mathbb{S}\mathbb{D}P_n\big]\\ &+ \mathbb{S}^2(\phi)\,\big[m_2(t)\mathbb{D}^4 P_n + 2m_1\,\mathbb{S}\mathbb{D}^3 P_n + \mathbb{D}^2 P_n\big] + m_2(t)\,\mathbb{D}\mathbb{S}(\phi)\,\big[\mathbb{S}\mathbb{D}^3 P_n + 2m_1\,\mathbb{D}^4 P_n\big]\\ &+ m_2(t)\,\mathbb{D}^2(\phi)\,\big[(m_2(t) + 2m_1^2)\,\mathbb{D}^4 P_n + 3m_1\,\mathbb{S}\mathbb{D}^3 P_n + \mathbb{D}^2 P_n\big]\\ &= \big[3m_1 m_2(t)\,\mathbb{S}\mathbb{D}(\phi) + m_2(t)\,\mathbb{S}^2(\phi) + m_2^2(t)\,\mathbb{D}^2(\phi) + 2m_1^2 m_2(t)\,\mathbb{D}^2(\phi)\\ &+ 2m_1 m_2(t)\,\mathbb{D}\mathbb{S}(\phi)\big]\,\mathbb{D}^4 P_n + \big[m_2(t)\,\mathbb{S}\mathbb{D}(\phi) + 2m_1^2\,\mathbb{S}\mathbb{D}(\phi) + 2m_1\,\mathbb{S}^2(\phi)\\ &+ 3m_1 m_2(t)\,\mathbb{D}^2(\phi) + m_2(t)\,\mathbb{D}\mathbb{S}(\phi)\big]\mathbb{S}\mathbb{D}^3 P_n + \big[m_1\,\mathbb{S}\mathbb{D}(\phi) + \mathbb{S}^2(\phi) + m_2(t)\,\mathbb{D}^2(\phi)\big]\,\mathbb{D}^2 P_n\\ &+ \mathbb{S}\mathbb{D}(\phi)\,\mathbb{S}\mathbb{D}P_n = \big[m_2(t)\,\big(m_1\,\mathbb{S}\mathbb{D}(\phi) + m_2(t)\,\mathbb{D}^2(\phi) + \mathbb{S}^2(\phi)\big)\\ &+ 2m_1\,\big(m_2(t)\,\mathbb{S}\mathbb{D}(\phi) + m_1 m_2(t)\,\mathbb{D}^2(\phi) + m_2(t)\,\mathbb{D}\mathbb{S}(\phi)\big)\big]\,\mathbb{D}^4 P_n\\ &+ \big[m_2(t)\,\big(\mathbb{S}\mathbb{D}(\phi) + \mathbb{D}\mathbb{S}(\phi) + m_1\,\mathbb{D}^2(\phi)\big) + 2m_1\,\big(m_2(t)\,\mathbb{D}^2(\phi) + m_1\,\mathbb{S}\mathbb{D}(\phi)\\ &+ \mathbb{S}^2(\phi)\big)\big]\,\mathbb{S}\mathbb{D}^3 P_n + \big[m_1\,\mathbb{S}\mathbb{D}(\phi) + \mathbb{S}^2(\phi) + m_2(t)\,\mathbb{D}^2(\phi)\big]\,\mathbb{D}^2 P_n + \mathbb{S}\mathbb{D}(\phi)\,\mathbb{S}\mathbb{D}P_n\\ &= m_2(t)\,\mathbb{D}^2\big[\phi\,\mathbb{D}^2 P_n\big] + 2m_1\,\mathbb{S}\mathbb{D}\big[\phi\,\mathbb{D}^2 P_n\big] - m_2(t)\,\mathbb{D}^2(\phi)\,\mathbb{D}^2 P_n - 2m_1\,\mathbb{S}\mathbb{D}(\phi)\,\mathbb{D}^2 P_n\\ &+ \big[m_1\,\mathbb{S}\mathbb{D}(\phi) + \mathbb{S}^2(\phi) + m_2(t)\,\mathbb{D}^2(\phi)\big]\,\mathbb{D}^2 P_n + \mathbb{S}\mathbb{D}(\phi)\,\mathbb{S}\mathbb{D}P_n.\end{aligned}$$

Thus,

$$\begin{aligned}\mathbb{SD}\big[\phi\,\mathbb{SD}P_n\big] &= m_2(t)\,\mathbb{D}^2\big[\phi\,\mathbb{D}^2P_n\big] + 2m_1\,\mathbb{SD}\big[\phi\,\mathbb{D}^2P_n\big] \\ &\quad + \big[m_2(t)\,\mathbb{D}^2(\phi) + \phi\big]\,\mathbb{D}^2P_n + \mathbb{SD}(\phi)\,\mathbb{SD}P_n. \qquad (50)\end{aligned}$$

Also,

$$\begin{aligned}\mathbb{SD}\big[\psi\, m_2(t)\,\mathbb{D}^2P_n\big] &= \mathbb{SD}(\psi\, m_2(t))\,\mathbb{S}^2\mathbb{D}^2P_n + \mathbb{S}^2(\psi\, m_2(t))\,\mathbb{SD}^3P_n \\ &\quad + m_2(t)\,\mathbb{D}^2(\psi\, m_2(t))\,\mathbb{DSD}^2P_n + m_2(t)\,\mathbb{DS}(\psi\, m_2(t))\,\mathbb{D}^4P_n \\ &= \big[m_2(t)\,\mathbb{SD}(\psi) + 2m_1^2\,\mathbb{SD}(\psi) + 2m_1\,\mathbb{S}^2(\psi)\big]\big[m_2(t)\mathbb{D}^4P_n + m_1\,\mathbb{SD}^3P_n + \mathbb{D}^2P_n\big] \\ &\quad + \big[4m_1m_2(t)\,\mathbb{SD}(\psi) + 2m_1^3\,\mathbb{SD}(\psi) + (2m_1^2 + m_2(t))\,\mathbb{S}^2(\psi)\big]\,\mathbb{SD}^3P_n \\ &\quad + 4m_1m_2(t)\,\mathbb{SD}(\psi)\,\big[\mathbb{SD}^3P_n + m_1\,\mathbb{D}^4P_n\big] \\ &\quad + m_2(t)\big[m_2(t)\,\mathbb{SD}(\psi) + 6m_1^2\,\mathbb{SD}(\psi) + 2m_1\,\mathbb{S}^2(\psi)\big]\,\mathbb{D}^4P_n \\ &= \big[2m_2^2(t)\,\mathbb{SD}(\psi) + 12m_1^2m_2(t)\,\mathbb{SD}(\psi) + 4m_1m_2(t)\,\mathbb{S}^2(\psi)\big]\,\mathbb{D}^4P_n \\ &\quad + \big[9m_1m_2(t)\,\mathbb{SD}(\psi) + 4m_1^3\,\mathbb{SD}(\psi) + 4m_1^2\,\mathbb{S}^2(\psi) + m_2(t)\,\mathbb{S}^2(\psi)\big]\,\mathbb{SD}^3P_n \\ &\quad + \big[m_2(t)\,\mathbb{SD}(\psi) + 2m_1^2\,\mathbb{SD}(\psi) + 2m_1\,\mathbb{S}^2(\psi)\big]\,\mathbb{D}^2P_n,\end{aligned}$$

and

$$\begin{aligned}\mathbb{SD}\big[\psi\, P_n\big] &= \mathbb{SD}(\psi)\,\mathbb{S}^2P_n + \mathbb{S}^2(\psi)\,\mathbb{SD}P_n + m_2(t)\,\mathbb{DS}(\psi)\,\mathbb{D}^2P_n \\ &= 2m_2(t)\,\mathbb{SD}(\psi)\,\mathbb{D}^2P_n + m_1\,\mathbb{SD}(\psi)\,\mathbb{SD}P_n + \mathbb{S}^2(\psi)\,\mathbb{SD}P_n + \mathbb{SD}(\psi)\,P_n.\end{aligned}$$

As a consequence,

$$\begin{aligned}\mathbb{SD}\big[\psi\, m_2(t)\,\mathbb{D}^2P_n\big] + \mathbb{SD}\big[\psi\, P_n\big] &= m_2(t)\,\mathbb{D}^2\big[\psi\,\mathbb{SD}P_n\big] + 2m_1\,\mathbb{SD}\big[\psi\,\mathbb{SD}P_n\big] \\ &\quad - m_1\,\mathbb{SD}(\psi)\,\mathbb{SD}P_n + m_2(t)\,\mathbb{SD}(\psi)\mathbb{D}^2P_n + \mathbb{S}^2(\psi)\,\mathbb{SD}P_n + \mathbb{SD}(\psi)\,P_n.\end{aligned}$$

Since $\mathbb{S}^2(\psi) = m_1\,\mathbb{SD}(\psi) + \psi$, we have

$$\begin{aligned}2\mathbb{SD}\big[\psi\, m_2(t)\,\mathbb{D}^2P_n\big] + \mathbb{SD}\big[\psi\, P_n\big] &= 2m_2(t)\,\mathbb{D}^2\big[\psi\,\mathbb{SD}P_n\big] \\ &\quad + 4m_1\,\mathbb{SD}\big[\psi\,\mathbb{SD}P_n\big] - 2m_1\,\mathbb{SD}(\psi)\,\mathbb{SD}P_n + \psi\,\mathbb{SD}P_n + \mathbb{SD}(\psi)\,P_n. \qquad (51)\end{aligned}$$

From (50) and (51) it yields

$$
\begin{aligned}
\mathbb{SD}\big[2\phi\,\mathbb{SD}P_n+2\psi\, m_2(t)\,\mathbb{D}^2P_n+\psi\, P_n\big] &= 2m_2(t)\,\mathbb{D}^2\big[\phi\,\mathbb{D}^2P_n\big]+4m_1\,\mathbb{SD}\big[\phi\,\mathbb{D}^2P_n\big]\\
&\quad + 2\big[m_2(t)\,\mathbb{D}^2(\phi)+\phi\big]\,\mathbb{D}^2P_n + 2\mathbb{SD}(\phi)\,\mathbb{SD}P_n + 2m_2(t)\,\mathbb{D}^2\big[\psi\,\mathbb{SD}P_n\big]\\
&\quad + 4m_1\,\mathbb{SD}\big[\psi\,\mathbb{SD}P_n\big] - 2m_1\,\mathbb{SD}(\psi)\,\mathbb{SD}P_n + \psi\,\mathbb{SD}P_n + \mathbb{SD}(\psi)\,P_n\\
&= 2m_2(t)\,\mathbb{D}^2\big[\lambda_nP_n\big]+4m_1\,\mathbb{SD}\big[\lambda_nP_n\big]+2\big[m_2(t)\,\mathbb{D}^2(\phi)+\phi\big]\,\mathbb{D}^2P_n+2\mathbb{SD}(\phi)\,\mathbb{SD}P_n\\
&\quad - 2m_1\,\mathbb{SD}(\psi)\,\mathbb{SD}P_n + \psi\,\mathbb{SD}P_n + \mathbb{SD}(\psi)\,P_n.
\end{aligned}
$$

Since $-m_1\,\mathbb{SD}(\psi)=\psi-\mathbb{S}^2(\psi)$, we have

$$
\begin{aligned}
&\mathbb{SD}\big[2\phi\,\mathbb{SD}P_n + 2\psi\, m_2(t)\,\mathbb{D}^2P_n + \psi\, P_n\big]\\
&\quad = 2m_2(t)\,\lambda_n\,\mathbb{D}^2P_n + 4m_1\,\lambda_n\,\mathbb{SD}P_n + 2m_2(t)\,\mathbb{D}^2(\phi)\,\mathbb{D}^2P_n + 2\lambda_n\,P_n\\
&\qquad + 2\mathbb{SD}(\phi)\,\mathbb{SD}P_n - m_1\,\mathbb{SD}(\psi)\,\mathbb{SD}P_n - \mathbb{S}^2(\psi)\,\mathbb{SD}P_n + \mathbb{SD}(\psi)\,P_n.
\end{aligned}
\tag{52}
$$

From (49) and (52) it yields

$$
\begin{aligned}
&\phi\,\mathbb{D}^2\big[2\phi\mathbb{S}\,\mathbb{D}P_n+2\psi\, m_2(t)\,\mathbb{D}^2P_n+\psi\, P_n\big]+\psi\,\mathbb{SD}\big[2\phi\,\mathbb{SD}P_n+2\psi\, m_2(t)\,\mathbb{D}^2P_n+\psi\ \ P_n\big]\\
&= 4m_1\lambda_n\phi\,\mathbb{D}^2P_n+2\lambda_n\phi\,\mathbb{SD}P_n+2\phi\,\mathbb{SD}(\phi)\,\mathbb{D}^2P_n+2\phi\,\mathbb{D}^2(\phi)\,\mathbb{SD}P_n-m_1\phi\,\mathbb{SD}(\psi)\,\mathbb{D}^2P_n\\
&\quad - \phi\,\mathbb{S}^2(\psi)\,\mathbb{D}^2P_n + 2m_2(t)\,\lambda_n\psi\,\mathbb{D}^2P_n + 4m_1\lambda_n\psi\,\mathbb{SD}P_n + 2m_2(t)\psi\,\mathbb{D}^2(\phi)\,\mathbb{D}^2P_n\\
&\quad +2\lambda_n\psi\, P_n+2\psi\,\mathbb{SD}(\phi)\,\mathbb{SD}P_n-m_1\psi\,\mathbb{SD}(\psi)\,\mathbb{SD}P_n-\psi\,\mathbb{S}^2(\psi)\,\mathbb{SD}P_n+\psi\,\mathbb{SD}(\psi)\,P_n\\
&= 4m_1\lambda_n\big[\phi\,\mathbb{D}^2P_n + \psi\,\mathbb{SD}P_n\big] + 2\lambda_n\phi\,\mathbb{SD}P_n + 2\mathbb{SD}(\phi)\big[\phi\,\mathbb{D}^2P_n + \psi\,\mathbb{SD}P_n\big]\\
&\quad + 2\phi\,\mathbb{D}^2(\phi)\,\mathbb{SD}P_n - m_1\mathbb{SD}(\psi)\big[\phi\,\mathbb{D}^2P_n + \psi\,\mathbb{SD}P_n\big] - \mathbb{S}^2(\psi)\big[\phi\,\mathbb{D}^2P_n + \psi\,\mathbb{SD}P_n\big]\\
&\quad + 2m_2(t)\,\lambda_n\psi\,\mathbb{D}^2P_n + 2\lambda_n\psi\, P_n + 2m_2(t)\psi\,\mathbb{D}^2(\phi)\,\mathbb{D}^2P_n + \psi\,\mathbb{SD}(\psi)\,P_n\\
&= \big[4m_1\,\lambda_n^2 + 2\lambda_n\,\mathbb{SD}(\phi) - m_1\,\lambda_n\,\mathbb{SD}(\psi) - \lambda_n\,\mathbb{S}^2(\psi) + 2\lambda_n\,\psi + \psi\,\mathbb{SD}(\psi)\big]\,P_n\\
&\quad + \big[2\lambda_n + 2\mathbb{D}^2(\phi)\big]\,\phi\,\mathbb{SD}P_n + 2\big[\lambda_n + \mathbb{D}^2(\phi)\big]\,m_2(t)\psi\,\mathbb{D}^2P_n.
\end{aligned}
$$

So,

$$
\begin{aligned}
&\mathrm{L}_{n-1}\left[2\phi\mathbb{S}\,\mathbb{D}P_n + 2\psi\, m_2(t)\,\mathbb{D}^2P_n + \psi\, P_n\right]\\
&= \big[4m_1\,\lambda_n^2+2\lambda_n\,\mathbb{SD}(\phi)-m_1\,\lambda_n\,\mathbb{SD}(\psi)-\lambda_n\,\mathbb{S}^2(\psi)+2\lambda_n\,\psi+\psi\,\mathbb{SD}(\psi)-\lambda_{n-1}\,\psi\big]\,P_n\\
&\quad + \big[2\lambda_n + 2\mathbb{D}^2(\phi) - 2\lambda_{n-1}\big]\,\phi\,\mathbb{SD}P_n + \big[2\lambda_n + 2\mathbb{D}^2(\phi) - 2\lambda_{n-1}\big]\,m_2(t)\psi\,\mathbb{D}^2P_n.
\end{aligned}
$$

From (45) and (46) we have

$$\begin{aligned}\mathrm{L}_{n-1}\left(U_n(\mu(t)\right) = \big[&4m_1\,\lambda_n^2 + 2\lambda_n\,\mathbb{SD}(\phi) - m_1\,\lambda_n\,\mathbb{SD}(\psi) - \lambda_n\,\mathbb{S}^2(\psi) + 2\lambda_n\,\psi \\ &+ \psi\,\mathbb{SD}(\psi) - \lambda_{n-1}\,\psi + \lambda_n(\lambda_n - \lambda_{n-1}) + (\lambda_n - \lambda_{n+1})(\lambda_n - \lambda_{n-1})(\mu(t) - \beta_n) \\ &+ (\lambda_n - \lambda_{n+1})\psi\big]\,P_n + \big[2\lambda_n + 2\mathbb{D}^2(\phi) - 2\lambda_{n-1} + 2(\lambda_n - \lambda_{n+1})\big]\,\phi\,\mathbb{SD}P_n \\ &+ \big[2\lambda_n + 2\mathbb{D}^2(\phi) - 2\lambda_{n-1} + 2(\lambda_n - \lambda_{n+1})\big]\,m_2(t)\psi\,\mathbb{D}^2 P_n.\end{aligned}$$

Since $\lambda_{n+1} = (n+1)(a_0 n + b_0)$, we have $\lambda_n - \lambda_{n-1} = 2a_0 n - 2a_0 + b_0$, $\lambda_n - \lambda_{n+1} = -2a_0 n - b_0$, and $\lambda_n + \mathbb{D}^2(\phi) - \lambda_{n-1} + \lambda_n - \lambda_{n+1} = 0$. As a consequence,

$$\begin{aligned}\mathrm{L}_{n-1}\left(U_n(\mu(t)\right) = \big[&4m_1\,\lambda_n^2 + 2\lambda_n\,\mathbb{SD}(\phi) - m_1\,\lambda_n\,\mathbb{SD}(\psi) - \lambda_n\,\mathbb{S}^2(\psi) \\ &+2\lambda_n\,\psi + \psi\,\mathbb{SD}(\psi) - \lambda_{n-1}\,\psi + \lambda_n(\lambda_n - \lambda_{n-1}) + (\lambda_n - \lambda_{n+1})(\lambda_n - \lambda_{n-1})(\mu(t) - \beta_n) \\ &+ (\lambda_n - \lambda_{n+1})\psi\big]\,P_n,\end{aligned}$$

i.e. we have that that $\mathrm{L}_{n-1}\left(U_n(\mu(t)\right)$ is a polynomial of degree at least n. Since U_n is a polynomial of degree $n-1$ and the operator L_{n-1} keeps the degree of the polynomials, $\mathrm{L}_{n-1}\left(U_n(\mu(t)\right) = 0$. ■

Proof of Lemma 6: The action of L_{n+1} on (38) gives

$$0 = \mathrm{L}_{n+1}\left(P_{n+1}\right) = U_n(\mu(t)) - \mathrm{L}_{n+1}(\gamma_n P_{n-1}).$$

Then, as

$$\mathrm{L}_{n+1}\left(P_{n-1}\right) = \left(\lambda_{n-1} - \lambda_{n+1}\right) P_{n-1},$$

and by (37), we get (39). ■

Proof of theorem 6: In order to determine the coefficients β_n and γ_n in the three-term recurrence relation (8) in terms of the coefficients a_0, a_1, a_2, b_0, and b_1 of the polynomials ϕ and ψ given in (28) of the divided-difference operator L_n given in (31), by using (36) and (35) we have

$$\begin{aligned}\mathrm{L}_n\left(P_n(\mu(t))\right) &= \mathrm{L}_n\left(\vartheta_n(t)\right) + p_{1,n}\,\mathrm{L}_n\left(\vartheta_{n-1}(t)\right) + p_{2,n}\,\mathrm{L}_n\left(\vartheta_{n-2}(t)\right) + \cdots \\ &= \big[k_{0,n}\,\vartheta_n(t) + k_{1,n}\,\vartheta_{n-1}(t) + k_{2,n}\,\vartheta_{n-2}(t)\big] \\ &\quad + p_{1,n}\big[k_{0,n-1}\,\vartheta_{n-1}(t) + k_{1,n-1}\,\vartheta_{n-2}(t) + k_{2,n-1}\,\vartheta_{n-3}(t)\big] \\ &\quad + p_{2,n}\big[k_{0,n-2}\,\vartheta_{n-2}(t) + k_{1,n-2}\,\vartheta_{n-3}(t) + k_{2,n-2}\,\vartheta_{n-4}(t)\big] + \cdots \\ &= k_{0,n}\,\vartheta_n(t) + \big[k_{1,n} + p_{1,n}k_{0,n-1}\big]\,\vartheta_{n-1}(t) + \big[k_{2,n} + p_{1,n}k_{1,n-1} + p_{2,n}k_{0,n-2}\big]\,\vartheta_{n-2}(t) + \cdots .\end{aligned}$$

Since

$$k_{1,n} + p_{1,n}k_{0,n-1} = 0, \qquad k_{2,n} + p_{1,n}k_{1,n-1} + p_{2,n}k_{0,n-2} = 0,$$

we obtain (40) and (41). Moreover, from the second-order linear divided-difference equation we derive (42), (43), and (44), which completes the proof. ∎

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