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EPIREFLECTIVE SUBCATEGORIES VIA EPI-CLOSURE OPERATORS

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Abstract: On a category $\mathscr{C}$ with a designated (well-behaved) class $\mathcal{M}$ of monomorphisms, a closure operator in the sense of D. Dikranjan and E. Giuli is a pointed endofunctor of $\mathcal{M}$, seen as a full subcategory of the arrow-category $\mathscr{C}^{\mathbf{2}}$ whose objects are morphisms from the class $\mathcal{M}$, which "commutes" with the codomain functor $\mathsf{cod} : \mathcal{M} \to \mathscr{C}$. In other words, a closure operator consists of a functor $C : \mathcal{M} \to \mathcal{M}$ and a natural transformation $c : 1_{\mathcal{M}} \to C$ such that $\mathsf{cod} \cdot C = C$ and $\mathsf{cod} \cdot c = 1_{\mathsf{cod}}$. In this paper we adapt this notion to the domain functor $\mathsf{dom} : \mathcal{E} \to \mathscr{C}$, where $\mathcal{E}$ is a class of epimorphisms in $\mathscr{C}$, and show that such closure operators can be used to classify $\mathcal{E}$-epireflective subcategories of $\mathscr{C}$, provided $\mathcal{E}$ is closed under composition and contains isomorphisms. Specializing to the case when $\mathcal{E}$ is the class of regular epimorphisms in a regular category, we obtain known characterizations of regular-epireflective subcategories of general and various special types of regular categories, appearing in the works of the second author and his coauthors. These results show the interest in investigating further the notion of a closure operator relative to an arbitrary functor.

Keywords: category of morphisms, category of epimorphisms, category of monomorphisms, domain functor, cartesian lifting, closure operator, codomain functor, epimorphism, epireflective subcategory, form, monomorphism, normal category, pointed endofunctor, reflection, reflective subcategory, regular category, subobject, quotient.
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Introduction

A classical result in the theory of abelian categories describes the correspondence between the localizations of a locally finitely presentable abelian category $\mathscr{C}$ and the universal closure operators on subobjects in $\mathscr{C}$ (see [3] for instance). Several related investigations in non-abelian contexts have been carried out during the last decade by several authors [5, 6, 8, 9, 10, 14, 15].

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In non-abelian algebraic contexts such as groups, rings, crossed modules and topological groups, regular-epireflections are much more interesting than localizations: not only they occur more frequently but also they have strong connections with non-abelian homological algebra and commutator theory [14, 15, 16]. In particular, in the pointed context of homological categories [4], the regular-epireflective subcategories were shown to bijectively correspond to a special type of closure operators on normal subobjects [8]. An analogous result was established later on in the non-pointed regular framework using closure operators on effective equivalence relations [6].

By carefully examining these similar results, it appeared that the crucial idea underlying the connection between regular-epireflective subcategories and closure operators could be expressed via a suitable procedure of "closing quotients". Indeed, in the above mentioned situations, both normal subobjects and effective equivalence relations were "representations" of regular quotients. The regularity of the base category was there to guarantee the good behavior of quotients, and the additional exactness conditions only provided the faithfulness of the representation of quotients by normal subobjects/effective equivalence relations. This led to the present article where we generalize these results after introducing a general notion of a closure operator which captures both procedures — "closing subobjects" and "closing quotients".

We now briefly describe the main content of the article. In the first section we introduce an abstract notion of a closure operator on a functor that enables us to give a common and simplified treatment of all the situations mentioned above. In the second section, we then prove our most general result, Theorem 1, relating some closure operators on a specific (faithful) functor with $\mathcal{E}$-reflective subcategories, for a suitable class $\mathcal{E}$ of epimorphisms. In the last section, we make use of the concept of a form [21, 22] to explain how this work extends and refines the main results concerning closure operators on normal subobjects and on effective equivalence relations.

1. The notion of a closure operator on a functor

Definition 1. A *closure operator* on a functor $F : \mathscr{B} \to \mathscr{C}$ is an endofunctor $C : \mathscr{B} \to \mathscr{B}$ of $\mathscr{B}$ together with a natural transformation $c : 1_{\mathscr{B}} \to C$ such that

$$FC = F \text{ and } F \cdot c = 1_F.$$

A closure operator will be written as an ordered pair (C, c) of the data above.

This notion is a straightforward generalization of the notion of a categorical closure operator in the sense of D. Dikranjan and E. Giuli [11]. Let $\mathcal{M}$ be a class of monomorphisms in a category $\mathscr{C}$ satisfying the conditions stated in [11]. Viewing $\mathcal{M}$ as the full subcategory of the arrow-category $\mathbb{C}^{\mathbf{2}}$, closure operators on the codomain functor $\mathsf{cod} : \mathcal{M} \to \mathbb{C}$ are precisely the Dikranjan-Giuli closure operators. A similar statement is true for Dikranjan-Tholen closure operators, as defined in [13], which generalize Dikranjan-Giuli closure operators by simply relaxing conditions on the class $\mathcal{M}$ (see also [12] and [23] for intermediate generalizations). For Dikranjan-Tholen closure operators, the class $\mathcal{M}$ is an arbitrary class of morphisms containing isomorphisms and being closed under composition with them; the closure operators are then required to satisfy an additional assumption that each component of the natural transformation c is given by a morphism from the class $\mathcal{M}$ — since this requirement is not expressible for an abstract functor F, in our definition of a closure operator we are forced to drop it.

Let us remark that every pointed endofunctor $(C : \mathscr{B} \to \mathscr{B}, c\colon 1_{\mathscr{B}} \to C)$ of $\mathscr{B}$ can be viewed as a closure operator on the functor $\mathscr{B} \to \mathbf{1}$ where $\mathbf{1}$ is a single-morphism category.

In this paper we will be concerned with a different particular instance of the notion of a closure operator, where instead of a class of monomorphisms, we work with a class of epimorphisms, and instead of the codomain functor, we work with the domain functor $\mathsf{dom} : \mathcal{E} \to \mathscr{C}$. Let us remark that these are not the same as *dual closure operators* studied in [13]. In the latter case, the functor to consider is the dual of the domain functor $\mathsf{dom}^{\mathsf{op}} : \mathcal{E}^{\mathsf{op}} \to \mathscr{C}^{\mathsf{op}}$.

There seems to be four fundamental types of functors on which closure operators are of interest. Given a class $\mathcal{A}$ of morphisms in a category $\mathscr{C}$, regarding $\mathcal{A}$ as the full subcategory of the arrow-category of $\mathscr{C}$, these four types of functors are the domain and the codomain functors and their duals:

$$\begin{tikzcd} \mathcal{A} \arrow[d, "\mathsf{cod}"'] & \mathcal{A}^{\mathsf{op}} \arrow[d, "\mathsf{dom}^{\mathsf{op}}"] \\ \mathscr{C} & \mathscr{C}^{\mathsf{op}} \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{A}^{\mathsf{op}} \arrow[d, "\mathsf{cod}^{\mathsf{op}}"'] & \mathcal{A} \arrow[d, "\mathsf{dom}"] \\ \mathscr{C}^{\mathsf{op}} & \mathscr{C} \end{tikzcd}$$

Horizontally, we have *categorical duality*, i.e., dualizing the construction of the functor gives the other functor in the same row. Vertically, we have *functorial duality*: to get the other functor in the same column, simply take the dual of the functor. The effects of closure of a morphism from the class $\mathcal{A}$ in each of the above four cases are as follows:

$$
\begin{tikzcd}
 & \bullet \arrow[dd, "C(a)"] \\
\bullet \arrow[ur] \arrow[dr, "a"'] & \\
 & \bullet
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \bullet \arrow[dd] \\
\bullet \arrow[ur, "C(a)"] \arrow[dr, "a"'] & \\
 & \bullet
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & \bullet \arrow[dd, "a"] \\
\bullet \arrow[ur] \arrow[dr, "C(a)"'] & \\
 & \bullet
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \bullet \arrow[dd] \\
\bullet \arrow[ur, "a"] \arrow[dr, "C(a)"'] & \\
 & \bullet
\end{tikzcd}
$$

Note that the closure operators in the top row factorize a morphism a, while those in the bottom row present it as part of a factorization. This gives a principal difference between the categorical closure operators considered in the literature (which are of the kind displayed in the top row) and those that we consider in the present paper (which are of the kind displayed in the bottom row). Let us also remark that a closure operator on a poset in the classical sense can be viewed as a categorical closure operator of the bottom-right type, when we take $\mathcal{A}$ to be the class of all morphisms in the poset (and dually, the bottom-left type captures interior operators on a poset). For a poset, the two types of closure operators in the top row become the same and they give precisely the binary closure operators in the sense of A. Abdalla [1]. In a poset all morphisms are both monomorphisms and epimorphisms, and it is interesting that in general, closure operators in the left column seem to be of interest when $\mathcal{A} = \mathcal{M}$ is a class of monomorphisms, and closure operators in the right column seem to be of interest when $\mathcal{A} = \mathcal{E}$ is a class of epimorphisms. In both cases the functors down to the base category are faithful. Note that another way to capture the classical notion of a closure operator on a preorder is to say that it is just a closure operator on a faithful functor $\mathscr{B} \to \mathbf{1}$.

Closure operators on a given functor F constitute a category in the obvious way, where a morphism $n : (C, c) \to (C', c')$ is a natural transformation $n : C \to C'$ such that $n \circ c = c'$ (and consequently $F \cdot n = 1_F$; note that when F is a faithful functor, this last equality is equivalent to the former). We will denote this category by $\mathsf{Clo}(F)$.

For a faithful functor $F : \mathscr{B} \to \mathscr{C}$ from a category $\mathscr{B}$ to a category $\mathscr{C}$, an object A in a fibre $F^{-1}(X)$ of F will be represented by the display

$$\begin{tikzcd} A \arrow[d, dotted, no head] \\ X \end{tikzcd}$$

and a morphism $A \to B$ which lifts a morphism $f : X \to Y$ by the display

$$\begin{tikzcd} A \arrow[r] \arrow[d, dotted, no head] & B \arrow[d, dotted, no head] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

Note that since the functor F is faithful, it is not necessary to label the top arrow in the above display. We will also interpret this display as a statement that the morphism f lifts to a morphism $A \to B$. When it is not clear which functor F do we have in mind, we will label the above square with the relevant F, as shown below:

$$\begin{tikzcd} A \arrow[r] \arrow[d, dotted, no head] \arrow[dr, phantom, "F"] & B \arrow[d, dotted, no head] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

We write $A \leqslant B$ to mean

$$\begin{tikzcd} A \arrow[r] \arrow[d, dotted, no head] & B \arrow[d, dotted, no head] \\ X \arrow[r, "1_X"'] & X \end{tikzcd}$$

and $A \approx B$ when we also have $B \leqslant A$. In the latter case, we say that A and B are *fibre-isomorphic*, since $A \approx B$ is equivalent to the existence of an isomorphism $A \to B$ which lifts the identity morphism 1_X. The relation of fibre-isomorphism is an equivalence relation.

Given a faithful functor $F : \mathscr{B} \to \mathscr{C}$ and a morphism $f : X \to Y$ in $\mathscr{C}$, we will write fA for the codomain of a cocartesian lifting of f at A, when it exists. The universal property of the cocartesian lifting can be expressed as the law

$$\begin{tikzcd} A \arrow[r] \arrow[d, dotted, no head] & fA \arrow[r] \arrow[d, dotted, no head] & C \arrow[d, dotted, no head] \\ X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z \end{tikzcd} \quad \Leftrightarrow \quad \begin{tikzcd} A \arrow[r] \arrow[d, dotted, no head] & C \arrow[d, dotted, no head] \\ X \arrow[r, "gf"'] & Z \end{tikzcd}$$

More precisely, a cocartesian lifting of f is the same as a lifting of f satisfying the above equivalence. Dually, we write Cg for the domain of a cartesian lifting of g at C, when it exists, and it is defined by the law

$$\begin{tikzcd} A \arrow[r] \arrow[d, dotted, no head] & Cg \arrow[r] \arrow[d, dotted, no head] & C \arrow[d, dotted, no head] \\ X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z \end{tikzcd} \quad \Leftrightarrow \quad \begin{tikzcd} A \arrow[r] \arrow[d, dotted, no head] & C \arrow[d, dotted, no head] \\ X \arrow[r, "gf"'] & Z \end{tikzcd}$$

We say fA *is defined* when a cocartesian lifting of f at A exists, and dually, we say Cg *is defined* when the cartesian lifting of f at C exists (this notation is taken from [21, 22]). When fA and Cg are used in an equation/diagram, we interpret this equation to subsume the statement that fA and Cg, respectively, are defined.

Liftings of identity morphisms can be represented by vertical arrows: the display

$$\begin{tikzcd} A' \\ A \arrow[u] \arrow[d, dotted, no head] \\ X \end{tikzcd}$$

shows two objects A and A' in the fibre $F^{-1}(X)$, and a morphism $A \to A'$ which by F is mapped to the identity morphism 1_X.

In the case of a faithful functor F, the natural transformation c in the definition of a closure operator is unique, when it exists, so a closure operator can be specified just by the functor C. In fact, it can even be given by a family $(C_X)_{X \in \mathscr{C}}$ of maps

$$C_X : F^{-1}(X) \to F^{-1}(X), \quad A \mapsto \overline{A},$$

such that for any morphism $f : X \to Y$ in $\mathscr{C}$, we have the following law:

$$\begin{tikzcd} A \arrow[r] \arrow[d, dotted, no head] & B \arrow[d, dotted, no head] \\ X \arrow[r, "f"'] & Y \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} \overline{A} \arrow[r] & \overline{B} \\ A \arrow[r] \arrow[u] \arrow[d, dotted, no head] & B \arrow[u] \arrow[d, dotted, no head] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

When F is faithful, $\mathsf{Clo}(F)$ is a preorder with $C \leqslant C'$ whenever $C(A) \leqslant C'(A)$ for all $A \in \mathscr{B}$. Note that the underlying pointed endofunctor of a closure operator on a faithful functor is always well-pointed, i.e., $C \cdot c = c \cdot C$. We shall say that a closure operator on a faithful functor is *idempotent* when the underlying pointed endofunctor is idempotent, i.e., $C \cdot c = c \cdot C$ is an isomorphism or, equivalently, $CC \approx C$.

2. Closure operators for epireflective subcategories

Let $\mathcal{E}$ be a class of epimorphisms in a category $\mathscr{C}$. We can view $\mathcal{E}$ as a full subcategory of the category of morphisms in $\mathscr{C}$ (the so-called "arrow-category"), where objects are morphisms belonging to the class $\mathcal{E}$, and a morphism is a commutative square

$$\begin{tikzcd} A \arrow[r, "r"] & B \\ X \arrow[r, "f"'] \arrow[u, "d"] & Y \arrow[u, "e"'] \end{tikzcd}$$

where $d \in \mathcal{E}$ and $e \in \mathcal{E}$ are the domain and the codomain, respectively, of the morphism. Since every morphism in the class $\mathcal{E}$ is an epimorphism, the top morphism in the above square is uniquely determined by the rest of the square. In other words, the domain functor $\mathcal{E} \to \mathscr{C}$, which maps the above square to its base, is faithful. We will use the above square to represent what we would have written as

$$\begin{tikzcd} d \arrow[r] \arrow[d, dotted, no head] & e \arrow[d, dotted, no head] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

for this faithful functor. By an *epi-closure operator* we mean a closure operator on the domain functor $\mathcal{E} \to \mathscr{C}$. Similarly, by a *mono-closure operator* we mean a closure operator defined on the codomain functor $\mathcal{M} \to \mathscr{C}$, where $\mathcal{M}$ is a class of monomorphisms in $\mathscr{C}$. Depending on how well-behaved is the class $\mathcal{M}$, we get the notions of closure operators introduced and studied in [11, 23]. The classical example of such a closure operator is the so-called Kuratowski closure operator on the category of topological spaces, which is given by defining the closure of an embedding $m : M \to X$ to be the embedding of the topological closure of the image of m in X.

Let us assume that the class $\mathcal{E}$ is closed under composition and contains identity morphisms. When f is in $\mathcal{E}$, it is not difficult to see that a cartesian lifting for

$$\begin{tikzcd} ef \arrow[r] \arrow[d, dotted, no head] & e \arrow[d, dotted, no head] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

under the domain functor $\mathcal{E} \to \mathscr{C}$, can be given by the square

$$\begin{tikzcd} B \arrow[r, "1_B"] & B \\ X \arrow[u, "ef"] \arrow[r, "f"'] & Y \arrow[u, "e"'] \end{tikzcd}$$

so that our notation ef agrees with composition of morphisms. We call these *canonical cartesian liftings.*

Theorem 1. *Let $\mathcal{E}$ be a class of epimorphisms in a category $\mathscr{C}$ such that it contains isomorphisms and is closed under composition. There is a bijection between full $\mathcal{E}$-reflective subcategories of $\mathscr{C}$ and closure operators C on the domain functor $\mathcal{E} \to \mathscr{C}$ satisfying the following conditions:*

(a) *C is (strictly) idempotent, i.e., for every object $e \in \mathcal{E}$ we have $C(C(e)) = C(e)$ (equivalently, $CC = C$);*
(b) *C preserves canonical cartesian liftings of morphisms f from the class $\mathcal{E}$, i.e., we have*
$$C(e)f = C(ef)$$
for arbitrary composable arrows $e, f \in \mathcal{E}$.

Under this bijection, the subcategory corresponding to a closure operator consists of those objects X for which $1_X = C(1_X)$, and for each object Y of $\mathscr{C}$ the morphism $C(1_Y)$ gives a reflection of Y in the subcategory.

Proof: First, we show that the correspondence described at the end of the theorem gives a bijection between the objects of the poset and the preorder in question. Let $\mathscr{X}$ be a full $\mathcal{E}$-reflective subcategory of $\mathscr{C}$, with G denoting the subcategory inclusion $G : \mathscr{X} \to \mathscr{C}$. Consider a left adjoint $L : \mathscr{C} \to \mathscr{X}$ of G, and the unit η of the adjunction. Since G is a subcategory inclusion, each component of η is a morphism $\eta_X : X \to L(X)$. Without loss of generality we may assume that the counit of the adjunction is an identity natural transformation. Then, an object X of $\mathscr{C}$ belongs to the subcategory $\mathscr{X}$ if and only if $\eta_X = 1_X$. We have

$$\begin{tikzcd} A \arrow[r, "r"] & B \\ X \arrow[u, "d"] \arrow[r, "f"'] & Y \arrow[u, "e"'] \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} L(A) \arrow[r, "L(r)"] & L(B) \\ A \arrow[u, "\eta_A"] \arrow[r, "r"] & B \arrow[u, "\eta_B"'] \\ X \arrow[u, "d"] \arrow[r, "f"'] & Y \arrow[u, "e"'] \end{tikzcd}$$

and this means that we can define a closure operator on the domain functor $\mathcal{E} \to \mathscr{C}$ by setting $C(e) = \eta_{\mathsf{cod}(e)}e$. It is easy to see that both (a) and (b) hold for such closure operator C. At the same time, the full subcategory $\mathscr{X}$ of $\mathscr{C}$ can be recovered from the corresponding closure operator C as the full subcategory of those objects X for which $C(1_X) = 1_X$.

Given a closure operator C on the domain functor $\mathcal{E} \to \mathscr{C}$, satisfying (a) and (b), we consider the full subcategory $\mathscr{X}$ of those objects X in $\mathscr{C}$ such that $C(1_X) = 1_X$. Consider the composite L of the three functors

$$\begin{tikzcd} \mathcal{E} \arrow[r, "C"] & \mathcal{E} \arrow[d, "\mathsf{cod}"] \\ \mathscr{C} \arrow[u, "I"] \arrow[r, "L"'] & \mathscr{C} \end{tikzcd}$$

where I maps every morphism $f : X \to Y$ in $\mathscr{C}$ to the morphism

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ X \arrow[u, "1_X"] \arrow[r, "f"'] & Y \arrow[u, "1_Y"'] \end{tikzcd}$$

in the category $\mathcal{E}$, and cod is the codomain functor from $\mathcal{E}$ to $\mathscr{C}$. We claim that the values of L lie in the subcategory $\mathscr{X}$. Indeed, we have

$$C(1_{L(X)})C(1_X) \overset{(b)}{=} C(1_{L(X)}C(1_X)) = C(C(1_X)) \overset{(a)}{=} C(1_X) = 1_{L(X)}C(1_X)$$

and since $C(1_X)$ is an epimorphism, we get $C(1_{L(X)}) = 1_{L(X)}$. So we can consider L as a functor $L : \mathscr{C} \to \mathscr{X}$. It follows from the construction that this functor is a right inverse of the subcategory inclusion $\mathscr{X} \to \mathscr{C}$. Since each morphism $C(1_X) : X \to L(X)$ is an epimorphism, it is easy to see that L is a left adjoint of the subcategory inclusion $\mathscr{X} \to \mathscr{C}$, with the $C(1_X)$'s being the components of the unit of adjunction.

To complete the proof of the bijection, it remains to show that $C(e) = C(1_{\mathsf{cod}(e)})e$. This we have by (b). ■

In the case of the domain functors $\mathsf{dom} : \mathcal{E} \to \mathscr{C}$, where objects in $\mathcal{E}$ are epimorphisms in $\mathscr{C}$, including the identity morphisms, cocartesian lifts are given by pushouts:

$$\begin{tikzcd} Y \arrow[r] & Y +_X Z \\ X \arrow[u, "f"] \arrow[r, "g"'] & Z \arrow[u, "gf"'] \end{tikzcd}$$

Unlike in the case of cartesian liftings, there are in general no canonical cocartesian liftings.

Theorem 2. *Let $\mathscr{C}$ and $\mathcal{E}$ be the same as in Theorem 1. If for any two morphisms $f : X \to Y$ and $g : X \to Z$ from the class $\mathcal{E}$, their pushout exists and the pushout injections belong to the class $\mathcal{E}$, then the bijection of Theorem 1 restricts to a bijection between:*

(a) *Full $\mathcal{E}$-reflective subcategories $\mathscr{X}$ of $\mathscr{C}$ closed under $\mathcal{E}$-quotients, i.e., those having the property that for any morphism $f\colon X \to Y$ in the class $\mathcal{E}$ with X in $\mathscr{X}$, the object Y also belongs to $\mathscr{X}$.*

(b) *Closure operators as in Theorem 1 having the additional property that*

$$fC(e) \approx C(fe)$$

for any morphisms $f\colon X \to Y$ and $e\colon X \to E$ in the class $\mathcal{E}$, and moreover, when $e = C(e)$ we have $fe = C(fe)$.

Proof: Thanks to the bijection in Theorem 1, it suffices to show that for a closure operator as in Theorem 1, and the corresponding full $\mathcal{E}$-reflective subcategory $\mathscr{X}$ of $\mathscr{C}$ constructed in the proof of Theorem 1, the following are equivalent:

(i) $\mathscr{X}$ is closed in $\mathscr{C}$ under $\mathcal{E}$-quotients.

(ii) The property on the closure operator C given in (b).

Let L and η be the functor and the natural transformation that give the reflection of $\mathscr{C}$ in $\mathscr{X}$, as in the proof of Theorem 1. As before, we choose L and η in such a way that an object X of $\mathscr{C}$ lies in $\mathscr{X}$ if and only if $\eta_X = 1_X$.

(i)$\Rightarrow$(ii): Let f and e be as in (ii), and consider the morphism g arising in a pushout giving a cocartesian lift of f at e, as displayed in the bottom left square in the following diagram:

$$\begin{tikzcd}
L(E) \arrow[r] & L(E) +_X (E +_X Y) \arrow[r, "h"] & L(E +_X Y) \\
E \arrow[u, "\eta_E"'] \arrow[r, "g"] & E +_X Y \arrow[u, "g\eta_E"'] \arrow[r, "1_{E+_X Y}"'] & E +_X Y \arrow[u, "\eta_{E+_X Y}"'] \\
X \arrow[u, "e"] \arrow[r, "f"'] & Y \arrow[u, "fe"']
\end{tikzcd}$$

Since $\eta_{E+_X Y} g = L(g)\eta_E$, we get a morphism h making the above diagram commute. The top left morphism in this diagram belongs to the class $\mathcal{E}$, by the assumption on $\mathcal{E}$ given in the theorem, and so by (i), the object $L(E) +_X (E +_X Y)$ belongs to the subcategory $\mathscr{X}$. We can then use the universal property of $\eta_{E+_X Y}$ to deduce that h is an isomorphism. We then get

$$fC(e) = f(\eta_E e) \approx (g\eta_E)(fe) \approx \eta_{E+_X Y}(fe) = C(fe).$$

If $C(e) = e$, then E lies in $\mathscr{X}$, and so $E +_X Y$ also lies in $\mathscr{X}$ by (i). Then $fe = C(fe)$.

For (ii)$\Rightarrow$(i), simply take $e = 1_X$ in (b). ■

The next result shows how the preorder structure of closure operators is carried over to full $\mathcal{E}$-reflective subcategories, under the bijection given by Theorem 1.

Theorem 3. *Let $\mathcal{E}$ and $\mathscr{C}$ be as in Theorem 1. Consider two full $\mathcal{E}$-reflective subcategories $\mathscr{X}_1$ and $\mathscr{X}_2$ of $\mathscr{C}$, and the closure operators C_1 and C_2 corresponding to them under the bijection established in Theorem 1. Then $C_1 \leqslant C_2$ if and only if every object in $\mathscr{X}_2$ is isomorphic to some object in $\mathscr{X}_1$.*

Proof: When $C_1 \leqslant C_2$, for an object X of $\mathscr{C}$ such that $1_X = C_2(1_X)$, we have:

$$1_X \leqslant C_1(1_X) \leqslant C_2(1_X) = 1_X.$$

This implies that $C_1(1_X)$ is an isomorphism, and since it is a reflection of X in the subcategory $\mathscr{X}_1$, we have the morphism $C_1(1_X)$ witnessing the fact that X is isomorphic to an object in $\mathscr{X}_1$. Suppose now every object in $\mathscr{X}_2$ is isomorphic to some object in $\mathscr{X}_1$. Then, for any morphism $e : X \to E$ from the class $\mathcal{E}$, we have $C_i(e) = C_i(1_E)e$, $i \in \{1, 2\}$, so to prove $C_1 \leqslant C_2$, it suffices to show that $C_1(1_E) \leqslant C_2(1_E)$ for any object E in $\mathscr{C}$. Since $C_2(1_E)$ is a reflection of E in $\mathscr{X}_2$, its codomain lies in $\mathscr{X}_2$ and subsequently, it is isomorphic to an object lying in $\mathscr{X}_1$. Now, we can use the universal property of the reflection $C_1(1_E)$ of E in $\mathscr{X}_1$ to ensure $C_1(1_E) \leqslant C_2(1_E)$. ■

Let us now look at how the axioms on closure operators appearing in Theorems 1 and 2 are affected by isomorphism of closure operators:

Theorem 4. *Let $\mathscr{C}$ and $\mathcal{E}$ be as in Theorem 1. For a closure operator D on the domain functor* $\mathsf{dom} : \mathcal{E} \to \mathscr{C}$*, we have:*

(i) *D is isomorphic to a closure operator C satisfying 1(a) and 1(b) if and only if $DD \approx D$ and $D(e)f \approx D(ef)$ for arbitrary composable arrows $e, f \in \mathcal{E}$ (this last condition expresses preservation by D of cartesian liftings of morphisms from the class $\mathcal{E}$).*

If further $\mathcal{E}$ satisfies the premise in Theorem 2, then we have:

(ii) *D is isomorphic to a closure operator C satisfying the condition stated in 2(b) if and only if D satisfies the conditions stated in the second part of (i) and D preserves cocartesian liftings of morphisms from the class $\mathcal{E}$, i.e., $fD(e) \approx D(fe)$ for arbitrary morphisms $f : X \to Y$ and $e : X \to E$ in the class $\mathcal{E}$.*

Proof: We first prove the only if part in each of (i) and in (ii). Suppose a closure operator D is isomorphic to a closure operator C. If C satisfies 1(a), then

$$D(D(e)) \approx D(C(e)) \approx C(C(e)) = C(e) \approx D(e)$$

for any morphism e in the class $\mathcal{E}$. If C satisfies 1(b), then

$$D(e)f \approx C(e)f = C(ef) \approx D(ef)$$

for arbitrary composable arrows $e, f \in \mathcal{E}$. Suppose now $\mathcal{E}$ satisfies the premise in Theorem 2. If C satisfies the condition stated in 2(b), then

$$fD(e) \approx fC(e) \approx C(fe) \approx D(fe),$$

for arbitrary morphisms $f : X \to Y$ and $e : X \to E$ in the class $\mathcal{E}$.

We will now prove the "if" parts in (i) and (ii). Consider a closure operator D on the domain functor $\mathsf{dom}\colon \mathcal{E} \to \mathscr{C}$. Suppose D satisfies the conditions stated in the second part of (i). Then the values of the map defined by

$$C(e) = \begin{cases} e & \text{if } D(1_E) \text{ is an isomorphism,} \\ D(1_E)e & \text{otherwise.} \end{cases}$$

are fibre-isomorphic to the values of D, so this gives a closure operator C isomorphic to D. Furthermore, it is easy to see that we have

$$C(ef) = C(1_E)ef = C(e)f,$$

as required in 1(b). Since

$$D(e') \approx D(D(e')) \approx D(1_{E'})D(e'),$$

for any morphism $e' \in \mathcal{E}$, where E' denotes the codomain of $D(e')$, we get that $D(1_{E'})$ is an isomorphism. We will use this fact for $e' = 1_E$ in what follows. Let $e \in \mathcal{E}$ and let E be the codomain of e. Write E' for the codomain of $D(1_E)$. If $D(1_E)$ is an isomorphism, then we trivially have $C(C(e)) = C(e)$. Suppose $D(1_E)$ is not an isomorphism. Since $D(1_{E'})$ is an isomorphism, we have

$$C(C(e)) = C(D(1_E)e) = D(1_E)e = C(e).$$

This completes the proof of the if part in (i). For the if part in (ii) we still use the same C. Suppose D satisfies the condition stated in the second part of (ii). In view of Theorems 1 and 2, it suffices to prove that for any morphism $f\colon X \to Y$ from the class $\mathcal{E}$, if $1_X = C(1_X)$ then $1_Y = C(1_Y)$. Suppose $1_X = C(1_X)$. Then $1_X \approx D(1_X)$ and since 1_Y is the codomain of a cocartesian lifting of f at 1_X, we have

$$1_Y \approx fD(1_X) \approx D(1_Y),$$

which implies that $D(1_Y)$ is an isomorphism. Then $1_Y = C(1_Y)$. ■

Recall that a full subcategory $\mathscr{X}$ of a category $\mathscr{C}$ is said to be replete when it contains all objects which are isomorphic to objects already contained in $\mathscr{X}$. Recall from Section 1 that a closure operator C is idempotent when $CC \approx C$. The work in this section leads to the following:

Theorem 5. *Let $\mathcal{E}$ be a class of epimorphisms in a category $\mathscr{C}$ such that it contains isomorphisms and is closed under composition.*

(a) *There is a bijection between full $\mathcal{E}$-reflective replete subcategories of $\mathscr{C}$ and isomorphism classes of idempotent closure operators C on the domain functor $\mathsf{dom}\colon \mathcal{E} \to \mathscr{C}$ which preserve cartesian liftings of morphisms from the class $\mathcal{E}$.*

(b) *The bijection above is given by assigning to a closure operator C the subcategory of $\mathscr{C}$ consisting of those objects X for which $C(1_X)$ is an isomorphism, and $C(1_Y)$ gives a reflection of each object Y from $\mathscr{C}$ into the subcategory.*

(c) *When the class $\mathcal{E}$ is closed under pushouts, the bijection above restricts to one where the subcategories are closed under $\mathcal{E}$-quotients and the closure operators also preserve cocartesian liftings of morphisms from the class $\mathcal{E}$.*

(d) *Each of the bijections above gives an equivalence between the (possibly large) poset of subcategories in question, where the poset structure is given by inclusion of subcategories, and the dual of the preorder of closure operators in question.*

3. Closure operators on forms

Recall that a functor is said to be amnestic when in each of its fibres, the only isomorphisms are the identity morphisms. Faithful amnestic functors were called *forms* in [22]. Any faithful functor gives rise to a form by identifying in it the fibre-isomorphic objects. The original faithful functor F and the corresponding form F' are related by a commutative triangle

$$\begin{tikzcd} \mathscr{B} \arrow[rr, "Q"] \arrow[dr, "F"'] & & \mathscr{B}' \arrow[dl, "F'"] \\ & \mathscr{C} & \end{tikzcd}$$

Writing $[A]_{\approx}$ for the equivalence class of an object A in $\mathscr{B}$ under the equivalence relation of fibre-isomorphism, we have:

$$\begin{tikzcd} A \arrow[r] \arrow[d, dotted, no head] \arrow[dr, phantom, "F"] & B \arrow[d, dotted, no head] \\ X \arrow[r, "f"'] & Y \end{tikzcd} \quad \Leftrightarrow \quad \begin{tikzcd} {[A]_{\approx}} \arrow[r] \arrow[d, dotted, no head] \arrow[dr, phantom, "F'"] & {[B]_{\approx}} \arrow[d, dotted, no head] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

The functor Q is an equivalence of categories, which is surjective on objects. The above display shows what the values of Q are: a morphism in $\mathscr{B}$ that fits in the left hand side display above is mapped by Q to a morphism in $\mathscr{B}'$ fitting the right hand side display. The fibres of a form are (possibly large) posets, and so the the preorder of closure operators on a form is a poset. The functor Q gives rise to an equivalence of categories

$$\mathsf{Clo}(F) \approx \mathsf{Clo}(F').$$

Under this equivalence, the closure operator C' on the form F' associated to a closure operator C on F is obtained by setting $C'_X([B]_\approx) = [C_X(B)]_\approx$. Notice that since $\mathsf{Clo}(F')$ is a poset, two closure operators on F correspond to the same closure operator on the associated form F', under the above equivalence, if and only if they are isomorphic.

Forms associated to the domain functors $\mathsf{dom} : \mathcal{E} \to \mathscr{C}$ that we have been considering in this paper, were called *forms of* $\mathcal{E}$*-quotients* in [22]. Theorem 5 gives us the following:

Theorem 6. *Let $\mathcal{E}$ be a class of epimorphisms in a category $\mathscr{C}$ such that it contains isomorphisms and is closed under composition. There is an antitone isomorphism between the poset of full $\mathcal{E}$-reflective replete subcategories of $\mathscr{C}$ and the poset of idempotent closure operators on the form of $\mathcal{E}$-quotients which preserve cartesian liftings of morphisms from the class $\mathcal{E}$. It is given by assigning to a closure operator the subcategory of $\mathscr{C}$ consisting of those objects X of $\mathscr{C}$ for which the initial $\mathcal{E}$-quotient is closed. When the class $\mathcal{E}$ is closed under pushouts, this isomorphism restricts to one where the subcategories are closed under $\mathcal{E}$-quotients and the closure operators also preserve cocartesian liftings of morphisms from the class $\mathcal{E}$.*

As in [22], we call the form corresponding to the codomain functor $\mathcal{M} \to \mathscr{C}$, where $\mathcal{M}$ is a class of monomorphisms in a category $\mathscr{C}$, the *form of* $\mathcal{M}$*-subobjects*. A normal category in the sense of [20] is a regular category [2] which is pointed and in which every regular epimorphism is a normal epimorphism. In a normal category, for the class $\mathcal{E}$ of normal epimorphisms and the class $\mathcal{M}$ of normal monomorphisms, the form of $\mathcal{E}$-quotients is isomorphic to the form of $\mathcal{M}$-subobjects, via the usual kernel-cokernel correspondence between normal quotients and normal subobjects. Theorem 6 then gives:

Theorem 7. *There is an antitone isomorphism between the poset of full normal-epi-reflective replete subcategories of a normal category $\mathscr{C}$ and the* *poset of idempotent closure operators on the form of normal subobjects which preserve cartesian liftings of normal epimorphisms. It is given by assigning to a closure operator the subcategory of $\mathscr{C}$ consisting of those objects X of $\mathscr{C}$ for which the null subobject of X is closed. Furthermore, when pushouts of normal epimorphisms along normal epimorphisms exist, this isomorphism restricts to one where the subcategories are closed under normal quotients and the closure operators also preserve cocartesian liftings of normal epimorphisms.*

This recovers Theorem 2.4 and Proposition 3.4 from [8], and moreover, slightly generalizes and refines them. Let us explain this in more detail. First of all, we remark that an *idempotent closure operator on kernels* defined in [8] is the same as an idempotent closure operator in the sense of the present paper, on the form of normal subobjects. The context in which these closure operators are considered in [8] is that of a homological category [4], which is the same as a pointed regular protomodular category [7]. Theorem 2.4 in [8] establishes, for a homological category, a bijection between such closure operators and normal-epi-reflective subcategories (which in [8] are simply called epi-reflective subcategories). This bijection is precisely the one established by the first half of Theorem 7 above. As this theorem shows, the bijection is there more generally for any normal category (a homological category is in particular a normal category, but the converse is not true). The last part of Theorem 7 similarly captures Proposition 3.4 from [8] characterizing Birkhoff subcategories [17] of a semi-abelian category. Once again, it reveals a more general context where the result can be stated, and namely that of a normal category with pushouts of normal epimorphisms along normal epimorphisms in the place of a semi-abelian category [18]. Thus, in particular, the characterization remains valid in any ideal determined category [19].

For a category $\mathscr{C}$, consider the full subcategory $\mathscr{B}$ of the category of parallel pairs of morphisms in $\mathscr{C}$, consisting with those parallel pairs of morphisms which arise as kernel pairs of a morphism f (i.e., projections in a pullback of f with itself). Thus, a morphism in $\mathscr{B}$ is a diagram

$$\begin{tikzcd} R \arrow[r, "g"] \arrow[d, shift right, "r_1"'] \arrow[d, shift left, "r_2"] & S \arrow[d, shift right, "s_1"'] \arrow[d, shift left, "s_2"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

where (R, r_1, r_2) and (S, s_1, s_2) are kernel pairs, and we have

$$f \circ r_1 = s_1 \circ g \text{ and } f \circ r_2 = s_2 \circ g.$$

Assigning to the above diagram the base morphism f defines a (faithful) functor $\mathscr{B} \to \mathscr{C}$. The form corresponding to the functor will be called the *congruence form* of $\mathscr{C}$ (when $\mathscr{C}$ is a variety of universal algebras, its fibres are isomorphic to congruence lattices of algebras). For a regular category, the congruence form is isomorphic to the form of regular quotients, and Theorem 6 can be rephrased as follows:

Theorem 8. *There is an antitone isomorphism between the poset of full regular-epi-reflective replete subcategories of a regular category $\mathscr{C}$ and the poset of idempotent closure operators on the congruence form which preserve cartesian liftings of regular epimorphisms. It is given by assigning to a closure operator the subcategory of $\mathscr{C}$ consisting of those objects X of $\mathscr{C}$ for which the smallest congruence on X is closed. Furthermore, when pushouts of regular epimorphisms along regular epimorphisms exist, this isomorphism restricts to one where the subcategories are closed under regular quotients and the closure operators also preserve cocartesian liftings of regular epimorphisms.*

The first part of the theorem above recovers Theorem 2.3 from [6]. Idempotent closure operators on the congruence form of a regular category are the same as idempotent closure operators on effective equivalence relations in the sense of [6]. The condition of preservation of cartesian liftings of regular epimorphisms defines precisely the effective closure operators in the sense of [6]. The last part of the above theorem includes Proposition 3.6 from [6] as a particular case.

Finally, let us remark that Theorem 7 can be deduced already from Theorem 8, since for a normal category the form of normal subobjects is isomorphic to the congruence form.

Applying Theorem 8 in the case when $\mathscr{C}$ is a variety of universal algebras, the first part of the theorem gives a characterization of quasi-varieties of algebras in the variety, and the second part — subvarieties of the variety.

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