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POLYNOMIAL INEQUALITIES AND THE NONNEGATIVITY OF THE COEFFICIENTS OF CERTAIN POWER SERIES

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Abstract: A natural and surprisingly often successful method to prove the positive semidefiniteness of multivariate polynomials $p(\underline{x}) \in \mathbb{R}[\underline{x}] = \mathbb{R}[x_1, \ldots, x_n]$ in subsets of $\mathbb{R}^n$ of the form $x_1 \geq x_2 \geq \cdots \geq x_n \geq r$, is to introduce new variables $h_i = x_i - x_{i+1}$, $i = 1, ..., n$, $x_{n+1} = r$, then to express the x_i via the h_i, and to show that the polynomial p in the h_i has only nonnegative coefficients. We show how such representations can be obtained systematically using partial derivatives. This method frequently lessens computational complexity and enhances insight considerably. It allows us to establish, for example, that for real nonnegative $\alpha_1, \ldots, \alpha_k$ of sum ≤ 1, the coefficients of the t^n, $n \geq 1$, of $(1-x_1t)^{\alpha_1}(1-x_2t)^{\alpha_2}\cdots(1-x_kt)^{\alpha_k}$, when developed into a power series, are nonpositive for nonnegative x_i. Similar (at the moment partial) results for power series coming from the harmonic mean of $1 - x_1t, ..., 1 - x_kt$ are established. These results were found by Laffey and Holland via more problem specific treatment. They used them in connection with the nonnegative inverse eigenvalue problem.

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0. Introduction

Assume $p \in \mathbb{R}[X_1, ..., X_n]$ is a real polynomial for which we wish to establish an inequality $p(\underline{x}) \geq 0$ for all $\underline{x} \in D$, D a certain subset in $\mathbb{R}^n$. An approach frequently leading to success is to cover D by polyhedral cones of the form $C_\pi = \{\underline{x} : x_{\pi 1} \geq x_{\pi 2} \geq \cdots \geq x_{\pi n} \geq r_\pi\}$, π a permutation of $\{1, 2, ..., n\}$, – which for want of a better terminology will be called simply *monotone cones* – and to prove the inequality on each such subset individually by introducing $h_i = x_{\pi i} - x_{\pi(i+1)}$, $h_n = x_{\pi n} - r_\pi$. Thus in effect one writes $x_{\pi i} = \sum_{l=i}^{n} h_i + r_\pi$, and verifies that p, written in terms of the h_i, is a polynomial having only nonnegative coefficients. With a few apparent exceptions, we shall assume all $r_\pi = 0$ and will say in such a case that p satisfies property pos on C_π;

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and it satisfies $-$pos if $-p$ is pos . Of course exploiting symmetries can often considerably reduce the necessary computational effort. We shall say that p satisfies pos if it satisfies pos on every cone C_π.

As an example consider the symmetric polynomial

$$p(x, y, z) = x^3 + y^3 + z^3 - 3xyz.$$

Introduce $h_1 = x - y, h_2 = y - z, h_3 = z$, and verify that

$$\begin{aligned} p(x, y, z) &= p(h_1 + h_2 + h_3, h_2 + h_3, h_1) \\ &= h_1^3 + 3h_1^2h_2 + 3h_1h_2^2 + 2h_2^3 + 3h_1^2h_3 + 3h_1h_2h_3 + 3h_2^2h_3. \end{aligned}$$

Since the coefficients are all nonnegative the inequality $p(x, y, z) \geq 0$ is established on $C_{\mathrm{id}} = \{(x, y, z) : x \geq y \geq z \geq 0\}$, and, by symmetry, everywhere on $\mathbb{R}^3_{\geq 0}$.

The mentioned elementary approach is perhaps advisable if nothing but an individual inequality with numerically known powers and coefficients is to be established. To show the relative ubiquity of property pos , in Section 1 we give a number of examples where the mentioned approach leads to success. We also give some codelines helpful for verifying property pos for nonsymmetric inequalities of this type and inform of other methods to prove individual polynomial inequalities.

The elementary method of substitution can be fearsome, though, if infinite families of inequalities, defined by a number of parameters are to be established. Even to prove the general arithmetic geometric (AG) mean inequality $\sum_{i=1}^n x_i^n - n \cdot x_1x_2 \cdots x_n \geq 0$ for all $n \in \mathbb{Z}_{\geq 0}$ and $\underline{x} \in \mathbb{R}^n_{\geq 0}$ in this direct way might make necessary some sophisticated combinatorial reasoning; more so the results on power series in Section 3.

In Section 2 we present a method using partial differentiation which turns such tasks often (not always) more manageable. It concentrates directly on computing the coefficients of expansions of polynomials in $h_1, h_2, ..., h_n$ as defined above.

In Section 3 we use the method to show that, given nonnegative α_i of sum ≤ 1, the power series

$$\sum_{n \geq 0} u_n(x_1, x_2, ..., x_k)t^n = (1 - x_1t)^{\alpha_1}(1 - x_2t)^{\alpha_2} \cdots (1 - x_kt)^{\alpha_k}$$

has polynomial coefficients $u_n = u_n(x_1, x_2, ..., x_k)$ for which for all $x_1, ..., x_k \geq 0$ there holds $u_n(x_1, ..., x_k) \leq 0$ whenever $n \geq 1$. In fact the nonconstant u_n are all $-$pos.

It was the special case $\alpha_1 = \cdots = \alpha_k = 1/k$ of this problem with which Professor Laffey challenged the author. This led to the method here proposed. Laffey himself had solved this problem in [L] and used it for getting information about necessary or sufficient conditions in order that a list of complex numbers represents the spectrum of a nonnegative matrix. Holland [Hol] observed that Laffey was actually looking at the geometric mean of $(1-x_1t), (1-x_2t), \cdots, (1-x_kt)$, and decided to investigate - also having the nonnegative inverse eigenvalue problem in sight - the coefficient polynomials that are obtained by looking at the harmonic mean of these quantities, developed as a power series. Here he could use some ideas of Kaluza [Kal] from the thirties. While we do not show Holland's coefficient inequalities in all generality we will see in Section 4 by means of examples that the proposed method allows with good reason to conjecture the stronger property that even nonsymmetric versions of these coefficient polynomials satisfy property $-$pos. We conclude in Section 5 by reporting further work in some aspects related to the present one and mentioning possible impact on determining the sign of certain recurrently defined sequences.

Some of our formulas are lengthy. So concerning notation we had to strike a balance between mnemonicity, precision, coherence, and lightness. We hope to have found an acceptable compromise.

1. Useful code, examples of properties pos and not pos , and other methods for proving polynomial inequalities.

By an individual polynomial we mean an expression that can be written down in the language of first order logic with function symbols $\cdot, +$, and constant symbols of usual interpretation in $\mathbb{Q}$. Thus $x^2 - 2xy + 5$ is an individual polynomial, but an informal claim like 'we have the polynomial inequality $x^{2n} + y^{2n} \geq 0$' is not a claim about an individual polynomial inequality but it is a claim about all members of an infinite class of first order expressions (one for each particular n).

In this section we give a few code lines which the reader might use to ascertain the claims about property pos for individual polynomial inequalities. We then inform briefly about other methods and theorems to establish such inequalities.

Since there seems not to exist any special merit in using the theoretical developments in Section 2 to base on them a code for automatization of this process, unless, perhaps, one wishes to examine extremely large polynomials, we content ourselves here with simple suggestions. The experiments one can make with this code indicate that the family of homogeneous polynomial inequalities that have property `pos` on certain simplicial cones seems to be quite large within the class of all homogeneous polynomial inequalities valid on such cones.

If one wishes to test a single given polynomial p in x, y, z, on the region $y \geq x \geq z \geq t$, say, then a quick way is to input the polynomial; write y, x, z as expressions in h_1, h_2, h_3; and output p. A Mathematica line for this is:

```
p=...; {y,x,z}={h1+h2+h3+t,h2+h3+t,h3+t};  p  .
```

If, however, one has many variables or has to test many regions because the inequality has not much symmetry, after inputting p, one may with advantage use the following lines.

```
0.  vars=Variables[p]; n=Length[vars];
1.  lsh = {};
    For[j = 1, j <= n, j++, AppendTo[ lsh, Subscript[h, j] ] ];
2.  ss=0;rpslsh={};
    For[j=n,j>=1,j--,ss=ss+lsh[[j]];PrependTo[rpslsh, ss];];
    rpslsh=rpslsh+t;
```

The meaning of line 0 is clear; line 1 produces the list of hs, `lsh`$= \{h_1, h_2, ..., h_n\}$; line 2 produces from this the list of its 'right partial sums plus t', i.e. `rpslsh`$= \{h_1 + \cdots + h_n + t, ..., h_{n-1} + h_n + t, h_n + t\}$.

After this preprocessing one may conveniently work by executing the following lines.

```
3. pi={...};  rules={};
4. For[j=1,j<=n,j++, AppendTo[rules,vars[[pi[[j]]]]->rpslsh[[j]]]];
5. p2=p/.rules//Expand
6. out=Apply[List,p2]/.{Subscript[h,_]->1}; Min[out]
```

In line 3, a user defining a permutation `pi` of the numbers $1, 2, ..., n$, orders the variables according to $x_{\pi 1}, x_{\pi 2}, ...$. The list `rules` produced in line 4 is used in line 5 to replace the variable $x_{\pi j}$ of p by $h_j + \cdots + h_n + t$, for $j = 1, ..., n$. Outputting this p yields the polynomial expanded in the h_i. This output may be too long for eye-examination of coefficients. One may wish to suppress its printing (terminating line 5 with a '`;`'), and execute line

6 instead: this produces from p the list of terms, puts all h_j in it equal to 1 so that only the list of coefficients survives in `out`. If the the minimum of these is nonnegative - this may depend on the value of `t` - one has the desired certificate of positivity on the region $x_{\pi 1} \geq \cdots \geq x_{\pi n} \geq t$.

Example 1.1. Let `p=x*y^2 - x*z + 2y*z^2.` Assume one wants some information about whether this polynomial is positive for $x, y, z \geq 0.5$? For this run lines 4,5,6 for the various permutations of x, y, z, always with $t = 0.5$. With `pi={1,2,3}` one gets a polynomial of form $0.125 - 0.25h_1 + \cdots$ without monomials h_1^k for $k \geq 2$. It shows that if $h_2 = h_3 = 0$, that is $y = z = 0.5$, but $x > 0.5 + (0.125/0.25) = 1$, the polynomial will assume negative values. At the other hand with `pi={3,1,2},` one gets a polynomial in the h_i with only positive coefficients. Consequently, on the set $z \geq x \geq y \geq 0.5$, the polynomial assumes only positive values.

Examples 1.2. After putting them in into the standard form $p(x_1, \ldots, x_n) \geq 0$, each of the following inequalities is confirmed by means of the above lines within seconds since the associated polynomials all have the property `pos` (on all nonnegative monotone cones).

i. For $a, b, c \geq 0$: $(a^2b + ab^2 + a^2c + ac^2 + b^2c + bc^2) \leq (a^3 + b^3 + c^3 + 3abc)$.

ii. For $x, y, z \geq 0$: $3xyz \leq x^2y + y^2z + z^2x$.

iii. For $a, b, c \geq 0$:

$$a^6b^3 + a^3b^6 - 2a^5b^2c^2 - 2a^2b^5c^2 + a^6c^3 + b^6c^3 - 2a^2b^2c^5 + a^3c^6 + b^3c^6 \geq 0.$$

This unwieldy but symmetric inequality comes from trying to prove

$$\left(1 + \frac{a}{b}\right)\left(1 + \frac{b}{c}\right)\left(1 + \frac{c}{a}\right) \geq 2\left(1 + \frac{a + b + c}{(a\,b\,c)^{\frac{1}{3}}}\right),$$

by putting everything over a common denominator $(abc)^{4/3}$ and inspecting the numerator of lhs-rhs, replacing a, b, c respectively by a^3, b^3, c^3 and dividing by abc.

iv. For $a, b, c, d \geq 0$: $abcd(a + b + c + d) \leq a^4b + b^4c + c^4d + d^4a$.

While the inequalities i and iii are symmetric under the group of permutations of a, b, c, the inequality ii is only invariant under the group of cyclic permutations of x, y, z and needs therefore be tested for orders $x > y > z$ and $y > x > z$, say. Inequality iv is only invariant under cyclic permutations of a, b, c, d, and needs to be tested under the order $a > b > c > d$ and, say, the six reorderings obtained herefrom by interchanging b, c, d.

v. For $x, y \geq 0$: $x^5 + x^4y - 2x^3y^2 - 2x^2y^3 + xy^4 + y^5 \geq 0$.

Now define the following polynomials all occurring with similar notation (in capitals) in [CLR2].

$$\begin{array}{rcl} d(x,y,z) & = & (x-y)^2(y-z)^2(z-x)^2; \\ l(x,y,z) & = & xyz - (y+z-x)(z+x-y)(x+y-z); \\ s(x,y,z) & = & l(x^2,y^2,z^2). \\ 2f(x,y,z) & = & (y+z-x)^2(z+x-y)^2(x+y-z)^2- \\ & & (y^2+z^2-x^2)(z^2+x^2-y^2)(x^2+y^2-z^2), \\ g(x,y,z) & = & 2(x^4(y-z)^2+y^4(x-z)^2+z^4(x-y)^2) - d(x,y,z). \end{array}$$

The following inequalities are then all valid since the associated standard form polynomials are all pos on all nonnegative monotone cones.

vi. For $x, y, z \geq 0: \quad s(x,y,z) \geq 8d(x,y,z)$.
vii. For $x, y, z \geq 0: \quad f(x,y,z) \geq 4d(x,y,z)$.
viii. For $x, y, z \geq 0: g(x,y,z) \geq d(x,y,z)$.
ix. For $x, y, z \geq 0:$

$$x^4(x-y)(x-z) + y^4(y-x)(y-z) + z^4(z-x)(z-y) \geq 5d(x,y,z).$$

The inequality iii above is a special case of a well known inequality of Muirhead which we will now show to follow from property pos of a suitable polynomial. Given a real nonnegative decreasing n-tuple a, $a_1 \geq a_2 \geq ... \geq a_n$, define as in [HLP] the symmetric polynomial

$$[a] = [a](x_1, ..., x_n) = \sum_{\sigma \in S_n} x_{\sigma 1}^{a_1} \cdots x_{\sigma n}^{a_n}.$$

Muirhead's inequality says that if two nonnegative decreasing real n-tuples a, b are such that the left partial sums of $a - b$ are nonnegative and the sum of its entries is 0 - a fact usually expressed by writing $a \succeq b$ and saying that *a majorizes* b - then one has for any $\underline{x} \in \mathbb{R}^n_{\geq 0}$ the inequality $[a] - [b] \geq 0$. Various proofs of Muirhead's inequality are known. It is a direct consequence of the following fact.

Theorem 1.1. *If a, b as above are integer, then the (symmetric) Muirhead polynomial $[a] - [b]$ is* pos .

Proof: We will show that substituting without loss of generality $x_i = h_i + \cdots + h_n$, $i = 1, ..., n$, we transform $[a] - [b]$ into a polynomial in the h_i with nonnegative coefficients. Parts of the proof follow closely ideas in Frenkel and Horvath [FH], where it is shown that under the conditions given, $([a]-[b])(x_1^2,\cdots,x_n^2)$ is a sum of squares (but we shall not use this fact as such). We write $a\cdot\succ b$ to say that there are indices $k<l$ such that $a_k=1+b_k>b_k\geq b_l>a_l=-1+b_l$, while for all $i\neq k,l$, $b_i=a_i$. In [HLP] it is shown that $a\succ b$ implies that there is a finite chain of decreasing nonnegative n-tuples a_i so that $a=a_0\cdot\succ a_1\cdot\succ a_2\cdot\succ\cdots\cdot\succ a_k=b$. Since $[a]-[b]=\sum_{i=0}^{k-1}([a_i]-[a_{i+1}])$, it is enough to show our claim under the additional hypothesis $a\cdot\succ b$. Also assume indices k,l be chosen like above. Now consider for $\sigma\in S_n$ the expression

$$*_1\colon\,(x_{\sigma k}^{a_k}x_{\sigma l}^{a_l}+x_{\sigma k}^{a_l}x_{\sigma l}^{a_k}-x_{\sigma k}^{b_k}x_{\sigma l}^{b_l}-x_{\sigma k}^{b_l}x_{\sigma l}^{b_k})\prod_{i\neq k,l}x_{\sigma i}^{e_i},$$

where $e_i:=b_i=a_i$. Note that such an expression is symmetric w.r.t interchanging k and l and the sum of all such expressions as σ ranges over S_n just gives $2([a]-[b])$. Now to simplify notation, write $x=x_{\sigma_l}$, and assume, without loss of generality, $x_{\sigma_k}=x+h$. Also let $b'=b_k, b=b_l$. Then the parenthesized expression is

$$\begin{aligned}(x+h)^{b'+1}\,x^{b-1}+(x+h)^{b-1}x^{b'+1}-(x+h)^{b'}x^{b}-(x+h)^{b}x^{b'}&\\ = x^{b-1}(x+h)^{b-1}\cdot h((x+h)^{b'-b+1}-x^{b'-b+1})&\\ = x^{b-1}(x+h)^{b-1}\textstyle\sum_{\nu\geq 1}\binom{b'-b+1}{\nu}h^{1+\nu}x^{b'-b+1-\nu}.&\end{aligned}$$

Now assume each x_i as written at the beginning. Then in particular $x=x_{\sigma l}=h_{\sigma l}+h_{1+\sigma_l}+\cdots+h_n$, $x_{\sigma k}=h_{\sigma k}+\cdots+h_n$, and $h=h_{\sigma k}+\cdots+h_{1+\sigma l}$. Thus it is obvious that expression $*_1$ above is a nonnegative linear combination in the h_is; and so, hence, $([a]-[b])(x_1,...,x_n)$ will be. ■

However, it is not surprising that we have:

Proposition 1.2. *In the class of homogeneous polynomials, to possess the property* pos *on some monotone cone is a genuinely stronger property than to be merely nonnegative on the cone.*

Proof: The Motzkin polynomial $m(x,y,z)=x^3+y^2z+yz^2-3xyz$ is nonnegative on $\mathbb{R}_{\geq 0}$ - use the AG inequality for $t_1=\sqrt[3]{x^3}, t_2=\sqrt[3]{y^2z}, t_3=\sqrt[3]{yz^2}$, to see this - but on the monotone cone $\{(x,y,z)\in\mathbb{R}^3_{\geq 0}: z\geq x\geq y\}$ it is not pos . Rather, substituting $z=h_1+h_2+h_3, x=h_2+h_3, y=h_3$, one gets $m=h_2^3+h_1^2h_3-h_1h_2h_3+h_2^2h_3$. ■

The Motzkin polynomial of 1967 is famous for the fact that $m(x^2,y^2,z^2)$ has been the first explicit example of a polynomial with the property to be positive semidefinite but not a sum of squares. The above mentioned sextic $s(x, y, z)$, discovered by R.M. Robinson in 1969, was the first symmetric psd form with this property [CLR1, p.561]. This shows that property `pos` is not too closely connected to the property of being sos.

Apart of clever tricks and the standard methods of differential calculus for minimization one has nowadays a number of other tools and theorems at one's disposal to prove individual polynomial inequalities.

• Whether an individual polynomial inequality holds on a semialgebraic set (in particular our monotone cones) can, in principle, be decided by methods like Tarski quantifier elimination [CJ] or modifications of it which are implemented in symbolic systems like Mathematica. Thus, strictly speaking, whether an individual polynomial inequality holds is not anymore a mathematical question, but a computational one. Because of its enormous complexity, however, this method works only for a few variables and low degree. Here are useful Mathematica© packages or commands.

· The package `Algebra‘AlgebraicInequalities‘` provides via `SemialgebraicComponents[{I1,...,In},{x1,...,xn}]` the possibility to find at least one point in each connected component of the open semialgebraic set defined by inequalities `{I1,...,In}`. Both sides of each inequality `Ij` should be polynomials in variables `{x1,...,xm}` with rational coefficients. For example, typing

```
SemialgebraicComponents[{y^3-2x*y^2+(9/10)x^2y+x^3>0,x>0,y>0},{x,y}]
```

one gets after a split of a second the response `{{1,1}}` which means that there is only one semialgebraic component and the polynomial has in the component the same sign as in the point $(1, 1)$, namely the $+1$.

· The package `Algebra’InequalitySolve’` has the command `InequalitySolve[expr, {x1, ..., xn}]`. This gives the solution set of an expression containing logical connectives and linear equations and inequalities in the variables `{x1, ..., xn}`. Using the above example, typing

```
InequalitySolve[y^3-2x*y^2+(9/10)x^2y+x^3>0 && x>0 && y>0,{x,y}]
```

one gets immediately the answer `x>0&&y>0` showing the truth of the inequality. Since the solution set sometimes leads to complicated logical descriptions - use the weak inequality $y^3 - 2xy^2 + (9/10)x^2y + x^3 \geq 0$ to see this - it may be better to type $y^3 - 2xy^2 + (9/10)x^2y + x^3 < 0$ getting the answer `False`.

· Still another automated method is to use Mathematica's standard command `FindInstance` . For example

```
FindInstance[y^3-2x*y^2+(9/10)x^2y+x^3<0&&x>0&&y>0,{x,y},Reals]
```

returns the empty set. If `FindInstance` finds surely an instance that makes the expression true if there is one, this yields another proof that the combination of inequalities inputted has no solutions. So we will have for $x \geq 0, y \geq 0$ always $y^3 - 2xy^2 + (9/10)x^2y + x^3 \geq 0$. This polynomial satisfies `pos` for $x \geq y > 0$, but not for $y \geq x \geq 0$. The inequality can also be established by classical methods or Polya's theorem below.

• Finally there are a number of more or less well known general theorems. Here is a selection. One of the older theorems along these lines is the following:

Theorem (Rado [R]). *Let G be a subgroup of the symmetric group on $\{1, 2, ..., n\}$. Let $a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n)$ be elements of $\mathbb{R}^n$ so that $a \in conv\{g.b : g \in G\}$, meaning the convex hull of the orbit of b under the action of G. Then for any $\underline{x} = (x_1, ..., x_n) \in \mathbb{R}^n_{\geq 0}$ there holds*

$$\sum_{g \in G} x_{g(1)}^{a_1} x_{g(2)}^{a_2} \cdots x_{g(n)}^{a_n} \leq \sum_{g \in G} x_{g(1)}^{b_1} x_{g(2)}^{b_2} \cdots x_{g(n)}^{b_n}.$$

Muirhead's inequality is the special case of this theorem arising for the case $G = S_n$. The inequality iv above can be shown as a special case of Rado's inequality (but not of Muirhead's).

Theorem (Pólya [P]). *Let $p \in \mathbb{R}[\underline{x}]$ be a homogeneous polynomial that is positive on $\Delta_n = \{\underline{x} \in \mathbb{R}^n_{\geq 0} : \sum_{i=1}^n x_i = 1\}$. Then there exists a positive integer m such that the polynomial $p \cdot (x_1 + \cdots + x_n)^m$ has only nonnegative coefficients.*

The polynomial we used to illustrate `FindInstance`, when multiplied with $(x + y)^{11}$, yields only positive coefficients. The exponent $m = 11$ is the smallest possible for this to happen.

The theorem can also be found in Hardy, Littlewood, Polya [HLP], Delzell and Prestel [DP], Polya [P2, paper 107] and, with a completely different proof, due to Wörmann, in Marshall [M, p.45]. A quantified Pólya type result was given by Reznick [Re]. He gives in dependency of the quotient formed from minimum and maximum that $p > 0$ (everywhere on $\mathbb{R}^n$) assumes on the unit sphere an exponent m that guarantees $p \cdot (x_1^2 + \cdots + x_n^2)^m$ is sum of squares. A difficulty in the application of the theorem is that it usually will not work if the polynomial p assumes the value zero. In [CPR] Castle, Powers and Reznick give the precise conditions in which one can expect that for some m nonnegative coefficients will show up in a situation like Pólya's original theorem but with p possibly assuming 0 on Δ_n.

Define $s_k(\underline{x}) = \sum_{i=1}^n x_i^k$. Then any symmetric homogeneous cubic f in n variables can be written in the form $f = as_3 + bs_1s_2 + cs_1^3$. In the next section we shall have occasion to refer to the following theorem.

Theorem (Choi, Lam, Reznick [CLR1, Theorem 3.7]. *In order that $f \geq 0$ on $\mathbb{R}^n_{\geq 0}$ it is necessary and sufficient that there holds the inequality $a+bt+ct^2 \geq 0$, for $t = 1, 2, ..., n$.*

To Timofte [T] we owe the following principle for symmetric polynomial inequalities. It has received a particularly nice proof by Riener [Rie].

Denote by $e_i(\underline{x})$ the i-th elementary symmetric polynomials in $\underline{x} = x_{1:n} = (x_1, \cdots, x_n)$, $i = 1, ..., n$, and, defining $\pi(\underline{x}) = (e_1(\underline{x}), ..., e_n(\underline{x}))$, combine the $e_i(\underline{x})$ into an n-tuple. The well known fundamental theorem for symmetric functions says that the (symmetric) polynomials $f \in \mathbb{R}[\underline{x}]^{S_n}$ are precisely the polynomials for which there exists a (unique) polynomial $g = g_f \in \mathbb{R}[z_1, \ldots, z_n]$ so that $f = g \circ \pi$. The proof of that theorem also shows that in g only monomials $z_1^{i_1} \cdots z_n^{i_n}$ can occur for which $i_1 + 2i_2 + ... + ni_n \leq d = \deg f$. We shall call g here the *associate* to f. Let us say that $f \in \mathbb{R}[\underline{x}]^{S_n}$ satisfies *hypothesis g.s* ($s \in \mathbb{Z}_{\geq 1}$) if its associate can be written in the form

$$g = g_1 + \sum_{i=s+1}^{n} g_i z_i \text{ with } g_1, g_{s+1}, ..., g_n \in \mathbb{R}[z_{1:s}].$$

Equivalent to this is saying that for $i = s+1, \ldots, n$, $\partial g/\partial z_i \in \mathbb{R}[z_{1:s}]$. Thus it is clear that $s \leq s'$ and $g.s$ implies $g.s'$. It is also easy to see that $g \in \mathbb{R}[\underline{x}]^{S_n}_{\leq d}$ (of degree d) implies that f satisfies $g.\lfloor d/2 \rfloor$, and furthermore then $g_{d+1} = \cdots = g_n = 0$. Let $v(\underline{x}) = \#\{x_1, ..., x_n\}$ be the number of distinct entries of $\underline{x}$.

Theorem (Timofte [T], Riener [Rie]). *Assume $f \in \mathbb{R}[x]^{S_n}$ has an associate g satisfying $g.s$, with some $s \in \{2, \ldots, n\}$. If $f(\underline{x}) \geq 0$ whenever $v(\underline{x}) \leq s$, then $f \geq 0$ on $\mathbb{R}^n$. An analogous claim holds with the symbols '$\geq$' here replaced by '$>$'.*

It is interesting to observe that the theorem above for symmetric cubics can also be stated as saying that $f \geq 0$ on $\mathbb{R}^n$ iff $f(1_k, 0_{n-k}) \geq 0$ holds for all

$k = 1, 2, ..., n$. Here $1_k = (1, 1, \cdots, 1) \in \mathbb{R}^k$, and 0_{n-k} is analogously defined. Thus [CLR1] as well as other authors at around that time might have had already inklings of the Timofte - Riener theorem.

Finally there are algorithms that decide whether a polynomial can be written as a sum of squares. The algorithm in Powers and Wörmann [PW] gives an easily accessible first idea; but at the end the question is reduced to whether or not certain semialgebraic sets are empty or not; to decide questions is as we noted, notoriously hard. In polynomial optimization sum of squares relaxations are used. These, by means of semidefinite programming are fast.

2. The Main Theorem

For explanatory reasons, in this section we sometimes use capital letters for variables.

Let $p \in \mathbb{R}[X_1, X_2, ..., X_n]$ and consider it as a polynomial in

$$\mathbb{R}[X_1, X_2, ..., X_n, T]$$

where $X_{n+1} = T$ is an additional indeterminate. Write $\partial^k_{X_i} p$ for the k-th derivative of p w.r.t. X_i.

We define the (first) *derived family of* p or, for short, the *$\mathcal{D}$-family of* p (w.r.t. the order $X_1, ..., X_n, T$ of variables) as the family of polynomials $\mathcal{D}(\{p\}) = \{(\partial^i_{X_1} p)(X_2, X_{2:n}) : i = 0, 1, 2, ..., \deg_{X_1}(p)\}$. If $\{p_1, p_2, ..., p_k\} \subseteq \mathbb{R}[X_1, X_2, ..., X_n]$, then we define $\mathcal{D}(\{p_1, p_2, ..., p_k\}) = \bigcup_{j=1}^k \mathcal{D}(\{p_j\})$.

So $\mathcal{D}(\{p\})$ is obtained by forming the derivatives of p of all orders$\leq \deg_{X_1}(p)$ with respect to X_1, and *then* replacing X_1 by X_2. It follows that

$$\mathcal{D}(\{p\}) \subseteq \mathbb{R}[X_2, ..., X_n] \subseteq \mathbb{R}[X_2, ..., X_n, T].$$

By choosing above $i = 0$, we see $p(X_2, X_{2:n}) \in \mathcal{D}(\{p\})$; furthermore note that in $\partial_{X_1}^{\deg_{X_1}(p)}(p)$ variable X_1 does not occur.

We can repeat the $\mathcal{D}$-process, forming successively

$\{p\} = \mathcal{D}^0(\{p\}), \mathcal{D}^1(\{p\}) = \mathcal{D}(\{p\}), ..., \mathcal{D}^k(\{p\}) = \mathcal{D}(\mathcal{D}^{k-1}(\{p\})), ..., \mathcal{D}^n(\{p\})$.

Since $\mathcal{D}^k(\{p\})$ is a family of polynomials in $X_{k+1}, ..., X_n, T$ (in which, for $k \leq n-1$, T does not occur), we get that $\mathcal{D}^n(\{p\})$ is a family of polynomials in T.

Theorem 2.1. *Let* $p \in \mathbb{R}[X_1, X_2, ..., X_n] \subseteq \mathbb{R}[X_1, X_2, ..., X_n, T]$. *Then:*

a. *p can be written as a polynomial in the differences $X_i - X_{i+1}$, $i = 1, ..., n$, $X_{n+1} = T$, with coefficients that are nonnegative rational multiples of the polynomials in $\mathcal{D}^n(\{p\}) \subseteq \mathbb{R}[T]$.*
b. *If the polynomials in $\mathcal{D}^n(\{p\})$ all assume nonnegative values in the point r then, whenever $x_1 \geq x_2 \geq \cdots \geq x_n = r$, there holds $p(x_1, x_2, ..., x_n) \geq 0$.*

Proof: a. We prove the claim by induction on n. So assume first $n = 1$. Write X for X_1. Let $p = p(X) \in \mathbb{R}[X]$, and let $d = \deg p = \deg_X p$. Then $\mathcal{D}(\{p\}) = \{p(T), p'(T), ..., p^{(d)}(T)\}$, where the primed p's indicate derivatives w.r.t. X. Since p is a polynomial, it is its own Taylor series around T, i.o.w. we have

$$p(X) = p(T) + \frac{p'(T)}{1!}(X - T) + \cdots + \frac{p^{(d)}(T)}{d!}(X - T)^d.$$

This identity can be derived from the classical Taylor series of real analysis by the 'polynomial argument': it is true for every real r in place of T, hence it is true on the formal level; but more natural is to show it true for each power X^l with $l \leq d$ writing $X^l = (T + (X - T))^l$ and using the binomial theorem; and then to extend this to arbitrary $p \in \mathbb{R}[X]$ of degree d, using linearity of the derivative. The identity shows the claim for $n = 1$.

Now assume $n \geq 2$ and the theorem already proved for less than n variables. Write X for X_1. By definition

$$\mathcal{D}(\{p\}) = \{p(X_2, X_{2:n}), (\partial_X p)(X_2, X_{2:n}), ..., (\partial_X^d p)(X_2, X_{2:n})\}.$$

Consequently, considering p as a polynomial in X of degree d, a formal Taylor expansion around X_2 yields

$$p(X, X_{2:n}) = p(X_2, X_{2:n}) + 1!^{-1}(\partial_X p)(X_2, X_{2:n})(X - X_2) + \cdots + \\ + d!^{-1}(\partial_X^d p)(X_2, X_{2:n})(X - X_2)^d.$$

Now the $(\partial_X^i p)(X_2, X_{2:n})$ are evidently polynomials of $n - 1$ variables. Furthermore, by definition,

$$\mathcal{D}^n(\{p\}) = \mathcal{D}^{n-1}(\{p(X_2, X_{2:n})\}) \cup \cdots \cup \mathcal{D}^{n-1}(\{(\partial_X^d p)(X_2, X_{2:n})\}).$$

By induction hypothesis, therefore, we can write for each $i = 0, 1, 2, ..., d$, the $(\partial_X^i p)(X_2, X_{2:n})$ as polynomials in $X_2 - X_3, ..., X_{n-1} - X_n, X_n - T$, with coefficients that are nonnegative rational multiples of polynomials in $\mathcal{D}^{n-1}(\{(\partial_X^i p)(X_2, X_{2:n})\}) \subset \mathcal{D}^n(\{p\})$. Substituting these elements in the Taylor series above we are done with part a.
b. Is an immediate consequence of part a. ■

Let us indicate the transformation a polynomial q suffers by taking k times the derivative with respect to X_i and then putting $X_{i+1} = X_i$ by

$$q \xrightarrow{\partial_i^k, X_i = X_{i+1}} \tilde{q},$$

where $\tilde{q}$ is the resulting polynomial. It will be also useful to observe that $\mathcal{D}^n(\{p\})$ equals the family of all possible endpoints of chains

$$p \xrightarrow{\partial_1^{i_1}, X_1 = X_2} \cdot \xrightarrow{\partial_2^{i_2}, X_2 = X_3} \cdot \cdots \cdot \xrightarrow{\partial_{n-1}^{i_{n-1}}, X_{n-1} = X_n} \cdot \xrightarrow{\partial_n^{i_n}, X_n = T} \tilde{p}$$

Sometimes it is more convenient or natural to use the Hasse derivatives $\partial^k = \partial^k/k!$ instead of the simple operators ∂^k. But be aware that $\partial^k \partial^l \neq \partial^{k+l}$. One sees from the reasoning in the proof of the theorem, that the following holds:

Corollary 2.2. *The coefficient of*

$$(X_1 - X_2)^{i_1}(X_2 - X_3)^{i_2} \cdots (X_{n-1} - X_n)^{i_{n-1}}(X_n - T)^{i_n}$$

in an expansion of p as proposed in part a of the Theorem 2.1 can be obtained in either of the following two ways:

a. *As $\tilde{p}(T)/(i_1! i_2! \cdots i_n!)$, where $\tilde{p}$ is the terminal polynomial in the chain above using ordinary derivatives.*
b. *As the terminal polynomial itself, in such a chain if Hasse derivatives are used instead.*

It is easy to see that if $i_1 + i_2 + \cdots + i_n$ is greater than the total degree of a polynomial, then $\tilde{p} = 0$. This also will follow from Lemma 2.3 below, recalling that binomial coefficients with negative upper index are 0.

Example 2.1. Here is a development of the AG polynomial for the case $n = 3$. It was found using the corollary. For $t = 0$ and putting $h_1 = x - y, h_2 = y - z, h_3 = z - t$ one recovers the development mentioned in the introduction.

$$\begin{aligned} x^3 + y^3 + z^3 - 3xyz = {} & 3t(y-z)^2 + 3(y-z)^2(z-t) + 2(y-z)^3 + \\ & 3t(x-y)(y-z) + 3(x-y)(y-z)(z-t) + \\ & 3(x-y)(y-z)^2 + 3t(x-y)^2 + 3(x-y)^2(z-t) + \\ & 1(x-y)^3 + 3(x-y)^2(y-z). \end{aligned}$$

Lemma 2.3. *Using the Hasse derivative, there holds*

$$X_1^{l_1}X_2^{l_2}\cdots X_n^{l_n} \xrightarrow{\partial_1^{k_1},X_1=X_2} \cdot\ \cdots\ \cdot \xrightarrow{\partial_n^{k_n},X_n=T}$$
$$\tbinom{l_1}{k_1}\tbinom{l_1+l_2-k_1}{k_2}\cdots\tbinom{l_1+l_2+\cdots+l_n-k_1-k_2-\cdots-k_{n-1}}{k_n}T^{l_1+l_2+\cdots+l_n-k_1-k_2-\cdots-k_n}.$$

Proof: The proof is done via induction on the number of variables involved. If $n=1$, we note

$$\partial_1^{k_1}X_1^{l_1}=\frac{1}{k_1!}(l_1\cdot(l_1-1)\cdots(l_1-k_1+1))X_1^{l_1-k_1}=\binom{l_1}{k_1}X_1^{l_1-k_1}.$$

Hence after the first step in the above chain we get $\binom{l_1}{k_1}X_2^{l_1+l_2-k_1}X_3^{l_3}\cdots X_n^{l_n}$. Thus supposing the lemma proved for $n-1$ variables, the proof is complete. ■

For illustration of these developments let us now prove the general AG mean inequality. We use the notation $\chi(P)=1$ or 0 according to whether or not a certain property P holds. According to Lemma 2.3 we have with the above chain (a.c.)

$$X_s^n=X_1^0\cdots X_s^n\cdots X_n^0\overset{\text{a.c.}}{\rightarrow}$$
$$\tbinom{0}{k_1}\tbinom{0-k_1}{k_2}\cdots\tbinom{n-k_1-k_2-\cdots-k_{s-1}}{k_s}\cdots\tbinom{n-k_1-k_2-\cdots-k_{n-1}}{k_n}T^{n-k_1-\cdots-k_n}.$$

The product of the leftmost $s-1$ factors at the right is

$$\tbinom{0}{k_1}\tbinom{0-k_1}{k_2}\cdots\tbinom{0-k_1-k_2-\cdots-k_{s-2}}{k_{s-1}}=\chi(k_1=\cdots=k_{s-1}=0),$$

since $\binom{0}{0}=1$ and binomial coefficients are zero whenever the lower entry exceeds the upper - see Lemma 3.1. The product of the remaining factors equals the multinomial coefficient $\binom{n-k_1-\cdots-k_{s-1}}{k_s,\cdots,k_n}$. We also have

$$X_1\cdots X_s\cdots X_n\overset{\text{a.c.}}{\rightarrow}\tbinom{1}{k_1}\tbinom{2-k_1}{k_2}\cdots\tbinom{s-k_1-k_2-\cdots-k_{s-1}}{k_s}\cdots\tbinom{n-k_1-k_2-\cdots-k_{n-1}}{k_n}T^{n-k_1-\cdots-k_n}.$$

Thus the AG-inequality, or more precisely the fact that $X_1^n+\cdots+X_n^n-nX_1X_2\cdots X_n$ is pos will follow from the following lemma.

Lemma 2.4. *Whenever $k_1,...,k_n\in\mathbb{Z}_{\geq 0}$ are such that $\sum_{i=1}^n k_i=n$, then*

$$\sum_{s=1}^n\chi(k_i=0,i=1,...,s-1)\binom{n-k_1-\cdots-k_{s-1}}{k_s,\dots,k_n}\geq$$
$$\geq n\binom{1}{k_1}\binom{2-k_1}{k_2}\cdots\binom{s-k_1-\cdots-k_{s-1}}{k_s}\cdots\binom{n-k_1-\cdots-k_{n-1}}{k_n}.$$

Proof: After multiplication with $k_1!k_2!\cdots k_n!$ the right hand side is

$$n\cdot\prod_{j=1}^{n}(j-\sum_{l=1}^{j-1}k_l)^{\underline{k_j}}.$$

This is a double product. Define sections of decreasing sequences of k_j integers

$$I_j=\{j-k_1-\cdots-k_{j-1},\ j-k_1-\cdots-k_{j-1}-1,\ \cdots,\ j+1-k_1-\cdots-k_{j-1}-k_j\}$$

and let $P(I_j)$ denote the product of the integers in this interval. Then

$$P(I_j)=(j-k_1-\cdots-k_{j-1})!\big/(j-k_1-\cdots-k_{j-1}-k_j)!.$$

Therefore

$$\prod_{j=1}^{n}(j-\sum_{l=1}^{j-1}k_l)^{\underline{k_j}}=\prod_{j=1}^{n}P(I_j)=$$

$$=\tfrac{1!}{(1-k_1)!}\cdot\tfrac{(2-k_1)!}{(2-k_1-k_2)!}\cdot\tfrac{(3-k_1-k_2)!}{(3-k_1-k_2-k_3)!}\cdots\tfrac{(n-k_1-k_2-\cdots-k_{n-1})!}{(n-k_1-k_2-\cdots-k_n)!}=$$

$$=1\cdot(2-k_1)(3-k_1-k_2)\cdots(n-k_1-k_2-\cdots-k_{n-1}).$$

This product is evidently $\leq n!$ and in fact except in the case that $k_1=k_2=\cdots=k_{n-1}=0$, it is even $\leq(n-1)!$. So in this case the right hand side of the inequality is not larger than the first term of the sum at the left alone,$\binom{n}{k_1,\ldots,k_n}$, and the inequality therefore proved. If $k_1=\cdots=k_{n-1}=0$ then the rhs of the inequality is n, while the left hand side degenerates to the sum $\sum_{s=1}^{n}1=n$. Thus the inequality is true again. ■

While this proof of the AG-inequality is certainly not particularly simple, it is probably simpler than its direct competitor that would consist of writing $X_i=\sum_{l=i}^{n}h_l$, and then showing that $X_1^n+\cdots+X_n^n-nX_1X_2\cdots X_n$ develops into a sum of monomials in $h_1,...,h_n$ whose coefficients are all nonnegative. The property pos of this polynomial can also be obtained as a consequence of the proof Muirhead's theorem, Theorem 1.1.

3. Laffey's inequalities related to the geometric mean of $(1-x_1t),...,(1-x_kt)$

We apologize that in this and the next section the number of variables x_i is k; but to change notation in complicated formulae already carefully verified is dangerous.

It is clear by a Taylor series developments around 0, that there exists a series representation

$$\sum_{n\geq 0} u_n(x_1, x_2, ..., x_k)t^n = (1-x_1t)^{\alpha_1}(1-x_2t)^{\alpha_2}\cdots(1-x_kt)^{\alpha_k}.$$

According to Professor Laffey it was discovered by computer that in the case $\alpha_1 = \cdots = \alpha_k = 1/k$, the coefficients of the power series around 0 are nonpositive for $n \geq 1$ if the x_i are nonnegative. In his article [L] a proof of this is given and in work [LLS], made together with Loewy and Šmigoc, related results are shown for polynomials $f(t) = \det(A - tI)$ with A being an entrywise positive square matrix A. In these cases there exists an N such that the coefficients of $t, t^2, ...$ of the power series around $t = 0$ of $f(t)^{1/N}$ are negative.

We shall show that in general if $0 \leq \alpha_i \leq 1$, and $\alpha_1 + \cdots + \alpha_k \leq 1$, then for $n \geq 1$ and $x_1, \cdots, x_k \geq 0$, we have $u_n(x_1, \cdots, x_k) \leq 0$.

Using the binomial series $(1-xt)^\alpha = \sum_{j\geq 0}\binom{\alpha}{j}(-xt)^j$ and Cauchy multiplication, one sees $u_n = (-1)^n U_n$, where

$$U_n = U_n(x_1, \cdots, x_k) = \sum_{j_1+\cdots+j_k=n} \binom{\alpha_1}{j_1}\binom{\alpha_2}{j_2}\cdots\binom{\alpha_k}{j_k} x_1^{j_1}\cdots x_k^{j_k}.$$

At this point we should recall the general binomial coefficients [GKP, Chapter 5]. One defines for $r \in \mathbb{R}$ and $k \in \mathbb{Z}$:

$$r^{\underline{k}} = \begin{cases} r(r-1)\cdots(r-k+1) & \text{if } k > 0 \\ 1 & \text{if } k = 0 \\ 0 & \text{if } k < 0 \end{cases};$$

and based on this

$$\binom{r}{k} = \frac{r^{\underline{k}}}{k!}.$$

The case $k = 0$ is actually a special case of the first case if one formulates the latter with $k \geq 0$ and understands as usual empty products as being 1. In particular $0! = 1$. Note that by virtue of these definitions we may - and will - let the j_i range over $\mathbb{Z}$. The sum for u_n above is finite, so u_n is a polynomial.

It is now clear that $u_0 = 1$ and $u_1 = -(\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_k x_k)$. We assume henceforth $n \geq 2$. Recall the notation $s^+ = \max\{0, s\}$.

Lemma 3.1. *Let $r, s \in \mathbb{R}$, $k, l, m, n \in \mathbb{Z}$. Then:*

(a) $k\binom{r}{k} = r\binom{r-1}{k-1}$;

(b) $k^{\underline{l}}\binom{r}{k} = r^{\underline{l}}\binom{r-l}{k-l}$;

(c) $\sum_k \binom{r}{m+k}\binom{s}{n-k} = \binom{r+s}{m+n}$.

(d) *Let* $k \geq 1$. *Then* $r^{\underline{k}} = 0$ *iff* $r \in \{0,1,2,...,k-1\}$. *For any* r, $r^{\underline{0}} = 1$.

(e) *If* $k \geq 1$ *and* $r \notin \{0,1,2,...,k-1\}$, *then*

$$\mathit{sign}(r^{\underline{k}}) = \mathit{sign}\binom{r}{k} = (-1)^{(k-\lceil r\rceil^+)^+}.$$

In particular, if $r < 0$, *then* $\mathit{sign}(r^{\underline{k}}) = (-1)^k$.

Proof: (a) is direct, (b) is a consequence of (a). For the extended Vandermonde identity (c) see [GKP]. For (d) look at the definition of $r^{\underline{k}}$. For (e) note that $\operatorname{sign}(r^{\underline{k}}) = (-1)^{\text{neg}}$, where neg=number of negative factors in $r(r-1)\cdots(r-k+1)$. Since neg $= \#\{l \in \{0,1,...,k-1\} : r-l < 0\} = \#\{... : \lceil r\rceil \leq l\} = \#\{\lceil r\rceil^+, ..., k-1\} = (k-\lceil r\rceil^+)^+$, the claim follows. ■

The essential step towards proving Theorem 3.3 is provided by the following proposition. To keep the formulae in formulation and proof at manageable size, allow for an n-tuple $a = (a_1, ..., a_n)$ and indices $1 \leq i \leq j \leq n$ the notations $a_{i:j} = (a_i, ..., a_j)$ and $Sa = a_1 + a_2 + \cdots + a_n$; so $Sa_{i:j} = a_i + \cdots + a_j$. Also, put $\partial_i = \partial_{x_i}$.

Proposition 3.2. *Let* $d_1, \ldots, d_k$ *be nonnegative integers and let* r *be a real. Subjecting the polynomial* $U_n(x_1, \ldots, x_k)$ *to the composition of maps*

$$U_n(x_1,\ldots,x_k) \xrightarrow{\partial_1^{d_1}, x_1=x_2} \cdot \xrightarrow{\partial_2^{d_2}, x_2=x_3} \cdots\cdots \xrightarrow{\partial_{k-1}^{d_{k-1}}, x_{k-1}=x_k} \cdot \xrightarrow{\partial_k^{d_k}, x_k=r} R,$$

as shown, results in the real number

$$R = R_n = R_n(r; d_1, ..., d_k) = \prod_{i=1}^{k} (S\alpha_{1:i} - Sd_{1:i-1})^{\underline{d_i}} \binom{S\alpha_{1:k} - Sd_{1:k}}{n - Sd_{1:k}} r^{n-Sd_{1:k}}.$$

Proof: We have

$$U_n(x_1,\ldots,x_k) = \sum_{Sj_{1:k}=n} \binom{\alpha_1}{j_1}\binom{\alpha_2}{j_2}\binom{\alpha_3}{j_3}\cdots\binom{\alpha_k}{j_k} x_1^{j_1}\cdots x_k^{j_k}.$$

Observe that

$$\sum_{Sj_{1:k}=n} \dots = \sum_{m} \sum_{j_1+j_2=m} \sum_{Sj_{3:k}=n-m} \dots$$

Now applying $\partial_1^{d_1}$, and then putting $x_1 = x_2$, the monomial $x_1^{j_1} x_2^{j_2}$, transforms into $j_1^{\underline{d_1}} x_2^{j_1+j_2-d_1}$. So, since by Lemma 3.1b, $j_1^{\underline{d_1}} \binom{\alpha_1}{j_1} = (\alpha_1)^{\underline{d_1}} \binom{\alpha_1-d_1}{j_1-d_1}$, the polynomial transforms through these operations into

$$(\alpha_1)^{\underline{d_1}} \sum_{m} \sum_{j_1+j_2=m} \binom{\alpha_1-d_1}{j_1-d_1} \binom{\alpha_2}{j_2} \sum_{Sj_{3:k}=n-m} \binom{\alpha_3}{j_3} \cdots \binom{\alpha_k}{j_k} x_2^{m-d_1} x_3^{j_3} \cdots x_k^{j_k},$$

which by using Vandermonde's identity of Lemma 2.1c in the middle summation yields

$$(\partial_1^{d_1} U_n)(x_2, x_{2:k}) = (\alpha_1)^{\underline{d_1}} \textstyle\sum_m \sum_{Sj_{3:k}=n-m} \binom{\alpha_1+\alpha_2-d_1}{m-d_1} \binom{\alpha_3}{j_3} \cdots \binom{\alpha_k}{j_k} x_2^{m-d_1} x_3^{j_3} \cdots x_k^{j_k}.$$

Assume we have carried out the operations 'apply $\partial_i^{d_i}$, then put $x_i = x_{i+1}$', for $i = 1, 2, ..., t$, in succession, beginning with U_n, and arrived at a polynomial

$$\underbrace{\prod_{i=1}^{t} (S\alpha_{1:i} - Sd_{1:i-1})^{\underline{d_i}}}_{=:P} \sum_{m} \sum_{Sj_{t+2:k} = n-m} \tbinom{S\alpha_{1:t+1}-Sd_{1:t}}{m-Sd_{1:t}} \tbinom{\alpha_{t+2}}{j_{t+2}} \cdots \tbinom{\alpha_k}{j_k} x_{t+1}^{m-Sd_{1:t}} x_{t+2}^{j_{t+2}} \cdots x_k^{j_k},$$

of which evidently the expression before is the case $t = 1$. We show that provided $1 \le t \le k-3$, the same holds for $t+1$ in place of t. Note that

$$\sum_{m} \sum_{Sj_{t+2:k} = n-m} \dots = \sum_{m'} \sum_{m+j_{t+2}=m'} \sum_{Sj_{t+3:k}=n-m'} \dots$$

Applying the operator $\partial_{t+1}^{d_{t+1}}$, and putting $x_{t+1} = x_{t+2}$, we transform the monomial $x_{t+1}^{m-Sd_{1:t}} x_{t+2}^{j_{t+2}}$ into $(m - Sd_{1:t})^{\underline{d_{t+1}}} x_{t+2}^{m-Sd_{1:t+1}+j_{t+2}}$. Now again by Lemma 3.1b, and using $Sd_{1:t} + d_{t+1} = Sd_{1:t+1}$,

$$(m - Sd_{1:t})^{\underline{d_{t+1}}} \binom{S\alpha_{1:t+1} - Sd_{1:t}}{m - Sd_{1:t}} = (S\alpha_{1:t+1} - Sd_{1:t})^{\underline{d_{t+1}}} \binom{S\alpha_{1:t+1} - Sd_{1:t+1}}{m - Sd_{1:t+1}}.$$

Thus after the referred two operations, there results the polynomial

$$P \cdot (S\alpha_{1:t+1} - Sd_{1:t})^{\underline{d_{t+1}}} \sum_{m'} \sum_{m+j_{t+2}=m'} \binom{S\alpha_{1:t+1} - Sd_{1:t+1}}{m - Sd_{1:t+1}} \binom{\alpha_{t+2}}{j_{t+2}} \times$$

$$\sum_{Sj_{t+3:k}=n-m'} \binom{\alpha_{t+3}}{j_{t+3}} \cdots \binom{\alpha_k}{j_k} x_{t+2}^{m'-Sd_{1:t+1}} x_{t+3}^{j_{t+3}} \cdots x_k^{j_k}.$$

By Vandermonde's identity, the middle sum is $\binom{S\alpha_{1:t+2}-Sd_{1:t+1}}{m'-Sd_{1:t+1}}$, and hence the whole expression is

$$P\cdot(S\alpha_{1:t+1}-Sd_{1:t})^{\underline{d_{t+1}}}\times$$
$$\sum_{m'}\sum_{Sj_{t+3:k}=n-m'}\binom{S\alpha_{1:t+2}-Sd_{1:t+1}}{m'-Sd_{1:t+1}}\binom{\alpha_{t+3}}{j_{t+3}}\cdots\binom{\alpha_k}{j_k}x_{t+2}^{m'-Sd_{1:t+1}}x_{t+3}^{j_{t+3}}\cdots x_k^{j_k},$$

as was to show.

According to this result after performing the operations 'apply $\partial_i^{d_i}$, then put $x_i = x_{i+1}$' for $i = 1, 2, ..., k-2$, we get

$$\underbrace{\prod_{i=1}^{k-2}(S\alpha_{1:i}-Sd_{1:i-1})^{\underline{d_i}}}_{=:P'}\sum_m\sum_{Sj_k=n-m}\binom{S\alpha_{1:k-1}-Sd_{1:k-2}}{m-Sd_{1:k-2}}\binom{\alpha_k}{j_k}x_{k-1}^{m-Sd_{1:k-2}}\cdot x_k^{j_k}$$
$$=P'\sum_m\binom{S\alpha_{1:k-1}-Sd_{1:k-2}}{m-Sd_{1:k-2}}\binom{\alpha_k}{n-m}x_{k-1}^{m-Sd_{1:k-2}}x_k^{n-m}.$$

Now we apply $\partial_{k-1}^{d_{k-1}}$ and put $x_{k-1} = x_k$. This transforms the monomial into $(m-Sd_{1:k-2})^{\underline{d_{k-1}}}x_k^{n-Sd_{1:k-1}}$ and so, for reasons we know already, the expression above into

$$P'\cdot(S\alpha_{1:k-1}-Sd_{1:k-2})^{\underline{d_{k-1}}}\sum_m\binom{S\alpha_{1:k-1}-Sd_{1:k-1}}{m-Sd_{1:k-1}}\binom{\alpha_k}{n-m}x_k^{n-Sd_{1:k-1}}$$
$$=\prod_{i=1}^{k-1}(S\alpha_{1:i}-Sd_{1:i-1})^{\underline{d_i}}\binom{S\alpha_{1:k}-Sd_{1:k-1}}{n-Sd_{1:k-1}}x_k^{n-Sd_{1:k-1}}.$$

Now apply $\partial_k^{d_k}$ to this, use

$$(n-Sd_{1:k-1})^{\underline{d_k}}\binom{S\alpha_{1:k}-Sd_{1:k-1}}{n-Sd_{1:k-1}}=(S\alpha_{1:k}-Sd_{1:k-1})^{\underline{d_k}}\binom{S\alpha_{1:k}-Sd_{1:k}}{n-Sd_{1:k}},$$

and put $x_k = r$. You then see the proposition proved. ■

Now we get the desired result.

Theorem 3.3. *Assume $\alpha_1, \alpha_2, ..., \alpha_k$ are nonnegative reals of sum≤ 1. Then the coefficients $u_n(x_1, x_2, ..., x_k)$, $n \geq 1$ in the power series development*

$$\sum_{n\geq 0}u_n(x_1,x_2,...,x_k)t^n=(1-x_1t)^{\alpha_1}(1-x_2t)^{\alpha_2}\cdots(1-x_kt)^{\alpha_k}$$

are homogeneous polynomials of degree n *that are nonpositive on the nonnegative orthant* $\mathbb{R}^k_{\geq 0}$: *For* $x_1, x_2, ..., x_k \geq 0$, *and* $n \geq 1$, $u_n(\underline{x}) \leq 0$; *in fact these* u_n *are* $-\mathsf{pos}$.

Proof: It is convenient to dispose off some easy cases first.

Case: $k = 1$. Then let $x = x_1$. We are speaking of the series $(1 - xt)^\alpha = \sum_{j\geq 0} \binom{\alpha}{j}(-xt)^j$. So then $u_n(x) = (-1)^n\binom{\alpha}{n}x^n$. If $\alpha = 0$, then for all $n \geq 1$, $u_n = 0$. If $\alpha > 0$, then by Lemma 3.1, $\mathrm{sign}\binom{\alpha}{n} = (-1)^{(n-\lceil\alpha\rceil^+)^+} = (-1)^{(n-1)^+}$. Thus if $n \geq 1$, $\mathrm{sign}((-1)^n\binom{\alpha}{n}) = (-1)^{2n-1} = -1$. Hence u_n is -pos , since x^n evidently is pos .

Case: Some αs are 0. Then the product at the right consists of less than k factors $(1 - x_it)^{\alpha_i} \neq 1$, and the claim can be assumed true inducting on k.

Case: Some α is 1. Then all other αs are 0 and the right hand side reduces to $(1 - x_it)$ for some i. So for $n \geq 1$, $u_n(x_1, ..., x_k) = -\delta_{ni}x_i$, showing the claim.

So we assume in what follows that $k \geq 2$ and $0 < \alpha_i < 1$ holds for $i = 1, ..., k$.

Putting $r = 0$ in the Proposition 3.2, we have $R(r) = R_n(r; d_1, .., d_k) = 0$ except possibly in cases where $n = d_1 + d_2 + \cdots + d_k = Sd_{1:k}$ which we assume from now on. Then the last two factors in $R(0)$ are $\binom{S\alpha_{1:k}-n}{0} = 1$ and $x_k^{n-Sd_{1:k}} = x_k^0 = 1$, and we have to compute

$$\mathrm{sign}(R(0)) = \prod_{i=1}^{k} \mathrm{sign}((S\alpha_{1:i} - Sd_{1:i-1})^{\underline{d_i}}).$$

Note that we have for $i = 1, ..., k-1$, $0 < S\alpha_{1:i} < 1$, and $0 < S\alpha_{1:k} \leq 1$. Now define s (possibly 1) by $d_1 = \cdots = d_{s-1} = 0 < 1 \leq d_s$. Then $Sd_{1:i-1} = 0$ for $i = 1, ..., s$, and so

$$\begin{aligned}\mathrm{sign}(R(0)) = &\prod_{i=1}^{s-1}\mathrm{sign}((S\alpha_{1:i})^{\underline{0}}) \cdot \mathrm{sign}((S\alpha_{1:s})^{\underline{d_s}}) \cdot \\ &\cdot \prod_{i=s+1}^{k-1} \mathrm{sign}((S\alpha_{1:i} - Sd_{1:i-1})^{\underline{d_i}}) \cdot \mathrm{sign}((S\alpha_{1:k} - Sd_{1:k-1})^{\underline{d_k}}).\end{aligned}$$

This product has four factors separated by the multiplication point '$\cdot$'. By Lemma 3.1, parts d and e, the first factor is 1, the second is $(-1)^{d_s-1}$; furthermore, since for $i \in \{s + 1, ..., k - 1\}$, $S\alpha_{1:i} - Sd_{1:i-1} < 0$, the third factor is $\prod_{i=s+1}^{k-1}(-1)^{d_i}$. Concerning the fourth factor, we usually will have $S\alpha_{1:k} - Sd_{1:k-1} < 0$, and then that factor is $(-1)^{d_k}$ so that $\operatorname{sign}R(0) = (-1)^{d_s-1}\cdot(-1)^{Sd_{s+1:k-1}}\cdot(-1)^{d_k} = (-1)^{Sd-1} = (-1)^{n-1}$. The other possible case for the last factor is $S\alpha_{1:k}-Sd_{1:k-1} = 0$. But then $Sd_{1:k-1} = 1, d_k = n-1$, and $R(0) = 0$ follows directly from Proposition 3.2.

Now $\mathcal{D}^k(\{U_n\})$ is evidently the set of all possible endpoints $R_n(0) = R(0; d_1, ..., d_k)$ of compositions like in the Proposition 3.2, as $d_1, ..., d_k$ vary over the elements in $\mathbb{Z}^k_{\geq 0}$. As we saw, each such endpoint is 0 or has sign $(-1)^{n-1}$. By the linearity of the operations in Proposition 3.2, if we applied them to u_n, the result would be that the endpoints are 0 or of sign $(-1)^n(-1)^{n-1} = -1$. Thus whenever $x_1 \geq x_2 \geq \cdots \geq x_k \geq 0$ we get $u_n(\underline{x}) \geq 0$. Since the hypothesis of the theorem is invariant under permutations of the indices i in the x_i and the α_i, $u_n(\underline{x}) \leq 0$ must hold for any $\underline{x} \in \mathbb{R}^k_{\geq 0}$ and by Corollary 2.2 the u_n are for $n \geq 1$ -pos on every C_π ∎

Remark 3.4. The fact that the polynomials $u_n(\underline{x})$ are $-\mathsf{pos}$ is (for the case all $\alpha_i = 1/k$) equivalent to Laffey's Remark 2.1 in [L]. Note, however, that in the general case these polynomials are not symmetric. Therefore, who wants $-\mathsf{pos}$-representations of a given u_n has to deal usually with various permutations π.

4. Holland's inequalities related to the harmonic mean of $(1 - x_1t), ..., (1 - x_kt)$

F. Holland proved another interesting theorem in the line of Theorem 3.3. In [Hol] he recalls that when approached by Laffey to prove Theorem 3.3, he decided to follow Pólya's advice to first solve a somewhat similar, but possibly simpler problem. He noted one can formulate Theorem 3.3 for the original case $\alpha_i = 1/n$ case as saying that the geometric mean of expressions $1 - x_1t, 1 - x_2t, ..., 1 - x_kt$ is a power series all whose coefficients pertaining to powers $t^n, n \geq 1$ are nonpositive and asked whether a similar claim holds for the harmonic mean of them.

As in Section 3, let us generalize the question to the weighted version. So with nonnegative reals $p_1, ..., p_k$ of sum equal to 1, we consider the weighted l-th power sum of $x_1, x_2, ..., x_k$. This is $s_l = s_l(\underline{p}, \underline{x}) = \sum_{i=1}^k p_i x_i^l$.

By definition, the referred harmonic mean is

$$(p_1(1-x_1t)^{-1}+p_2(1-x_2t)^{-1}+\cdots+p_k(1-x_kt)^{-1})^{-1} = \left(\sum_{i=1}^k p_i \sum_{l\geq 0}(x_it)^l\right)^{-1}$$

$$= \left(1 + \sum_{i=1}^{k} p_i \sum_{l\geq 1} x_i^l t^l\right)^{-1}$$
$$= \left(1 + \sum_{l\geq 1}(\sum_{i=1}^{k} p_i x_i^l) t^l\right)^{-1} = \left(1 + \sum_{l\geq 1} s_l(p, \underline{x}) t^l\right)^{-1}$$
$$= \left(1 - \sum_{l\geq 1} s_l(\underline{p}, \underline{x}) t^l + (\sum_{l\geq 1} s_l(\underline{p}, \underline{x}) t^l)^2 - (\sum_{l\geq 1} s_l(\underline{p}, \underline{x}) t^l)^3 + \cdots\right).$$

From this follows that the coefficient of t^l is,

$$q_l(\underline{p}, \underline{x}) = -s_l(\underline{p}, \underline{x}) + \sum_{l_1+l_2=l} s_{l_1}(\underline{p}, \underline{x}) s_{l_2}(\underline{p}, \underline{x})$$
$$- \sum_{l_1+l_2+l_3=l} s_{l_1}(\underline{p}, \underline{x}) s_{l_2}(\underline{p}, \underline{x}) s_{l_3}(\underline{p}, \underline{x}) + \cdots + (-1)^l s_1(\underline{p}, \underline{x})^l,$$

where it is assumed that the $l_i \in \mathbb{Z}_{\geq 1}$. It is clear that q_l is a homogeneous, polynomial of degree l in $x_1, ..., x_k$.

Holland considered the case $\underline{p} = \frac{1}{k}1_k$ and proved that for any $\underline{x} \in \mathbb{R}^k_{\geq 0}$, $q(\frac{1}{k}1_k, \underline{x}) \leq 0$. He showed that the sequence of power sums $s_1, s_2, ...$ is log-convex and noticed with Kaluza [Kal] that the s_i and q_l are related by means of a convolution. From this by an induction he gets that the q_l are ≤ 0. As follows from this description, Holland's respective Proposition 2.2 has a non-algebraic proof and perhaps is not easily generalizable to an arbitrary probability vector $\underline{p}$; in fact upon entering Section 4, the author specializes his considerations to the case $\underline{p} = \frac{1}{k}1_k$.

While we cannot presently give a general proof of this result, we give in this section hints that polynomials $q_l(\frac{1}{k}1_k, \underline{x})$ are not only nonnegative, but in fact the polynomials $q_l(\underline{p}, \underline{x})$ are $-\mathsf{pos}$.

We prove this for degree 3 and any number k of variables in detail and then relate similar results for degree 4. So we wish now to consider

$$q_3(\underline{p}, \underline{x}) = -s_3(\underline{p}, \underline{x}) + 2s_1(\underline{p}, \underline{x}) s_2(\underline{p}, \underline{x}) - s_1(\underline{p}, \underline{x})^3,$$

and to show that any chain of differential operators of the form mentioned before Corollary 2.2 with $i_1 + i_2 + i_3 = 3$, leads to a nonpositive real. To see this, a notation better adapted to many variables and low degree seems to be convenient. In such a situation, in many successive of the elementary operations '$\xrightarrow{\partial_\nu^{i_\nu}, x_\nu = x_{\nu+1}}$' introduced in Section 2 we might actually find that $i_\nu = 0$, that is, the partial differentiation is not active. This means that a number of successive variables are mapped to the same variable. So we will write simply '$\xrightarrow{\text{to } i}$' for saying that we map the current variables of index $\leq i$

to x_i. For example,

$$s_3(p,\underline{x}) \xrightarrow{\text{to } i} p_1x_i^3+\cdots+p_ix_i^3+p_{i+1}x_{i+1}^3+\cdots+p_kx_k^3$$
$$= Sp_{1:i}x_i^3+\sum_{l=i+1}^{k}p_lx_l^3 = Sp_{1:i}x_i^3+s_3(1+i:k),$$

where for reasons of space, as in Section 3, we write $Sp_{1:i}$ for $p_1+\cdots+p_i$ and use additionally the notation $s_3(i:k)$ for $\sum_{l=i}^{k}p_ix_i$.

Assume now $1\le i\le j\le m\le k$(=number of variables). We examine the effect of applying the operator $\cdot \xrightarrow{\text{to } i,\partial_i,\text{to } j,\partial_j,\text{to } m,\partial m} \cdot$ to polynomials $s_3(\underline{p},\underline{x})$, $s_1(\underline{p},\underline{x})s_2(\underline{p},\underline{x})$, and $s_1(\underline{p},\underline{x})^3$, respectively. We get

$$s_3(\underline{p},\underline{x}) \xrightarrow{\text{to } i} Sp_{1:i}x_i^3+s_3(\underline{p},x_{i+1:k}) \xrightarrow{\partial_i} 3Sp_{1:i}x_i^2 \xrightarrow{\text{to } j} 3Sp_{1:i}x_j^2 \xrightarrow{\partial_j} 6Sp_{1:i}x_j$$
$$\xrightarrow{\text{to } m} 6Sp_{1:i}x_m \xrightarrow{\partial_m} 6Sp_{1:i}.$$

$$s_1(\underline{p},\underline{x})s_2(\underline{p},\underline{x}) \xrightarrow{\text{to } i} (Sp_{1:i}x_i+s_1(i+1:k))(Sp_{1:i}x_i^2+s_2(i+1:k))$$
$$\xrightarrow{\partial_i} Sp_{1:i}(Sp_{1:i}x_i^2+s_2(i+1:k))+(Sp_{1:i}x_i+s_1(i+1:k))2Sp_{1:i}x_i$$
$$\xrightarrow{\text{to } j} Sp_{1:i}(Sp_{1:j}x_j^2+s_2(j+1:k))+(Sp_{1:j}x_j+s_1(j+1:k))2Sp_{1:i}x_j$$
$$\xrightarrow{\partial_j} 2Sp_{1:i}Sp_{1:j}x_j+Sp_{1:j}\cdot 2Sp_{1:i}x_j+(Sp_{1:j}x_j+s_1(1+j:k))2Sp_{1:i}$$
$$\xrightarrow{\text{to } m} 2Sp_{1:i}Sp_{1:j}x_m+Sp_{1:j}\cdot 2Sp_{1:i}x_m+(Sp_{1:m}x_m+s_1(1+m:k))2Sp_{1:i}$$
$$\xrightarrow{\partial_m} 4Sp_{1:i}Sp_{1:j}+2Sp_{1:i}Sp_{1:m}$$

$$s_1(\underline{p},\underline{x})^3 \xrightarrow{\text{to } i} (Sp_{1:i}x_i+s_1(i+1:k))^3 \xrightarrow{\partial_i} 3(Sp_{1:i}x_i+s_1(i+1:k))^2Sp_{1:i}$$
$$\xrightarrow{\text{to } j} 3(Sp_{1:j}x_j+s_1(j+1:k))^2Sp_{1:i} \xrightarrow{\partial_j} 6(Sp_{1:j}x_j+s_1(j+1:k))^1Sp_{1:i}Sp_{1:j}$$
$$\xrightarrow{\text{to } m} 6(Sp_{1:m}x_m+s_1(m+1:k))^1Sp_{1:i}Sp_{1:j} \xrightarrow{\partial_m} 6Sp_{1:i}Sp_{1:j}Sp_{1:m}.$$

Now recall that $\underline{p}$ is a probability vector and that $1\le i\le j\le m\le k$. So we get with $a=Sp_{1:i}$, $b=Sp_{1:j}$, $c=Sp_{1:m}$, that $0\le a\le b\le c\le 1$. From this and the formula for $q_3(\underline{p},\underline{x})$ above it follows that

$$q_3(\underline{p},\underline{x}) \xrightarrow{\text{to } i,\partial_i,\text{to } j,\partial_j,\text{to } m,\partial m} -6a+8ab+4ac-6abc = 2a(-3+4b+2c-3bc).$$

Assume $-3+4b+2c-3bc=-3+2c+b(4-3c)$ is positive for some admissible a,b,c. Then necessarily $4-3c>0$. Hence by choosing $b=c$ we again get a positive expression. It is $-3+6c-3c^2=-3(1-c)^2\le 0$, a contradiction. This proves that in a development of q in the sense of Corollary 2.2, the coefficient of any monomial product $(x_i - x_{i+1})(x_j - x_{j+1})(x_m - x_{m+1})$ is nonpositive; in other words, q_3 is $-\mathsf{pos}$. This proves Holland's inequality for degree 3 and any number of variables. The reader can also derive the symmetric case of the inequality we just proved via the theorem of Choi, Lam, Reznick mentioned in Section 1.

One can similarly investigate

$$q_4(\underline{p}, \underline{x}) = -s_4 + (2s_1s_3 + s_2^2) - 3s_1^2s_2 + s_1^4.$$

Upon applying the operator

$$\xrightarrow{\text{to } i, \partial_i, \text{to } j, \partial_j, \text{ to } m, \partial_m, \text{ to } l, \partial l}$$

and introducing a, b, c as above, and $d = Sp_{1:l}$, one will find that $s_4 \to 24a$, $s_3s_1 \to 12ab + 6ac + 6ad$, $s_2s_1^2 \to 12abc + 8abd + 4acd$, $s_2^2 \to 16ab + 8ac$, $s_1^4 \to 24abcd$, from where it follows that

$$q_4 \to -24a + (40ab + 20ac + 12ad) - (36abc + 24abd + 12acd) + 24abcd.$$

Again one can prove for the admissible $0 \leq a \leq b \leq c \leq d \leq 1$ that the expression on the right cannot be positive; hence q_4 is $-\mathsf{pos}$.

As of the closing date we had to impose on ourselves for this pre-print, we had made quite a number of manual computations of the above type – we do not know a good way to automatize them – and on base of these, we think to be now in the possession of a method that permits us to compute the reduction of $s_n(\underline{p}, \underline{x})s_k(\underline{p}, \underline{x})$ with relatively much less effort than the first few initial ones. In particular – dependent still on verification – we know some explicit formulae. For example, s_ns_3 reduces to

$$3n!(2n(n+1)a_1a_2 + n(n+1)a_1a_3 + (n^2 - 3n + 4)a_1a_4 + \sum_{l=5}^{n+1}(n-l+3)(n-l+2)a_1a_l),$$

where $a_i = Sp_{1:i}$, for $l = 1, ..., n$, if $n \geq 4$, and for smaller n if suitably truncated. The number of variables in a general weighted version can be choosen larger than the degree since the weights of unwanted variables can be choosen to be 0. However, we have not yet attacked the general reduction of products with three or more factors s_i; nor of course do we yet know to what expression the $q_l(\underline{p}, \underline{x})$ in general reduce. With some luck these things will be the object of somebody's future publication.

5. Final Remarks

This paper is a contribution to the general problem to write a polynomial nonnegative on a certain region of $\mathbb{R}^n$ in a way that turns its nonnegativity manifest. The most well-known contributions of this type are of course those that deal with sum-of-squares representations of everywhere nonnegative rational functions. More closely related to the results here proved, are the author's unpublished thesis and a publication of Handelman [H].

In the thesis a quite different, combinatorial, way to deal with questions of above sort is given. It works by noting that, given a real $m \times n$ matrix $A = (a_{ij}) = [a_{*,1}, a_{*,2}, ..., a_{*,n}]$, where $a_{*,j}$ is the j-th column of A and a function like $\tau(A) = \sum_{j=1}^{n} \prod_{i=1}^{m} a_{ij}$, then one can use iteratively the inequality

$$\tau(A) \leq \tau([a_{*,1}, ..., a_{*,j} \wedge a_{*,j+1}, a_{*,j} \vee a_{*,j+1}, ..., a_{*,n}])$$

to achieve a stepwise transformation of a left hand side of an inequality to the right (larger side) of the same inequality. Here $a \wedge a'$ and $a \vee a'$ means to form the entrywise minimum and maximum of columns a, a' respectively. Section 2 of [K] provides a glimpse of the method.

The main result in [H] is this:

Theorem (Handelman [H]) *Let $p \in \mathbb{R}[x_1, ..., x_n]$ and let $g_1, g_2, ..., g_m(x)$ be linear polynomials which define a (bounded) polytope $K = \{x \in \mathbb{R}^n : g_1(x) \geq 0, \cdots, g_m(x) \geq 0\}$. If $p > 0$ on K then p can be written in the form*

$$p = \sum_{\beta \in \mathbb{N}^m} c_\beta \prod_{j=1}^{m} g_j^{\beta_j}.$$

In our case the polynomials g_i would be $g_i(x) = x_i - x_{i+1}$; but these define only a non-compact polyhedron. Also note that in Handelman's theorem, $p > 0$ is required. Our example for Motzkin's polynomial shows that one cannot hope for a Handelman-like-theorem if one relaxes the conditions above to '$p \geq 0$ and K polyhedron': representations via the $h_i = x_i - x_{i+1}$ are of course unique. Handelman's paper of course has further examples showing inadmissibility of certain relaxations on K and positivity necessary for his theorem to hold.

Investigations into the signs of the coefficients of another type of power series have apparent origin in a question of Friedrich's and Lewy on the wave equation and a result of Szegö according to which there holds positivity of the numerical coefficients a_{klm} of the power series

$$((1-x)(1-y)+(1-x)(1-z)+(1-y)(1-z))^{-1} = \sum_{k,l,m\geq 0} a_{k,l,m} x^k y^l z^m.$$

A very impressive paper that deals with power series of this type is the paper of Scott and Sokal [SS] who show that expressions of the above type can be viewed as inverses of $T_G(1-x_1,...,1-x_n)$ where T_G is a spanning tree polynomial of a graph G on n vertices. With their methods Scott and Sokal solve questions that have been open for decades. It might pay to see whether there is a connection between these lines of investigation and our type of power series, but to the present author they are not apparent.

When we began working on Laffey's problem, we looked into what is known about taking roots of power series and discovered the paper by Gould [G]. This paper formulates and proves the following

Theorem. *A power series $f(t) = \sum_{j\geq 0} a_j t^j$ with $a_0 \neq 0$ is related to its p-th power $f^p = \sum_{j\geq 0} b_j t^j$ by the recursion*

$$\sum_{k=0}^{n} (k(p+1)-n) a_k b_{n-k} = 0.$$

If we start with the fact that, e.g. $f(t) = ((1-x_1 t)(1-x_2 t)(1-x_3 t))^{1/3}$ is a power series in t, whose third power is evidently $1 - e_1(\underline{x})t + e_2(\underline{x})t^2 - e_3(\underline{x})t^3$, where $\underline{x} = x_{1:3}$ and $e_i(\underline{x})$ is the i-th elementary symmetric polynomial in x, then this theorem implies that the coefficient polynomials $u_n(\underline{x})$ can be computed by the recursion

$$u_{\text{negative}} = 0, \quad u_0 = 1, \quad u_n = (1-\frac{4}{3n})e_1 u_{n-1} + (\frac{8}{3n}-1)e_2 u_{n-2} + (1-\frac{4}{n})e_3.$$

We originally hoped to use this recursion to prove that for all $n \geq 1$, $\underline{x} \geq 0$, $u_n(\underline{x}) \leq 0$, but now we see instead that results as in Section 2 prove at the same time results about positivity or negativity of certain recursively defined sequences.

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References

[CJ] Caviness, B.F., Johnson, J. R. (eds.), Quantifier Elimination and Cylindrical Algebraic Decomposition, Springer 1995.

[CLR1] Choi, M.D., Lam, T.Y., Reznick, B., *Even symmetric sextics,* Math. Z. 195 (1987) 559-580.

[CLR2] Choi, M.D., Lam, T.Y., Reznick, B., *Positive Sextics and Schur's Inequalities,* J. Algebra, 141 (1991) 36-78.

[CPR] Castle, M., Powers, V., Reznick, B., *Polya's Theorem with zeros,* J. of Symb. Computation 46 (2011) 1039-1048.

[DP] Delzell, C., Prestel, A., Positive Polynomials, Springer 2001.

[FH] Frenkel, P., Horvath, P., *Minkowski's Inequalities and Sums of Squares,* Centr. Eur. J. Mathematics 12 (3) (2014) 510-516.

[G] Gould, H.W., *Coefficient identities for Powers of Taylor and Dirichlet Series,* Amer. Math. Monthly 81(1) (1974) 3-14.

[GKP] Graham, R., Knuth, D., Patashnik, O., Concrete Mathematics, Addison Wesley 1989.

[HLP] Hardy G.H., Littlewood, J.E., and Pólya G., Inequalities, 2nd. ed. Cambridge Univ. Press, Cambridge 1952.

[H] Handelman, D.W., *Representing polynomials by positive linear functions on compact convex polyhedra,* Pacific J. Math. 132(1) 1988, 35-62.

[Hol] Holland, F., *A contribution to the nonnegative inverse eigenvalue problem,* Mathematical Proceedings of the Royal Irish Academy 113A (2013) (2) 81-96.

[Kal] Kaluza, T., *Über die Koeffizienten reziproker Potenzreihen,* Math. Z. 38 (1928) 161-70.

[K] Kovačec, A., *On an algorithmic method to prove inequalities,* General inequalities 3, Proc. of the 3rd int. Conf. on Inequalities, Oberwolfach 1981, ISNM 64, 69-89, Birkhaeuser 1983.

[L] Laffey, T.J., *Formal power series and the spectra of nonnegative real matrices,* Mathematical Proceedings of the Royal Irish Academy 113A (2013) (2) 97-106.

[LLS] Laffey, T.J., Loewy, R., Šmigoc, H., *Power series with positive coefficients arising from the characteristic polynomials of positive matrices,* Math. Ann. 364 (2015) 687-707.

[M] Marshall, M., Positive Polynomials and Sums of Squares, Mathematical Surveys and Monographs Vol. 146, Amer.Math. Soc. 2008.

[P] Pólya, G., *Über positive Darstellung von Polynomen,* Vierteljschr. Naturforsch. Ges. Zürich (1928) 141-145, also in [P2], pages 309-313.

[P2] Pólya, G., Collected Papers of G. Pólya, Vol II, Location of zeros (R.P. Boas, Ed.), MIT Press 1974.

[PR] Powers, V., Reznick, B., *A new bound for Pólya's Theorem with applications to polynomials positive on polyhedra,* J. Pure Appl. Algebra 164 (2001) 221-229.

[PW] Powers, V., Woermann, T., *An algorithm for sums of squares of real polynomials,* J. Pure and Appl. Algebra 127 (1998) 99-104.

[R] Rado, R., *An inequality,* J. London Math. Soc. 27 (1952) 1-6.

[Re] Reznick, B., *Uniform denominators in Hilbert's 17th problem,* Math. Z. 220 (1995) 75-98.

[Rie] Riener, C., *On the degree and half degree principle for symmetric polynomials,* J. Pure Appl. Algebra, 216 (2012) 850-856.

[SS] Scott, A.D., Sokal, A.D., *Complete monotonicity for inverse powers of some combinatorially defined polynomials,* Acta Mathematica, 213 (2014) 323-329.

[T] Timofte, V., *On the positivity of symmetric polynomial functions. Part I: General results.* J. Math. Anal. Appl. 284, (2003) 174-190.

[VB] Vandenberghe,L., Boyd, S., *Semidefinite Programming,* SIAM Rev. 38 (1996) 49-95.

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