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# EXISTENCE OF SUBLATTICE POINTS IN LATTICE POLYGONS

NIKOLAI BLIZNYAKOV AND STANISLAV KONDRATYEV

Abstract: We state the formula for the critical number of vertices of a convex lattice polygon that guarantees that the polygon contains at least one point of a given sublattice and give a partial proof of the formula. We show that the proof can be reduced to finding upper bounds on the number of vertices in certain classes of polygons. To obtain these bounds, we establish inequalities relating the number of edges of a broken line and the coordinates of its endpoints within a suitable class of broken lines.

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## 1. Introduction

The study of lattice point in convex sets is a classical subject. The starting point was Minkowski's Convex Body Theorem, which became the foundation of the geometry of numbers. The theorem states that if a compact set in $\mathbb{R}^d$ is symmetric with respect to origin and has volume at least $2^d$, then it contains a point of the integer lattice $\mathbb{Z}^d$. The constant $2^d$ cannot be improved. This theorem has quite a few modificatioins and generalisations, see e. g. the nice short survey [22].

There are numerous results concerning lattice points in various regions, see e. g. [6, 9, 10, 12]. The regions at issue can be either general convex and nonconvex or polyhedral sets. Among more recent works we note the following that are close to ours. The papers [2,5,16,17] deal with the largest possible number of facets of maximal lattice-free polytopes. The papers [3,14,15,19] study properties of lattice polytopes having a specified (positive) number of interior lattice points such as upper bounds for the volume and the number of sublattice points and a classification of such polytopes. The papers [20,21] deal with similar issues for polygons.

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Besides, there are other interesting results about lattice polygons, such as [1, 22, 23], not to mention the well-known Pick's theorem.

In this paper we consider the natural problem of relating the existence of sublattice points in a convex lattice polygon to the number of vertices (or edges) of the polygon.

In higher dimensions, a large number of faces cannot guarantee that the polytope contains a point of a given sublattice. For instance, there is no upper bound for the number of vertices and facets of polytopes in $\mathbb{R}^3$ free of points of $(2\mathbb{Z})^3$.

Surprisingly, things are different in two dimensions. It was noticed in [7] that any convex integer pentagon on the plain contains a point of the lattice $(2\mathbb{Z})^2$. In this paper we show that any convex integer polygon with many enough vertices contains at least one point of a given sublattice (of maximal rank) of $\mathbb{Z}^2$.

In the spirit of the Minkowski Convex Body Theorem, our main goal is to explicitly find the critical number of vertices that ensures that the polygon contains a point of a given sublattice. The Main Theorem stated in Section 2.1 provides this formula.

To put it the other way around, the Main Theorem gives an optimal upper bound on the number of vertices of a convex lattice polygon free of points of a given sublattice. Clearly, convexity is essential for this bound to exist, but we do not impose other requirements on the lattice polygons.

The proof of the Main Theorem can be naturally reduced to estimating the number of vertices of integer polygons free of points of the lattice $n\mathbb{Z}^2$. This can be broken up into two major steps.

First, we would like to obtain a feasible description of integer polygons free of points of $n\mathbb{Z}^2$. A crucial property of such polygons is each of them lies in a $n\mathbb{Z}^2$-slab of $n\mathbb{Z}^2$-width 3 (Proposition 2.8). Using this as basis, we classify such polygons up to affine transformations preserving the lattice $n\mathbb{Z}^2$ into six types differing by imposed geometric constraints (Definition 2.11 and Theorem 2.12).

The second step is to estimate the number of vertices for each type. This requires a subtle geometric analysis and can be quite technical in terms of estimates. In this paper we develop necessary tools in Section 3 and apply them to one particular class of polygons, where the estimates can be derived immediately. More technical cases are the subject of [8].

[8] Nikolai Bliznyakov and Stanislav Kondratyev. Bounds on the number of vertices of convex integer polygons, 2016. Pré-Publicações do Departamento de Matemática, Universidade de Coimbra, Preprint Number 16-28.
[9] John William Scott Cassels. *An introduction to the geometry of numbers*. Springer Science & Business Media, 2012.
[10] Paul Erdös, Peter M Gruber, and Joseph Hammer. *Lattice points*. Longman scientific & technical Harlow, 1989.
[11] Peter Gruber. *Convex and discrete geometry*, volume 336. Springer Science & Business Media, 2007.
[12] Peter M Gruber and Cornelis Gerrit Lekkerkerker. *Geometry of numbers*. North-Holland, 1987.
[13] Branko Grünbaum. Convex polytopes, volume 221 of graduate texts in mathematics, 2003.
[14] Douglas Hensley. Lattice vertex polytopes with interior lattice points. *Pacific Journal of Mathematics*, 105(1):183–191, 1983.
[15] Jeffrey C Lagarias and Günter M Ziegler. Bounds for lattice polytopes containing a fixed number of interior points in a sublattice. *Canadian Journal of Mathematics*, 43:1022–1035, 1991.
[16] László Lovász. Geometry of numbers and integer programming. *Mathematical Programming: Recent Developments and Applications*, pages 177–210, 1989.
[17] Diego AR Morán and Santanu S Dey. On maximal s-free convex sets*. *SIAM Journal on Discrete Mathematics*, 25(1):379, 2011.
[18] C. Norman. *Finitely Generated Abelian Groups and Similarity of Matrices over a Field*. Springer Undergraduate Mathematics Series. Springer, 2012.
[19] Oleg Pikhurko. Lattice points in lattice polytopes. *Mathematika*, 48(1-2):15–24, 2001.
[20] Stanley Rabinowitz. A census of convex lattice polygons with at most one interior lattice point. *Ars Combin*, 28:83–96, 1989.
[21] Stanley Rabinowitz. On the number of lattice points inside a convex lattice n-gon. *Congr. Numer*, 73:99–124, 1990.
[22] Paul R Scott. Modifying Minkowski's theorem. *Journal of Number Theory*, 29(1):13–20, 1988.
[23] RJ Simpson. Convex lattice polygons of minimum area. *Bulletin of the Australian Mathematical Society*, 42(03):353–367, 1990.
[24] Günter M Ziegler. *Lectures on polytopes*, volume 152. Springer Science & Business Media, 1995.

Nikolai Bliznyakov
Faculty of Mathematics, Voronezh State University, 1 Universitetskaya pl., Voronezh, 394006, Russia
*E-mail address*: bliznyakov@vsu.ru

Stanislav Kondratyev
CMUC, Department of Mathematics, University of Coimbra, 3001-501 Coimbra, Portugal
*E-mail address*: kondratyev@mat.uc.pt



The general idea is to break up the boundary of a polygon into several broken lines and to translate geometrical constraints imposed on the polygon into Diaphantine inequalities. Resulting inequalities relate the numbers of edges of the broken lines and the coordinates of their endpoints, which are also the overall dimensions of the polygon. The most valuable tools are Theorem 3.7, Corollary 3.8, and Theorem 3.9. They are an important contribution of this paper.

The rest of the paper is organised as follows.

Section 2 is devoted to the overview of the results, the Main Theorem being stated in Section 2.1 and Section 2.3 containing a detailed synopsis of the proof. For convenience, the statement of the Main Theorem is split into Sub-Theorems A, B, and C.

In Section 3 we study a class of broken lines we call slopes. The first two subsections contain definitions and statements estimating the number of edges of a slope. Then we show how these estimates can be applied to polygons and conclude the section by obtaining estimates for a particular class of polygons.

We tried to keep to a minimum the number of proofs in Sections 2 and 3. Rather, we collected technical proofs in subsequent sections.

In Section 4 we give fairly simple proofs of Sub-Theorems A and B that make use of the so-called parity argument and Pick's formula. An easy particular case of Sub-Theorem C is also proved there.

In Section 5 we study certain properties of the lattice diameter of polygons that allow us to prove Proposition 2.8 and Theorem 2.12.

Section 6 provides the proofs of theorems of Section 3.

## 2. Main theorem

**2.1. Main theorem.** Suppose the system of vectors $\mathbf{a}_1,\ \mathbf{a}_2 \in \mathbb{R}^2$ is linearly independent; then the set

$$\{u_1\mathbf{a}_1 + u_2\mathbf{a}_2 \colon u_1, u_2 \in \mathbb{Z}\}$$

is called *a lattice* spanned by $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_1$, $\mathbf{a}_2$ are called the *basis* of the lattice.

**Example 2.1.** The vectors $\mathbf{e}_1 = (1,0)$, $\mathbf{e}_2 = (0,1)$ span the *integer lattice* denoted by $\mathbb{Z}^2$. It is the set of points with both integral coordinates. Those are called *integer points*.

A lattice $\Gamma$ is called *a sublattice* of a lattice $\Lambda$ if $\Gamma \subset \Lambda$. If moreover $\Gamma \neq \Lambda$, $\Gamma$ is called *a proper sublattice* of $\Lambda$. In what follows we only consider sublattices of the integer lattice.

A lattice $\Lambda \subset \mathbb{Z}^2$ is spanned by the columns of a matrix $A = (a_{ij}) \in GL_2(\mathbb{Z})$ if and only if $\Lambda = A\mathbb{Z}^2 = \{A\mathbf{u}\colon \mathbf{u} \in \mathbb{Z}^2\}$. Given $\Lambda$, the matrix $A$ is not uniquely defined. However, the numbers

$$\delta = \gcd(a_{ij}),\ n = |\det A|/\delta$$

are independent of $A$. They are called *invariant factors* of $\Lambda$, and the pair $(\delta, n)$ is *the invariant factor sequence of* $\Lambda$ (see e. g. [18]).

**Example 2.2.** The lattice $n\mathbb{Z}^2 = (n\mathbb{Z}) \times (n\mathbb{Z}) = \{(nu_1, nu_2)\colon u_1, u_2 \in \mathbb{Z}\}$, where $n$ is a positive integer, has invariant factor sequence $(n, n)$.

**Example 2.3.** The lattice $\delta\mathbb{Z} \times n\mathbb{Z}^2 = \{(\delta u_1, nu_2)\colon u_1, u_2 \in \mathbb{Z}^2\}$, where $\delta$ and $n$ are positive integers and $\delta$ divides $n$, has invariant factor sequence $(\delta, n)$.

The *convex polygon* is a two-dimensional polytope, i. e. the convex hull of a finite set of points that has nonempty interior. In what follows we only consider convex polygons, so we often drop the word 'convex'. We assume that the reader is familiar with basic terminology such as vertex and edge, see [13,24] for details. A polygon with $N$ vertices, $N \geq 3$, is called an $N$-gon. The vertices of an *integer polygon* belong to $\mathbb{Z}^2$. More generally, if all the vertices of a polygon belong to a lattice $\Gamma$, it is called a $\Gamma$-*polygon*. Integer polygons are also called *lattice polygons*, but to avoid misunderstanding, we prefer the first term, since we consider $\Gamma$-polygons with different lattices $\Gamma$.

Given a sublattice $\Lambda$ of $\mathbb{Z}^2$ with invariant factor sequence $(\delta, n)$, define

$$\nu(\Lambda) = \nu(\delta, n) = 2n + 2\min\{\delta, 3\} - 3.$$

**Main Theorem.** *Let $\Lambda$ be a proper sublattice of $\mathbb{Z}^2$. Then any convex integer polygon with $\nu(\Lambda)$ vertices contains a point of $\Lambda$.*

It is easily seen that the constant $\nu(\Lambda)$ in the Main Theorem is sharp, i. e. if $\nu(\Lambda) > 3$ for given $\Lambda$, then there exist $(\nu(\Lambda)-1)$-gons containing no points of $\Lambda$. This is very clear in case $\Lambda = \delta\mathbb{Z} \times n\mathbb{Z}$ (see Figure 1). The general case follows from the fact that any lattice with invariant factor sequence $(\delta, n)$ is the image of $\delta\mathbb{Z} \times n\mathbb{Z}$ under a linear transformation preserving the integer lattice, see Section 2.2.

For a synopsis of the proof, see Section 2.3.



As $\hat{\pi}(E)$ is an integer, this yields

$$\hat{\pi}(E) \geq t - s + \left\lceil \frac{-w_2}{2} \right\rceil - 1.$$

As $\pi_1(E) + \pi_2(E) = v_2 + w_1$ by Remark 6.4, now it remains to use (6.5) in order to obtain (3.14)

Now assume that the frame does not form small angle with $Q$. Let us check that in this case (3.11)–(3.14) hold with $t = s = 0$.

Inequality (3.11) becomes trivial, and (3.12) follows from the definition of a splitting frame. Inequality (3.13) becomes

$$-v_{N1} < v_{N2}.$$

It is true, since otherwise by Proposition 3.6 the frame would form small angle with $Q$. To prove (3.14), it suffices to apply the proved part of the theorem to $Q$ and the frame $(\mathbf{o}; \mathbf{f}_2, \mathbf{f}_1)$, which forms small angle with $Q$ by Proposition 3.6. Indeed, with certain $\tilde{t}$ and $\tilde{s}$ by virtue of (3.14) and (3.11) we have:

$$2N \leq w_1 + v_2 - \tilde{t} + \tilde{s} \leq v_2 + w_1,$$

so (3.14) holds for $(\mathbf{o}; \mathbf{f}_2, \mathbf{f}_1)$ with $t = s = 0$ as claimed. ∎

*Proof of Theorem 3.9*: It suffices to apply Lemma 6.7, write inequality (6.14) in the form

$$\pi_1(E) + \pi_2(E) - 2N \geq 1,$$

and substitute $\pi_1(E) = w_1$ and $\pi_2(E) = v_2$ according to Remark 6.4. ∎

## References


[1] Dragan M Acketa and Jovia D unić. On the maximal number of edges of convex digital polygons included into an m× m-grid. *Journal of Combinatorial Theory, Series A*, 69(2):358–368, 1995.

[2] Gennadiy Averkov. On maximal s-free sets and the helly number for the family of s-convex sets. *SIAM Journal on Discrete Mathematics*, 27(3):1610–1624, 2013.

[3] Gennadiy Averkov, Jan Krümpelmann, and Benjamin Nill. Largest integral simplices with one interior integral point: Solution of hensley's conjecture and related results. *Advances in Mathematics*, 274:118–166, 2015.

[4] Imre Bárány and Zoltán Füredi. On the lattice diameter of a convex polygon. *Discrete Mathematics*, 241(1):41–50, 2001.

[5] Amitabh Basu, Michele Conforti, Gérard Cornuéjols, and Giacomo Zambelli. Maximal lattice-free convex sets in linear subspaces. *Mathematics of Operations Research*, 35(3):704–720, 2010.

[6] Matthias Beck and Sinai Robins. *Computing the continuous discretely*. Springer, 2007.

[7] Nikolai Bliznyakov. On an extremal property of integer polygons (Russian). In *Abstracts of the international conference 'Nonlinear analysis and functional differential equations'*, page 58, Voronezh, 2000. Voronezh State University.

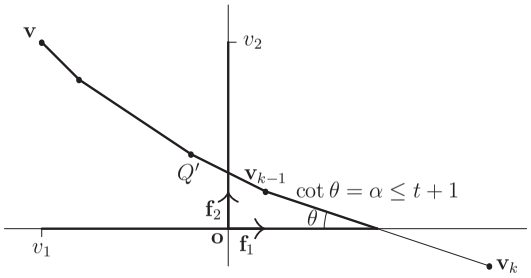


FIGURE 10. The intersection of $Q$ with the upper half-space is a (possibly non-integer) slope $Q'$. The length of the vertical projection of $Q'$ is $v_2$ and that of its horizontal projection is strictly greater than $-v_1$. All the edges belonging to $S$ are edges of $Q'$. They all contribute $s$ to the length of the vertical projection of $Q'$, whence (3.12). Also, they all contribute at most $t + (t-1) + \cdots + (t-s+1) = ts - s(s-1)/2$ to the horizontal projection. The horizontal contribution of any other edge is less than $t+1$ times its vertical contribution, totalling at most $(t+1)(v_2 - s)$ for all the edges not in $S$, and (3.13) follows.

Let us prove (3.14). Since the sets $E_1$ and $E_2$ are disjoint and their union is $E$, we have $\hat{\pi}(E) = \hat{\pi}(E_1) + \hat{\pi}(E_2)$. Evoking Lemmas 6.5 and 6.6, we obtain

$$\hat{\pi}(E) = (v^+_{k-1,1} + v_{k-1,2} - 1) + \delta + (t-s) + \lfloor(-v_{k2}-1)\alpha\rfloor + \frac{1}{2}(v_{k2} - w_2 - 1), \quad (6.21)$$

where $\delta$ is defined by (6.7). By definition, $\delta \ge 0$. The first term on the right-hand side of (6.21) is nonnegative, since at least one coordinate of $\mathbf{v}_{k-1}$ must be positive (Remark 3.5. Let us estimate the fourth term on the right-hand side of (6.21). By the definition of $k$ we have $v_{k2} < 0$, so $-v_{k2} - 1 \ge 0$, and by assumption, $\alpha \ge 1$; thus, we have:

$$\lfloor(-v_{k2}-1)\alpha\rfloor \ge -v_{k2} - 1 \ge \frac{1}{2}(-v_{k2}-1).$$

Consequently, from (6.21) we obtain

$$\hat{\pi}(E) \ge t - s + \frac{-w_2}{2} - 1.$$



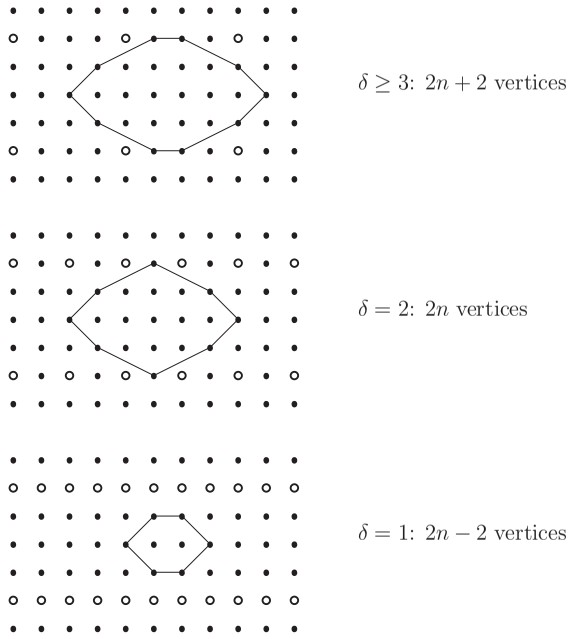


FIGURE 1. If $\delta \ge 3$, it is easy to construct a polygon lying in the slab $0 \le x_2 \le n$ and having two vertices on each of the lines $x_2 = j$, where $j = 0, \ldots, n$, such that the vertices belonging to the lines $x_2 = 0$ and $x_2 = n$ lie between adjacent points of $\Lambda$. Clearly, such a polygon is free from points of $\Lambda$ and has $2n+2 = \nu(\delta, n) - 1$ vertices. If $\delta = 1$, the construction is similar, only the polygon should have one vertex on each of the lines $x_2 = 0$ and $x_2 = n$ not belonging to $\Lambda$. If $\delta = 0$, it suffices to take any integer polygon lying in the slab $1 \le x_2 \le n-1$ having two vertices on each of the lines $x_2 = j$, where $j = 1, \ldots, n-1$.

**2.2. Preliminaries.** In this section we list a few familiar properties of lattices. The proofs can be found in [6, 9, 11, 12].

We always denote the vectors of the standard basis of $\mathbb{R}^2$ by $\mathbf{e}_1 = (1,0)$ and $\mathbf{e}_2 = (0,1)$ and the standard coordinates in $\mathbb{R}^2$ by $x_1$, $x_2$.

Note that any lattice is a subgroup of the additive group of the linear space $\mathbb{R}^2$ and a free abelian group of rank 2.

A matrix $A \in M_2(\mathbb{Z})$ is called *unimodular*, if $\det A = \pm 1$.

**Proposition 2.4.** *Let $(\mathbf{f}_1, \mathbf{f}_2)$ be a basis of a lattice $\Lambda$; then the vectors $a_{i1}\mathbf{f}_j + a_{i2}\mathbf{f}_2$, where $i = 1, 2$, form a basis of $\Lambda$ if and only if the matrix $(a_{ij})$ is unimodular.*

For brevity, we write that $\Lambda$ is a $(\delta, n)$-lattice if it is a sublattice of $\mathbb{Z}^2$ with invariant factor sequence $(\delta, n)$. The number $\delta n$ is called the *determinant* of $\Gamma$ and denoted $\det \Gamma$.

We use the term '$\Lambda$-point' as a synonym of 'point of $\Lambda$'.

A linear transformation of the plane is called a (linear) automorphism of a lattice if it maps the lattice onto itself. It is easily seen that a linear transformation is an automorphism of a lattice if and only if it maps some (hence, any) basis of the lattice onto another basis. Consequently, given a matrix $A \in M_2(\mathbb{R})$, the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is an automorphism of $\mathbb{Z}^2$ if and only if the matrix $A$ is unimodular. We call such transformation *unimodular*. For any positive integer $n$, the automorphisms of $n\mathbb{Z}^2$ are exactly unimodular transformations.

Clearly, linear automorphisms of a lattice form a group.

Let $\Lambda$ be a lattice. A vector $\mathbf{f} \in \Lambda$ is called $\Lambda$-*primitive*, if any representation $\mathbf{f} = u\mathbf{g}$ with $g \in \Lambda$ and $u \in \mathbb{Z}$ implies $u = \pm 1$.

**Proposition 2.5.** *Suppose that $\Lambda$ is a lattice and $\mathbf{f}$ and $\mathbf{g}$ are $\Lambda$-primitive vectors; then there exists an automorphism $A$ of $\Lambda$ such that $A\mathbf{f} = \mathbf{g}$.*

If $\Lambda$ is a sublattice of $\mathbb{Z}^2$ and $A$ is a unimodular transformation, the image $A\Lambda$ is a lattice with the same invariant factors as $\Lambda$.

The following proposition is a fundamental result about unimodular transformations. It is a geometric version of the Smith normal form of integral matrices [18].

**Proposition 2.6.** *For any sublattice of $\mathbb{Z}^2$ with invariant factors $(\delta, n)$ there exists a unimodular transformation mapping it onto the lattice $\delta\mathbb{Z} \times n\mathbb{Z}$.*

An *affine frame* of a lattice $\Lambda$ is a pair $(\mathbf{o}; \mathbf{f}_1, \mathbf{f}_2)$ consisting of a point $\mathbf{o} \in \Lambda$ and a basis $(\mathbf{f}_1, \mathbf{f}_2)$ of $\Lambda$. An *integer frame* is an affine frame of $\mathbb{Z}^2$.

An *affine automorphism* of a lattice $\Lambda$ is an affine transformation of $\mathbb{R}^2$ mapping $\Lambda$ onto itself. It is not hard to see that given $A \in M_2(\mathbb{R})$ and $\mathbf{b} \in \mathbb{R}^2$, the mapping $\mathbf{x} \mapsto A\mathbf{x}+\mathbf{b}$ is an affine automorphism of $\Lambda$ if and only if $\mathbf{x} \mapsto A\mathbf{x}$ is an automorphism of $\Lambda$ and $\mathbf{b} \in \Lambda$. In particular, affine automorphisms



The second one is nonnegative by definition, the third one by Lemma 6.3, and the last one by the definition of $k$. Thus, (6.15) can hold only if

$$v^+_{k-1,1} + v_{k-1,2} = 1, \tag{6.16}$$

$$\delta = 0, \tag{6.17}$$

$$t = s, \tag{6.18}$$

$$v_{k2} = -1, \tag{6.19}$$

where we recover (6.19) due to the fact that $\alpha \geq 1$. Also, (6.17) implies that $\alpha > 1$, since we are assuming $v_{k-1,2} > 0$ if $\alpha = 1$. Thus, we have $t \geq 1$, and by Lemma 6.3 we conclude from (6.18) that

$$t = s = 1 \tag{6.20}$$

and the set $S$ consists of a single edge $\varepsilon_i$ having the associated vector $\mathbf{a}_i = \mathbf{f}_1 - \mathbf{f}_2$. Thus, we necessarily have $\mathbf{f}_1 - \mathbf{f}_2 \in \Gamma$.

It follows from (6.16) that either $v_{k-1,2} = 0$ or $v_{k-1,2} = 1$.

In the former case case we use (6.19) to get $a_{k2} = v_{k2} - v_{k-1,2} = -1$; moreover, $t = 1$ translates into $1 < \alpha \leq 2$. Thus, $a_{k1} = -\alpha a_{k2} = \alpha$, and consequently, $a_{k1} = 2$. We conclude that $\Gamma$ contains the vector $\mathbf{a}_k = 2\mathbf{f}_1 - \mathbf{f}_2$. But then by Proposition 2.4 the vectors $\mathbf{a}_k$ and $\mathbf{f}_1 - \mathbf{f}_2$ form a basis of $\mathbb{Z}^2$, which is impossible, since they belong to its proper sublattice.

In the latter case $v_{k-1,2} = 1$ we again use (6.19) to get $a_{k2} = -2$. Moreover, as $t = 1$ and $\alpha \notin \mathbb{Z}$ by virtue of (6.17), we have $1 < \alpha < 2$. Consequently, the integer $a_{k1} = -\alpha a_{k2}$ belongs to the interval $(2, 4)$, i. e. $-a_{k1} = 3$. Thus, $\mathbf{a}_k = 3\mathbf{f}_1 - 2\mathbf{f}_2$. But in this case we see once again that the vectors $\mathbf{a}_k$ and $\mathbf{f}_1 - \mathbf{f}_2$ form a basis of $\mathbb{Z}^2$, which is impossible since they belong to its proper sublattice $\Gamma$. ∎

### 6.3. Proof of Theorems 3.7 and 3.9.

*Proof of Theorem 3.7*: First assume that the frame $(\mathbf{o}; \mathbf{f}_1, \mathbf{f}_2)$ forms small angle with $Q$, or, equivalently, $\alpha \geq 1$ (Remark 6.1). We show that in these case inequalities (3.11)–(3.15) hold with $t$ and $s$ defined in Section 6.1.

Inequalities (3.11)–(3.13) follow from Lemma 6.3 and simple combinatorial arguments, see Figure 10.

Using (3.3), we see that

$$\alpha = \frac{a_{k1}}{-a_{k2}} < \frac{a_{k+1,1}}{-a_{k+1,2}} < \cdots < \frac{a_{N1}}{-a_{N2}};$$

besides, as $(\mathbf{o}; \mathbf{f}_1, \mathbf{f}_2)$ forms small angle with $Q$, we have $\alpha \geq 1$ (Remark 6.1; consequently,

$$a_{i1} + a_{i2} > 0.$$

Now it is not hard to check that $g$ takes nonnegative values in all the points of the set

$$\{(m_1, m_2) \in \mathbb{Z}^2 \colon m_1 > 0, m_2 < 0, m_1 + m_2 > 0\}$$

except $(2, -1)$, and $g(2, -1) = -1$. Consequently, (6.13) gives

$$\hat{\pi}(\varepsilon_i) \geq \frac{-a_{i2} - \delta_i}{2},$$

where

$$\delta_i = \begin{cases} 1, & \text{if } \mathbf{a}_i = 2\mathbf{f}_1 - \mathbf{f}_2, \\ 0, & \text{otherwise.} \end{cases}$$

The vectors $\mathbf{a}_i$ are distinct, so at most one $\delta_i$ is nonzero. Thus, we have:

$$\hat{\pi}(E_1) = \sum_{i=k+1}^{N} \hat{\pi}(\varepsilon_i) \geq \frac{1}{2}\left(-\sum_{i=k+1}^{N} a_{i2} - \sum_{i=k+1}^{N} \delta_i\right) \geq \frac{1}{2}(v_{k-1,\,2} - v_{02} - 1),$$

and (6.12) is proved. ∎

**Lemma 6.7.** *Suppose that $Q$ is a $\Gamma$-slope, where $\Gamma$ is a proper sublattice of $\mathbb{Z}^2$; then*

$$\hat{\pi}(E) \geq 1. \tag{6.14}$$

*Proof*: In view of Remark 6.2, there is no loss of generality in assuming that $(\mathbf{o}; \mathbf{f}_1, \mathbf{f}_2)$ forms small angle with $Q$, and moreover, that $v_{k-1,2} > 0$ if $\alpha = 1$.

Suppose, contrary to our claim, that $\hat{\pi}(E) \leq 0$. As $E = E_1 \cup E_2$ and $E_1 \cap E_2 = \varnothing$, we have $\hat{\pi}(E) = \hat{\pi}(E_1) + \hat{\pi}(E_2)$. It follows from Lemma 6.6 that $\hat{\pi}(E_2) \geq 0$, so we conclude that $\hat{\pi}(E_1) \leq 0$. Together with Lemma 6.5 this gives

$$(v^+_{k-1,1} + v_{k-1,2} - 1) + \delta + (t - s) + \lfloor(-v_{k2} - 1)\alpha\rfloor \leq 0, \tag{6.15}$$

where $\delta$ is defined by (6.7). Observe that the summands on the left-hand side are nonnegative. Indeed, the vertex $\mathbf{v}_{k-1}$ cannot simultaneously satisfy $v_{k-1,1} \leq 0$ and $v_{k-1,2} \leq 0$ (Remark 3.5), so the first summand is nonnegative.



of $n\mathbb{Z}^2$, where $n$ is a positive integer, are exactly the transformations of the form $\mathbf{x} \mapsto A\mathbf{x} + \mathbf{b}$, where $A$ is unimodular and $\mathbf{b} \in n\mathbb{Z}^2$.

Of course, if $P$ is a convex integer $N$-gon and $\varphi$ is an affine automorphism of $\mathbb{Z}^2$, the image $\varphi(P)$ is still a convex integer $N$-gon. Obviously, is $P$ is free from points of a lattice $\Lambda$, then so is its image under any affine automorphism of $\Lambda$.

We conclude with a nonstandard definition.

Let $\Lambda$ be a sublattice of $\mathbb{Z}^2$ and $(\mathbf{f}_1, \mathbf{f}_2)$ be a basis of $\mathbb{Z}^2$. Clearly, $\{u \in \mathbb{Z} \colon u\mathbf{f}_1 \in \Lambda\}$ is a subgroup of $\mathbb{Z}$. It is generated by a positive integer, which we call the *large* $\mathbf{f}_1$*-step* of $\Lambda$ with respect to $(\mathbf{f}_1, \mathbf{f}_2)$. Further, $\{u_1 \in \mathbb{Z} \colon \exists\, u_2 \in z,\ u_1 f_1 + u_2 f_2 \in \Lambda\}$ is a subgroup of $\mathbb{Z}$, too. We call its positive generator the *small* $\mathbf{f}_1$*-step* of $\Lambda$. Alternatively, the small $\mathbf{f}_1$-step can be defined as the largest $s$ such that all the points of $\Lambda$ lie on the lines $\{ks\mathbf{f}_1 + t\mathbf{f}_2\}$, $k \in \mathbb{Z}$. Obviously, the small step is smaller then the large step. We can define the large and small $\mathbf{f}_2$-steps of $\Lambda$ with respect to $(\mathbf{f}_1, \mathbf{f}_2)$ in the same way.

In what follows we nearly always consider small and large steps of lattices with respect to bases made up of the vectors $\pm\mathbf{e}_1$, $\pm\mathbf{e}_2$, and we usually omit the reference to the basis when there is no ambiguity.

**Proposition 2.7.** *Let $\Lambda$ be a sublattice of $\mathbb{Z}^2$ and $(\mathbf{f}_1, \mathbf{f}_2)$ be a basis of $\mathbb{Z}^2$. Then the product of the small $\mathbf{f}_1$-step and the large $\mathbf{f}_2$-step of $\Lambda$ equals* $\det\Lambda$.

The proof is left to the reader.

In what follows we use standard notations $\lfloor\cdot\rfloor$ for the floor function, $\lceil\cdot\rceil$ for the ceiling function, $^+$ for the positive part, and $|\cdot|$ for the cardinality of a finite set. By $[\mathbf{a}, \mathbf{b}]$ we denote the segment with the endpoints $\mathbf{a}$ and $\mathbf{b}$.

**2.3. Synopsis of the proof.** It turns out that the Main Theorem can be fairly easily proved for $(1, 2)$- and $(2, 2)$-lattices.

In the case of the lattice $\Lambda = \mathbb{Z} \times 2\mathbb{Z}$, the Main Theorem becomes

**Sub-Theorem A.** *Any convex integer polygon contains a point with an even ordinate.*

Sub-Theorem A implies the Main Theorem for arbitrary $(1, 2)$-lattices $\Lambda$, because if $\Lambda$ is such a lattice and $P$ is an integer polygon, we can find a unimodular transformation $A$ such that $A\Lambda = \mathbb{Z} \times 2\mathbb{Z}$ (Proposition 2.6); Sub-Theorem A asserts that $AP$ contains a $A\Lambda$-point, so $P$ contains a point of $\Lambda$.

In the case of the lattice $2\mathbb{Z}^2$, the Main Theorem becomes

**Sub-Theorem B.** *Any convex integer pentagon contains a point of the lattice* $2\mathbb{Z}^2$.

This statement was announced in [7].

We prove Sub-Theorems A and B in Section 4.

In proving the Main Theorem we adopt the strategy of estimating the number of vertices (equivalently, of edges) of polygons not containing points of given lattices. We will presently see that we can concentrate on integer polygons free of $n\mathbb{Z}^2$-points. As we can always substitute such a polygon by its image under an affine automorphism of $n\mathbb{Z}^2$, our first goal is to find out how drastically we can reduce the set of polygons to consider applying such automorphisms. We use the following proposition as our basis.

**Proposition 2.8.** *Given a convex integer polygon $P$ free of $n\mathbb{Z}^2$-points, where $n \in \mathbb{Z}$, $n \geq 2$, there exists an automorphism $\psi$ of $n\mathbb{Z}^2$ such that $\psi(P)$ lies in the slab*

$$-n + 1 \leq x_1 \leq 2n - 1.$$

The proposition is proved in Section 5.

*Remark* 2.9. The proof of Proposition 2.8 ensures that $\psi$ can be chosen in such a way that $\psi(P)$ contains a segment with $\ell(P) + 1$ integer points lying on a line of the form $x_1 = c$ with $0 \leq c \leq n$, where $\ell(P)$ is the lattice diameter of $P$ (see Section 5). Moreover, $\psi$ can be chosen in such a way that if $\psi(P)$ has common points with the lines $x_1 = 0$ and $x_1 = n$, they lie on the segments $[\mathbf{0}, (0, n)]$ and $[(n, 0), (n, n)]$, respectively.

*Remark* 2.10. If $\Lambda$ is a $(\delta, n)$-lattice, it is not hard to prove that $n\mathbb{Z}^2 \subset \Lambda$. Proposition 2.8 immediately implies that the number of vertices of a polygon free of $\Lambda$-points cannot be greater than $2(3n - 2)$. Of course, in view of the Main Theorem this fairly simple estimate is not optimal.

Proposition 2.8 allows for a classification of polygons free of points of $\mathbb{Z}^2$ into feasible classes.

We say that a line or a segment *splits* a polygon, if it divides the polygon into two parts with nonempty interior.

Let $P$ be an integer polygon free of points of $n\mathbb{Z}^2$ and $n$ be an integer, $n \geq 2$.

**Definition 2.11.** We say that $P$ is a



Assume that $\mathbf{z}$ is an interior point of $\varepsilon_k$, then $v_{k-1,2} > 0$. As in this case $\mathbf{z}$ is not the leftmost point of $\varepsilon_k$, it is clear that $\pi_1(\varepsilon_k)$ is strictly greater than $v_{k1} - z_1 = \alpha(-v_{k2})$. Thus,

$$\pi_1(\varepsilon_k) \geq \lfloor \alpha(-v_{k2}) \rfloor + 1 \geq \lfloor \alpha \rfloor + \lfloor (-v_{k2} - 1)\alpha \rfloor + 1 = \lceil \alpha \rceil + \delta + \lfloor (-v_{k2} - 1)\alpha \rfloor,$$

and (6.8) follows. The inequality is proved.

Since $v_{k-1,2} \geq 0$ and $v_{k2} < 0$, we have $\pi_2(\varepsilon_k) = v_{k-1,2}$. This and (6.8) implies

$$\hat{\pi}(\varepsilon_k) \geq v_{k-1,2} + \delta - 1 + t + \lfloor (-v_{k2} - 1)\alpha \rfloor. \tag{6.9}$$

Let $\tilde{Q}$ be the subslope of $Q$ with the vertices $\mathbf{v}_0 = \mathbf{v}$, $\mathbf{v}_1$, …, $\mathbf{v}_{k-1}$ and $\tilde{E}$ be the set of its edges, then $\varepsilon_k \notin \tilde{E}$, and $\tilde{E} \cup \{\varepsilon_k\} = E_1$. As $\tilde{Q}$ lies in the upper half-plane, for any $\varepsilon_i \in \tilde{E}$ we have $\pi_2(\varepsilon_i) = v_{i-1,2} - v_{i2} = -a_{i2}$. Consequently, $\pi_2(\varepsilon_i) = 1$ if $\varepsilon_i \in S$ and $\pi_2(\varepsilon_i) \geq 2$ otherwise, whence

$$\pi_2(\tilde{E}) \geq 2|\tilde{E}| - s. \tag{6.10}$$

Further, by Remark 6.4, we have $\pi_1(\tilde{E}) = v^+_{k-1,1} - v^+_{k1} = v^+_{k-1,1}$. Combining this with (6.10) and using (6.5), we obtain

$$\hat{\pi}(\tilde{E}) \geq v^+_{k-1,1} - s. \tag{6.11}$$

As $\hat{\pi}(E_1) = \hat{\pi}(\varepsilon_k) + \hat{\pi}(\tilde{E})$, we sum (6.9) and (6.11) and obtain (6.6). ∎

**Lemma 6.6.** *Suppose that $(\mathbf{o}; \mathbf{f}_1, \mathbf{f}_2)$ forms small angle with $Q$; then*

$$\hat{\pi}(E_2) \geq \frac{1}{2}(v_{k2} - w_2 - 1). \tag{6.12}$$

*Proof*: If $E_2 = \varnothing$, we have $\mathbf{v}_k = \mathbf{w}$ and (6.12) is obvious.

Assume that $E_2 \neq \varnothing$ and take $\varepsilon_i \in E_2$. It is easy to see that all the edges belonging to $E_2$ lie in the right half-plane, so $\pi_1(\varepsilon_i) = v_{i-1,1} - v_{i1} = a_{i1}$, and since $\pi_2(\varepsilon_i) \geq 0$, we obtain

$$\hat{\pi}(\varepsilon_i) \geq a_{i1} - 2$$

(actually, the equality holds here). Write the last inequality in form

$$\hat{\pi}(\varepsilon_i) \geq \frac{-a_{i2} + g(a_{i1}, a_{i2})}{2}, \tag{6.13}$$

where

$$g(m_1, m_2) = 2m_1 + m_2 - 4.$$

and extend the functions $\pi_1$, $\pi_2$, and $\hat{\pi}$ to the set of all subsets of $E$ by additivity. Observe that these functions take integer values.

*Remark* 6.4. Obviously, for any $F \subset E$ we have

$$\pi_j(F) \geq 0 \qquad (j = 1, 2), \tag{6.4}$$

$$\hat{\pi}(F) = \pi(F) - 2|F|. \tag{6.5}$$

Moreover, it is clear that if $\tilde{Q}$ is a subslope of $Q$ (not necessarily integer) with the endpoints $\tilde{\mathbf{v}} = \mathbf{o} + \tilde{v}_1\mathbf{f}_1 + \tilde{v}_2\mathbf{f}_2$ and $\tilde{\mathbf{w}} = \mathbf{o} + \tilde{w}_1\mathbf{f}_1 + \tilde{w}_2\mathbf{f}_2$, then

$$\pi_1(\tilde{Q}) = |\tilde{v}_1^+ - \tilde{w}_1^+|, \qquad \pi_2(\tilde{Q}) = |\tilde{v}_2^+ - \tilde{w}_2^+|.$$

Set

$$E_1 = \{\varepsilon_1, \dots, \varepsilon_k\}, \qquad E_2 = \{\varepsilon_{k+1}, \dots, \varepsilon_N\}.$$

Clearly, $E_1 \cap E_2 = \varnothing$ and $E_1 \cup E_2 = E$.

### 6.2. Auxiliary statements.

**Lemma 6.5.** *We have*

$$\hat{\pi}(E_1) \geq (v_{k-1,1}^+ + v_{k-1,2} - 1) + \delta + (t - s) + \lfloor(-v_{k2} - 1)\alpha\rfloor, \tag{6.6}$$

*where*

$$\delta = \begin{cases} 1, & \text{if } v_{k-1,2} > 0 \text{ and } \alpha \in \mathbb{Z}, \\ 0, & \text{otherwise.} \end{cases} \tag{6.7}$$

*Proof*: Let us show the inequality

$$\pi_1(\varepsilon_k) \geq \delta + 1 + t + \lfloor(-v_{k2} - 1)\alpha\rfloor. \tag{6.8}$$

Since the frame $(\mathbf{o}; \mathbf{f}_1, \mathbf{f}_2)$ splits the slope $Q$, it follows from the definition that the ray $\{\mathbf{o} + \lambda\mathbf{f}_1 \colon \lambda \geq 0\}$ meets $Q$ at a point $\mathbf{z} = \mathbf{o} + z_1\mathbf{f}_1$, where $z_1 > 0$. As $v_{k2} < 0 \leq v_{k-1,2}$, it is easily seen that $\mathbf{z}$ belongs to the edge $\varepsilon_k$ and either coincides with $\mathbf{v}_{k-1}$ or is an inner point of the edge. We consider these cases separately.

If $\mathbf{z} = \mathbf{v}_{k-1}$, then $v_{k-1,2} = 0$ and $v_{k-1,1} > 0$, so

$$\pi_1(\varepsilon_k) = v_{k1} - v_{k-1,1} = a_{k1} = \alpha a_{k2} = \alpha(-v_{k2}) = \lceil\alpha\rceil + \lfloor(-v_{k2} - 1)\alpha\rfloor,$$

and as in this case $\delta = 0$, (6.8) follows.



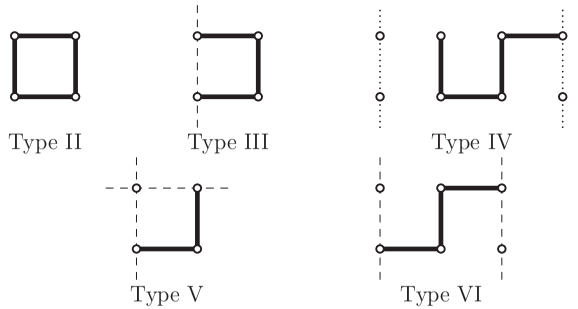


FIGURE 2. Definition 2.11 introduces the types of polygons in terms of intersection with segments and lines. Here thick segments split polygons of the specified type, thin lines do not split them, and dotted lines have no common points with them.

- *type $I_n$ polygon*, if no line of the form $x_1 = jn$ or $x_2 = jn$ where $j \in \mathbb{Z}$, splits $P$, or, equivalently, if $P$ lies in a slab of the form $jn \leq x_1 \leq (j+1)n$ or $jn \leq x_2 \leq (j+1)n$, where $j \in \mathbb{Z}$;
- *type $II_n$ polygon*, if each of the segments $[\mathbf{0}, (n, 0)]$, $[(n, 0), (n, n)]$, $[(0, n), (n, n)]$, and $[\mathbf{0}, (0, n)]$ splits $P$;
- *type $III_n$ polygon*, if each of the segments $[\mathbf{0}, (n, 0)]$, $[(n, 0), (n, n)]$, and $[(n, n), (0, n)]$ splits $P$, and the line $x_1 = 0$ does not split $P$;
- *type $IV_n$ polygon*, if each of the segments $[\mathbf{0}, (0, n)]$, $[\mathbf{0}, (n, 0)]$, $[(n, 0), (n, n)]$, and $[(n, n), (2n, n)]$ splits $P$ and $P$ has no common points with the lines $x_1 = -n$ and $x_n = 2n$;
- *type $V_n$ polygon*, if each of the segments $[\mathbf{0}, (-n, 0)]$ and $[\mathbf{0}, (0, n)]$, and the lines $x_1 = -n$ and $x_2 = n$ do not split $P$;
- *type $VI_n$ polygon*, if each of the segments $[\mathbf{0}, (-n, 0)]$, $[\mathbf{0}, (0, n)]$, and $[(0, n), (n, n)]$ split $P$, and the lines $x_1 = \pm n$ do not split $P$.

The polygon types are illustrated on Figure 2.

**Theorem 2.12.** *Suppose that an integer polygon $P$ is free of points of the lattice $n\mathbb{Z}^2$, where $n \in \mathbb{Z}$, $n \geq 2$; then there exists an affine automorphism $\varphi$ of $n\mathbb{Z}^2$ such that $\varphi(P)$ is a polygon of one of the types $I_n$–$VI_n$.*

*Proof*: Let $\psi$ an automorphism as in Proposition 2.8 and Remark 2.9 and $P' = \psi(P)$.

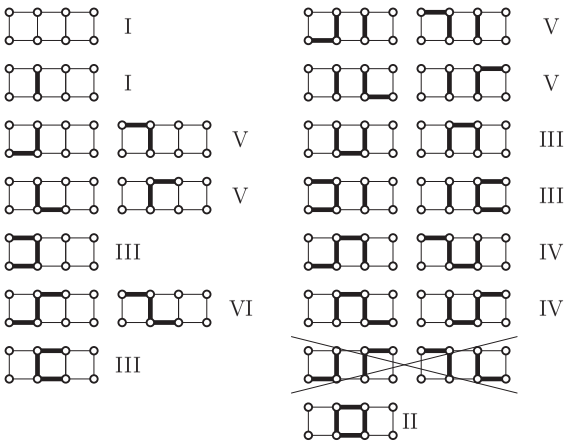


FIGURE 3. The polygon $P'$ obtained in the proof of Theorem 2.12 has no common points with the lines $x_1 = -n$ and $x_1 = 2n$ and may or may not be split by the eight segments. Two configurations are ruled out by Lemma 5.5. For any remaining combination of splitting segments (drawn as thick lines), $P'$ can be easily mapped onto a polygon of specified type.

As $P'$ is free of $n\mathbb{Z}^2$-points, it is clear that $P'$ may be split by at most one of the three segments $I_1 = [\mathbf{0}, (-n, 0)]$, $I_2 = [\mathbf{0}, (n, 0)]$, and $I_3 = [(n, 0), (2n, 0)]$, and at most one of the three segments $J_1 = [(-n, n), (0, n)]$, $J_2 = [(0, n), (n, n)]$, and $J_3 = [(n, n), (2n, n)]$.

If the lines $x_1 = 0$ and $x_1 = n$ do not split $P'$, it is a type $\mathrm{I}_n$ polygon.

If exactly one of the lines $x_1 = 0$ and $x_1 = n$, then there is no loss of generality in assuming it is the former, because otherwise we can replace $P'$ by its reflection about the line $x_1 = n/2$, the reflection being an automorphism of $n\mathbb{Z}^2$. Thus, the segment $[\mathbf{0}, (0, n)]$ splits $P'$ and the segments $I_3$ and $J_3$ have no common points with $P'$. Individually examining the possibilities according to which of the segments $I_{1,2}$ and $J_{1,2}$ split $P'$, we see that in each case the polygon either is of one of the types $\mathrm{I}_n$–$\mathrm{VI}_n$ or can be trivially mapped onto such a polygon by an automorphism of $n\mathbb{Z}^2$ (Figure 3).

If both lines $x_1 = 0$ and $x_1 = n$ split $P'$, we likewise consider the possibilities according to which of the segments $I_{1,2,3}$ and $J_{1,2,3}$ split $P'$ (Lemma 5.5 rules out two of them) and draw the same conclusion (Figure 3). ∎



$(\mathbf{o}; \mathbf{f}_1, \mathbf{f}_2)$. The statement of Proposition 3.6 saying that at least one of those frames forms small angle with $Q$ can be equivalently expressed in the form of the inequality

$$\min\{\alpha, \tilde{\alpha}\} \geq 1.$$

Moreover, it is not hard to prove that equality holds if and only if $\alpha = \tilde{\alpha} = 1$, in which case $k = \tilde{k}$, and consequently $v_{k-1,1} < 0$ and $v_{k-1,2} > 0$.

**Lemma 6.3.** *The cardinality $s$ of $S$ satisfies*

$$s \leq t, \tag{6.1}$$

*and, moreover,*

$$\sum_{\varepsilon_i \in S} a_{i1} \leq (t - s)s + \frac{s(s+1)}{2}. \tag{6.2}$$

*If the vertices of $Q$ belong to a proper sublattice of $\mathbb{Z}^2$, then $s = t$ only if both equal 0 or 1, and in the latter case the only edge $\varepsilon_i \in S$ has the associated vector $\mathbf{a}_i = \mathbf{f}_1 - \mathbf{f}_2$.*

*Proof*: Let $S = \{\varepsilon_{i_1}, \ldots, \varepsilon_{i_s}\}$, where $i_1 < \cdots < i_s < k$. It follows from (3.2) and (3.3) that

$$\frac{a_{11}}{-a_{12}} < \frac{a_{21}}{-a_{22}} < \cdots < \frac{a_{k1}}{-a_{k2}} = \alpha.$$

Hence, as $a_{i_p 2} = -1$, we see that

$$0 < a_{i_1 1} < a_{i_2 1} < \cdots < a_{i_s 1} \leq \lceil \alpha \rceil - 1 = t. \tag{6.3}$$

This implies (6.1). Moreover, (6.3) implies that $a_{i_p 1} \leq t - s + p$, where $p = 1, \ldots, s$, and upon summation, we recover (6.2). Now suppose that $s = t$, and the vertices of $Q$ belong to a proper sublattice $\Gamma$ of $\mathbb{Z}^2$. Let us show that either $s = 0$ or $s = 1$. If $s \neq 0$, then $S \neq \varnothing$, and as the vectors $\mathbf{a}_i$ belong to $\Gamma$, we see that the small $\mathbf{f}_2$-step of $\Gamma$ is 1. By Proposition 2.7, $\Gamma$ has large $\mathbf{f}_1$-step $m \geq 2$. The differences $\mathbf{a}_{i_p} - \mathbf{a}_{i_q} \in \Gamma$ are proportional to $\mathbf{f}_1$, so the numbers $a_{i_p}$, where $p = 1, \ldots, s$, differ by multiples of $m$. Thus, in view of (6.3), we can only have $s = t$ if $s = t = 1$, as claimed. In this case $a_{i_1 1} = 1$, so that $\mathbf{a}_{i_1} = \mathbf{f}_1 - \mathbf{f}_2$. ∎

Given an edge $\varepsilon_i \in E$, define

$$\pi_1(\varepsilon_i) = v^+_{i1} - v^+_{i-1,1},$$
$$\pi_2(\varepsilon_i) = v^+_{i-1,2} - v^+_{i2},$$
$$\hat{\pi}(\varepsilon_i) = \pi_1(\varepsilon_i) + \pi_2(\varepsilon_i) - 2$$

Thus, $\mathbf{p}$ lies on the segment $[\mathbf{a}, \mathbf{b}]$.

The points $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{w}$ are the vertices of a triangle $T$. The line $l$ passing through $\mathbf{v}$ and $\mathbf{p}$ intersects the side $[\mathbf{a}, \mathbf{b}]$ of $T$, so it intersects another side as well. The line $l$ is parallel to the line $x_1 + x_2 = n$ and lies below it; on the contrary, neither $\mathbf{b}$ nor $\mathbf{w}$ lie below the latter line, so $l$ has no common points with the segment $[\mathbf{b}, \mathbf{w}]$. Consequently, $l$ intersects the side $[\mathbf{a}, \mathbf{w}]$ of $T$ at a possibly non-integer point $\mathbf{z} = (z_1, z_2)$. The segment $[\mathbf{p}, \mathbf{z}]$ is contained in $T$, so the whole segment $[\mathbf{v}, \mathbf{z}]$ is contained in $P$.

The slope of the line passing through $\mathbf{a}$ and $\mathbf{w}$ is negative (this can be seen by e. g. comparing the coordinates of those points using (5.15)), so $z_2 \le a$. Using (5.14) and (5.16), we can estimage the number of integer points lying on the segment $[\mathbf{v}, \mathbf{z}]$ as follows:

$$|[\mathbf{v}, \mathbf{z}] \cap \mathbb{Z}^2| = \lfloor v_2 - z_2 \rfloor + 1 \ge n - a + 2 \ge b - a + 3 = \ell(P) + 3,$$

which contradicts the definition of the lattice diameter. The contradiction proves the lemma. ∎

## 6. Properties of slopes

In this section we prove Theorems 3.7 and 3.9.

**6.1. Preliminaries.** Throughout this section $(\mathbf{o}; \mathbf{f}_1, \mathbf{f}_2)$ is an integer frame splitting an integer slope $Q$. Let $\mathbf{v}_0 = \mathbf{v}$, $\mathbf{v}_1, \dots, \mathbf{v}_N = \mathbf{w}$ be the vertices of $Q$. We define $\mathbf{a}_i$ by (3.1) and assume that (3.2) and (3.3) hold. By $\varepsilon_i = [\mathbf{v}_{i-1}, \mathbf{v}_i]$ $(i = 1, \dots, N)$ denote the edges of $Q$ and by $E$, the set of edges. Set

$$\mathbf{v}_i - \mathbf{o} = v_{i1}\mathbf{f}_1 + v_{i2}\mathbf{f}_2 \qquad (i = 0, \dots, N).$$

Note that $v_{ij}$ and $a_{ij}$ are integers.

Further, set

$$k = \min\{i \colon v_{i2} < 0\}, \quad \alpha = -\frac{a_{k1}}{a_{k2}}, \quad t = \lceil \alpha \rceil - 1,$$
$$S = \{\varepsilon_i \in E \colon i < k,\ a_{i2} = -1\}, \quad s = |S|.$$

All these are well-defined.

*Remark* 6.1. It is easily seen that $(\mathbf{o}; \mathbf{f}_1, \mathbf{f}_2)$ forms small angle with $Q$ if and only if $\alpha \ge 1$.

*Remark* 6.2. Let us define $\tilde{\alpha} = -a_{\tilde{k}2}/a_{\tilde{k}1}$, where $\tilde{k} = \min\{i \colon v_{k1} \ge 0\}$. The coefficient $\tilde{\alpha}$ is related to the frame $(\mathbf{o}; \mathbf{f}_2, \mathbf{f}_1)$ in the same way as $\alpha$ is to



Now we recast the Main Theorem for lattices with the largest invariant factor greater than 2 as follows:

**Sub-Theorem C.** *Let $P$ be a convex integer $N$-gon of one of the types $I_n$–$VI_n$, where $n$ is an integer, $n \ge 3$. Then:*

*(i) the following inequality holds:*

$$N \le 2n + 2;$$

*(ii) if the vertices of $P$ belong to a $(1, n/2)$-lattice, then*

$$N \le 2n;$$

*(iii) if the vertices of $P$ belong to a $(1, n)$-lattice, then*

$$N \le 2n - 2.$$

Let us make sure that having proved Sub-Theorem C, we in fact establish the Main Theorem for $(\delta, n)$-lattices $\Lambda$ with $n \ge 3$. Indeed, let $\Lambda$ be such a lattice and $P$ be an integer $N$-gon with $N \ge \nu(\Lambda)$. Suppose that contrary to our expectations, $P$ is free of points of $\Lambda$. Let $A$ be a unimodular transformation mapping $\Lambda$ onto $\delta\mathbb{Z} \times n\mathbb{Z}$ and $S$ be the scaling $\mathrm{diag}(n/\delta, 1)$. The superposition $SA$ maps $\Lambda$ onto $n\mathbb{Z}^2$ and $P$, onto an integer polygon $P'$ free of points of $n\mathbb{Z}^2$. Let $\varphi$ be an affine automorphism of $n\mathbb{Z}^2$ mapping $P'$ onto a polygon $P''$ of one of the types $\mathrm{I}_n$–$\mathrm{VI}_n$. Note that the vertices of $P'$ belong to $S\mathbb{Z}^2 = (n/\delta)\mathbb{Z} \times \mathbb{Z}$, so the vertices of $P''$ belong to a $(1, n/\delta)$-lattice. Now the assumption $N \ge \nu(\delta, n)$ contradicts Sub-Theorem C applied to $P''$.

Thus, the Main Theorem is the sum of Sub-Theorems A, B and C.

In the case of type I polygons the proof of Sub-Theorem C is a simple combinatorial argument, see Section 4. However, the rest types require a fine geometric analysis. In Section 3 we collect necessary tools and apply them to type II polygons. The rest types require more technical treatment carried out in [8].

## 3. Slopes

**3.1. Slopes.** Let $(\mathbf{f}_1, \mathbf{f}_2)$ be a basis of $\mathbb{R}^2$, and let $\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_N$ $(N \ge 0)$ be a finite sequence of points on the plane. If $N \ge 1$, set

$$\mathbf{v}_i + \mathbf{v}_{i-1} = \mathbf{a}_i = a_{i1}\mathbf{f}_1 + a_{i2}\mathbf{f}_2 \qquad (i = 1, \dots, N). \tag{3.1}$$

If

$$a_{i1} > 0,\ a_{i2} < 0 \qquad (i = 1, \dots, N) \tag{3.2}$$

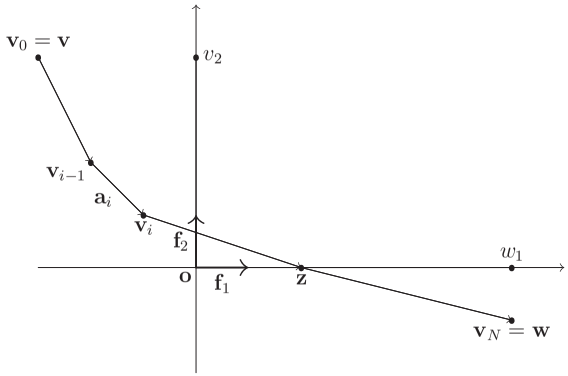


FIGURE 4. The broken line is a slope with respect to the basis $(\mathbf{f}_1, \mathbf{f}_2)$. It is convex and the vectors associated with its edges point down and to the right. The frame $(\mathbf{o}; \mathbf{f}_1, \mathbf{f}_2)$ splits the slope and forms small angle with it, since there is a supporting line passing through the point $\mathbf{z}$ and forming an angle $\le \pi/4$ with the axis.

and

$$\begin{vmatrix} a_{i1} & a_{i+1,1} \\ a_{i2} & a_{i+1,2} \end{vmatrix} > 0 \qquad (i = 1, \dots, N-1), \tag{3.3}$$

we say that the union $Q$ of the segments $[\mathbf{v}_0, \mathbf{v}_1]$, $[\mathbf{v}_1, \mathbf{v}_2]$, $\dots$, $[\mathbf{v}_{N-1}, \mathbf{v}_N]$ is a *slope* with respect to the basis $(\mathbf{f}_1, \mathbf{f}_2)$. These segments are called the *edges* of the slope, and the points $\mathbf{v}_0$, $\mathbf{v}_1$, $\dots$, $\mathbf{v}_N$, its *vertices*, $\mathbf{v}_0$ and $\mathbf{v}_N$ being the *endpoints*. If $N = 1$, we call the segment $[\mathbf{v}_0, \mathbf{v}_1]$ a slope if (3.2) holds, and if $N = 0$, we still call the one-point set $\{\mathbf{v}_0\}$ a slope. If all the vertices of $Q$ belong to a lattice $\Gamma$, we call it a $\Gamma$-*slope*. A $\mathbb{Z}^2$-slope is called *integer*, and it is the only kind of slopes we are interested in.

It is not hard to prove that the vertices and edges of a slope are uniquely defined, and that the basis induces a unique ordering of vertices.

Figure 4 illustrates the concepts of a slope and of an affine frame splitting a slope, to be considered below.

*Remark* 3.1. If $Q$ is a slope with respect to a basis $(\mathbf{f}_1, \mathbf{f}_2)$, then it is a slope with respect to the basis $(\mathbf{f}_2, \mathbf{f}_1)$, too.



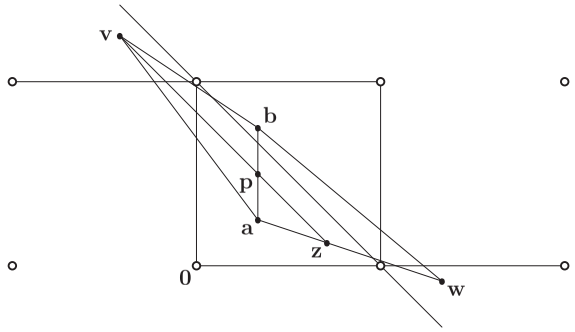


FIGURE 9. Lemma 5.5

(see Figure 9).

Let $\mathbf{a} = (c, a)$ and $\mathbf{b} = (c, b)$ be the endpoints of the segment mentioned in the hypothesis, so that $b - a = \ell(P)$. Observe that

$$a \ge 1, \ b \le n - 1. \tag{5.16}$$

Indeed, $P$ cannot have common points with the segments $[\mathbf{0}, (n, 0)]$ and $[(n, 0), (n, n)]$, for otherwise it would contain an $n\mathbb{Z}^2$-point. These segments are the sides of the square that clearly has common points with $P$. Then the segment $[\mathbf{a}, \mathbf{b}]$ must lie in the slab $0 < x_2 < n$, whence (5.16).

By Lemma 5.2 applied to the polygon $P - \mathbf{a}$, we have

$$v_1 \ge c - b + a - 2, \tag{5.17}$$

$$w_1 \le c + b - a + 2. \tag{5.18}$$

It follows from (5.17), (5.14), and (5.16) that the the coordinates of $\mathbf{a}$ satisfy $a + c \le n$, i. e. $\mathbf{a}$ cannot lie above the line $x_1 + x_2 = n$. In the same way, it follows from (5.18), (5.15), and (5.16) that $\mathbf{b}$ cannot lie below this line. As a consequence, $\mathbf{v}$ must lie below this line, for else $P$ would have a common point with the segment $[(0, n), (n, n)]$; likewise, $\mathbf{w}$ must lie above this line.

Let $\mathbf{p} = (c, p)$, where $p = v_1 + v_2 - c$, be the projection of $\mathbf{v}$ on the line $x_1 = c$ along the vector $\mathbf{e}_1 - \mathbf{e}_2$. As $\mathbf{v}$ lies below the line $x_1 + x_2 = n$, so does $\mathbf{p}$, and therefore $\mathbf{p}$ lies below $\mathbf{b}$. Moreover, $\mathbf{p}$ cannot lie below $\mathbf{a}$, since using (5.14) and (5.17) and then (5.16), we have

$$p = v_1 + v_2 - c \ge a - b + n - 1 \ge a.$$

These inequalities and (5.11) imply a contradiction exactly as in the case $v_1 = 2n$.

The contradictions show that in fact $P$ lies in the half-plane $x_1 \le 2n-1$. To show that it lies in the half-plane $x_1 \ge -n+1$ as well, it suffices to apply the established part of the lemma to the reflection of $P$ about the line $x_1 = n/2$. ∎

*Proof of Proposition 2.8*: Let $[\mathbf{a},\mathbf{b}] \in P$ be a segment containing $\ell(P)+1$ integer points. By Proposition 2.5, there exists a unimodular transformation $A$ such that $A(\mathbf{b}-\mathbf{a}) = t\mathbf{e}_2$, then the segment $A[\mathbf{a},\mathbf{b}]$ lies on a line $x_1 = m$, where $m$ is an integer. Let $m = nq + c$, where $q, c \in \mathbb{Z}$ and $0 \le c \le n-1$ and let $T$ be the translation by the vector $-nq\mathbf{e}_1 \in n\mathbb{Z}^2$. Then the segment $TA[\mathbf{a},\mathbf{b}]$ contains $\ell(P)+1 = \ell(TAP)+1$ integer points and lies on the line $x_1 = c$. In view of Lemma 5.3, $\psi = TAP$ is the required automorphism. ∎

*Remark* 5.4. Let $P$ be an integer polygon free of $n\mathbb{Z}^2$-points and let $\psi$ be an automorphism of $n\mathbb{Z}^2$ constructed in the proof of Proposition 2.8. As the polygon $\psi(P)$ does not contain points of the lattice $n\mathbb{Z}^2$, its intersection with the line $x_1 = 0$ lies between two adjacent points of $n\mathbb{Z}^2$, i. e. it is a subset of a segment $I_1 = [(0,u_1 n),(0,u_1 n+n)]$, where $u_1 \in \mathbb{Z}$ (if the intersection is empty, $u_1$ can be chosen arbitrarily). Likewise, the intersection of $\psi(P)$ with the line $x_1 = n$ is a subset of a segment $I_2 = [(n,u_2 n),(n,u_2 n+n)]$, where $u_2 \in \mathbb{Z}$. Define the affine automorphism of $n\mathbb{Z}^2$ by

$$\tilde{\psi}(x_1,x_2) = (x_1, x_2 + (u_1-u_2)x_1 - u_1 n).$$

Since it preserves the first coordinate and maps $I_1$ onto $[\mathbf{0},(0,n)]$ and $I_2$ onto $[(n,0),(n,n)]$, it is easily seen that the automorphism $\tilde{\psi}\psi$ satisfies the requirements of Proposition 2.8 and Remark 2.9.

The following lemma is used in the proof of Theorem 2.12.

**Lemma 5.5.** *Under the hypotheses of Lemma 5.3, the segments* $[(0,n),(-n,n)]$ *and* $[(n,0),(2n,0)]$ *cannot simultaneously split* $P$. *The same is true about the pair of segments* $[\mathbf{0},(-n,0)]$ *and* $[(n,n),(2n,n)]$.

*Proof*: By symmetry, it suffices to consider only the first pair of segments. To obtain a contradiction, we assume that both segments split $P$. Then we see that $P$ has vertices $\mathbf{v} = (v_1,v_2)$ and $\mathbf{w} = (w_1,w_2)$ satisfying

$$v_1 \le -1, v_2 \ge n+1; \tag{5.14}$$

$$w_1 \ge n+1,\ w_2 \le -1 \tag{5.15}$$



Although the following statement is simple, it provides some important tools for estimating the number of edges of a slope. We are interesting in comparing the doubled number of edges with the 'width' of the slope, i. e. its projection on the axis spanned by $\mathbf{f}_1$. The general point is that the edges with projection 1 contribute quadratic growth to the 'height' of the slope.

**Proposition 3.2.** *Let* $(\mathbf{f}_1,\mathbf{f}_2)$ *be a basis of* $\mathbb{Z}^2$ *and* $\mathbf{v}$ *and* $\mathbf{w}$ *be the endpoints of an integer slope (with respect to* $(\mathbf{f}_1,\mathbf{f}_2)$*) having* $N$ *edges. Let*

$$\mathbf{w} - \mathbf{v} = b_1\mathbf{f}_1 + b_2\mathbf{f}_2.$$

*Then there exists an integer* $s$ *such that*

$$2N \le |b_1| + s, \tag{3.4}$$

$$|b_2| \ge \frac{s(s+1)}{2}, \tag{3.5}$$

$$0 \le s \le N. \tag{3.6}$$

*If the vertices of the slope belong to a lattice with small* $\mathbf{f}_1$*-step greater then 1, one can take* $s = 0$, *so that*

$$2N \le |b_1|. \tag{3.7}$$

*If the vertices of the slope belong to a lattice having the basis* $(\mathbf{f}_1 - a\mathbf{f}_2, m\mathbf{f}_2)$, *where* $1 \le a \le m$, *then* (3.5) *can be replaced by*

$$|b_2| \ge \frac{2a + (s-1)m}{2}s. \tag{3.8}$$

*Proof*: Let $\mathbf{v}_0 = v$, $\mathbf{v}_1, \ldots, \mathbf{v}_N = \mathbf{w}$ be the vertices of the slope and assume that (3.1)–(3.3) hold. It follows from (3.2) and (3.3) that $\mathbf{a}_i \ne \mathbf{a}_j$ for $i \ne j$. Set $A = \{\mathbf{a}_i \colon a_{i1} = 1\}$ and $s = |A|$. Observe that $s$ satisfies (3.6) and $s = 0$ if the vertices of the slope belong to a lattice with small $\mathbf{f}_1$-step greater then 1.

Let us prove (3.4). If $\mathbf{a}_i \notin A$, we have $a_{i1} \ge 2$, so

$$|b_1| = \sum_{i=1}^{N} a_{i1} = \sum_{\mathbf{a}\in A}^{N} a_{i1} + \sum_{\mathbf{a}\notin A}^{N} a_{i1} \ge |S| + 2(N - |S|) = 2N - s,$$

and (3.4) follows.

Let us prove (3.8) assuming that the slope satisfies correspondent hypothesis. It is easily seen that the vectors belonging to $A$ are of the form

$\mathbf{f}_1 - (a+um)\mathbf{f}_2$, where $u \in \mathbb{Z}$, $u \geq 0$. Thus,

$$|b_2| = \sum_{i=1}^{N}(-a_{i1}) \geq \sum_{\mathbf{a}_i \in A}(-a_{i1})$$
$$\geq a + (a+m) + \cdots + (a+(s-1)m) = \frac{2a+(s-1)m}{2}s,$$

as claimed.

In the case of a generic integer slope, letting $m = a = 1$, we recover (3.5) from (3.8). ∎

**3.2. Splitting frames.** Let $(\mathbf{o}; \mathbf{f}_1, \mathbf{f}_2)$ be an integer frame and $Q$ be a slope with respect to $(\mathbf{f}_1, \mathbf{f}_2)$.

**Definition 3.3.** We say that the frame $(\mathbf{o}; \mathbf{f}_1, \mathbf{f}_2)$ *splits* the slope $Q$ if

(1) one endpoint $\mathbf{v} = \mathbf{o} + v_1\mathbf{f}_1 + v_2\mathbf{f}_2$ of $Q$ satisfies

$$v_1 < 0,\ v_2 > 0, \tag{3.9}$$

while the other endpoint $\mathbf{w} = \mathbf{o} + w_1\mathbf{f}_1 + w_2\mathbf{f}_2$ satisfies

$$w_1 > 0,\ w_2 < 0; \tag{3.10}$$

(2) there exists a point on $Q$ having both positive coordinates in the frame $(\mathbf{o}; \mathbf{f}_1, \mathbf{f}_2)$.

*Remark* 3.4. Obviously, a frame can only split a slope if the slope has at least one edge.

*Remark* 3.5. If an integer frame $(\mathbf{o}; \mathbf{f}_1, \mathbf{f}_2)$ splits a slope $Q$, it is obvious that $Q$ has no points in the quadrant $\{\mathbf{o} + \lambda_1\mathbf{f}_1 + \lambda_2\mathbf{f}_2 \colon \lambda_1, \lambda_2 \leq 0\}$.

Suppose that a frame $(\mathbf{o}; \mathbf{f}_1, \mathbf{f}_2)$ splits a slope $Q$ and let $\mathbf{z}$ be the point where $Q$ meets the ray $\{\mathbf{o} + \lambda\mathbf{f}_1 \colon \lambda \geq 0\}$. If there is a supporting line for $Q$ passing through $\mathbf{z}$ that forms an angle $\leq \pi/4$ with the ray, we say that the frame $(\mathbf{o}; \mathbf{f}_1, \mathbf{f}_2)$ *forms small angle* with the slope $Q$.

**Proposition 3.6.** *Suppose that an integer frame $(\mathbf{o}; \mathbf{f}_1, \mathbf{f}_2)$ splits a slope $Q$; then the frame $(\mathbf{o}; \mathbf{f}_2, \mathbf{f}_1)$ splits it as well, and at least one of the frames forms small angle with $Q$. If there exists a point $\mathbf{y} = \mathbf{o} + y_1\mathbf{f}_1 + y_2\mathbf{f}_2 \in Q$ such that $y_2 > 0$ and $y_1 + y_2 \leq 0$, then $(\mathbf{o}; \mathbf{f}_1, \mathbf{f}_2)$ forms small angle with $Q$.*



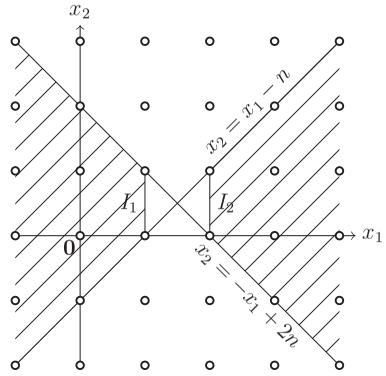


FIGURE 8. Proof of Lemma 5.3. The lines passing through $\mathbf{a}$ and $\mathbf{v}$ and through $\mathbf{b}$ and $\mathbf{v}$ belong to the set of lines joining interior points of $I_1$ with interior points of $I_2$, so the $\mathbf{a}$ and $\mathbf{b}$ belong to the hatched region on the left and $\mathbf{v}$ belongs to the one on the right.

whereas from (5.9) and (5.10) we obtain

$$a + \ell \geq c - n + \ell + 1, \tag{5.12}$$
$$a \leq -c + 2n - \ell - 1. \tag{5.13}$$

Assume that $v_1 = 2n$. According to (5.7), we have

$$\ell = 2n - c - 2,$$

so (5.12) and (5.13) yield

$$a \leq 1,\ a + \ell \geq n - 1.$$

These inequalities and (5.11) imply that the integer point $(c, v_2)$ belongs to $[\mathbf{a}, \mathbf{b}]$. But then $P$ contains the integer segment $[(c, v_2), \mathbf{v}]$ having $v_1 - c + 1 = \ell + 3$ integer points (according to (5.7)). This contradicts the definition of the lattice diameter.

Now assume $v_1 = 2n + 1$. From (5.7) we get

$$\ell = 2n - c - 1,$$

and from (5.12) and (5.13) it follows that

$$a \leq 0,\ a + \ell \geq n.$$

Let us show the inequality

$$v_1 \le 2n+1. \tag{5.5}$$

The intersection of the line $x_1 = n$ with the triangle with the vertices $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{v}$ is a segment $J \subset I_1$. We have:

$$\frac{\ell(v_1-n)}{v_1-c} < n, \tag{5.6}$$

where the left-hand side is the length of $J$.

Assuming that (5.5) is not valid, we have $v_1 \ge 2n+2$. By (5.4), we have $v_1 - c \le \ell + 2$ and $\ell \ge v_1 - c - 2 \ge n$; thus,

$$\begin{aligned}\frac{\ell(v_1-n)}{v_1-c} &\ge \frac{\ell(n+2)}{\ell+2}\\ &= \left(1-\frac{2}{\ell+2}\right)(n+2) \ge \left(1-\frac{2}{n+2}\right)(n+2) = n,\end{aligned}$$

contrary to (5.6), and (5.5) is proved.

In view of (5.4) and (5.5) we have only two possible values for $v_1$: $v_1 = 2n$ and $v_1 = 2n+1$.

Further, (5.6) gives

$$\ell < \frac{n}{v_1-n}(v_1-c) \le v_1 - c,$$

whence

$$\ell \le v_1 - c - 1.$$

Comparing this with (5.4), we see that necessarily

$$\ell = v_1 - c - 2. \tag{5.7}$$

Let us estimate $v_2$ and $a$. It is easily seen (see Figure 8) that the coordinates of $\mathbf{v}$, $\mathbf{a}$, and $\mathbf{b}$ satisfy

$$-v_1 + 2n < v_2 < v_1 - n, \tag{5.8}$$

$$a > c - n, \tag{5.9}$$

$$a + \ell < -c + 2n. \tag{5.10}$$

From (5.8) we get

$$\left.\begin{array}{ll} 1 \le v_2 \le n-1 & \text{if } \ v_1 = 2n, \\ 0 \le v_2 \le n & \text{if } \ v_1 = 2n+1, \end{array}\right\} \tag{5.11}$$



The proof is left to the reader.

The following theorems provide much more sophisticated estimates of the number of edges of a slope than those of Preposition 3.2. This time we are comparing the doubled number of edges with the length of the projection of the slope on the positive half-axes of the frame, where by the projection on a half-axis we mean the intersection of the projection on the axis with the half-axis. It turns out that the doubled number of edges is always less then or equal to the total length of the projection.

**Theorem 3.7.** *Suppose that an integer frame* $(\mathbf{o};\mathbf{f}_1,\mathbf{f}_2)$ *splits an integer slope* $Q$ *having* $N$ *edges and the endpoints* $\mathbf{v} = \mathbf{o}+v_1\mathbf{f}_1+v_2\mathbf{f}_2$ *and* $\mathbf{w} = \mathbf{o}+w_1\mathbf{f}_1+w_2\mathbf{f}_2$ *satisfying* (3.9) *and* (3.10). *Then there exist* $s \in \mathbb{Z}$ *and* $t \in \mathbb{Z}$ *such that*

$$0 \le s \le t, \tag{3.11}$$

$$v_2 - s \ge 0, \tag{3.12}$$

$$-v_1 < ts - \frac{s^2-s}{2} + (v_2-s)(t+1), \tag{3.13}$$

$$2N \le v_2 + w_1 - t + s. \tag{3.14}$$

*Moreover, if* $(\mathbf{o};\mathbf{f}_1,\mathbf{f}_2)$ *forms small angle with* $Q$, *we have*

$$2N \le v_2 + w_1 - t + s - \left\lceil \frac{-w_2}{2} \right\rceil + 1. \tag{3.15}$$

**Corollary 3.8.** *Under the hypotheses of Theorem 3.7,*

$$2N \le v_2 + w_1,$$

*and if* $(\mathbf{o};\mathbf{f}_1,\mathbf{f}_2)$ *forms small angle with* $Q$, *then*

$$2N \le v_2 + w_1 - \left\lceil \frac{-w_2}{2} \right\rceil + 1.$$

**Theorem 3.9.** *Under the hypotheses of Theorem 3.7, if the vertices of* $Q$ *belong to a proper sublattice of* $\mathbb{Z}^2$, *then*

$$2N \le v_2 + w_1 - 1.$$

The proofs of Theorems 3.7 and 3.9 are rather technical. We give them in Section 6.

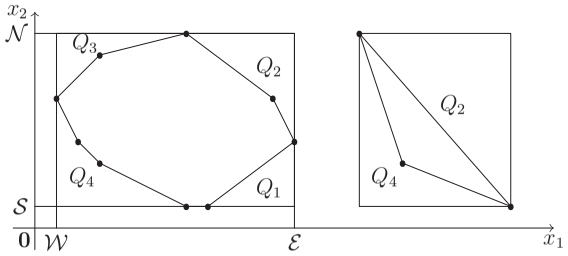


FIGURE 5. The edges of a polygon not belonging to the bounding box form four maximal slopes $Q_k$. These slopes may degenerate into a point, as is the case for the triangle on the right, which has only two nontrivial maximal slopes.

**3.3. The boundary of a convex polygon.** Let $P$ be a convex integer polygon. Define

$$\begin{aligned}
&\mathcal{N} = \max\{x_2 :\ (x_1, x_2) \in P\}, && \mathcal{S} = \min\{x_2 :\ (x_1, x_2) \in P\},\\
&\mathcal{N}_- = \min\{x_1 :\ (x_1, \mathcal{N}) \in P\}, && \mathcal{S}_- = \min\{x_1 :\ (x_1, \mathcal{S}) \in P\},\\
&\mathcal{N}_+ = \max\{x_1 :\ (x_1, \mathcal{N}) \in P\}, && \mathcal{S}_+ = \max\{x_2 :\ (x_2, \mathcal{S}) \in P\},\\
&\mathcal{W} = \min\{x_1 :\ (x_1, x_2) \in P\}, && \mathcal{E} = \max\{x_1 :\ (x_1, x_2) \in P\},\\
&\mathcal{W}_- = \min\{x_2 :\ (\mathcal{W}, x_2) \in P\}, && \mathcal{E}_- = \min\{x_2 :\ (\mathcal{E}, x_2) \in P\},\\
&\mathcal{W}_+ = \max\{x_2 :\ (\mathcal{W}, x_2) \in P\}, && \mathcal{E}_+ = \max\{x_2 :\ (\mathcal{E}, x_2) \in P\}.
\end{aligned}$$

All these are integers. Note that $(\mathcal{S}_-, \mathcal{S})$, $(\mathcal{S}_+, \mathcal{S})$, $(\mathcal{N}_-, \mathcal{N})$, $(\mathcal{N}_+, \mathcal{N})$, $(\mathcal{W}, \mathcal{W}_-)$, $(\mathcal{W}, \mathcal{W}_+)$, $(\mathcal{E}, \mathcal{E}_-)$, and $(\mathcal{E}, \mathcal{E}_+)$ are (not necessarily distinct) vertices of $P$.

There are four slopes naturally associated with a given polygon $P$.

Let us enumerate the vertices of $P$ starting from $\mathbf{v}_0 = (\mathcal{W}, \mathcal{W}_-)$ and going in the positive direction until we come to $v_{N_4} = (\mathcal{S}_-, \mathcal{S})$. Clearly, the sequence $\mathbf{v}_0, \dots, \mathbf{v}_{N_4}$ gives rise to a slope with respect to the basis $(\mathbf{e}_1, \mathbf{e}_2)$. We denote it by $Q_4$. Obviously, $Q_4$ is an inclusion-wise maximal slope with respect to $(\mathbf{e}_1, \mathbf{e}_2)$ contained in the boundary of $P$. Likewise, we define the slope $Q_1$ with respect to $(\mathbf{e}_2, -\mathbf{e}_1)$ having the endpoints $(\mathcal{S}_+, \mathcal{S})$ and $(\mathcal{E}, \mathcal{E}_-)$, the slope $Q_2$ with respect to $(-\mathbf{e}_1, -\mathbf{e}_2)$ having the endpoints $(\mathcal{E}, \mathcal{E}_+)$ and $(\mathcal{N}_+, \mathcal{N})$, and the slope $Q_3$ with respect to $(-\mathbf{e}_2, \mathbf{e}_1)$ having the endpoints $(\mathcal{N}_-, \mathcal{N})$ and $(\mathcal{W}, \mathcal{W}_+)$. We call those *maximal slopes* of the polygon $P$ and denote by $N_k$ the number of edges of $Q_k$.



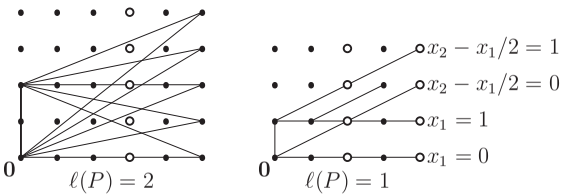


FIGURE 7. Proof of Lemma 5.2. Assuming that $\ell(P) = 2$ and $v_1 = 5$, we consider all possible cases for $\mathbf{v}$ and see that the triangle $T$ invariably contains an integer point lying on the line $x_1 = 3$, which is impossible. Now assume that $\ell(P) = 1$. As the common points of $T$ and the line $x_1 = 2$ must lie between adjacent integer points, we see that $\mathbf{v}$ must lie either in the slab $0 < x_2 < 1$ or in the slab $0 < x_2 - x_1/2 < 1$. The former case is clearly impossible. In the latter case, because $T$ cannot contain a segment with more than two integer points, we see that $v_1 \le 3$, as claimed.

*Proof*: Set $\ell = \ell(P)$ and let $\mathbf{a} = (c, a)$ and $\mathbf{b} = (c, a + \ell)$ be the endpoints of the segment mentioned in the hypothesis of the lemma.

Applying Lemma 5.2 to the polygon $P - \mathbf{a}$, we see that $P$ lies in the slab $|x_1 - c| \le \ell + 2$ and has no common integer points with the lines $x_1 = c \pm (\ell + 1)$.

Let us show that $P$ lies in the half-plane $x_1 \le 2n - 1$. Assume the converse. Then the rightmost vertex $\mathbf{v} = (v_1, v_2)$ of $P$ satisfies

$$2n \le v_1 \le c + \ell + 2, \qquad v_1 \ne c + \ell + 1. \tag{5.4}$$

Consequently, the lines $x_1 = n$ and $x_1 = 2n$ intersect $P$. Clearly, the intersections must lie between pairs of adjacent points of $n\mathbb{Z}^2$ belonging to respective lines, i. e. on some segments $I_1 = [(n, u_1 n), (n, (u_1 + 1)n)]$ and $I_2 = [(2n, u_2 n), (2n, (u_2 + 1)n)]$, where $u_1, u_2 \in \mathbb{Z}$. There is no loss of generality in assuming that $I_1 = [(n, 0), (n, n)]$ and $I_2 = [(2n, 0), (2n, n)]$, since otherwise we could replace $P$ by its image under the affine automorphism $\varphi$ of $n\mathbb{Z}^2$ given by

$$\varphi(x_1, x_2) = (x_1, (u_1 - u_2)x_1 + x_2 + (u_2 - 2u_1)n),$$

which does not affect the first coordinate and maps $I_1$ onto $[(n, 0), (n, n)]$ and $I_2$ onto $[(2n, 0), (2n, n)]$.

by its image under a suitable unimodular transformation having a lower-triangular matrix (the line $x_1 = 0$ is invariant under such transformations, so the hypotheses of the theorem persist).

We must prove that $v_1 \le \ell + 2$. If $v_1 \le \ell + 1$, there is nothing to prove, so assume that $v_1 \ge \ell + 2$. By convexity, $P$ contains the triangle $T$ having the vertices $\mathbf{0}$, $\mathbf{b}$, and $\mathbf{v}$. The line $x_1 = \ell + 1$ intersects $T$ by a segment $I$ of length

$$d = \frac{\ell(v_1 - \ell - 1)}{v_1}.$$

First, suppose that $\ell \ge 2$. If $d \ge 1$, $I$ necessarily contains at least one integer point of the line $x_1 = \ell + 1$, which is impossible by the above. Consequently, we have

$$\frac{\ell(v_1 - \ell - 1)}{v_1} < 1.$$

The numerator and the denominator are positive integers, so we get

$$\ell(v_1 - \ell - 1) \le v_1 - 1,$$

whence

$$v_1 \le \ell + 2 + \frac{1}{\ell - 1}. \tag{5.2}$$

As $v_1$ is an integer, this yields the desired inclusion provided that $\ell \ge 3$.

If $\ell = 2$, inequality (5.2) becomes $v_1 \le 5$. However, setting $v_1 = 5$ and checking possible values $v_2 = 0, 1, \dots, 4$, we see that $T$ invariably contains an integer point lying on the line $x_1 = \ell + 1$ (Figure 7). This is impossible, so actually $v_1 \le 4$, as claimed.

For the case $\ell = 1$, see Figure 7.

To prove that $P$ is contained in the half-plane $x_1 \ge -\ell - 2$, it suffices to reflect $P$ about the line $x_1 = 0$ and apply the established part of the theorem. ∎

**Lemma 5.3.** *Suppose that an integer polygon $P$ is free of $n\mathbb{Z}^2$-points, where $n \in \mathbb{Z}$, $n \ge 2$, and that $P$ has an integer segment with $\ell(P)+1$ integer points lying on the line $x_1 = c$, where*

$$0 \le c \le n; \tag{5.3}$$

*then $P$ is contained in the slab*

$$-n + 1 \le x_1 \le 2n - 1.$$



*Remark* 3.10. For each of the mentioned bases, the boundary of the polygon contains single-point maximal slopes apart from correspondent $Q_k$. However, we single $Q_k$ out by explicitly indicating its endpoints. For a given polygon, some of the maximal slopes $Q_k$ may have but one vertex.

Define

$$M_1 = \begin{cases} 0, & \text{if } \mathcal{S}_- = \mathcal{S}_+, \\ 1, & \text{otherwise;} \end{cases} \quad M_2 = \begin{cases} 0, & \text{if } \mathcal{E}_- = \mathcal{E}_+, \\ 1, & \text{otherwise;} \end{cases}$$

$$M_3 = \begin{cases} 0, & \text{if } \mathcal{N}_- = \mathcal{N}_+, \\ 1, & \text{otherwise;} \end{cases} \quad M_4 = \begin{cases} 0, & \text{if } \mathcal{W}_- = \mathcal{W}_+, \\ 1, & \text{otherwise.} \end{cases}$$

**Proposition 3.11.** *Let $P$ be an $N$-gon; then each edge of $P$ either lies on a horizontal or a vertical line or it is the edge of exactly one of the maximal slopes of $P$; thus,*

$$N = \sum_{k=1}^{4} N_k + \sum_{k=1}^{4} M_k.$$

The point of Proposition 3.11 is that if we want to estimate the number of edges of a polygon, we can do so by considering its maximal slopes and applying the techniques presented above. The following statement is a helpful sufficient condition for a frame to split a maximal slope.

**Proposition 3.12.** *Let $P$ be a convex integer polygon and $(\mathbf{o}; \mathbf{f}_1, \mathbf{f}_2)$ be an integer frame such that $\mathbf{f}_1, \mathbf{f}_2 \in \{\pm\mathbf{e}_1, \pm\mathbf{e}_2\}$. Suppose that $\mathbf{o}$ does not belong to $P$ and the rays $\{\mathbf{c} + \lambda\mathbf{f}_j \colon \lambda \ge 0\}$ $(j = 1, 2)$ split $P$; then $(\mathbf{o}; \mathbf{f}_1, \mathbf{f}_2)$ splits $Q_k$, where*

$$k = \begin{cases} 1, & \text{if } (\mathbf{f}_1, \mathbf{f}_2) = (-\mathbf{e}_1, \mathbf{e}_2) \quad \text{or } (\mathbf{f}_1, \mathbf{f}_2) = (\mathbf{e}_2, -\mathbf{e}_1), \\ 2, & \text{if } (\mathbf{f}_1, \mathbf{f}_2) = (-\mathbf{e}_2, -\mathbf{e}_1) \quad \text{or } (\mathbf{f}_1, \mathbf{f}_2) = (-\mathbf{e}_1, -\mathbf{e}_2), \\ 3, & \text{if } (\mathbf{f}_1, \mathbf{f}_2) = (\mathbf{e}_1, -\mathbf{e}_2) \quad \text{or } (\mathbf{f}_1, \mathbf{f}_2) = (-\mathbf{e}_2, \mathbf{e}_1), \\ 4, & \text{if } (\mathbf{f}_1, \mathbf{f}_2) = (\mathbf{e}_2, \mathbf{e}_1) \quad \text{or } (\mathbf{f}_1, \mathbf{f}_2) = (\mathbf{e}_1, \mathbf{e}_2). \end{cases}$$

The following simple statement also proves useful.

**Proposition 3.13.** *Let $P$ be a $\Gamma$-polygon, $S_1$ be the large $\mathbf{e}_1$-step of $\Gamma$, and $S_2$ be the large $\mathbf{e}_2$-step of $\Gamma$. Then*

$$\mathcal{S}_+ - \mathcal{S}_- \ge S_1 M_1, \ \mathcal{E}_+ - \mathcal{E}_- \ge S_2 M_2,$$
$$\mathcal{N}_+ - \mathcal{N}_- \ge S_1 M_3, \ \mathcal{W}_+ - \mathcal{W}_- \ge S_2 M_4.$$

The proofs of Propositions 3.11, 3.12, and 3.13 are left to the reader.

**3.4. Application to type II polygons.** In this section we present a demonstration of the tools developed above by proving Sub-Theorem C for type II polygons.

**Lemma 3.14.** *Suppose that $n \geq 3$ is an integer and $P$ is a type $II_n$ polygon; then*

*(i)* $((n,0); -\mathbf{e}_1, \mathbf{e}_2)$ *splits* $Q_1$*;*
*(ii)* $((n,n); -\mathbf{e}_1, -\mathbf{e}_2)$ *splits* $Q_2$*;*
*(iii)* $((0,n); \mathbf{e}_1, -\mathbf{e}_2)$ *splits* $Q_3$*;*
*(iv)* $(\mathbf{0}; \mathbf{e}_1, \mathbf{e}_2)$ *splits* $Q_4$*;*
*(v) if all the vertices of $P$ belong to a $(1,n)$-lattice $\Gamma$, then the large $\mathbf{e}_1$-step and large $\mathbf{e}_2$-step of $\Gamma$ are greater then or equal to 2.*

*Proof*: Statements (i)–(iv) immediately follow from the definition of a type $\mathrm{II}_n$ polygon and Proposition 3.12. To prove (v), note that the $\det\Gamma = n$, so by Proposition 2.7 it suffices to show that the small $\mathbf{e}_1$- and $\mathbf{e}_2$-steps of $\Gamma$ are less then $n$. Obviously, the vertex $(\mathcal{S}_-, \mathcal{S})$ of $P$ lies in the slab $0 < x_1 < n$ (this follows, for example, from (i) and (ii)) and belongs to $\Gamma$, so the small $\mathbf{e}_1$-step of $\Gamma$ is indeed less than $n$. Likewise, the vertex $(\mathcal{W}, \mathcal{W}_-)$ lies in the slab $0 < x_2 < n$, so the small $\mathbf{e}_2$-step of $\Gamma$ is less than $n$ as well. ∎

*Proof of Sub-Theorem C for type $II_n$ polygons*: Assume that $P$ is a type $\mathrm{II}_n$ $N$-gon whose vertices belong to $\Gamma$, where either $\Gamma = \mathbb{Z}^2$, or $\Gamma$ is a $(1, n/2)$-lattice (which is only possible if $n$ is even), or a $(1,n)$-lattice. Define $b$ as follows:

$$b = \begin{cases} 0, & \text{if } \Gamma = \mathbb{Z}^2, \\ 1, & \text{if } \Gamma \text{ is a } (1,n/2)\text{-lattice}, \\ 2, & \text{if } \Gamma \text{ is a } (1,n)\text{-lattice}. \end{cases}$$

It suffices to show that

$$N \leq 2n + 2 - 2b. \tag{3.16}$$

We begin by translating the geometrical constraints on $P$ into inequalities.



## 5. Lattice diameter and lattice width

In this section we study the properties of integer polygons free of $n\mathbb{Z}^2$-points related to their lattice diameter and lattice width. Our aim is to prove Proposition 2.8 and supply what is necesary for the proof of Theorem 2.12.

Following [4], let us introduce

**Definition 5.1.** The *lattice diameter* of an integer polygon $P$ is

$$\ell(P) = \max\{|P \cap \mathbb{Z}^2 \cap L| - 1\}, \tag{5.1}$$

where the maximum is taken over all the straight lines $L$ in the plane.

Clearly, $\ell(P)+1$ is the greatest possible length of a string of integer points in a line contained in $P$.

Affine automorphisms of $\mathbb{Z}^2$ preserve the lattice diameter of polygons.

The following lemma provides a simple estimate of the lattice width of a an integer polygon $P$ in terms of its lattice diameter. For the sake of completeness, we include the proof, even though it is implied by the reasoning used in the proof of Theorem 2 of [4].

**Lemma 5.2.** *Suppose that the convex integer polygon $P$ contains the points $\mathbf{0}$ and $(0, \ell(P))$. Then $P$ lies in the slab*

$$|x_1| \leq \ell(P) + 2.$$

*Moreover, no integer point lying on the lines $x_1 = \pm(\ell(P)+1)$ belongs to $P$.*

*Proof*: Set $\ell = \ell(P)$ and $\mathbf{b} = (0, \ell)$. By convexity, $P$ contains the segment $[\mathbf{0}, \mathbf{b}]$ having the integer points $\mathbf{0}$, $(0,1)$, …, $(0,\ell)$.

Let us show that no integer point of the lines $x_1 = \pm(\ell+1)$ belongs to $P$. Consider the point $\mathbf{z} = (\varepsilon(\ell+1), z)$, where $\varepsilon = \pm 1$ and $z$ is an arbitrary integer. Let

$$z = (\ell+1)q + r, \qquad q, r \in \mathbb{Z},\ 0 \leq r \leq \ell.$$

If the point $\mathbf{z}$ belonged to $P$, by convexity the polygon would contain the segment $[r\mathbf{e}_2, \mathbf{z}]$ having $\ell+2$ integer points $r\mathbf{e}_2 + j(\varepsilon\mathbf{e}_1 + q\mathbf{e}_2)$, where $j = 0, \ldots, \ell+1$, which contradicts the definition of the lattice diameter. Consequently, $\mathbf{z} \notin P$, as claimed.

Let us show that $P$ is contained in the half-plane $x_1 \leq \ell + 2$.

Let $\mathbf{v} = (v_1, v_2) \in \mathbb{Z}^2$ be the rightmost vertex of $P$. There is no loss of generality in assuming that $0 \leq v_1 < v_2$, for otherwise we could replace $P$

half-planes $x_1 < 1$ and $x_1 > 1$ (for definitiveness, the former) contains at least two vertices of $P'$. These are integer points, so actually they lie in the half-plane $x_1 \leq 0$. Consequently, $P'$ has common points with the line $x_1 = 0$, all lying between a pair of adjacent $2\mathbb{Z}^2$-points, say, on the segment $I = [(0, 2m), (0, 2m + 2)]$, where $m$ is an integer. We can certainly assume that $m = 0$, for if not, we replace $P'$ by $P' - (0, 2m)$. Moreover, without loss of generality, $\mathbf{a}' = (1, 0)$, because if $\mathbf{a}' = (1, a'_2)$, we can replace $P'$ by $BP'$, where

$$B = \begin{pmatrix} 1 & 0 \\ -a'_2 & 1 \end{pmatrix}$$

(note that $B$ preserves the first component of the vectors, so applying it we do not break previous assumptions). To sum up, there is no loss of generality in assuming that the polygon $P'$ contains the points $\mathbf{a}' = (1, 0)$, and $\mathbf{a}' + 2\mathbf{e}_2 = (1, 2)$, has common points with the segment $[\mathbf{0}, (0, 2)]$ and has two vertices satisfying $x_1 \leq 0$. However, this is impossible, since the vertices of $P'$ belonging to the said half-plane must lie on the line $x_2 = 1$ (see Figure 6). ∎

*Proof of Sub-Theorem C for type $I_n$ polygons*: For simplicity, assume that $P$ lies in the slab $0 \leq x_1 \leq n$.

All the integer points of the slab lie on $n + 1$ lines, so $N \leq 2(n + 1)$.

If the vertices of $P$ belong to a lattice having small $\mathbf{e}_1$-step $s \geq 2$, we have

$$N \leq 2\left(\frac{n}{s} + 1\right) \leq 2\left(\left\lfloor\frac{n}{2}\right\rfloor + 1\right) \leq 2n - 2.$$

Now assume that $P$ is a $\Gamma$-polygon, where $\Gamma$ is a lattice with small $\mathbf{e}_1$-step 1.

If $\Gamma$ is a $(1, n)$-lattice, by Proposition 2.7 the large $\mathbf{e}_2$-step of $\Gamma$ is $n$. Consequently, all the points of $\Gamma$ lying on the lines $x_1 = 0$ and $x_1 = n$ belong to $n\mathbb{Z}^2$. Thus, all the vertices of $P$ lie on the $n - 1$ lines

$$x_1 = j \quad (j = 1, \ldots, n - 1), \tag{4.1}$$

whence $N \leq 2(n - 1)$.

If $\Gamma$ is a $(1, n/2)$-lattice, then the large $\mathbf{e}_2$-step of $\Gamma$ is $n/2$. This implies that on the lines $x_1 = 0$ and $x_1 = n$ there is a single point of $\Gamma$ between adjacent points of $n\mathbb{Z}^2$. Thus, $P$ has at most 1 vertex on each of these lines and at most 2 vertices on each of the lines (4.1), totalling at most $2(n-1)+2 = 2n$ vertices. ∎



Evoking Corollary 3.8 and Theorem 3.9 for the maximal slopes of $P$ and correspondent frames indicated in Lemma 3.14, we obtain:

$$\begin{aligned} 2N_1 &\leq -\mathcal{S}_+ + \mathcal{E}_- + n + \frac{b^2 - 3b}{2}, \\ 2N_2 &\leq -\mathcal{N}_+ - \mathcal{E}_+ + 2n + \frac{b^2 - 3b}{2}, \\ 2N_3 &\leq \mathcal{N}_- - \mathcal{W}_+ + n + \frac{b^2 - 3b}{2}, \\ 2N_4 &\leq \mathcal{S}_- + \mathcal{W}_- + \frac{b^2 - 3b}{2}, \end{aligned}$$

where the term $(b^2 - 3b)/2$ is chosen in such a way that it vanishes at $b = 0$ and equals $-1$ at $b = 1$ and $b = 2$. Further, by Proposition 3.13 we obtain

$$\begin{aligned} \mathcal{S}_+ - \mathcal{S}_- &\geq \frac{b^2 - b + 2}{2} M_1, \\ \mathcal{E}_+ - \mathcal{E}_- &\geq \frac{b^2 - b + 2}{2} M_2, \\ \mathcal{N}_+ - \mathcal{N}_- &\geq \frac{b^2 - b + 2}{2} M_3, \\ \mathcal{W}_+ - \mathcal{W}_- &\geq \frac{b^2 - b + 2}{2} M_4, \end{aligned}$$

since if $b = 0$ or $b = 1$, the large $\mathbf{e}_1$- and $\mathbf{e}_2$-steps of $\Gamma$ are at least 1, and if $b = 2$, by Lemma 3.14 we have that those steps are at least 2.

Using the above inequalities, we obtain:

$$\begin{aligned}2N = \sum_{k=1}^{4} 2N_k + \sum_{k=1}^{4} 2M_k &\le \left(-\mathcal{S}_+ + \mathcal{E}_- + n + \frac{b^2-3b}{2}\right) \\ &+ \left(-\mathcal{N}_+ - \mathcal{E}_+ + 2n + \frac{b^2-3b}{2}\right) + \left(\mathcal{N}_- - \mathcal{W}_+ + n + \frac{b^2-3b}{2}\right) \\ &+ \left(\mathcal{S}_- + \mathcal{W}_- + \frac{b^2-3b}{2}\right) + 2M_1 + 2M_2 + 2M_3 + 2M_4 = 4n + 2b^2 - 6b \\ &+ \left(\frac{b^2-b+2}{2}M_1 - (\mathcal{S}_+ - \mathcal{S}_-)\right) + \left(\frac{b^2-b+2}{2}M_2 - (\mathcal{E}_+ - \mathcal{E}_-)\right) \\ &+ \left(\frac{b^2-b+2}{2}M_3 - (\mathcal{N}_+ - \mathcal{N}_-)\right) + \left(\frac{b^2-b+2}{2}M_4 - (\mathcal{W}_+ - \mathcal{W}_-)\right) \\ &+ \frac{-b^2+b+2}{2}(M_1+M_2+M_3+M_4) \\ &\le 4n + 2b^2 - 6b + \frac{-b^2+b+2}{2}(M_1+M_2+M_3+M_4).\end{aligned}$$

Observe that $(-b^2+b+2)/2 \ge 0$ for $b = 0, 1, 2$, so we can proceed as follows:

$$2N \le 4n + 2b^2 - 6b + \frac{-b^2+b+2}{2} \cdot 4 = 4n + 4 - 4b,$$

which yields (3.16). ∎

## 4. Proof of Sub-Theorems A and B and of Sub-Theorem C for type I polygons

*Proof of Sub-Theorem A*: Assume that, contrary to our claim, there exists an integer polygon containing no integer points with even ordinates. Let $T \subset P$ be an integer triangle having no integer points apart from its vertices $\mathbf{a} = (a_1, a_2)$, $\mathbf{b} = (b_1, b_2)$, and $\mathbf{c} = (c_1, c_2)$. The choice of $P$ ensures that the numbers $a_2$, $b_2$, and $c_2$ are odd. Consequently, the area of $T$ is an integer number, as up to sign it equals

$$\frac{1}{2}\begin{vmatrix} b_1 - a_1 & b_2 - a_2 \\ c_1 - a_1 & c_2 - a_2 \end{vmatrix} = \begin{vmatrix} b_1 - a_1 & (b_2 - a_2)/2 \\ c_1 - a_1 & (c_2 - a_2)/2 \end{vmatrix}.$$



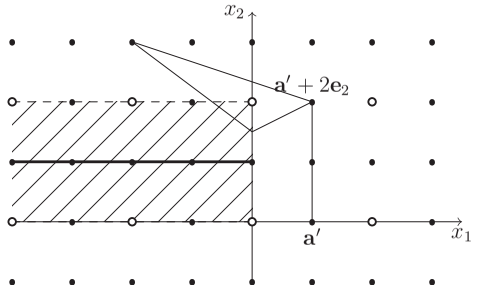


FIGURE 6. The convex hull of the point $\mathbf{a}' = (1, 0)$, any point satisfying $x_1 < 0$ and $x_2 \le 0$, and any point of the segment $[(0, 0), (0, 2)]$ contains the point $(0, 0)$. Likewise, the convex hull of the point $\mathbf{a}' + 2\mathbf{e}_2 = (1, 2)$, any point satisfying $x_1 < 0$ and $x_2 \ge 2$, and any point of the segment $[(0, 0), (0, 2)]$ contains the point $(0, 2)$. Thus, if a polygon is free of $2\mathbb{Z}^2$-points, contains $(1, 0)$ and $(1, 2)$ and has common points with $[\mathbf{0}, (0, 2)]$, it cannot contain any point from the left half-plane not belonging to the hatched slab. Consequently, all the integer points of such a polygon belonging to the left half-plane lie on the line $x_2 = 1$.

However, by Pick's theorem the area of $T$ equals $i + b/2 - 1 = 1/2$, where $i = 0$ is the number of integer points belonging to the interior of $T$ and $b = 3$ is the number of integer points on the boundary; a contradiction. ∎

*Proof of Sub-Theorem B*: Conversely, suppose that $P$ is an integer pentagon free of $2\mathbb{Z}^2$-points. We use the so-called parity argument based on the fact that the index of $2\mathbb{Z}^2$ in $\mathbb{Z}^2$ is 4. This implies that the pentagon has two (distinct) vertices $\mathbf{a} \equiv \mathbf{b} \pmod{2\mathbb{Z}^2}$. Consequently, $\mathbf{b} - \mathbf{a} = u\mathbf{f}$, where $\mathbf{f}$ is a $\mathbb{Z}^2$-primitive vector, and $u \ge 2$ is an integer. Therefore, the segment $[\mathbf{a}, \mathbf{b}]$ contains at least three integer points. By Proposition 2.5, there exists a unimodular transformation $A$ such that $A\mathbf{f} = \mathbf{e}_2$, then the segment $A[\mathbf{a}, \mathbf{b}]$ lies on a line $x_1 = c$, where $c \in \mathbb{Z}$. Observe that $c$ is odd, since otherwise every second integer point of the line would belong to $2\mathbb{Z}^2$, and thus the segment would contain a point of this lattice.

Let $T$ be the translation by the vector $(1 - c, 0) \in 2\mathbb{Z}^2$, then the points $\mathbf{a}' = TA\mathbf{a}$ and $\mathbf{b}' = TA\mathbf{b}$ lie on the line $x_1 = 1$. They are vertices of the pentagon $P' = TAP$, which is still free of $2\mathbb{Z}^2$-points. Clearly, one of the