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Nikolai Bliznyakov
Faculty of Mathematics, Voronezh State University, 1 Universitetskaya pl., Voronezh, 394006, Russia
E-mail address: bliznyakov@vsu.ru

Stanislav Kondratyev
CMUC, Department of Mathematics, University of Coimbra, 3001-501 Coimbra, Portugal
E-mail address: kondratyev@mat.uc.pt

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BOUNDS ON THE NUMBER OF VERTICES OF SUBLATTICE-FREE LATTICE POLYGONS

NIKOLAI BLIZNYAKOV AND STANISLAV KONDRATYEV

Abstract: In this paper we establish bounds on the number of vertices for a few classes of sublattice-free lattice polygons. The bounds are essential for proving the formula for the critical number of vertices of a lattice polygon that ensures the existence of a sublattice point in the polygon. To obtain the bounds, we use relations between the number of edges of lattice broken lines and the coordinates of their endpoints.

Keywords: integer lattice, integer polygons, lattice-free polygons, Diophantine inequalities, integer broken lines.
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1. Introduction

There are quite numerous papers studying geometric and combinatorial properties of convex lattice polytopes and polygons: [1–4, 13–16, 18–22] to cite a few; see also the monographs [5, 9–11]. Our present study is motivated by our paper [7].

Remember that a *lattice* in $\mathbb{R}^2$ is the set of integral linear combinations of a linearly independent system of two vectors. The system itself is called the *basis* of the lattice, and the lattice is said to be *spanned* by the bases. The *integer lattice* $\mathbb{Z}^2$ is the lattice spanned by the standard basis of $\mathbb{R}^2$. The points of $\mathbb{Z}^2$ are called *integer points*. We are primarily interested in the integer lattice and the ones contained in it, i. e. its sublattices.

By a convex polygon we understand the convex hull of a finite set of points in $\mathbb{R}^2$ that has nonempty interior. We assume that the reader is familiar with basic terminology such as vertices and edges, see [12,23] for reference. As we never consider nonconvex polygons, we occasionally drop the word 'convex'. If a polygon has N vertices, we refer to it as an N-gon. An *integer polygon*, or a *lattice polygon* is a polygon, whose vertices are integer points. Generally,

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we prefer the former term, as it is more unambiguous in a context where a few lattices are considered simultaneously.

It was noted in [6] that any convex integer pentagon contains a point of the lattice $2\mathbb{Z}^2$. The paper [7] raises the following question: given a sublattice Λ of the integer lattice $\mathbb{Z}^2$, what is the smallest number of vertices of an integer polygon that ensures that the polygon contains at least one point of Λ? The answer is the Main Theorem of [7]. To state it, we recall that any sublattice of $\mathbb{Z}^2$ is characterised by two positive integers called invariant factors (see the definition in Section 2.1). Let δ, n be the invariant factors of Λ and set

$$\nu(\Lambda) = \nu(\delta, n) = 2n + 2\min\{\delta, 3\} - 3.$$

Theorem 1.1 (Main Theorem of [7])**.** *Let Λ be a proper sublattice of $\mathbb{Z}^2$ having the invariant factors δ, n. Then any convex integer polygon with $\nu(\Lambda)$ vertices contains a point of Λ.*

The constant $\nu(\Lambda)$ in Theorem 1.1 cannot be improved.

The paper [7] does not give a full proof of this theorem. It is shown that the the proof can be reduced to obtaining upper bounds on the number of vertices of integer polygons free of points of the lattice $n\mathbb{Z}^2 = (n\mathbb{Z}) \times (n\mathbb{Z})$. The paper [7] introduces the following classification of such polygons.

We say that a line or a segment *splits* a polygon, if it divides the polygon into two parts with nonempty interior. Here and in the sequel $[\mathbf{a}, \mathbf{b}]$ denotes the segment with the endpoints $\mathbf{a}$ and $\mathbf{b}$.

Let P be an integer polygon free from points of $n\mathbb{Z}^2$, where $n \geq 2$ is an integer.

Definition 1.2. We say that P is a

- *type I_n polygon*, if no line of the form $x_1 = jn$ or $x_1 = jn$ where $j \in \mathbb{Z}$, splits P, or, equivalently, if P lies in a slab of the form $jn \leq x_1 \leq (j+1)n$ or $jn \leq x_1 \leq (j+1)n$, where $j \in \mathbb{Z}$;
- *type II_n polygon*, if each of the segments $[\mathbf{0}, (n, 0)]$, $[(n, 0), (n, n)]$, $[(0, n), (n, n)]$, and $[\mathbf{0}, (0, n)]$ splits P;
- *type III_n polygon*, if each of the segments $[\mathbf{0}, (n, 0)]$, $[(n, 0), (n, n)]$, and $[(n, n), (0, n)]$ splits P, and the line $x_1 = 0$ does not split P;
- *type IV_n polygon*, if each of the segments $[\mathbf{0}, (0, n)]$, $[\mathbf{0}, (n, 0)]$, $[(n, 0), (n, n)]$, and $[(n, n), (2n, n)]$ splits P and P has no common points with the lines $x_1 = -n$ and $x_n = 2n$;

and the midpoints of Δ' do not belong to P'. Therefore, if P' has common points with a side of Δ', they lie between the midpoint of the side and one of its endpoints. This and the fact that the vertices of Δ' are integer points, imply

$$n_0 < n' + \frac{b'}{2}.$$

Let s' be the area of Δ'. By Pick's theorem,

$$s' = n' + \frac{b'}{2} - 1,$$

whence

$$n_0 < s' + 1.$$

On the other hand, $s' = 4n$, because the area of Δ_n is $2n^2$ and $|\det A| = 2/n$. Finally, we obtain

$$n_0 < 4n + 1.$$

Comparing the last inequality and (7.4), we get

$$(n-1)^2 < 4n + 1.$$

which cannot hold if $n \geq 6$. The contradiction proves the lemma for $n \geq 6$. ∎

As a corollary of Lemmas 7.2 and Lemma 5.3 we obtain that Theorem 1.4 holds for type Va polygons. In view of Lemma 5.2 we conclude that it also holds for type V polygons. Now, Lemma 6.1 implies that Theorem 1.4 is true for type VI polygons as well.

Thus, Theorem 1.4 is proved in full generality. Combining it with other results of [7], we conclude that Theorem 1.1—the Main Theorem of [7]—is proved as well.

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If the small $\mathbf{e}_1$-step of Γ is 2, it is easily seen that all the points of Γ lying in Δ_4 and not belonging to $4\mathbb{Z}^2$ lie on the lines

$$x_1 = 0, \quad x_1 = 2, \quad x_1 = 4, \quad x_1 = 6.$$

In particular, all the vertices of P lies on these lines. Each of the four lines contains at most two vertices, and (7.2) follows.

The case when the small $\mathbf{e}_2$-step of Γ is 2 is handled in the same way.

Suppose that the small $\mathbf{e}_1$-step of Γ is 1. By Proposition 2.4 it has a basis $(\mathbf{e}_1 + a\mathbf{e}_2, 2\mathbf{e}_2)$, where $a = 0$ or $a = 1$. In the former case the small $\mathbf{e}_1$-step of Γ is 2 and (7.2) is proved. In the latter case define the automorphism of $4\mathbb{Z}^2$ by

$$\psi(x_1, x_2) = (x_1, x_1 - x_2 + 8).$$

The polygon $\psi(P)$ lies in the triangle $\Delta_4 = \psi(\Delta_4)$ and its vertices belong to a $(1, 2)$-lattice $\psi(\Gamma)$ spanned by $(\mathbf{e}_1, -2\mathbf{e}_2)$. Obviously, the small $\mathbf{e}_2$-step of $\psi(\Gamma)$ is 2, so we obtain (7.2) applying the proved part of the lemma to $\psi(\widehat{P})$.

Now assume that $n \geq 6$ and, contrary to our assertion,

$$N > 2n. \tag{7.3}$$

It follows from Proposition 2.2 that there exists a unimodular transformation B such that $B\Gamma = \mathbb{Z} \times (n/2)\mathbb{Z}$. Clearly, the transformation $A = \operatorname{diag}(1, 2/n)B$ maps Γ onto $\mathbb{Z}^2$, so $P' = AP$ is an integer N-gon contained in the integer triangle $\Delta' = A\Delta_n$.

Let us estimate the number of integer points in P', which we denote by n_0.

We claim that

$$n_0 \geq (n-1)^2. \tag{7.4}$$

Indeed, if P' contained less then $(n-1)^2$ integer points, we could find two integers i_1 and i_2 such no integer point $(u_1, u_2) \in P'$ would satisfy

$$x_k \equiv i_k \pmod{(n-1)}$$

simultaneously for $k = 0$ and $k = 1$. In other words, the shifted N-gon $P' - (i_1, i_2)$ would be free of $(n-1)\mathbb{Z}^2$-points, which is impossible due to Lemma 7.1 and inequality (7.3). Thus, (7.4) is proved.

To estimate n_0 from above we use the inclusion $P' \subset \Delta'$. Let n' be the number of integer points in the interior of Δ' and b' be the number of integer points on its boundary. Observe that the vertices and midpoints of the sides of Δ_n are $n\mathbb{Z}^2$-points, so they do not belong to P; consequently, the vertices

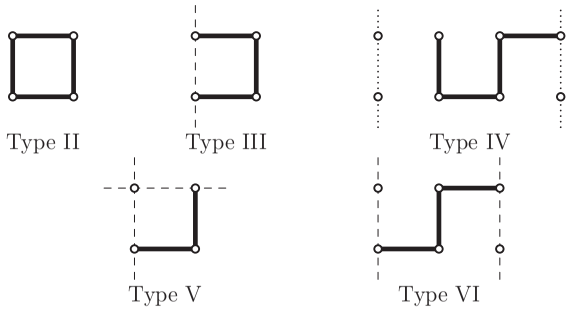

FIGURE 1. Definition 1.2 introduces the types of polygons in terms of intersection with segments and lines. Here thick segments split polygons of the specified type, thin lines do not split them, and dotted lines have no common points with them.

- *type V_n polygon*, if each of the segments $[\mathbf{0}, (-n, 0)]$ and $[\mathbf{0}, (0, n)]$, and the lines $x_1 = -n$ and $x_2 = n$ do not split P;
- *type VI_n polygon*, if each of the segments $[\mathbf{0}, (-n, 0)]$, $[\mathbf{0}, (0, n)]$, and $[(0, n), (n, n)]$ split P, and the lines $x_1 = \pm n$ do not split P.

The polygon types are illustrated on Figure 1.

The following theorem is proved in [7].

Theorem 1.3. *Suppose that an integer polygon P is free of points of the lattice $n\mathbb{Z}^2$, where $n \in \mathbb{Z}$, $n \geq 2$; then there exists an affine transformation φ of $\mathbb{R}^2$ preserving $n\mathbb{Z}^2$ such that $\varphi(P)$ is a polygon of one of the types I_n–VI_n.*

The hard part of Theorem 1.1 is encapsulated in the following assertion.

Theorem 1.4 (Sub-Theorem C of [7])**.** *Let P be a convex integer N-gon of one of the types I_n–VI_n, where n is an integer, $n \geq 3$. Then:*

(i) the following inequality holds:

$$N \leq 2n + 2;$$

(ii) if the vertices of P belong to a $(1, n/2)$-lattice, then

$$N \leq 2n;$$

(iii) if the vertices of P belong to a $(1, n)$-lattice, then

$$N \leq 2n - 2.$$

It is shown in [7] that Theorem 1.4 together with other results of that paper imply Theorem 1.1. In [7] Theorem 1.4 is proved for type I and II polygons. The present paper aims to prove this theorem for the rest cases, systematically applying the approach developed in [7]. Thus, the proof of Theorem 1.1 will be also completed.

The method we adopt can be summarised as follows. Geometric constraints imposed on the polygon are translated into Diophantine inequalities relating the numbers of vertices of certaing broken lines contained in the boundary of the polygon, the parameters of its bounding box and, possibly, auxiliary integral parameters. The analysis of the inequalities, however, may prove rather technical due to the number of parameters and nonlinearities. The core of the method is constituted by a few statements about certain classes of broken lines. These statements provide relations between the number of edges of the broken lines and the coordinates of their endpoints.

The rest of the paper is organised as follows.

In Section 2 we collect familiar facts concerning the geometry of lattices and convex polygons as well as results of [7] useful for estimating the number of edges of lattice broken lines and polygons.

In Sections 3 and 4 we prove Theorem 1.4 for type III and IV polygons, respectively, applying the method described above. In the case of type III polygons the transition from geometric constraints to Diaphantine equations is fairly straightforward, but the analysis of the equations is rather involved. Type IV polygons are somewhat more technical from the geometric point of view.

In Section 5 we consider type V polygons. We introduce so called lifts for a class of polygons and show that iterating lifting in combination with certain affine transformations we can map any type V polygon either onto a type III polygon, thus reducing the problem to a known case, or onto a polygon lying in a certain triangle (we call them type Va polygons). Thus, it suffices to consider type Va polygons instead of type V polygons. We establish a few bounds on the number of vertices of type Va polygons using various algebraic and geometric tricks; however, at this point we are unable to obtain all the estimates required by Theorem 1.4. We revisit type Va polygons in Section 7.

In Section 6 we reuse the lifts introduced in the previous section and show that any type VI polygon can be mapped onto a polygon of another type. However, type V is not excluded, so at this point we are unable to prove Theorem 1.4 for type VI polygons.

and consider the polygon $\psi_1(\widehat{P}')$. It is not split by the lines $x_1 = 0$ and $x_1 = n$, being the images of $x_1 - x_2 = 0$ and $x_1 - x_2 = -n$, respectively, and $\psi_1(\widehat{P}')$ has points inside the slab $0 \le x_1 \le n$, e. g. on the segment

$$[(n, 0), (0, n)] = \psi_1([\mathbf{0}, (0, n)]).$$

Consequently, $\psi_1(\widehat{P}')$ is a type I_n polygon.

In Cases 2 and 3 we use the same automorphism ψ_1. It is not hard to check that in Case 2, $\psi_1(\widehat{P}')$ is a type II_n polygon and in Case 3, it is a type III_n polygon

In Case 4, define the automorphism of $n\mathbb{Z}^2$ by

$$\psi_2(x_1, x_2) = (x_1 - x_2 + n, x_2).$$

It is easily seen that $\psi_2(\widehat{P}')$ is a type III_n polygon. ∎

7. Proof of Theorem 1.4 for polygons of types V and VI

We have already proved Theorem 1.4 for polygons of types I_n–IV_n. Type Va_n polygons have been our stumbling block so far, because we have not proved the estimate $N \le 2n$ for such N-gons, in case their vertices belong to a $(1, n/2)$-lattice. However, for type Va_n N-gons we have the estimate $N \le 2n + 2$ (Lemma 5.3). In view of Lemmas 5.2 and 6.1, this estimate is valid for type V_n and VI_n N-gons as well. Combining this with Theorem 1.3, we obtain the following particular case of Theorem 1.1:

Lemma 7.1. *Let n be an integer, $n \ge 3$; then any integer polygon free of $n\mathbb{Z}^2$-points has no more than $2n + 2$ vertices.*

Now we can prove the missing estimate for type Va polygons.

Lemma 7.2. *Let n be an even integer, $n \ge 4$, and P be a type Va_n N-gon. Suppose that the vertices of P belong to a $(1, n/2)$-lattice. Then*

$$N \le 2n. \tag{7.1}$$

Proof: Let Γ be a $(1, n/2)$-lattice containing the vertices of P.

By definition, P is contained in the triangle Δ_n defined by (5.1).

Assume that $n = 4$, then we must prove that

$$N \le 8. \tag{7.2}$$

In this case Γ is a $(1, 2)$-lattice, and by Proposition 2.3, the only possible values for the small $\mathbf{e}_1$- and $\mathbf{e}_2$-steps of Γ are 1 and 2.

It remains to consider the case of the segment $[(0,n),(n,n)]$ splitting $\widehat{P}$. Let φ be an affine automorphism of $n\mathbb{Z}^2$ defined by

$$\varphi(x_1,x_2)=(-x_1,n-x_2),$$

(the symmetry with respect to $(0,n/2)$) and set $P'=\varphi(\widehat{P})$. The polygon P' lies in the slab (6.1), which is invariant under φ; also P' is split by the segments

$$[(-n,0),\mathbf{0}]=\varphi([(n,n),(0,n)]),$$
$$[\mathbf{0},(0,n)]=\varphi([(0,n),\mathbf{0}])$$

and is not split by the segment

$$[(0,n),(n,2n)]=\varphi([\mathbf{0},(-n,-n)]).$$

Thus, the lift $\widehat{P}'$ is well-defined. By the definition of the lift and by Lemma 5.5, the polygon $\widehat{P}'$ is split by the segments $[\mathbf{0},(-n,0)]$ and $[\mathbf{0},(0,n)]$ and is not split by the segments $[\mathbf{0},(-n,-n)]$ and $[(0,n),(n,2n)]$; moreover, $\widehat{P}'$ lies in the slab (6.1). Consequently, the line $x_1=-n$ does not split $\widehat{P}'$. If the line $x_2=n$ does not split it either, it is a type V_n polygon, and we are done. Otherwise, as before, we infer that either $[(-n,n),(0,n)]$ splits $\widehat{P}'$, and we conclude by noticing that $T\widehat{P}'$ is a type III_n polygon, or $[(0,n),(n,n)]$ splits $\widehat{P}'$, which we assume in what follows.

The intersection of the line $x_1-x_2=-n$ and the slab (6.1) is the union of the segments $[(-n,0),(0,n)]$ and $[(0,n),(n,2n)]$. The latter segment does not split $\widehat{P}'$, the line $x_1-x_2=-n$ splits $\widehat{P}'$ if and only if the segment $[(-n,0),(0,n)]$ does so. Likewise, the line $x_1-x_2=0$ splits $\widehat{P}'$ if and only if the segment $[\mathbf{0},(n,n)]$ does so, for $\widehat{P}'$ is not split by $[(-n,-n),\mathbf{0}]$. Thus, we have four logical possibilities.

Case 1. The lines $x_1-x_2=-n$ and $x_1-x_2=0$ do not split $\widehat{P}'$.

Case 2. The segments $[(-n,0),(0,n)]$ and $[\mathbf{0},(n,n)]$ split $\widehat{P}'$.

Case 3. The segment $[(-n,0),(0,n)]$ splits $\widehat{P}'$ and the line $x_1-x_2=0$ does not.

Case 4. The segment $[\mathbf{0},(n,n)]$ splits $\widehat{P}'$ and the line $x_1-x_2=-n$ does not.

In Case 1 define the affine automorphism of $n\mathbb{Z}^2$ by

$$\psi_1(x_1,x_2)=(-x_1+x_2,x_2)$$

In Section 7 we finally prove the missing inequality for type Va polygons, which automatically gives the proof of Theorems 1.4 and, eventually, 1.1. What enables us to carry out this proof is a bound on the number of vertices of an arbitrary integer polygon free of points of $n\mathbb{Z}^2$, obtained as a combination of established estimates and Theorem 1.3.

2. Preliminaries

2.1. Lattices and polygons. In this section we collect a few definitions and facts concerning the geometry of the integer lattice and convex polytops. For reference, see [5,8,10,11].

A lattice $\Lambda\subset\mathbb{Z}^2$ is spanned by the columns of a matrix $A=(a_{ij})\in GL_2(\mathbb{Z})$ if and only if $\Lambda=A\mathbb{Z}^2=\{A\mathbf{u}\colon \mathbf{u}\in\mathbb{Z}^2\}$. Given Λ, the matrix A is not uniquely defined. However, the numbers

$$\delta=\gcd(a_{ij}),\ n=|\det A|/\delta$$

are independent of A. They are called *invariant factors* of Λ, and the pair (δ,n) is *the invariant factor sequence of* Λ (see [17]). The product of the invariant factors equals the determinant of A; it is called the *determinant* of the lattice and denoted $\det\Lambda$. Clearly, proper sublattices of $\mathbb{Z}^2$ (i. e. the ones that do not coincide with $\mathbb{Z}^2$) have determinants ≥ 2.

For brevity, we write that Λ is a (δ,n)-lattice if it is a sublattice of $\mathbb{Z}^2$ with invariant factor sequence (δ,n).

As an example, the lattice $n\mathbb{Z}^2=\{(nu_1,nu_2)\colon u_1,u_2\in\mathbb{Z}^2\}$, where n is a positive integer, has invariant factor sequence (n,n), and the lattice $\delta\mathbb{Z}\times n\mathbb{Z}^2=\{(\delta u_1,nu_2)\colon u_1,u_2\in\mathbb{Z}^2\}$, where δ and n are positive integers and δ divides n, has invariant factor sequence (δ,n).

If a point belongs to a lattice Λ, we call it a Λ-point.

A matrix $A\in M_2(\mathbb{Z})$ is called *unimodular*, if $\det A=\pm1$.

Proposition 2.1. *Let $(\mathbf{f}_1,\mathbf{f}_2)$ be a basis of a lattice Λ; then the vectors $a_{i1}\mathbf{f}_j+a_{i2}\mathbf{f}_2$, where $i=1,2$, form a basis of Λ if and only if the matrix (a_{ij}) is unimodular.*

A linear transformation of the plane is called a linear automorphism of a lattice if it maps the lattice onto itself. A linear transformation is an automorphism of a lattice if and only if it maps some (hence, any) basis of the lattice onto another basis. Clearly, linear automorphism of a lattice form a group.

Given a matrix $A \in M_2(\mathbb{R})$, the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is an automorphism of $\mathbb{Z}^2$ if and only if the matrix A is unimodular. We call such transformation *unimodular*. What is more, for any positive integer n, the linear automorphisms of $n\mathbb{Z}^2$ are exactly the unimodular transformations.

If Λ is a sublattice of $\mathbb{Z}^2$ and A is a unimodular transformation, the image $A\Lambda$ is a lattice with the same invariant factors as Λ.

The following proposition is a geometric version of the Smith normal form of integral matrices [17].

Proposition 2.2. *For any sublattice of $\mathbb{Z}^2$ with invariant factors (δ, n) there exists a unimodular transformation mapping it onto the lattice $\delta\mathbb{Z} \times n\mathbb{Z}$.*

An *affine frame* of a lattice Λ is a pair $(\mathbf{o}; \mathbf{f}_1, \mathbf{f}_2)$ consisting of a point $\mathbf{o} \in \Lambda$ and a basis $(\mathbf{f}_1, \mathbf{f}_2)$ of Λ. An *integer frame* is an affine frame of $\mathbb{Z}^2$.

An *affine automorphism* of a lattice Λ is an affine transformation of $\mathbb{R}^2$ mapping Λ onto itself. It is not hard to see that given $A \in M_2(\mathbb{R})$ and $\mathbf{b} \in \mathbb{R}^2$, the mapping $\mathbf{x} \mapsto A\mathbf{x}+\mathbf{b}$ is an affine automorphism of Λ if and only if $\mathbf{x} \mapsto A\mathbf{x}$ is an automorphism of Λ and $\mathbf{b} \in \Lambda$. In particular, affine automorphisms of $n\mathbb{Z}^2$, where n is a positive integer, are exactly the transformations of the form $\mathbf{x} \mapsto A\mathbf{x} + \mathbf{b}$, where A is unimodular and $\mathbf{b} \in n\mathbb{Z}^2$.

Of course, if P is a convex integer N-gon and φ is an affine automorphism of $\mathbb{Z}^2$, the image $\varphi(P)$ is still a convex integer N-gon. Obviously, is P is free from points of a lattice Λ, then so is its image under any affine automorphism of Λ.

Following [7], we introduce the following definition.

Let Λ be a sublattice of $\mathbb{Z}^2$ and $(\mathbf{f}_1, \mathbf{f}_2)$ be a basis of $\mathbb{Z}^2$. Clearly, $\{u \in \mathbb{Z}\colon u\mathbf{f}_1 \in \Lambda\}$ is a subgroup of $\mathbb{Z}$. It is generated by a positive integer, which we call the *large* $\mathbf{f}_1$*-step* of Λ with respect to $(\mathbf{f}_1, \mathbf{f}_2)$. Further, $\{u_1 \in \mathbb{Z}\colon \exists\, u_2 \in z,\ u_1 f_1 + u_2 f_2 \in \Lambda\}$ is a subgroup of $\mathbb{Z}$, too. We call its positive generator the *small* $\mathbf{f}_1$*-step* of Λ. Alternatively, the small $\mathbf{f}_1$-step can be defined as the largest s such that all the points of Λ lie on the lines $\{ks\mathbf{f}_1 + t\mathbf{f}_2\}$, $k \in \mathbb{Z}$. Obviously, the small step is smaller then the large step. We can define the large and small $\mathbf{f}_2$-steps of Λ with respect to $(\mathbf{f}_1, \mathbf{f}_2)$ in the same way.

In what follows we consider small and large steps of lattices with respect to bases made up of the vectors $\pm\mathbf{e}_1$, $\pm\mathbf{e}_2$, where $(\mathbf{e}_1, \mathbf{e}_2)$ is the standard basis of $\mathbb{R}^2$, and we usually omit the reference to the basis when there is no ambiguity.

We note two simple properties of the steps.

Thus, the polygon $\varphi(P)$ satisfies (5.53), and applying the already proved part of the lemma to $\varphi(P)$, we obtain (5.52).

The part of the lemma concerning polygons with vertices belonging to a $(1, n)$-lattice is given by Lemma 5.9. ∎

6. Type VI polygons

It turns out that any type VI polygon can be mapped onto a polygon of another type by an automorphism of $n\mathbb{Z}^2$. The following lemma is the main result of this section.

Lemma 6.1. *Suppose that P is a type VI_n polygon; then there exists an affine automorphism ψ of $n\mathbb{Z}^2$ such that $\psi(P)$ is a polygon of one of the types I_n, II_n, III_n, or V_n.*

Proof: The polygon P is split by the segments $[\mathbf{0}, (-n, 0)]$ and $[\mathbf{0}, (0, n)]$, so the lift $\widehat{P}$ (see Section 5.2) is well-defined. The polygon $\widehat{P}$ is split by the segment $[\mathbf{0}, (-n, 0)]$ by the definition of the lift and by the segment $[\mathbf{0}, (0, n)]$ by Lemma 5.5. By the same lemma, the segment $[\mathbf{0}, (-n, -n)]$ does not split $\widehat{P}$. The lines $x_1 = \pm n$ are invariant under the lift transformation, so they do not split $\widehat{P}$ either. Besides, $\widehat{P}$ has points in the slab

$$-n \le x_1 \le n, \tag{6.1}$$

(e. g. on the segment $[\mathbf{0}, (0, n)]$), so we conclude that it is contained in this slab.

If the line $x_2 = n$ does not split $\widehat{P}$, the latter is a type V_n polygon. Otherwise, $\widehat{P}$ is split by one of the segments $[(-n, n), (0, n)]$ and $[(0, n), (n, n)]$, since their union is exaclty the set of common points of the line $x_2 = n$ and the slab (6.1).

Suppose that the segment $[(-n, n), (0, n)]$ splits $\widehat{P}$. Let T be the translation by the vector $(n, 0) \in n\mathbb{Z}^2$. The polygon $T\widehat{P}$ is split by the segments

$$[(0, n), (n, n)] = T[(-n, n), (0, n)],$$
$$[(n, 0), \mathbf{0}] = T[\mathbf{0}, (-n, 0)],$$
$$[(n, 0), (n, n)] = T[\mathbf{0}, (0, n)]$$

and is not split by the line $x_1 = 0$ being the image of $x_1 = -n$ under T. In other words, $T\widehat{P}$ is a type III_n polygon, and we are done.

or

$$\mathcal{S}_+ \geq n+1, \quad \mathcal{W}_+ \leq n. \tag{5.54}$$

If P lies in a slab of the form

$$0 \leq x_1 \leq n, \quad n \leq x_1 \leq 2n, \quad 0 \leq x_2 \leq n, \quad n \leq x_2 \leq 2n,$$

it is a type I_n polygon, and the estimate (5.53) follows from Theorem 1.4 for type I polygons. Otherwise, P is split by the segments $[(n,0),(n,n)]$ and $[(0,n),(n,n)]$, which are the intersections of the lines $x_1 = n$ and $x_2 = n$ with Δ_n. Therefore, by Proposition 2.16 the frame $((n,n);-\mathbf{e}_2,-\mathbf{e}_1)$ splits the slope Q_2. If this frame forms small angle with the slope, Lemma 5.8 provides (5.52). If not, it follows from Proposition 2.10 that the frame $((n,n);-\mathbf{e}_1,-\mathbf{e}_2)$ forms small angle with Q_2. Let P' be the reflection of P about the line $x_1 = x_2$. It is not hard to check that $((n,n);-\mathbf{e}_2,-\mathbf{e}_1)$ forms small angle with $Q_2(P')$; moreover, P' is a minimal type Va_n polygon, and since

$$\mathcal{S}_+(P') = \mathcal{W}_+(P), \qquad \mathcal{W}_+(P') = \mathcal{S}_+(P),$$

we see that P' satisfies (5.53) or (5.54). Applying the already proved part of the lemma to P', we obtain (5.52).

Now suppose that P satisfies neither (5.53), nor (5.54). Thus, in particular,

$$\mathcal{S}_+(P) \geq n+1.$$

Consider the affine automorphism of $n\mathbb{Z}^2$ given by

$$\varphi(x_1,x_2) = (-x_1 - x_2 + 2n, x_2).$$

By Proposition 5.12, the polygon $\varphi(P)$ is minimal. Moreover, it lies in the triangle Δ_n, since $\Delta_n = \varphi(\Delta_n)$. Obviously, we have

$$\mathcal{S}(\varphi(P)) = \mathcal{S}(P).$$

A straightforward computation gives

$$\mathcal{S}_-(\varphi(P)) = -\mathcal{S}_+(P) - \mathcal{S}(P) + 2n.$$

As the points of P satisfy system (5.1), we have $\mathcal{S}(P) \geq 0$, so

$$\mathcal{S}_-(\varphi(P)) \leq -\mathcal{S}_+(P) + 2n \leq n-1.$$

By Proposition 5.11,

$$\mathcal{S}_+(\varphi(P)) \leq \mathcal{S}_-(\varphi(P)) + 1,$$

so we have

$$\mathcal{S}_+(\varphi(P)) \leq n.$$

Proposition 2.3. *Let Λ be a sublattice of $\mathbb{Z}^2$ and $(\mathbf{f}_1,\mathbf{f}_2)$ be a basis of $\mathbb{Z}^2$. Then the product of the small $\mathbf{f}_1$-step and the large $\mathbf{f}_2$-step of Λ equals the determinant of Λ.*

Proposition 2.4. *Let $(\mathbf{f}_1,\mathbf{f}_2)$ be a basis of $\mathbb{Z}^2$, Λ be a sublattice of $\mathbb{Z}^2$ such that the small $\mathbf{f}_1$-step of Λ is 1, and R be a complete residue system modulo $\det\Lambda$. Then there exists a unique $r \in R$ such that $(\mathbf{f}_1 + r\mathbf{f}_2, (\det\Lambda)\mathbf{f}_2)$ is a basis of Λ.*

The following Lemma and its corollary are geometrically obvious. An application of Helly's theorem provides an immediate proof of the proposition.

Lemma 2.5. *If a convex polygon has common points with each of the four angles formed by intersecting lines, it contains the intersection point.*

Corollary 2.6. *If a convex polygon has common points with both sides of one of the vertical angles and does not contain its vertex, it has no common points with the other vertical angle.*

We always denote the vectors of the standard basis of $\mathbb{R}^2$ by $\mathbf{e}_1 = (1,0)$ and $\mathbf{e}_2 = (0,1)$ and the standard coordinates in $\mathbb{R}^2$ by x_1, x_2. We also use usual notations $\lfloor\cdot\rfloor$ for the floor function, $\lceil\cdot\rceil$ for the ceiling function, and $|\cdot|$ for the cardinality of a finite set.

2.2. Slopes. This section summarises the results of [7] about a class of broken lines called slopes. These are our main tool for obtaining bounds on the number of vertices of polygons. The proofs can be found in [7].

Let $(\mathbf{f}_1,\mathbf{f}_2)$ be a basis of $\mathbb{R}^2$, and let $\mathbf{v}_0,\mathbf{v}_1,\ldots,\mathbf{v}_N$ ($N \geq 0$) be a finite sequence of points on the plane. If $N \geq 1$, set

$$\mathbf{v}_i + \mathbf{v}_{i-1} = \mathbf{a}_i = a_{i1}\mathbf{f}_1 + a_{i2}\mathbf{f}_2 \qquad (i = 1,\ldots,N). \tag{2.1}$$

If

$$a_{i1} > 0,\ a_{i2} < 0 \qquad (i = 1,\ldots,N) \tag{2.2}$$

and

$$\begin{vmatrix} a_{i1} & a_{i+1,1} \\ a_{i2} & a_{i+1,2} \end{vmatrix} > 0 \qquad (i = 1,\ldots,N-1), \tag{2.3}$$

we say that the union Q of the segments $[\mathbf{v}_0,\mathbf{v}_1]$, $[\mathbf{v}_1,\mathbf{v}_2]$, …, $[\mathbf{v}_{N-1},\mathbf{v}_N]$ is a *slope* with respect to the basis $(\mathbf{f}_1,\mathbf{f}_2)$. These segments are called the *edges* of the slope, and the points $\mathbf{v}_0$, $\mathbf{v}_1$, …, $\mathbf{v}_N$, its *vertices*, $\mathbf{v}_0$ and $\mathbf{v}_N$ being the *endpoints*. If $N = 1$, we call the segment $[\mathbf{v}_0,\mathbf{v}_1]$ a slope if (2.2) holds, and if $N = 0$, we still call the one-point set $\{\mathbf{v}_0\}$ a slope. If all the vertices of Q

belong to a lattice Γ, we call it a Γ-*slope*. A $\mathbb{Z}^2$-slope is called *integer*, and it is the only kind of slopes we are interested in.

If Q is a slope with respect to a basis $(\mathbf{f}_1, \mathbf{f}_2)$, then it is a slope with respect to the basis $(\mathbf{f}_2, \mathbf{f}_1)$ as well.

Proposition 2.7. *Let $(\mathbf{f}_1, \mathbf{f}_2)$ be a basis of $\mathbb{Z}^2$ and $\mathbf{v}$ and $\mathbf{w}$ be the endpoints of an integer slope (with respect to $(\mathbf{f}_1, \mathbf{f}_2)$) having N edges. Let*

$$\mathbf{w} - \mathbf{v} = b_1\mathbf{f}_1 + b_2\mathbf{f}_2.$$

Then there exists an integer s such that

$$2N \le |b_1| + s, \tag{2.4}$$

$$|b_2| \ge \frac{s(s+1)}{2}, \tag{2.5}$$

$$0 \le s \le N. \tag{2.6}$$

If the vertices of the slope belong to a lattice with small $\mathbf{f}_1$-step greater then 1, one can take $s = 0$, so that

$$2N \le |b_1|. \tag{2.7}$$

If the vertices of the slope belong to a lattice having the basis $(\mathbf{f}_1 - a\mathbf{f}_2, m\mathbf{f}_2)$, where $1 \le a \le m$, then (2.5) can be replaced by

$$|b_2| \ge \frac{2a + (s-1)m}{2}s. \tag{2.8}$$

Let $(\mathbf{o}; \mathbf{f}_1, \mathbf{f}_2)$ be an affine frame of $\mathbb{Z}^2$ and Q be a slope with respect to $(\mathbf{f}_1, \mathbf{f}_2)$.

Definition 2.8. We say that the frame $(\mathbf{o}; \mathbf{f}_1, \mathbf{f}_2)$ *splits* the slope Q, if

(1) one endpoint $\mathbf{v} = \mathbf{o} + v_1\mathbf{f}_1 + v_2\mathbf{f}_2$ of Q satisfies

$$v_1 < 0,\ v_2 > 0, \tag{2.9}$$

while the other endpoint $\mathbf{w} = \mathbf{o} + w_1\mathbf{f}_1 + w_2\mathbf{f}_2$ satisfies

$$w_1 > 0,\ w_2 < 0; \tag{2.10}$$

(2) there exists a point on Q having both positive coordinates in the frame $(\mathbf{o}; \mathbf{f}_1, \mathbf{f}_2)$.

Remark 2.9. Obviously, a frame can only split a slope if the slope has at least one edge.

so $(2n-2, -n+2)$ is the only point of the line $x_2 = -n+2$ that could be a vertex of $\widehat{P}$.

As in our case $p = -n+1$ and $q = 2$, we see that each of the $n-2$ lines (5.49) corresponding to $j = p+2, \dots, q-2$ contains at most two vertices of $\widehat{P}$; each of the lines corresponding to $j = p+1$ and $j = q-1$ contains at most one vertex; finally, as $a \ne 0$ and $a \ne 1$, the lines corresponding to $j = p$ and $j = q$ contain no vertices. This amounts to a total of at most $2n-2$ vertices, so (5.46) is proved.

It only remains to check the case $n = 3$. Then the vertices of the triangle $\widehat{\Delta}$ are the points $\mathbf{0}$, $(0, 2)$, and $(6, -2)$. It is easy to check that $\widehat{\Delta}$ contains only 4 integer points not belonging to Λ, so the inequality $N \le 4 = 2n - 2$ is trivial. ∎

Definition 5.10. We call an integer polygon *minimal* if it does not contain other integer polygon with the same number of vertices.

We note two simple properties of minimal polygons.

Proposition 5.11. *Any edge of a minimal polygon contains precisely two interger points—its endpoints.*

Proof: If $\mathbf{v}_1, \dots, \mathbf{v}_N$ are the vertices of an integer N-gon and its edge $[\mathbf{v}_1, \mathbf{v}_2]$ contains an integer point $\mathbf{v}$ different from $\mathbf{v}_1$ and $\mathbf{v}_2$, then it is easily seen that the convex hull of the points $\mathbf{v}$, $\mathbf{v}_2$, $\dots$, $\mathbf{v}_n$ is an integer N-gon contained in the original one and different from it. This means that the original polygon is not minimal. ∎

Proposition 5.12. *Affine automorphisms of the integer lattice map minimal polygons onto minimal polygons.*

This proposition is obvious.

Proof of Lemma 5.3: Let us prove the inequality

$$N \le 2n + 2. \tag{5.52}$$

We can certainly assume that P is minimal, for if not, we replace P by a minimal polygon it contains (which is, of course, again a type Va_n polygon).

First, assume that P satisfies either

$$\mathcal{S}_+ \le n \tag{5.53}$$

According to (5.47) and (5.48), we have

$$q-p=\begin{cases}2a, & \text{if } a\geq 1,\\ -2a+2, & \text{if } a\leq 0.\end{cases}$$

Using this formula, we see that if $1\leq a\leq \frac{n}{2}$, then

$$q-p\leq 2\cdot\frac{n}{2}=n,$$

and if $n/2-n+1\leq a\leq 0$, then

$$q-p\leq -2(n/2-n+1)+2=n.$$

Thus, inequality (5.50) holds provided that $n/2-n+1\leq a\leq n/2$, so for these a inequality (5.46) is proved. Taking into account the range of possible values of a, we see that it only remains to check the case $a=\lfloor n/2\rfloor-n+1$ when $\lfloor n/2\rfloor<n/2$, i. e. $a=(-n+1)/2$ for an odd n.

In this case the vertices of $\widehat{\Delta}$ are the points $\mathbf{0}$, $(0,2)$, and $(2n,-n+1)$, and this triangle is the solution set of the system

$$\begin{cases} x_1\geq 0,\\ x_1\leq -\frac{2n}{n+1}(x_2-2),\\ x_1\geq -\frac{2n}{n-1}x_2.\end{cases}\tag{5.51}$$

We will show that if $n\geqslant 5$, each of the lines $x_2=1$ and $x_2=-n+2$ contains at most one vertex of $\widehat{P}$.

It follows from system (5.51) that points of $\widehat{\Delta}$ lying on the line $x_2=1$ satisfy

$$0\leq x_1\leq\frac{2n}{n+1}.$$

As $2n/(n+1)<2$ and all the integer points of the line $x_1=0$ belong to Λ, we see that the line $x_2=1$ has only one point that could be a vertex of $\widehat{P}$: the point $(1,1)$.

It also follows from (5.51) that the points of $\widehat{\Delta}$ lying on the line $x_2=-n+2$ satisfy

$$\frac{2n(n-2)}{n-1}\leq x_1\leq\frac{2n^2}{n+1}.$$

Given that $n\geq 5$, we have

$$2n-3<\frac{2n(n-2)}{n-1}<\frac{2n^2}{n+1}<2n-1,$$

Suppose that a frame $(\mathbf{o};\mathbf{f}_1,\mathbf{f}_2)$ splits a slope Q and let $\mathbf{z}$ be the point where Q meets the ray $\{\mathbf{o}+\lambda\mathbf{f}_1\colon \lambda\geq 0\}$. If there is a supporting line for Q passing through $\mathbf{z}$ that forms an angle $\leq\pi/4$ with the ray, we say that the frame $(\mathbf{o};\mathbf{f}_1,\mathbf{f}_2)$ *forms small angle* with the slope Q.

Proposition 2.10. *Suppose that an integer frame $(\mathbf{o};\mathbf{f}_1,\mathbf{f}_2)$ splits a slope Q; then the frame $(\mathbf{o};\mathbf{f}_2,\mathbf{f}_1)$ splits it as well, and at least one of the frames forms small angle with Q. If there exists a point $\mathbf{y}=\mathbf{o}+y_1\mathbf{f}_1+y_2\mathbf{f}_2\in Q$ such that $y_2>0$ and $y_1+y_2\leq 0$, then $(\mathbf{o};\mathbf{f}_1,\mathbf{f}_2)$ forms small angle with Q.*

Theorem 2.11. *Suppose that an integer frame $(\mathbf{o};\mathbf{f}_1,\mathbf{f}_2)$ splits an integer slope Q having N edges and the endpoints $\mathbf{v}=\mathbf{o}+v_1\mathbf{f}_1+v_2\mathbf{f}_2$ and $\mathbf{w}=\mathbf{o}+w_1\mathbf{f}_1+w_2\mathbf{f}_2$ satisfying* (2.9) *and* (2.10)*. Then there exist $s\in\mathbb{Z}$ and $t\in\mathbb{Z}$ such that*

$$0\leq s\leq t,\tag{2.11}$$

$$v_2-s\geq 0,\tag{2.12}$$

$$-v_1<ts-\frac{s^2-s}{2}+(v_2-s)(t+1),\tag{2.13}$$

$$2N\leq v_2+w_1-t+s.\tag{2.14}$$

Moreover, if $(\mathbf{o};\mathbf{f}_1,\mathbf{f}_2)$ forms small angle with Q, we have

$$2N\leq v_2+w_1-t+s-\left\lceil\frac{-w_2}{2}\right\rceil+1.\tag{2.15}$$

Corollary 2.12. *Under the hypotheses of Theorem 2.11,*

$$2N\leq v_2+w_1,$$

and if $(\mathbf{o};\mathbf{f}_1,\mathbf{f}_2)$ forms small angle with Q, then

$$2N\leq v_2+w_1-\left\lceil\frac{-w_2}{2}\right\rceil+1.$$

Theorem 2.13. *Under the hypotheses of Theorem 2.11, if the vertices of Q belong to a proper sublattice of $\mathbb{Z}^2$, then*

$$2N\leq v_2+w_1-1.$$

There are four slopes naturally associated with a given convex polygon P.

Let P be an integer polygon in the plane. Define

$$\begin{aligned}
\mathcal{N} &= \max\{x_2 : (x_1, x_2) \in P\}, & \mathcal{S} &= \min\{x_2 : (x_1, x_2) \in P\},\\
\mathcal{N}_- &= \min\{x_1 : (x_1, \mathcal{N}) \in P\}, & \mathcal{S}_- &= \min\{x_1 : (x_1, \mathcal{S}) \in P\},\\
\mathcal{N}_+ &= \max\{x_1 : (x_1, \mathcal{N}) \in P\}, & \mathcal{S}_+ &= \max\{x_2 : (x_2, \mathcal{S}) \in P\},\\
\mathcal{W} &= \min\{x_1 : (x_1, x_2) \in P\}, & \mathcal{E} &= \max\{x_1 : (x_1, x_2) \in P\},\\
\mathcal{W}_- &= \min\{x_2 : (\mathcal{W}, x_2) \in P\}, & \mathcal{E}_- &= \min\{x_2 : (\mathcal{E}, x_2) \in P\},\\
\mathcal{W}_+ &= \max\{x_2 : (\mathcal{W}, x_2) \in P\}, & \mathcal{E}_+ &= \max\{x_2 : (\mathcal{E}, x_2) \in P\}.
\end{aligned}$$

All these are integers. Note that $(\mathcal{S}_-, \mathcal{S})$, $(\mathcal{S}_+, \mathcal{S})$, $(\mathcal{N}_-, \mathcal{N})$, $(\mathcal{N}_+, \mathcal{N})$, $(\mathcal{W}, \mathcal{W}_-)$, $(\mathcal{W}, \mathcal{W}_+)$, $(\mathcal{E}, \mathcal{E}_-)$, and $(\mathcal{E}, \mathcal{E}_+)$ are (not necessarily distinct) vertices of P.

Let us enumerate the vertices of P starting from $\mathbf{v}_0 = (\mathcal{W}, \mathcal{W}_-)$ and going in the positive direction until we come to $v_{N_4} = (\mathcal{S}_-, \mathcal{S})$. Clearly, the sequence $\mathbf{v}_0, \ldots, \mathbf{v}_{N_4}$ gives rise to a slope with respect to the basis $(\mathbf{e}_1, \mathbf{e}_2)$. We denote it by Q_4. Obviously, Q_4 is an inclusion-wise maximal slope with respect to $(\mathbf{e}_1, \mathbf{e}_2)$ contained in the boundary of P. Likewise, we define the slope Q_1 with respect to $(\mathbf{e}_2, -\mathbf{e}_1)$ having the endpoints $(\mathcal{S}_+, \mathcal{S})$ and $(\mathcal{E}, \mathcal{E}_-)$, the slope Q_2 with respect to $(-\mathbf{e}_1, -\mathbf{e}_2)$ having the endpoints $(\mathcal{E}, \mathcal{E}_+)$ and $(\mathcal{N}_+, \mathcal{N})$, and the slope Q_3 with respect to $(-\mathbf{e}_2, \mathbf{e}_1)$ having the endpoints $(\mathcal{N}_-, \mathcal{N})$ and $(\mathcal{W}, \mathcal{W}_+)$. We call those *maximal slopes* of the polygon P and denote by N_k the number of edges of Q_k.

Remark 2.14. For each of the mentioned bases, the boundary of the polygon contains single-point maximal slopes apart from correspondent Q_k. However, we single Q_k out by explicitly indicating its endpoints. For a given polygon, some of the maximal slopes Q_k may have but one vertex.

Define

$$M_1 = \begin{cases} 0, & \text{if } \mathcal{S}_- = \mathcal{S}_+,\\ 1, & \text{otherwise;} \end{cases} \quad M_2 = \begin{cases} 0, & \text{if } \mathcal{E}_- = \mathcal{E}_+,\\ 1, & \text{otherwise;} \end{cases}$$

$$M_3 = \begin{cases} 0, & \text{if } \mathcal{N}_- = \mathcal{N}_+,\\ 1, & \text{otherwise;} \end{cases} \quad M_4 = \begin{cases} 0, & \text{if } \mathcal{W}_- = \mathcal{W}_+,\\ 1, & \text{otherwise.} \end{cases}$$

Proposition 2.15. *Let P be an N-gon; then each edge of P either lies on a horizontal or a vertical line or it is the edge of exactly one of the maximal slopes of P; thus,*

$$N = \sum_{k=1}^{4} N_k + \sum_{k=1}^{4} M_k.$$

We have thus proved the lemma under the hypothesis that the small $\mathbf{e}_1$-step of Γ is greater then 1.

Assume that Γ has small $\mathbf{e}_1$-step 1. Then by Proposition 2.4 the lattice Γ admits a basis of the form $(\mathbf{e}_1 - a\mathbf{e}_2, n\mathbf{e}_2)$, where $\lfloor n/2 \rfloor - n + 1 \le a \le \lfloor n/2 \rfloor$. Consider the linear transformation

$$A = \begin{pmatrix} 1 & 0 \\ a/n & 1/n \end{pmatrix}.$$

It is easily seen that $A\Gamma = \mathbb{Z}^2$ and $\Lambda = A(n\mathbb{Z}^2)$ is a lattice having the basis $(n\mathbf{e}_1, \mathbf{e}_2)$; the image $\widehat{P}$ of P is an integer N-gon free from points of Λ and contained in the triangle $\widehat{\Delta} = A\Delta_n$ with the vertices $\mathbf{0}$, $(0, 2)$, and $(2n, 2a)$.

Set

$$p = \min\{x_2 \colon (x_1, x_2) \in \widehat{\Delta}\} = \min\{2a, 0\}, \tag{5.47}$$

$$q = \max\{x_2 \colon (x_1, x_2) \in \widehat{\Delta}\} = \max\{2a, 2\}. \tag{5.48}$$

Note two obvious facts. Firstly, all the integer points of the triangle $\widehat{\Delta}$ lie on the $q - p + 1$ lines

$$x_2 = j \qquad (j = p, p+1, \ldots, q), \tag{5.49}$$

so *all the vertices of $\widehat{P}$ lie on the lines* (5.49)*, each line containing at most two vertices.* Secondly, *if $a \ne 0$, there are no vertices of $\widehat{P}$ on the line $x_2 = p$,* as it is easy to check that in this case the line has the only common point with $\widehat{\Delta}$—the vertex of the triangle, which belongs to Λ. By the same argument, *if $a \ne 1$, there are no vertices of $\widehat{P}$ on the line $x_2 = q$.*

As a consequence, we see that (5.46) holds, provided that

$$q - p \le n. \tag{5.50}$$

Indeed, if additionally $a \ne 0$ and $a \ne 1$, then the vertices of $\widehat{P}$ lie on the $q - p - 1$ lines (5.49) corresponding to $j = p + 1, \ldots, q - 1$, whence

$$N \le 2(q - p - 1) \le 2n - 2.$$

On the other hand, if $a = 0$ or $a = 1$, then according to (5.47) and (5.48) we have $p = 0$, $q = 2$, and the vertices of $\widehat{P}$ lie on the lines (5.49) (excluding $j = p$ and $j = q$), whence

$$N \le 4 \le 2n - 2$$

as $n \ge 3$.

Using (5.31), (5.17), (5.19), and (5.29), we deduce

$$\mathcal{N}_+ = \mathcal{W} + (\mathcal{N}_- - \mathcal{W}) + (\mathcal{N}_+ - \mathcal{N}_-) \geq 2N_3 - s_3 + M_3 \geq s_3 + M_3,$$

while by virtue of (5.40) and (5.18) we obtain

$$\mathcal{N} = \mathcal{W}_+ + (\mathcal{N} - \mathcal{W}_+) \geq (-s_3 - s' + 2n + 5 - M_1 - M_2 - M_3 - M_4) + \frac{1}{2}s_3(s_3+1).$$

Combining (5.44) with two last estimates, we get

$$s' \geq \frac{1}{2}s_3(s_3+1) + 5 - M_1 - M_2 - M_4 \geq \frac{1}{2}s_3(s_3+1) + 2 = \frac{1}{2}(s_3^2 + s_3 + 4),$$

and finally

$$s' \geq \frac{1}{2}(s_3^2 + s_3 + 4). \tag{5.45}$$

It is not hard to check that the inequalities (5.43) and (5.45) are incompatible. This contradiction proves (5.13) in case (5.12) holds. ∎

Lemma 5.9. *Suppose that the vertices of a type Va_n N-gon P belong to a $(1,n)$-lattice Γ; then*

$$N \leq 2n - 2. \tag{5.46}$$

Proof: Note that by Proposition 2.3 the product of the small $\mathbf{e}_1$-step and the large $\mathbf{e}_2$-step of Γ equals n.

First, assume that Γ has small $\mathbf{e}_1$-step $s \geq 2$. In this case s divides n and all the points of Γ belonging to Δ_n lie on the lines

$$x_1 = js \qquad (j = 0, \ldots, 2n/s).$$

Consequently, the vertices of P lie on the same lines as well. Each of the $2n/s$ lines corresponding to $j = 0, \ldots, 2n/s - 1$ contains at most two vertices while the line $x_1 = 2n$ corresponding to $j = 2n/s$ does not contain any vertex, since its only common point with Δ_n is $(2n, 0) \in n\mathbb{Z}^2$. Thus, if $s \geq 3$, we can estimate the number of vertices of P as follows:

$$N \leq 2 \cdot \frac{2n}{s} \leq \frac{4n}{3} \leq 2n - 2$$

(since $n \geq 3$). If $s = 2$, the large $\mathbf{e}_2$-step of Γ is $n/2$, so each of the lines the lines $x_1 = 0$ and $x_1 = n$ has a single point of Γ between adjacent points of $n\mathbb{Z}^2$. Consequently, each of these lines (corresponding to $j = 0$ and $j = n/2$) contains at most one vertex of P and there are $n - 2$ other lines containing up to two vertices. Thus, the total number of vertices does not exceed

$$N \leq 2 + 2 \cdot (n - 2) = 2n - 2.$$

Proposition 2.16. *Let P be a convex integer polygon and $(\mathbf{o}; \mathbf{f}_1, \mathbf{f}_2)$ be an integer frame such that $\mathbf{f}_1, \mathbf{f}_2 \in \{\pm\mathbf{e}_1, \pm\mathbf{e}_2\}$. Suppose that $\mathbf{o}$ does not belong to P and the rays $\{\mathbf{o} + \lambda\mathbf{f}_j \colon \lambda \geq 0\}$ $(j = 1, 2)$ split P; then $(\mathbf{o}; \mathbf{f}_1, \mathbf{f}_2)$ splits Q_k, where*

$$k = \begin{cases} 1, & \text{if } (\mathbf{f}_1, \mathbf{f}_2) = (-\mathbf{e}_1, \mathbf{e}_2) \quad \text{or } (\mathbf{f}_1, \mathbf{f}_2) = (\mathbf{e}_2, -\mathbf{e}_1), \\ 2, & \text{if } (\mathbf{f}_1, \mathbf{f}_2) = (-\mathbf{e}_2, -\mathbf{e}_1) \quad \text{or } (\mathbf{f}_1, \mathbf{f}_2) = (-\mathbf{e}_1, -\mathbf{e}_2), \\ 3, & \text{if } (\mathbf{f}_1, \mathbf{f}_2) = (\mathbf{e}_1, -\mathbf{e}_2) \quad \text{or } (\mathbf{f}_1, \mathbf{f}_2) = (-\mathbf{e}_2, \mathbf{e}_1), \\ 4, & \text{if } (\mathbf{f}_1, \mathbf{f}_2) = (\mathbf{e}_2, \mathbf{e}_1) \quad \text{or } (\mathbf{f}_1, \mathbf{f}_2) = (\mathbf{e}_1, \mathbf{e}_2). \end{cases}$$

Proposition 2.17. *Let P be a Γ-polygon, S_1 be the large $\mathbf{e}_1$-step of Γ, and S_2 be the large $\mathbf{e}_2$-step of Γ. Then*

$$\mathcal{S}_+ - \mathcal{S}_- \geq S_1M_1, \ \mathcal{E}_+ - \mathcal{E}_- \geq S_2M_2,$$
$$\mathcal{N}_+ - \mathcal{N}_- \geq S_1M_3, \ \mathcal{W}_+ - \mathcal{W}_- \geq S_2M_4.$$

3. Type III polygons

In this section we prove Theorem 1.4 for type III polygons.

Lemma 3.1. *Given an integer $n \geq 2$ and a type III_n polygon P, the following assertions hold:*

(i) The frame $((n,0); \mathbf{e}_2, -\mathbf{e}_1)$ splits Q_1.
(ii) The frame $((n,n); -\mathbf{e}_2, -\mathbf{e}_1)$ splits Q_2.
(iii) The following inequalities hold:

$$\mathcal{W} \geq 0, \tag{3.1}$$
$$\mathcal{N}_+ \leq n - 1, \tag{3.2}$$
$$\mathcal{S}_+ \leq n - 1, \tag{3.3}$$
$$\mathcal{E} \geq n + 1, \tag{3.4}$$
$$\mathcal{S} < \mathcal{E}_- < 0. \tag{3.5}$$

(iv) The intersection of P with the open half-plane $x_1 < n$ is contained in the slab $0 \leq x_1 < n$. All the vertices of P belonging to the closed half-plane $x_1 \geq n$ lie on the lines

$$x_2 = k \qquad (k = 1, \ldots, n-1), \tag{3.6}$$

each line containing at most one vertex.
(v) If the vertices of P belong to a $(1,n)$-lattice Γ, then the large $\mathbf{e}_2$- and $\mathbf{e}_1$-steps of Γ are greater then or equal to 2.

Proof: Assertions (i) and (ii) follow from the definition of a type III_n polygon and Proposition 2.16. Assertions (iii) and (iv) are obvious. According to Proposition 2.3, to prove (v), it suffices to show that the small $\mathbf{e}_1$- and $\mathbf{e}_2$-steps of Γ are less then n. As the vertex $(\mathcal{E}, \mathcal{E}_-)$ belongs to Γ and satisfies (3.5), we see that indeed the small $\mathbf{e}_2$-step of Γ is less than n. Further, it follow from (3.2), (3.3), and (3.5) that the vertices $(\mathcal{N}_+,\mathcal{N})$ and $(\mathcal{S}_+,\mathcal{S})$ lie in the slab $0 \le x_1 < n$. Suppose, contrary to our claim, that the small $\mathbf{e}_1$-step of Γ equals n. Then we see that $\mathcal{N}_+ = \mathcal{S}_+ = 0$ and consequently, P contains the segment $[(0,\mathcal{S}),(0,\mathcal{N})]$. However, we obviusly have $\mathcal{S} < 0$ and $\mathcal{N} > n$, so the segment contains the points $\mathbf{0}, (0,n) \in n\mathbb{Z}^2$, which is impossible, as P is free of $n\mathbb{Z}^2$-points. ∎

Lemma 3.2. *Suppose that $n \ge 3$. Let Γ and b be a lattice and a number such that either $\Gamma = \mathbb{Z}^2$ and $b = 0$ or Γ is a $(1, n/2)$-lattice with the basis $(\mathbf{e}_1 + a\mathbf{e}_2, (n/2)\mathbf{e}_2)$, where the integer a satisfies $1 \le a \le n/2 - 1$, and $b = 1$. Let P be a type III_n N-gon with the vertices belonging to Γ. Then*

$$N \le 2n + 2 - 2b. \tag{3.7}$$

Proof: We begin by translating the geometrical constraints on P into inequalities.

The frame $((n,0); \mathbf{e}_2, -\mathbf{e}_1)$ splits Q_1, so by Theorem 2.11 there exist integers s_1 and t_1 such that

$$2N_1 \le \mathcal{E}_- - \mathcal{S}_+ + n - t_1 + s_1, \tag{3.8}$$

$$-\mathcal{S}_+ + n - s_1 \ge 0, \tag{3.9}$$

$$-\mathcal{S} < t_1 s_1 - \frac{s_1^2 - s_1}{2} + (-\mathcal{S}_+ + n - s_1)(t_1 + 1), \tag{3.10}$$

$$0 \le s_1 \le t_1. \tag{3.11}$$

Likewise, $((n,n); -\mathbf{e}_2, -\mathbf{e}_1)$ splits Q_2, so there exist integers s_2 and t_2 such that

$$2N_2 \le -\mathcal{E}_+ - \mathcal{N}_+ + 2n - t_2 + s_2, \tag{3.12}$$

$$-\mathcal{N}_+ + n - s_2 \ge 0, \tag{3.13}$$

$$\mathcal{N} - n < t_2 s_2 - \frac{s_2^2 - s_2}{2} + (-\mathcal{N}_+ + n - s_2)(t_2 + 1), \tag{3.14}$$

$$0 \le s_2 \le t_2. \tag{3.15}$$

Now estimate $2N$ using the last inequality and (5.17), (5.23), (5.29), (5.30), (5.31), and (5.32):

$$\begin{aligned} 2N &= (2N_1 + 2N_2 + 2M_1 + 2M_2) + 2N_3 + 2N_4 + 2M_3 + 2M_4 \\ &\le (2n - \mathcal{N}_+ + M_1 + M_2 + 1) + (\mathcal{N}_- - \mathcal{W} + s_3) \\ &\quad + (\mathcal{W}_- - \mathcal{S} + s') + 2M_3 + 2M_4 \\ &\le 2n + 1 + s_3 + s' + (M_3 - (\mathcal{N}_+ - \mathcal{N}_-)) \\ &\quad + (\mathcal{W}_- + M_4) + M_1 + M_2 + M_3 + M_4 \\ &\le 2n + 1 + s_3 + s' + \mathcal{W}_+ + M_1 + M_2 + M_3 + M_4. \end{aligned}$$

Comparing this estimate with (5.33) we obtain

$$\mathcal{W}_+ \ge -s_3 - s' + 2n + 5 - M_1 - M_2 - M_3 - M_4. \tag{5.40}$$

Together with (5.12) this implies

$$n \le s_3 + s' - 5 + M_1 + M_2 + M_3 + M_4. \tag{5.41}$$

The triangle Δ_n lies in the half-plane $x_1 \le 2n$, so we have

$$\mathcal{S}_+ \le 2n. \tag{5.42}$$

We can estimate the left-hand side by means of (5.31), (5.24), and (5.27) as follows:

$$\mathcal{S}_+ = \mathcal{W} + (\mathcal{S}_- - \mathcal{W}) + (\mathcal{S}_+ - \mathcal{S}_-) \ge \frac{1}{2}s'(s'+1) + M_1.$$

Using this estimate and (5.41), we obtain from (5.42):

$$\frac{1}{2}s'(s'+1) + M_1 \le 2s_3 + 2s_4 - 10 + 2M_1 + 2M_2 + 2M_3 + 2M_4,$$

whence

$$\begin{aligned} 2s_3 &\ge \frac{1}{2}(s'^2 - 3s') + 10 - M_1 - 2M_2 - 2M_3 - 2M_4 \ge \\ &\ge \frac{1}{2}(s'^2 - 3s') + 3 = \frac{1}{2}(s'^2 - 3s' + 6), \end{aligned}$$

and finally

$$s_3 \ge \frac{1}{4}(s'^2 - 3s' + 6). \tag{5.43}$$

The vertices of P solve (5.1), so

$$\mathcal{N}_+ + \mathcal{N} \le 2n. \tag{5.44}$$

Let us estimate $2N$ from above starting with the sum $2N_1 + 2N_2 + 2M_1 + 2M_2$. Using (5.14), (5.26), (5.28), and (5.15), we obtain

$$\begin{aligned}
&2N_1 + 2N_2 + 2M_1 + 2M_2 \\
&\quad \le (\mathcal{E}_- - \mathcal{S} + s_1) + \left(2n - \mathcal{N}_+ - \mathcal{E}_+ - \left\lceil \frac{\mathcal{E} - n}{2} \right\rceil + 1\right) + 2M_1 + 2M_2 \\
&\quad \le 2n - \mathcal{N}_+ - \mathcal{S} + 2M_1 + M_2 + 1 - \left\lceil \frac{\mathcal{S}_+ - n}{2} + \frac{s_1^2 - 3s_1}{4} \right\rceil
\end{aligned}$$

Estimating $(s_1^2 - 3s_1)/4 \ge -1/2$, we get

$$\begin{aligned}
&2N_1 + 2N_2 + 2M_1 + 2M_2 \\
&\quad \le 2n - \mathcal{N}_+ - \mathcal{S} + 2M_1 + M_2 + 1 - \left\lceil \frac{\mathcal{S}_+ - n - 1}{2} \right\rceil \\
&\quad = 2n - \mathcal{N}_+ - \mathcal{S} + 2M_1 + M_2 + 1 - \left\lfloor \frac{\mathcal{S}_+ - n}{2} \right\rfloor.
\end{aligned}$$

Write the estimate in the form

$$\begin{aligned}
&2N_1 + 2N_2 + 2M_1 + 2M_2 \\
&\quad \le 2n - \mathcal{N}_+ + M_1 + M_2 + 1 - \left(\left\lfloor \frac{\mathcal{S}_+ - n}{2} \right\rfloor + \mathcal{S} - M_1\right). \quad (5.38)
\end{aligned}$$

Let us show that

$$\left\lfloor \frac{\mathcal{S}_+ - n}{2} \right\rfloor + \mathcal{S} - M_1 \ge 0. \quad (5.39)$$

Assume that $M_1 = 1$. According to (5.12), we have either $\mathcal{S}_+ \ge n + 2$ or $\mathcal{S}_+ = n + 1$. In the former case we use (5.32) and obtain (5.39). In the latter case by (5.27) we have $\mathcal{S}_- \le \mathcal{S}_+ - M_1 = n$, so the edge $[(\mathcal{S}_-, \mathcal{S}), (\mathcal{S}_+, \mathcal{S})]$ of P contains the point $(n, \mathcal{S})$. Thus, we cannot have $\mathcal{S} = 0$, since P is free of points of $n\mathbb{Z}^2$. Thus, we must have $\mathcal{S} \ge 1$, and (5.39) follows.

If $M_1 = 0$, inequality (5.39) follows from (5.12) and (5.32).

Thus, we have proved (5.39) for all possible cases. Combining it with (5.38), we obtain

$$2N_1 + 2N_2 + 2M_1 + 2M_2 \le 2n - \mathcal{N}_+ + M_1 + M_2 + 1.$$

As Q_3 is a slope with respect to $(\mathbf{e}_1, -\mathbf{e}_2)$, by Proposition 2.7 there exists $s_3 \in \mathbb{Z}$ such that

$$2N_3 \le \mathcal{N}_- - \mathcal{W} + s_3, \quad (3.16)$$

$$\mathcal{N} - \mathcal{W}_+ \ge \frac{1}{2} s_3(s_3 + 1), \quad (3.17)$$

$$0 \le s_3 \le N_3. \quad (3.18)$$

Likewise, applying Proposition 2.7 to Q_4 and $(\mathbf{e}_1, \mathbf{e}_2)$, we conclude that there exists $s_4 \in \mathbb{Z}$ such that

$$2N_4 \le \mathcal{S}_- - \mathcal{W} + s_4, \quad (3.19)$$

$$\mathcal{W}_- - \mathcal{S} \ge \frac{1}{2} s_4(s_4 + 1), \quad (3.20)$$

$$0 \le s_4 \le N_4. \quad (3.21)$$

Further, by Proposition 2.17,

$$\mathcal{S}_+ - \mathcal{S}_- \ge (1 + b)M_1, \quad (3.22)$$

$$\mathcal{E}_+ - \mathcal{E}_- \ge (1 + b)M_2, \quad (3.23)$$

$$\mathcal{N}_+ - \mathcal{N}_- \ge (1 + b)M_3, \quad (3.24)$$

$$\mathcal{W}_+ - \mathcal{W}_- \ge (1 + b)M_4, \quad (3.25)$$

$$\mathcal{W} \ge bM_4. \quad (3.26)$$

Indeed, if $b = 0$, (3.22)–(3.25) immediately follow from the proposition. Suppose that $b = 1$; then the large $\mathbf{e}_2$-step of Γ is $n/2 \ge 2$, and as $a \ge 1$, by virtue of Proposition 2.4 we have $\mathbf{e}_1 \notin \Gamma$, and consequently, the large $\mathbf{e}_2$-step of Γ is ≥ 2 as well.

By Lemma 3.1, we have $\mathcal{W} \ge 0$, so to prove (3.26) it suffices to show that $M_4 = 0$ provided that $b = 1$ and $\mathcal{W} = 0$. Indeed, in this case Γ has large $\mathbf{e}_2$-step $n/2$, so every other point of Γ lying on the line $x_1 = 0$ belongs to $n\mathbb{Z}^2$. Therefore, P cannot have two vertices on this line and $M_4 = 0$ if $\mathcal{W} = 0$.

To prove the lemma, we argue by contradiction, assuming that

$$2N \ge 4n + 6 - 4b. \quad (3.27)$$

Summing (3.8), (3.12), (3.16), and (3.19) and subsequently using (3.22)–(3.26), we obtain:

$$\begin{aligned}2N = \sum_{k=1}^{4} 2N_k + \sum_{k=1}^{4} 2M_k \\ &\leq 3n + s_1 + s_2 + s_3 + s_4 - t_1 - t_2 \\ &- (\mathcal{S}_+ - \mathcal{S}_-) - (\mathcal{E}_+ - \mathcal{E}_-) - (\mathcal{N}_+ - \mathcal{N}_-) \\ &- 2\mathcal{W} + 2M_1 + 2M_2 + 2M_3 + 2M_4 \\ &\leq 3n + s_1 + s_2 + s_3 + s_4 - t_1 - t_2 \\ &+ (1-b)M_1 + (1-b)M_2 + (1-b)M_3 + (2-2b)M_4.\end{aligned}$$

Comparing this with (3.27), we deduce

$$\begin{aligned}&n - s_1 - s_2 - s_3 - s_4 + t_1 + t_2 \\ &- (1-b)M_1 - (1-b)M_2 - (1-b)M_3 - (2-2b)M_4 + 6 - 4b \leq 0. \quad (3.28)\end{aligned}$$

Now we use (3.10) and (3.14) to estimate $\mathcal{N} - \mathcal{S}$ from above:

$$\begin{aligned}\mathcal{N} - \mathcal{S} < n + t_1 s_1 - \frac{s_1^2 - s_1}{2} + t_2 s_2 - \frac{s_2^2 - s_2}{2} \\ + (-\mathcal{S}_+ + n - s_1)(t_1 + 1) + (-\mathcal{N}_+ + n - s_2)(t_2 + 1). \quad (3.29)\end{aligned}$$

Let us estimate $\mathcal{S}_+$ and $\mathcal{N}_+$. Using (3.22), (3.19), (3.21), and (3.1), we obtain

$$\begin{aligned}\mathcal{S}_+ \geq \mathcal{S}_- + (1+b)M_1 \geq 2N_4 + \mathcal{W} - s_4 + (1+b)M_1 \\ \geq s_4 + \mathcal{W} + (1+b)M_1 \geq s_4 + M_1,\end{aligned}$$

whence

$$-\mathcal{S}_+ + n - s_1 \leq n - s_1 - s_4 - M_1. \quad (3.30)$$

Incidentally, note that the left-hand side is nonnegative by virtue of (3.9), so

$$n - s_1 - s_4 - M_1 \geq 0. \quad (3.31)$$

Likewise, from (3.24), (3.17), (3.18), and (3.1) we derive

$$-\mathcal{N}_+ + n - s_2 \leq n - s_2 - s_3 - M_3, \quad (3.32)$$

which together with (3.13) implies

$$n - s_2 - s_3 - M_3 \geq 0. \quad (3.33)$$

whence

$$n \leq s_3 + s + M_3 + 2M_4 - 3. \quad (5.36)$$

As $M \subset \Delta_n$, we have

$$\mathcal{N}_+ + \mathcal{N} \leq 2n.$$

Let us estimate the terms on the left-hand side. Using (5.31), (5.17), (5.19), and (5.29), we get

$$\mathcal{N}_+ = \mathcal{W} + (\mathcal{N}_- - \mathcal{W}) + (\mathcal{N}_+ - \mathcal{N}_-) \geq 2N_3 - s_3 + M_3 \geq s_3 + M_3.$$

Using (5.32), (5.21), (5.30), and (5.18), we obtain

$$\mathcal{N} = \mathcal{S} + (\mathcal{W}_- - \mathcal{S}) + (\mathcal{W}_+ - \mathcal{W}_-) + (\mathcal{N} - \mathcal{W}_+) \geq \frac{1}{2}s(s+1) + M_4 + \frac{1}{2}s_3(s_3+1).$$

Thus,

$$(s_3 + M_3) + \left(\frac{1}{2}s(s+1) + M_4 + \frac{1}{2}s_3(s_3+1)\right) \leq 2n,$$

or, equivalently,

$$\frac{1}{2}(s_3^2 + 3s_3) + \frac{1}{2}(s^2 + s) + M_3 + M_4 \leq 2n. \quad (5.37)$$

Now we use (5.36) to estimate n on the right-hand side of (5.37):

$$\frac{1}{2}(s_3^2 + 3s_3) + \frac{1}{2}(s^2 + s) + M_3 + M_4 \leq 2(s_3 + s + M_3 + 3M_4 - 3).$$

Hence

$$\frac{1}{2}(s_3^2 - s_3) + \frac{1}{2}(s^2 - 3s) \leq M_3 + 3M_4 - 6 \leq -2,$$

so

$$s_3^2 - s_3 + s^2 - 3s \leq -4.$$

Completing the squares, we obtain a contradiction:

$$\left(s_3 - \frac{1}{2}\right)^2 + \left(s - \frac{3}{2}\right)^2 \leq -\frac{3}{2},$$

Thus, we have proved (5.13) provided that (5.11) holds.

Now assume that (5.12) holds.

Moreover, as the points of P satisfy (5.1), we have

$$\mathcal{W} \geq 0, \tag{5.31}$$

$$\mathcal{S} \geq 0. \tag{5.32}$$

Assume that (5.13) does not hold. Then

$$2N \geq 4n + 6. \tag{5.33}$$

First, assume that (5.11) holds.

Let us estimate $2N$ from above. First, estimate the sum $2N_1 + 2N_2 + 2M_2$. Using (5.14), (5.26), (5.28), and (5.32), we have

$$2N_1 + 2N_2 + 2M_2 \leq 2n + s_1 - \mathcal{N}_+ + M_2 - \left\lceil \frac{\mathcal{E} - n}{2} \right\rceil + 1. \tag{5.34}$$

Estimating the ceiling by means of (5.15), we obtain:

$$\left\lceil \frac{\mathcal{E} - n}{2} \right\rceil \geq -n + \mathcal{S}_+ + s_1 + \left\lceil \frac{n - \mathcal{S}_+}{2} + \frac{1}{4}(s_1^2 - 3s_1) \right\rceil. \tag{5.35}$$

It follows from (5.11) that $(n - \mathcal{S}_+)/2 \geq 0$, and because s_1 is an integer, we have $1/4(s_1^2 - 3s_1) \geq -1/2$, so we get

$$\left\lceil \frac{n - \mathcal{S}_+}{2} + \frac{1}{4}(s_1^2 - 3s_1) \right\rceil \geq \left\lceil -\frac{1}{2} \right\rceil = 0.$$

Combining this with (5.35), we get

$$\left\lceil \frac{\mathcal{E} - n}{2} \right\rceil \geq -n + \mathcal{S}_+ + s_1,$$

and further combining this with (5.34) and the inequality $M_2 \leq 1$, we obtain

$$2N_1 + 2N_2 + 2M_2 = 3n - \mathcal{N}_+ - \mathcal{S}_+ + M_2 + 1 \leq 3n - \mathcal{N}_+ - \mathcal{S}_+ + 2.$$

By means of the last estimate and (5.17), (5.20), (5.27), (5.29), and (5.31), we obtain

$$\begin{aligned} 2N &= (2N_1 + 2N_2 + 2M_2) + 2N_3 + 2N_4 + 2M_1 + 2M_3 + 2M_4 \\ &\leq (3n - \mathcal{N}_+ - \mathcal{S}_+ + 2) + (\mathcal{N}_- - \mathcal{W} + s_3) + (\mathcal{S}_- - \mathcal{W} + s) + 2M_1 + 2M_3 + 2M_4 \\ &\leq 3n + 2 + s_3 + s + M_1 + M_3 + 2M_4 \leq 3n + 3 + s_3 + s + M_3 + 2M_4. \end{aligned}$$

Comparing this estimate with (5.33), we get

$$3n + 3 + s_3 + s + M_3 + 2M_4 \geq 4n + 6,$$

As $t_1 + 1 > 0$ and $t_2 + 1 > 0$, we can use (3.30) and (3.32) to obtain from (3.29)

$$\begin{aligned} \mathcal{N} - \mathcal{S} < n + t_1 s_1 - \frac{s_1^2 - s_1}{2} + t_2 s_2 - \frac{s_2^2 - s_2}{2} \\ + (n - s_1 - s_4 - M_1)(t_1 + 1) + (n - s_2 - s_3 - M_3)(t_2 + 1). \end{aligned} \tag{3.34}$$

Now we estimate $\mathcal{N} - \mathcal{S}$ from below by summing (3.17), (3.20), and (3.25):

$$\mathcal{N} - \mathcal{S} \geq (1 + b)M_4 + \frac{1}{2}s_3(s_3 + 1) + \frac{1}{2}s_4(s_4 + 1). \tag{3.35}$$

Consider the second term on the right-hand side. Inequality (3.28) gives

$$\begin{aligned} s_3 - 1 \geq (n - s_1 - s_4 + t_1 - M_1) + (t_2 - s_2) \\ + (5 - 4b + bM_1 - (1 - b)M_2 - (1 - b)M_3 - (2 - 2b)M_4). \end{aligned}$$

The second term on the right-hand side is nonnegative by virtue of (3.15) and the third one is also nonnegative (even positive), which is easily seen by separately checking $b = 0$ and $b = 1$. Consequently, we have

$$s_3 - 1 \geq n - s_1 - s_4 + t_1 - M_1. \tag{3.36}$$

By virtue of (3.31) we have $n - s_1 - s_4 + t_1 - M_1 \geq t_1 \geq 0$, so using (3.36), we get

$$\begin{aligned} \frac{1}{2}s_3(s_3 + 1) &= s_3 + \frac{1}{2}s_3(s_3 - 1) \\ &\geq s_3 + \frac{1}{2}(n - s_1 - s_4 + t_1 - M_1 + 1)(n - s_1 - s_4 + t_1 - M_1). \end{aligned}$$

Set

$$A = n - s_1 - s_4 - M_1, \quad B = t_1 + 1$$

(A and B are integers) and continue as follows:

$$\begin{aligned} \frac{1}{2}s_3(s_3 + 1) &\geq s_3 + \frac{1}{2}(A + B)(A + B - 1) \\ &= s_3 + \frac{1}{2}(A^2 - A) + \frac{1}{2}(B^2 - B) + AB \geq s_3 + \frac{1}{2}(B^2 - B) + AB. \end{aligned}$$

For the terms on the right-hand side we have

$$\frac{1}{2}(B^2-B)=\frac{1}{2}(t_1^2+t_1)=t_1s_1-\frac{s_1^2-s_1}{2}+\frac{1}{2}(t_1-s_1)(t_1-s_1+1)$$
$$\geq t_1s_1-\frac{s_1^2-s_1}{2}$$

(since $t_1-s_1\geq 0$ according to (3.11)), and

$$AB=(n-s_1-s_4-M_1)(t_1+1),$$

and we finally obtain

$$\frac{1}{2}s_3(s_3+1)\geq s_3+t_1s_1-\frac{s_1^2-s_1}{2}+(n-s_1-s_4-M_1)(t_1+1). \tag{3.37}$$

One can estimate the third term on the right-hand side of (3.35) in much the same way by making use of (3.11), (3.33), and (3.15). Eventually,

$$\frac{1}{2}s_4(s_4+1)\geq s_4+t_2s_2-\frac{s_2^2-s_2}{2}+(n-s_2-s_3-M_3)(t_2+1). \tag{3.38}$$

Now, using (3.37) and (3.38), we derive from (3.35) the following estimate:

$$\mathcal{N}-\mathcal{S}\geq(1+b)M_4+s_3+s_4+t_1s_1-\frac{s_1^2-s_1}{2}+t_2s_2-\frac{s_2^2-s_2}{2}$$
$$+(n-s_1-s_4-M_1)(t_1+1)+(n-s_2-s_3-M_3)(t_2+1). \tag{3.39}$$

Comparing (3.34) with (3.39), we obtain

$$-n+s_3+s_4+(1+b)M_4<0.$$

Summing this inequality with (3.28), we get

$$(t_1-s_1)+(t_2-s_2)+(6-4b-(1-b)M_1-(1-b)M_2-(1-b)M_3-(1-3b)M_4)<0.$$

However, the summands on the left-hand side are nonnegative. Indeed, in the case of the first and the second ones it follows from (3.11) and (3.15), respectively. In the case of the third summand for $b=0$ we have

$$6-4b-(1-b)M_1-(1-b)M_2-(1-b)M_3-(1-3b)M_4$$
$$=6-M_1-M_2-M_3-M_4\geq 2,$$

while for $b=1$ we have

$$6-4b-(1-b)M_1-(1-b)M_2-(1-b)M_3-(1-3b)M_4=2+2M_4\geq 2.$$

This contradiction proves the lemma. ∎

Proof: As Q_1 is a slope with respect to the basis $(\mathbf{e}_2,-\mathbf{e}_1)$, by Proposition 2.7 there exists an integer s_1 such that

$$2N_1\leq\mathcal{E}_--\mathcal{S}+s_1, \tag{5.14}$$
$$\mathcal{E}-\mathcal{S}_+\geq\frac{1}{2}s_1(s_1+1), \tag{5.15}$$
$$0\leq s_1\leq N_1. \tag{5.16}$$

The same proposition applied to Q_3 and $(\mathbf{e}_1,-\mathbf{e}_2)$ ensures the existence of an integer s_3 such that

$$2N_3\leq\mathcal{N}_--\mathcal{W}+s_3, \tag{5.17}$$
$$\mathcal{N}-\mathcal{W}_+\geq\frac{1}{2}s_3(s_3+1), \tag{5.18}$$
$$0\leq s_3\leq N_3. \tag{5.19}$$

As Q_4 is a slope with respect to the bases $(\mathbf{e}_1,\mathbf{e}_2)$ and $(\mathbf{e}_2,\mathbf{e}_1)$, by the same proposition there exist integers s and s' such that

$$2N_4\leq\mathcal{S}_--\mathcal{W}+s, \tag{5.20}$$
$$\mathcal{W}_--\mathcal{S}\geq\frac{1}{2}s(s+1), \tag{5.21}$$
$$0\leq s\leq N_4, \tag{5.22}$$
$$2N_4\leq\mathcal{W}_--\mathcal{S}+s', \tag{5.23}$$
$$\mathcal{S}_--\mathcal{W}\geq\frac{1}{2}s'(s'+1), \tag{5.24}$$
$$0\leq s'\leq N_4. \tag{5.25}$$

The frame $((n,n);-\mathbf{e}_2,-\mathbf{e}_1)$ forms small angle with Q_2, so by Corollary 2.12

$$2N_2\leq 2n-\mathcal{N}_+-\mathcal{E}_+-\left\lceil\frac{\mathcal{E}-n}{2}\right\rceil+1. \tag{5.26}$$

By Proposition 2.17,

$$\mathcal{S}_+-\mathcal{S}_-\geq M_1, \tag{5.27}$$
$$\mathcal{E}_+-\mathcal{E}_-\geq M_2, \tag{5.28}$$
$$\mathcal{N}_+-\mathcal{N}_-\geq M_3, \tag{5.29}$$
$$\mathcal{W}_+-\mathcal{W}_-\geq M_4. \tag{5.30}$$

We claim that $\widehat{P}$ lies in the triangle Δ having the vertices $(-n,-n)$, $(-n,n)$, and (n,n), which is the solution set of the system

$$\begin{cases} x_1 \geq -n, \\ x_2 \leq n, \\ x_1 - x_2 \leq 0. \end{cases}$$

Since $\widehat{P}$ is a type V_n polygon, it lies in the angle

$$\begin{cases} x_1 \geq -n, \\ x_2 \leq n. \end{cases}$$

The intersection of the line $x_1 - x_2 = 0$ with this angle is the segment $[(-n,-n),(n,n)]$, so we only need to show that neither of the segments $I_1 = [(-n,-n),\mathbf{0}]$ and $I_2 = [\mathbf{0},(n,n)]$ splits $\widehat{P}$. It the case of the former this is true by Lemma 5.5, as $\widehat{P}$ is the lift of P_{k_0}. Likewise, I_1 does not split $\widehat{P}'_{k_0}$, so I_2, being the reflection of I_1 about the line $x_1+x_2=0$, does not split $P_{k_0+1} = \widehat{P}$, as claimed.

By construction, $\widehat{P} = BP$, where B is a unimodular transformation. The affine automorphism of $n\mathbb{Z}^2$ defined by

$$\psi(x_1,x_2) = (x_1+n, -x_2+n)$$

maps Δ onto Δ_n. Consequently, the polygon $\psi(BP)$ lies in Δ_n, i. e. $\varphi = \psi B$ is the required automorphism. ∎

5.3. Bounds on the number of vertices of type Va_n polygons. Here we establish a few estimates of the number of vertices of type Va_n polygons and eventually prove Lemma 5.3.

Lemma 5.8. *Suppose that P is a type Va_n polygon, the frame $((n,n);-\mathbf{e}_2,-\mathbf{e}_1)$ splits the slope Q_2 and forms small angle with it and either*

$$\mathcal{S}_+ \leq n \tag{5.11}$$

or

$$\mathcal{S}_+ \geq n+1, \qquad \mathcal{W}_+ \leq n. \tag{5.12}$$

Then

$$N \leq 2n+2. \tag{5.13}$$

Lemma 3.3. *Suppose that $n \geq 4$ is even and P is a type III_n N-gon with vertices belonging to a $(1,n/2)$-lattice Γ having the basis $(\mathbf{e}_1,(n/2)\mathbf{e}_2)$; then*

$$N \leq 2n.$$

Proof: According to Lemma 3.1, all the vertices of P belonging to the open half-plane $x_1 < n$ lie in the slab $0 \leq x_1 < n$. All the points of Γ belonging to this slab lie on the lines $x_1 = i$ $(i = 0,1,\ldots,n-1)$, and each of these lines contains at most two vertices except for $x_1 = 0$, which contain at most one (since every other point of Γ lying on this line belongs to $n\mathbb{Z}^2$). This gives the maximum of $2n-1$ lying in the said half-plane.

It remains to prove that at most one vertex lies in the half-plane $x_1 \geq n$. According to Lemma 3.1, each of the lines (3.6) contains at most one vertex, and there are no other vertices. But among these only the line $x_2 = n/2$ has points belonging to Γ. ∎

Lemma 3.4. *Suppose that $n \geq 3$ and P is a type III_n N-gon with vertices belonging to a $(1,n)$-lattice Γ having the basis $(\mathbf{e}_1 + a\mathbf{e}_2, n\mathbf{e}_2)$, where $1 \leq a \leq n-1$. Then*

$$N \leq 2n-2. \tag{3.40}$$

Proof: The frame $((n,0),\mathbf{e}_2,-\mathbf{e}_1)$ splits Q_1, and the vertices of Q_1 belong to a proper subset of $\mathbb{Z}^2$, so by Theorem 2.13 we have

$$2N_1 \leq \mathcal{E}_- - \mathcal{S}_+ + n - 1. \tag{3.41}$$

Applying the same theorem to $((n,n),-\mathbf{e}_2,-\mathbf{e}_1)$ and Q_2, we obtain

$$2N_2 \leq -\mathcal{E}_+ - \mathcal{N}_+ + 2n - 1. \tag{3.42}$$

As Q_3 is a slope with respect to $(\mathbf{e}_1,-\mathbf{e}_2)$ and $(\mathbf{e}_1 - a(-\mathbf{e}_2), n(-\mathbf{e}_2))$ is a basis of Γ (Proposition 2.1), we evoke Proposition 2.7 and conclude that there exists an integer s_3 such that

$$2N_3 \leq \mathcal{N}_- - \mathcal{W} + s_3, \tag{3.43}$$

$$\mathcal{N} - \mathcal{W}_+ \geq \frac{2a + (s_3-1)n}{2} s_3, \tag{3.44}$$

$$0 \leq s_3 \leq N_3. \tag{3.45}$$

Likewise, applying Proposition 2.7 to Q_4, the basis $(\mathbf{e}_1, \mathbf{e}_2)$ and the basis $(\mathbf{e}_1 - (n-a)\mathbf{e}_2, n\mathbf{e}_2)$ of Γ, we obtain an integer s_4 such that

$$2N_4 \le \mathcal{S}_- - \mathcal{W} + s_4, \tag{3.46}$$

$$\mathcal{W}_- - \mathcal{S} \ge \frac{2(n-a) + (s_4 - 1)n}{2} s_4, \tag{3.47}$$

$$0 \le s_4 \le N_4. \tag{3.48}$$

Finally, we have

$$\mathcal{S}_+ - \mathcal{S}_- \ge 2M_1, \tag{3.49}$$

$$\mathcal{E}_+ - \mathcal{E}_- \ge nM_2, \tag{3.50}$$

$$\mathcal{N}_+ - \mathcal{N}_- \ge 2M_3, \tag{3.51}$$

$$\mathcal{W}_+ - \mathcal{W}_- \ge nM_4, \tag{3.52}$$

$$\mathcal{W} \ge 1. \tag{3.53}$$

Indeed, the large $\mathbf{e}_2$-step of Γ is n, so by Proposition 2.17 we have (3.50) and (3.52). Because $1 \le a \le n-1$, we have $\mathbf{e}_1 \notin \Gamma$ and the large $\mathbf{e}_1$-step of Γ is greater then or equal to 2, so inequalities (3.49) and (3.51) hold by virtue of the same proposition. Finally, (3.53) follows from the fact that $\mathcal{W}$ is nonnegative by Lemma 3.1, and the fact that all the points of Γ lying on the line $x_1 = 0$ belong to $n\mathbb{Z}^2$.

Assuming that (3.40) does not hold, we have

$$2N \ge 4n - 2. \tag{3.54}$$

Estimate $2N$ from above by summing (3.41), (3.42), (3.43), and (3.46) and subsequently using (3.49)–(3.51) and (3.53):

$$\begin{aligned} 2N &= \sum_{k=1}^{4} 2N_k + \sum_{k=1}^{4} 2M_k \\ &\le 3n - 2 + s_3 + s_4 + (2M_1 - (\mathcal{S}_+ - \mathcal{S}_-)) + (nM_2 - (\mathcal{E}_+ - \mathcal{E}_-)) \\ &\quad + (2M_3 - (\mathcal{N}_+ - \mathcal{N}_-)) - (n-2)M_2 + 2(M_4 - \mathcal{W}) \\ &\le 3n - 2 + s_3 + s_4. \end{aligned}$$

Comparing this with (3.54), we get

$$s_3 + s_4 \ge n. \tag{3.55}$$

On the other hand,

$$\begin{aligned} &\max\{x_2 \colon (x_1, x_2) \in \widehat{P}, x_1 \ge 0\} \\ &\qquad = \max\{-a_0 x_1' + x_2' \colon (x_1', x_2') \in P, x_1 \le 0\} \\ &\qquad\quad \le \max\{x_2' \colon (x_1', x_2') \in P, x_1 \le 0\} \le \mathcal{N}(P) \le n. \end{aligned} \tag{5.10}$$

Estimates (5.9) and (5.10) imply that $\mathcal{N}(\widehat{P}) \le n$, as claimed. ∎

Proof of Lemma 5.2: Take a type V_n polygon P_0, and let $\widehat{P}_0$ be its lift. If the translation by the vector $(n, 0)$ maps $\widehat{P}_0$ onto a type III_n polygon, we are done. Otherwise, by Lemma 5.7, $\widehat{P}_0$ is a type V_n polygon. Let P_0' be the reflection of $\widehat{P}_0$ about the line $x_1 + x_2 = 0$. It is easy to check that it is again a type V_n polygon. Let $\widehat{P}_0'$ be its lift. As before, either the translation of $\widehat{P}_0'$ by $(n, 0)$ is a type III_n polygon and we are done, or $\widehat{P}_0'$ is a type V_n polygon, in which case we define the type V_n polygon P_1 to be the reflection of $\widehat{P}_0'$ about the line $x_1 + x_2 = 0$.

Iterating this procedure, we either find an affine automorphism of $n\mathbb{Z}^2$ mapping P_0 onto a type III_n polygon, or construct the sequences of type V_n polygons $\{P_k\}$, $\{\widehat{P}_k\}$, $\{P_k'\}$, and $\{\widehat{P}_k'\}$. In the latter case consider the sequence of integers $\{\mathcal{S}(P_k)\}_{k=0}^{\infty}$. As P_k are type V_n polygons, it is easily seen that the the members of this sequence are negative (this follows e. g. from the fact that by Proposition 2.16 the frame $(\mathbf{0}; -\mathbf{e}_1, -\mathbf{e}_2)$ splits any type V_n polygon). Observe that the sequence increases. Indeed, it is easy to check that

$$\mathcal{S}(P_{k+1}) = -\mathcal{E}(\widehat{P}_k') = -\mathcal{E}(P_k') = \mathcal{S}(\widehat{P}_k);$$

furthermore, by Lemma 5.6 we have

$$\mathcal{S}(\widehat{P}_k) \ge \mathcal{S}(P_k).$$

Thus, we see that the sequence $\{\mathcal{S}(P_k)\}$ increases; moreover, we have $\mathcal{S}(P_{k+1}) = \mathcal{S}(P_k)$ if and only if $\mathcal{S}(\widehat{P}_k) = \mathcal{S}(P_k)$, which by Lemma 5.6 is equivalent to $\widehat{P}_k = P_k$.

The sequence of integers $\{\mathcal{S}(P_k)\}$ increases and is bounded from above, so it stabilises. We show in the same way that the sequence $\{\mathcal{S}(P_k')\}$ stabilises, too. Consequently, there exists k_0 such that $P_{k_0} = \widehat{P}_{k_0}$ and $P_{k_0}' = \widehat{P}_{k_0}'$. Then also $P_{k_0} = P_{k_0+1}$. Set $\widehat{P} = P_{k_0}$.

Using these representations and (5.7), we get

$$\begin{aligned}\mathcal{S}(\widehat{P}) &= \min\{\hat{x}_2\colon (\hat{x}_1,\hat{x}_2)\in\widehat{P},\ \hat{x}_1\le -1\}\\ &= \min\{-a_0x_1+x_2\colon (x_1,x_2)\in P,\ x_1\le -1\}\\ &\ge \min\{x_2+1\colon (x_1,x_2)\in P,\ x_1\le -1\} = \mathcal{S}(P)+1,\end{aligned}$$

so (5.8) is proved. ∎

Lemma 5.7. *Let P be a type V_n polygon and $\widehat{P}$ be its lift. Then either $\widehat{P}$ is a type V_n polygon, or the translation of $\widehat{P}$ by the vector $(n,0)$ is a type III_n polygon.*

Proof: Let T be the translation by the vector $(n,0)$. Note that $\widehat{P}$ and $T\widehat{P}$ are obtained by applying affine automorphisms os $n\mathbb{Z}^2$ to P, so they are free of points of this lattice.

The polygon $\widehat{P}$ is split by the segments $[\mathbf{0},(0,n)]$ (by Lemma 5.5) and $[\mathbf{0},(-n,0)]$ (by the definition of lift), but not by the line $x_1=-n$ (because by the definition of a type V_n polygon this line does not split P and it is invariant under the lift transformation). Assume for a moment that the segment $[(0,n),(-n,n)]$ splits $\widehat{P}$. Then the segments

$$\begin{aligned}[\mathbf{0},(n,0)] &= T[(-n,0),\mathbf{0}],\\ [(n,0),(n,n)] &= T[\mathbf{0},(0,n)],\\ [(0,n),(n,n)] &= T[(-n,n),(0,n)].\end{aligned}$$

split TP, while the line $x_1=0$, being the image of $x_1=-n$ under T, does not. Consequently, TP is a type III_n polygon.

It remains to show that if the segment $[(0,n),(-n,n)]$ does not split $\widehat{P}$, the latter is a type V_n polygon. We know already that the segments $[\mathbf{0},(-n,0)]$ and $[\mathbf{0},(0,n)]$ split $\widehat{P}$, while the line $x_1=-n$ does not, so we only need to show that the line $x_2=n$ does not split $\widehat{P}$ either, or, equivalently, that $\mathcal{N}(\widehat{P})\le n$.

As $\widehat{P}$ lies in the half-plane $x_1\ge -n$ and the segment $[(-n,n),(0,n)]$ does not split $\widehat{P}$, it is clear that

$$\max\{x_2\colon (x_1,x_2)\in\widehat{P}, x_1\le 0\}\le n. \tag{5.9}$$

Summing (3.44), (3.47), and (3.52) and discarding the nonnegative term nM_4, we obtain a lower estimate of $\mathcal{N}-\mathcal{S}$:

$$\mathcal{N}-\mathcal{S}\ge \frac{2a+(s_3-1)n}{2}s_3+\frac{2(n-a)+(s_4-1)n}{2}s_4. \tag{3.56}$$

Now, a simple geometrical reasoning provides the upper estimate

$$\mathcal{N}-\mathcal{S}\le n^2-1. \tag{3.57}$$

Indeed, note that the triangle with the vertices $(\mathcal{N}_+,\mathcal{N})$, $(\mathcal{S}_+,\mathcal{S})$, and $(\mathcal{E},\mathcal{E}_-)$ is contained in P, so the segment being the intersection of the triangle with the line $x_1=n$ lies between two adjacent points of $n\mathbb{Z}^2$. As the distance from $(\mathcal{E},\mathcal{E}_-)$ to the line is greater than or equal to 1, it is not hard to see that the projection of the segment $[(\mathcal{N}_+,\mathcal{N}),(\mathcal{S}_+,\mathcal{S})]$ onto $x_1=0$ has length strictly less than n^2, whence (3.57) follows.

Comparing (3.56) and (3.57), we obtain

$$\frac{2a+(s_3-1)n}{2}s_3+\frac{2(n-a)+(s_4-1)n}{2}s_4\le n^2-1. \tag{3.58}$$

Let us estimate the second term on the left-hand side. It follows from (3.43) and (3.45) that

$$2s_3\le \mathcal{N}_- -\mathcal{W}+s_3,$$

whence using (3.53) and (3.2) we obtain

$$s_3\le \mathcal{N}_- -\mathcal{W}\le \mathcal{N}_+-1\le n-2.$$

Combining this with (3.55) we deduce

$$0<n-s_3\le s_4.$$

Consequently, we have

$$\frac{2(n-a)+(s_4-1)n}{2}s_4\ge \frac{2(n-a)+(n-s_3-1)n}{2}(n-s_3),$$

and from (3.58) we obtain

$$\frac{2a+(s_3-1)n}{2}s_3+\frac{2(n-a)+(n-s_3-1)n}{2}(n-s_3)-n^2+1\le 0.$$

Transforming the left-hand side, we can write the inequality in the form

$$n\left(s_3+\frac{2a-n-n^2}{2n}\right)^2+\frac{a(n-a)}{n}+\frac{1}{4}(n+1)(n-1)(n-4)\le 0.$$

Clearly, this inequality cannot hold with $n \geq 4$. This contradiction proves the lemma in the case $n \geq 4$.

If $n = 3$, it follows from (3.2) and (3.3) that

$$\mathcal{N}_+ + \mathcal{S}_+ \leq 4, \tag{3.59}$$

inequalities (3.43), (3.45), and (3.53) give

$$\mathcal{N}_- \geq s_3 + \mathcal{W} + (2N_3 - 2s_3) \geq s_3 + 1,$$

and inequalities (3.46), (3.48), and (3.53) similarly imply

$$\mathcal{S}_- \geq s_4 + 1.$$

From these estimates and (3.55) we derive

$$\mathcal{N}_+ + \mathcal{S}_+ \geq \mathcal{N}_- + \mathcal{S}_- \geq s_3 + s_4 + 2 \geq n + 2 = 5,$$

which contradicts (3.59). ∎

Lemma 3.5. *Suppose that $n \geq 3$. Let Γ and b be a lattice and a number such that either Γ has invariant factors $(1, n)$ and $b = 0$ or Γ has invariant factors $(1, n/2)$, and $b = 1$. Suppose that the small $\mathbf{e}_1$-step of Γ is greater then 1, and let P be a type III_n N-gon with the vertices belonging to Γ. Then*

$$N \leq 2n - 2 + 2b. \tag{3.60}$$

Proof: As in the proof of Lemma 3.4, we have (3.41) and (3.42). Applying Proposition 2.7 to Q_3 and Q_4, we obtain

$$2N_3 \leq \mathcal{N}_- - \mathcal{W}, \tag{3.61}$$
$$2N_4 \leq \mathcal{S}_- - \mathcal{W}. \tag{3.62}$$

As the small $\mathbf{e}_1$-step of Γ is greater then 1, we still have (3.49) and (3.51). By Proposition 2.17, we also have

$$\mathcal{E}_+ - \mathcal{E}_- \geq (2 - b)M_2 \tag{3.63}$$

(in the case $b = 0$ this follows from assertion (v) of Lemma 3.1).

Let $a_0 \geq 0$ be the greatest integer such that P_{a_0} is split by the segment $[\mathbf{0}, (-n, 0)]$. We say that the polygon $\widehat{P} = P_{a_0}$ is the *lift* of P and that A_{a_0} is the *lift transformation* of P.

Lemma 5.5. *Let that $\widehat{P}$ be the lift of P; then the segment $[\mathbf{0}, (0, n)]$ splits $\widehat{P}$ and the segment $[\mathbf{0}, (-n, -n)]$ does not. If the segment $[(0, n), (n, 2n)]$ does not split P, it does not split $\widehat{P}$ either.*

Proof: The line $x_1 = 0$ is invariant under the lift transformation and since the segment $[\mathbf{0}, (0, n)] = A_{a_0}[\mathbf{0}, (0, n)]$ splits P, it splits $A_{a_0}P = \widehat{P}$ as well.

By the definition of a_0, the segment $[\mathbf{0}, (-n, 0)]$ does not split the polygon $A_{a_0+1}P = A_1\widehat{P}$. Consequently, the segment $[\mathbf{0}, (-n, -n)] = A_1^{-1}[\mathbf{0}, (-n, 0)]$ does not split $\widehat{P}$, as claimed.

Suppose that the segment $[(0, n), (n, 2n)]$ does not split P. The the intersection $P \cap \{0 \leq x_1 \leq n\}$ lies in the half-plane $x_2 \leq x_1 + n$. It suffices to check that the intersection $\widehat{P} \cap \{0 \leq x_1 \leq n\}$ lies in the same half-plane. Indeed, let $(\hat{x}_1, \hat{x}_2) \in \widehat{P}$ and $0 \leq \hat{x}_1 \leq n$; then $\hat{x}_1 = x_1$ and $\hat{x}_2 = -a_0x + x_2$ for some $(x_1, x_2) \in P \cap \{0 \leq x_1 \leq n\}$, so $\hat{x}_2 \leq x_2 \leq x_1 + n = \hat{x}_1 + n$, as claimed. ∎

Lemma 5.6. *Suppose that $\widehat{P}$ is the lift of P; then $\mathcal{S}(\widehat{P}) \geq \mathcal{S}(P)$ and $\mathcal{S}(\widehat{P}) = \mathcal{S}(P)$ if and only if $P = \widehat{P}$.*

Proof: If $a_0 = 0$, we have $P = \widehat{P}$, so $\mathcal{S}(\widehat{P}) = \mathcal{S}(P)$. It remains to show that

$$a_0 \geq 1 \tag{5.7}$$

implies

$$\mathcal{S}(\widehat{P}) > \mathcal{S}(P). \tag{5.8}$$

As the segments $[\mathbf{0}, (-n, 0)]$ and $[\mathbf{0}, (0, n)]$ split both P and $\widehat{P}$, by Proposition 2.16 the frame $(\mathbf{0}; -\mathbf{e}_1, \mathbf{e}_2)$ splits the slopes $Q_1(P)$ and $Q_1(\widehat{P})$, whence

$$\mathcal{S}_+(P) \leq -1, \quad \mathcal{S}_+(\widehat{P}) \leq -1.$$

Thus,

$$\mathcal{S}(P) = \min\{x_2 \colon (x_1, x_2) \in P,\ x_1 \leq -1\},$$

$$\mathcal{S}(\widehat{P}) = \min\{x_2 \colon (x_1, x_2) \in \widehat{P},\ x_1 \leq -1\}.$$

5.2. The lift. Let P be an integer polygon free of $n\mathbb{Z}^2$-points, where $n \geq 3$ is an integer. Assume that the segments $[\mathbf{0}, (-n, 0)]$ and $[\mathbf{0}, (0, n)]$ split P. In particular, P can be any type V_n polygon. Given $a \in \mathbb{Z}$, consider the unimodular transformation

$$A_a = \begin{pmatrix} 1 & 0 \\ -a & 1 \end{pmatrix}$$

and the polygon $P_a = A_a P$.

Lemma 5.4. *The set of such $a \in \mathbb{Z}$ that P_a is split by the segment $[\mathbf{0}, (-n, 0)]$ is nonempty and has a nonnegative maximal element.*

Proof: Obviously, $P_0 = P$, so the set in question contains 0 and its maximal element, if it exists, is nonnegative. To prove the lemma, it remains to show that the set is bounded from above, i. e. that the segment $[\mathbf{0}, (-n, 0)]$ does not split the polygon P_a for large a.

As P does not contain the point $\mathbf{0} \in n\mathbb{Z}^2$, there exists a linear form $\ell(x_1, x_2) = \alpha_1 x_1 + \alpha_2 x_2$ such that

$$\ell(\mathbf{x}) > 0, \ \mathbf{x} \in P. \tag{5.2}$$

Choosing points $\check{\mathbf{x}} \in P \cap [\mathbf{0}, (0, n)]$ and $\hat{\mathbf{x}} \in P \cap [\mathbf{0}, (-n, 0)]$, so that $\check{\mathbf{x}} = (0, \check{x}_2)$, $\check{x}_2 > 0$, and $\hat{\mathbf{x}} = (\hat{x}_1, 0)$, $\hat{x}_1 < 0$ and computing $\ell(\check{\mathbf{x}})$ and $\ell(\hat{\mathbf{x}})$, we see that in view of (5.2),

$$\alpha_1 < 0, \ \alpha_2 > 0. \tag{5.3}$$

Fix an integer a such that

$$a \geq -\frac{\alpha_1}{\alpha_2}. \tag{5.4}$$

Consider the linear form

$$\tilde{\ell}(\tilde{x}_1, \tilde{x}_2) = (\alpha_1 + a\alpha_2)\tilde{x}_1 + \alpha_2 \tilde{x}_2.$$

It is easy to check that $\ell(\mathbf{x}) = \tilde{\ell}(\tilde{\mathbf{x}})$ whenever $\mathbf{x} = A_a^{-1}\tilde{\mathbf{x}}$. In particular, if $\tilde{\mathbf{x}} \in P_a$, we have $\mathbf{x} = A_{a_0}^{-1}\tilde{\mathbf{x}} \in P$, and according to (5.2), we obtain

$$\tilde{\ell}(\tilde{\mathbf{x}}) > 0, \quad \tilde{\mathbf{x}} \in P_a. \tag{5.5}$$

On the other hand, if $\tilde{\mathbf{x}} = (\tilde{x}_1, 0) \in [0, (-n, 0)]$, then $\tilde{x}_1 \leq 0$, and

$$\tilde{\ell}(\tilde{\mathbf{x}}) = (\alpha_1 + a\alpha_2)\tilde{x}_1 < 0, \quad \tilde{\mathbf{x}} \in [\mathbf{0}, (-n, 0)] \tag{5.6}$$

according to the choice of a. Comparing (5.5) and (5.7), we see that P_a has no common points with the segment $[\mathbf{0}, (-n, 0)]$. This is true for any a satisfying (5.4), so the set in question is bounded from above, as claimed. ∎

Summing (3.41), (3.42), (3.61), and (3.62) and subsequently applying (3.49), (3.51), (3.63), and (3.1), we obtain the estimate

$$\begin{aligned} 2N = \sum_{k=1}^{4} 2N_k + \sum_{k=1}^{4} 2M_k \\ \leq 3n + (2M_1 - (\mathcal{S}_+ - \mathcal{S}_-)) + ((2-b)M_2 - (\mathcal{E}_+ - \mathcal{E}_-)) \\ + (2M_3 - (\mathcal{N}_+ - \mathcal{N}_-)) + 2(M_4 - 1) + bM_2 \leq 3n + b, \end{aligned}$$

whence

$$N \leq \frac{3}{2}n + \frac{b}{2} = (2n - 2 + 2b) + \frac{-n - 3b + 4}{2}.$$

Since $n \geq 3$ and $b \geq 0$, we can write

$$N \leq (2n - 2 + 2b) + \frac{1}{2},$$

which yields (3.60). ∎

Proof of Theorem 1.4 for type III polygons: Let P be a type III_n N-gon. By Lemma 3.2, its number of vertices satisfies $N \leq 2n + 2$.

Suppose that the vertices of P belong to a $(1, n)$-lattice Γ. If the small $\mathbf{e}_1$-step of Γ is greater then 1, by Lemma 3.5 we have $N \leq 2n - 2$. If the small $\mathbf{e}_1$-step of Γ equals 1, by Proposition 2.4 this lattice admits a basis of the form $(\mathbf{e}_1 + a\mathbf{e}_2, n\mathbf{e}_2)$, where $0 \leq a \leq n - 1$. According to assertion (v) of Lemma 3.1, we cannot have $a = 0$, so Lemma 3.4 provides the same bound on the number of vertices.

Finally, suppose that n is even and the vertices of P belong to a $(1, n/2)$-lattice Γ. If the small $\mathbf{e}_1$-step of Γ is greater than 1, by Lemma 3.5 we have $N \leq 2n$. Otherwise, Γ has a basis of the form $(\mathbf{e}_1 + a\mathbf{e}_2, (n/2)\mathbf{e}_2)$, where $0 \leq a \leq n/2 - 1$; then Lemma 3.2 gives the same estimate in case $a \neq 0$ and Lemma 3.3, in case $a = 0$. ∎

4. Type IV polygons

In this section we prove Theorem 1.4 for type IV polygons.

Throughout the section, we fix an integer $n \geq 3$.

Lemma 4.1. *Suppose that the line $x_1 - x_2 = n$ splits a type IV_n polygon; then so does one of the segments $[(0, -n), (n, 0)]$ and $[(n, 0), (2n, n)]$.*

Proof: All the points of the line $x_1 - x_2 = n$ belonging to the slab $-n + 1 \leq x_1 \leq 2n - 1$ lie on the segments $[(-n, -2n), (0, -n)]$, $[(0, -n), (n, 0)]$, and

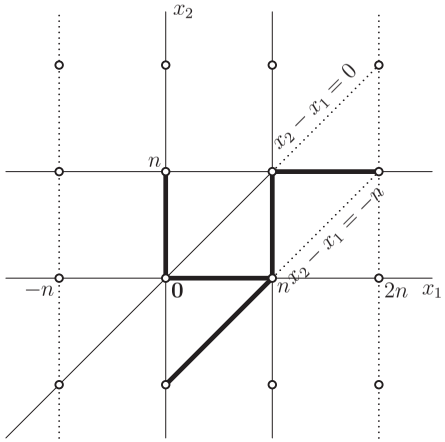

FIGURE 2. The segments splitting the polygon in the hypothesis of Lemma 4.2 are thick, and P does not intersect dotted lines. The inequalities (4.3)–(4.6) are obvious.

$[(n,0),(2n,n)]$, and as the polygon is free of $n\mathbb{Z}^2$-points, exactly one of the segments splits it. However, it cannot be the first one, because it follows from Corollary 2.6 that the polygon has no points with both nonpositive coordinates. ∎

Lemma 4.2. *Suppose that P is a type IV_n polygon and the segment $[(0,-n),(n,0)]$ splits it. Then the following assertions hold:*

(i) The intersection of P with the half-plane $x_1 \geq n$ lies in the slab

$$-n < x_2 - x_1 < 0. \tag{4.1}$$

(ii) The frame $((n,0);\mathbf{e}_2,-\mathbf{e}_1)$ splits the slope Q_1 and forms small angle with it.

(iii) The frame $((n,n);\mathbf{e}_1,-\mathbf{e}_2)$ splits the slope Q_3 and forms small angle with it.

(iv) The frame $(\mathbf{0};\mathbf{e}_1,\mathbf{e}_2)$ splits the slope Q_4.

(v) The slope Q_1 has a vertex $\mathbf{v} = (v_1,v_2)$ satisfying

$$v_2 - v_1 \leq -n - 1. \tag{4.2}$$

By definition, the segments $[\mathbf{0},(n,0)]$ and $[(n,n),(2n,n)]$ split the polygon P, and so does $[\mathbf{0},(n,n)]$ by virtue of Lemma 4.1. Consequently, the images of those segments under φ—i. e., the segments $[(n,0),\mathbf{0}]$, $[(n,n),(0,n)]$, $[(n,n),(n,2n)]$—split $\varphi(P)$. If P is also split by $[(n,0),(2n,n)]$, then $\varphi(P)$ is split by $[\mathbf{0},(0,n)]$, and consequently, $\varphi(P)$ is a type II_n polygon. Otherwise, the line $x_1 - x_2 = n$ does not split P, so the line $x_1 = 0$ does not split $\varphi(P)$ either, and the latter is a type III_n polygon. ∎

5. Type V and Va polygons

5.1. Main results. In this section we prove that for any type V_n polygon there exists an affine automorphism of $n\mathbb{Z}^2$ mapping it on a type III_n or a type Va_n polygon, the latter to be defined presently. We find certain bounds for the number of vertices of type Va polygons, which are not sufficient, however, to prove Theorem 1.4 for this class of polygons. We revisit type Va polygons in Section 7, establishing the missing estimate and completing the proof of Theorem 1.4.

Fix an integer $n \geq 3$.

We will denote by Δ_n the triangle with the vertices $\mathbf{0}$, $(2n,0)$, and $(0,2n)$. It is easy to check that Δ_n is given by the following system of linear inequalities:

$$\begin{cases} x_1 \geq 0, \\ x_2 \geq 0, \\ x_1 + x_2 \leq 2n. \end{cases} \tag{5.1}$$

Definition 5.1. We say that P is a *type Va_n polygon*, if it is free of $n\mathbb{Z}^2$-points and lies in Δ_n.

Two following lemmas are the main results of the section.

Lemma 5.2. *For any type V_n polygon there exists an affine automorphism of $n\mathbb{Z}^2$ mapping it onto a type III_n or a type Va_n polygon.*

The proof is given in Section 5.2.

Lemma 5.3. *Suppose that P is a type Va_n N-gon; then*

$$N \leq 2n + 2,$$

and if the vertices of P belong to a $(1,n)$-lattice, then

$$N \leq 2n - 2.$$

The proof is given in Section 5.3.

according to (4.6) we have

$$1 \le \mathcal{W}_+ \le 3.$$

Given that P has the vertices $\mathbf{u}_1$ and $\mathbf{u}_2$ with said properties as well as a common point with the segment

$$[(n,0),(n,n)] = [(4,0),(4,4)],$$

it is not hard to check that P and the lines

$$x_2 = \frac{1}{2}x_1 + 2,\ x_1 = 4$$

satisfy the hypotheses of Lemma 2.5. Consequently, P contains the point $(4,4) \in 4\mathbb{Z}^2 = n\mathbb{Z}^2$, which is impossible. The contradiction means that the vertices of a type IV_4 polygon cannot belong to a lattice Γ having said properties.

Thus, (4.64) holds for any $(1, n/2)$-lattice Γ.

Finally, suppose that the vertices of P belong to a $(1,n)$-lattice Γ (now there is no need to assume that n is even). Let us prove that

$$N \le 2n - 2. \tag{4.67}$$

Note that as P has the vertex $(\mathcal{N}_+, \mathcal{N})$ satisfying (4.3) and (4.4), it is clear that the both the small $\mathbf{e}_1$-step and the small $\mathbf{e}_2$-step of Γ are not equal to n. By Proposition 2.3, the product of the small $\mathbf{e}_1$-step and the large $\mathbf{e}_2$-step of Γ equals n, so Γ has large $\mathbf{e}_2$-step different from 1. Likewise, Γ has large $\mathbf{e}_2$-step different from 1 as well.

Suppose that the small $\mathbf{e}_1$-step of Γ equals 1. Then by Proposition 2.4 the lattice admits a basis of the form $(\mathbf{e}_1 + a\mathbf{e}_2, n\mathbf{e}_2)$, where $0 \le a \le n-1$. The equality $a = 0$ is impossible, as the large $\mathbf{e}_1$-step of the lattece is not 1. Consequently, we can apply Lemma 4.6, which gives (4.67).

Otherwise, the small $\mathbf{e}_1$-step of Γ is greater than 1, and as its large $\mathbf{e}_2$-step is greater than 1 as well, we can apply Lemma 4.4 and obtain (4.67). ∎

Proof of Theorem 1.4 for type IV_n polygons. Let P be an arbitrary type IV_n polygon. If the segment $[(0,-n),(n,0)]$ splits it, we complete the proof by evoking Lemma 4.7. Otherwise it suffices to show that there is an affine automorphism of $\mathbb{Z}^2$ mapping P on a type II_n or a type III_n polygon, as the required estimates have already been proved for those kinds of polygons.

Define the automorphism φ by

$$\varphi(x_1, x_2) = (-x_1 + x_2 + n, x_2).$$

(vi) The following inequalities hold:

$$n < \mathcal{N}_+ \le \mathcal{E} \le 2n - 1, \tag{4.3}$$

$$n < \mathcal{N} \le \mathcal{N}_+ - 1, \tag{4.4}$$

$$-n < \mathcal{W} < 0, \tag{4.5}$$

$$0 < \mathcal{W}_+ < n. \tag{4.6}$$

Proof: To prove (i), it suffices to observe that P cannot have common points with the segments $[(n,0),(2n,n)]$ and $[(n,n),(2n,2n)]$. Indeed, if P had common points with the former segment, by convexity it would containt the point $(n,0)$; if it had common points with the latter, it would contain the point (n,n) by Lemma 2.5 applied to P and the lines $x_2 - x_1 = 0$ and $x_2 = n$. Thus, the part of P contained in the half-plane $x_1 \ge n$ must lie between the lines $x_2 - x_1 = -n$ and $x_2 - x_1 = 0$.

Let us prove (v). Clearly, the functional $x_2 - x_1$ attains its maximum on P on a vertex $\mathbf{v} \in Q_1$. As the line $x_2 - x_1 = -n$ splits P, this minimum is less than n, and (v) follows.

The fact that the frames split correspondent slopes in assertions (ii)–(iv) follows from Proposition 2.16. To prove that $((n,0); \mathbf{e}_2, \mathbf{e}_1)$ forms small angle with Q_1, we apply Proposition 2.10 taking the vertex from assertion (v) as $\mathbf{y}$. To prove that $((n,n); \mathbf{e}_1, -\mathbf{e}_2)$ splits Q_3, we use the same theorem with $\mathbf{y} = (\mathcal{W}, \mathcal{W}_+)$.

The inequalities in (vi) are is fairly intuitive, see Figure 4. ∎

Lemma 4.3. *Let Γ and b be a sublattice of $\mathbb{Z}^2$ and a number such that either $\Gamma = \mathbb{Z}^2$ and $b = 0$ or Γ has a basis of the form $(\mathbf{e}_1 + a\mathbf{e}_2,\ (n/2)\mathbf{e}_2)$, where $1 \le a \le n/2 - 1$, and $b = 1$ (this is only possible if n is even). Let P be a type IV_n N-gon with vertices belonging to Γ, and suppose that the segment $[(0,-n),(n,0)]$ splits P. Then*

$$N \le 2n + 2 - 2b. \tag{4.7}$$

Proof: By Lemma 4.2, the frame $((n,0); \mathbf{e}_2, -\mathbf{e}_1)$ forms small angle with the slope Q_1, so by Corollary 2.12 we have

$$2N_1 \le \mathcal{E}_- - \mathcal{S}_+ + n - \left\lceil \frac{\mathcal{E} - n}{2} \right\rceil + 1. \tag{4.8}$$

Likewise, as $((n,n);\mathbf{e}_1,-\mathbf{e}_2)$ forms small angle with Q_3, we obtain

$$2N_3 \le \mathcal{N}_- - \mathcal{W}_+ - \left\lceil \frac{\mathcal{N}-n}{2} \right\rceil + 1. \tag{4.9}$$

Applying Proposition 2.7 to the basis $(-\mathbf{e}_1,-\mathbf{e}_2)$ and the slope Q_2, we see that there exists an integer s_2 such that

$$2N_2 \le \mathcal{E} - \mathcal{N}_+ + s_2, \tag{4.10}$$

$$\mathcal{N} - \mathcal{E}_+ \ge \frac{s_2^2 + s_2}{2}, \tag{4.11}$$

$$0 \le s_2 \le N_2. \tag{4.12}$$

As the frame $(\mathbf{0};\mathbf{e}_1,\mathbf{e}_2)$ splits Q_4, by Corollary 2.12 and Theorem 2.13 we have

$$2N_4 \le \mathcal{S}_- + \mathcal{W}_- - b. \tag{4.13}$$

Finally, by Proposition 2.17 we have

$$\mathcal{S}_+ - \mathcal{S}_- \ge (1+b)M_1, \tag{4.14}$$

$$\mathcal{E}_+ - \mathcal{E}_- \ge (1+b)M_2, \tag{4.15}$$

$$\mathcal{N}_+ - \mathcal{N}_- \ge (1+b)M_3, \tag{4.16}$$

$$\mathcal{W}_+ - \mathcal{W}_- \ge (1+b)M_4. \tag{4.17}$$

Indeed, if $b=0$, these inequalities immediately follow from the proposition. If $b=1$, Γ has large $\mathbf{e}_2$-step $n/2 \ge 2$ and as $\mathbf{e}_1 \notin \Gamma$ due to the restriction $1 \le a \le n-1$, we see that Γ has large $\mathbf{e}_1$-step greater then or equal to 2.

We estimate $2N$ by means of (4.8)–(4.10), and (4.13):

$$\begin{aligned} 2N &= \sum_{k=1}^{4} 2N_k + \sum_{k=1}^{4} 2M_k \\ &\le n + 2 - b - \left\lceil \frac{\mathcal{E}-n}{2} \right\rceil + \mathcal{E} - \left\lceil \frac{\mathcal{N}-n}{2} \right\rceil + s_2 + \mathcal{E}_- + 2M_2 \\ &\quad - (\mathcal{S}_+ - \mathcal{S}_-) - (\mathcal{N}_+ - \mathcal{N}_-) - (\mathcal{W}_+ - \mathcal{W}_-) + 2M_1 + 2M_3 + 2M_4. \end{aligned}$$

Dropping the ceilings, using (4.3) and (4.14)–(4.17) and subsequently estimating $M_k \le 1$, we obtain

$$2N \le 3n + \frac{9}{2} - 4b + \left(-\frac{\mathcal{N}}{2} + s_2 + \mathcal{E}_- + 2M_2 \right). \tag{4.18}$$

Let s_1 be the small $\mathbf{e}_1$-step of Γ and S_2 be its large $\mathbf{e}_2$-step. By Proposition 2.3, we have

$$s_1 S_2 = \frac{n}{2}. \tag{4.65}$$

First, assume that $s_1 = 1$. Then by Proposition 2.4, Γ admits a basis of the form $(\mathbf{e}_1 + a\mathbf{e}_2, (n/2)\mathbf{e}_2)$, where $0 \le a \le n/2-1$. If $a \ge 1$, the estimate (4.64) follows from Lemma 4.3. Assume that $a=0$. It is easily seen that in this case all the points of Γ lie on the lines $x_2 = (n/2)r$, where $r \in \mathbb{Z}$. In particular, inequalities (4.4) and (4.6) become

$$\mathcal{N} = \frac{3n}{2}, \quad \mathcal{W}_+ = \frac{n}{2}.$$

Consider the vertices $\mathbf{w_1} = (\mathcal{N}_+, 3n/2)$ and $\mathbf{w_2} = (\mathcal{W}, n/2)$ of P. By (4.3) and (4.5), their first components satisfy

$$n < \mathcal{N}_+ < 2n, \quad \mathcal{W} < 0.$$

Taking into account that P has such vertices as well as a common point with the segment $[(n,0),(n,n)]$, it is not hard to check that P and the lines

$$x_2 = \frac{1}{2}x_1 + \frac{n}{2},\ x_1 = n$$

satisfy the hypothesis of Lemma 2.5. Consequently, P contains the point $(n,n) \in n\mathbb{Z}^2$, which is impossible. Thus, we cannot have $a=0$ and estimate (4.12) is proved for the case $s_1 = 1$.

Now assume that $s_1 \ge 2$. If additionally $S_2 \ge 2$, Lemma 4.4 provides an even stronger estimate than (4.64). Assume that $S_2 = 1$, then (4.65) gives $s_1 = n/2$. Consequently, all the points of Γ belonging to the slab $-n+1 \le x_1 \le 2n-1$, which contains P, lie on the five lines

$$x_1 = \frac{nr}{2} \qquad (r = 0, \pm 1, \pm 2). \tag{4.66}$$

Thus, P has no more than 10 vertices, and (4.64) is true for $n \ge 6$ (remember that we are considering even n at the moment). Assume that $n = 4$. The vertex $(\mathcal{N}_+, \mathcal{N})$ lies on one of the lines (4.66), so taking into account (4.3), we see that necessarily $\mathcal{N}_+ = 6$. Inequalities (4.3) and (4.4) imply that

$$4 < \mathcal{N} \le \mathcal{N}_+ - 1 = 5,$$

so necessarily $\mathcal{N} = 5$ and $\mathbf{u}_1 = (6,5)$ is a vertex of P. Likewise, (4.5) can only hold if $\mathcal{W} = -2$. Consequently, $\mathbf{u}_2 = (-2, \mathcal{W}_+)$ is a vertex of P, and

This and (4.59) give

$$nM_2 \le n-2,$$

which can only hold if

$$M_2 = 0. \tag{4.62}$$

Let us estimate the difference $\mathcal{N}-\mathcal{E}_+$ from above using (4.55), (4.56), and the evident inequality $\mathcal{E} \ge \mathcal{N}_+$. We have:

$$\mathcal{N}-\mathcal{E}_+ \le (\mathcal{N}_+ - 1) - (\mathcal{E} - n + 1) = n - 2 - (\mathcal{E} - \mathcal{N}_+) \le n-2.$$

Comparing this with (4.51), we obtain

$$\frac{2+n(s_2-1)}{2} s_2 \le n-2,$$

which can only hold if

$$s_2 \le 1.$$

This and (4.49) give

$$2N_2 \le \mathcal{E} - \mathcal{N}_+ + 1. \tag{4.63}$$

Now we estimate N by means of (4.48), (4.52), (4.53), (4.57), (4.58), and (4.60)–(4.63). We have:

$$\begin{aligned} 2N = \sum_{k=1}^{4} 2N_k + \sum_{k=1}^{4} 2M_k &\le 2n - 2 + \mathcal{E} + (2M_1 - (\mathcal{S}_+ - \mathcal{S}_-)) \\ &+ (2M_3 - (\mathcal{N}_+ - \mathcal{N}_-)) + (nM_4 - (\mathcal{W}_+ - \mathcal{W}_-)) - (n-2)M_4 \\ &\le 4n - 3. \end{aligned}$$

Dividing by 2 and taking floor, we obtain (4.47). ∎

Now we are in position to prove Theorem 1.4 for type IV_n polygons split by the segment $[(0,-n),(n,0)]$.

Lemma 4.7. *Theorem 1.4 holds for type IV_n polygons split by the segment $[(0,-n),(n,0)]$.*

Proof: Let P be a N-gon satisfying the hypothesis of the lemma.

Lemma 4.3 grants the estimate

$$N \le 2n+2.$$

Assume that n is even and that the vertices of P belong to a $(1, n/2)$-lattice Γ. Let us show that

$$N \le 2n. \tag{4.64}$$

Let us estimate the term in parentheses on the right-hand side. From (4.11) and (4.15) we get

$$\mathcal{N} \ge \mathcal{E}_+ + \frac{s_2^2+s_2}{2}, \quad \mathcal{E}_- \le \mathcal{E}_+ - (1+b)M_2 \le \mathcal{E}_+ - M_2,$$

whence

$$-\frac{\mathcal{N}}{2} + s_2 + \mathcal{E}_- + 2M_2 \le \frac{\mathcal{E}_+}{2} - \frac{s_2^2 - 3s_2}{4} + M_2. \tag{4.19}$$

It follows from assertion (iv) of Lemma 4.2 that the vertex $(\mathcal{E}_+, \mathcal{E})$ of P lies in the half-plane $x_1 \ge n$, so using assertion (i) and (4.3), we get $\mathcal{E}_+ \le \mathcal{E} - 1 \le 2n-2$. Moreover, $M_2 \le 1$ and $s_2^2 - 3s_2 \ge -2$, since s_2 is an integer, so from (4.19) we obtain

$$-\frac{\mathcal{N}}{2} + s_2 + \mathcal{E}_- + 2M_2 \le n + \frac{1}{2}.$$

Combining this with (4.18), we get

$$2N \le 4n + 5 - 4b.$$

Dividing both sides by 2 and taking floor, we obtain (4.7). ∎

Lemma 4.4. *Suppose that P is a type IV_n N-gon, the segment $[(0,-n),(n,0)]$ splits P, the vertices of P belong to a $(1,m)$-lattice Γ, where m divides n, and Γ has small $\mathbf{e}_1$-step and large $\mathbf{e}_2$-step greater than or equal to 2. Then*

$$N \le 2n-2.$$

Proof: Applying Proposition 2.7 to the basis $(-\mathbf{e}_1, \mathbf{e}_2)$ and the slope Q_1 and to the basis $(-\mathbf{e}_1, -\mathbf{e}_2)$ and the slope Q_2, we obtain

$$2N_1 \le \mathcal{E} - \mathcal{S}_+, \tag{4.20}$$

$$2N_2 \le \mathcal{E} - \mathcal{N}_+. \tag{4.21}$$

Applying Theorem 2.13 to the frame $((n,n), \mathbf{e}_1, -\mathbf{e}_2)$ and the slope Q_3 and to the frame $(\mathbf{0}, \mathbf{e}_1, -\mathbf{e}_2)$ and the slope Q_4, we obtain

$$2N_3 \le \mathcal{N}_- - \mathcal{W}_+ - 1, \tag{4.22}$$

$$2N_4 \le \mathcal{S}_- + \mathcal{W}_- - 1. \tag{4.23}$$

By Proposition 2.17,

$$\mathcal{S}_+ - \mathcal{S}_- \geq 2M_1, \tag{4.24}$$
$$\mathcal{E}_+ - \mathcal{E}_- \geq 2M_2, \tag{4.25}$$
$$\mathcal{N}_+ - \mathcal{N}_- \geq 2M_3, \tag{4.26}$$
$$\mathcal{W}_+ - \mathcal{W}_- \geq 2M_4. \tag{4.27}$$

Observe that $n\mathbb{Z}^2$ is a sublattice of Γ. Indeed, $n\mathbb{Z}^2$ is a sublattice of $\mathbb{Z}\times m\mathbb{Z}$, since m divides n, and the unimodular transformation mapping $\mathbb{Z}\times m\mathbb{Z}$ onto Γ maps $n\mathbb{Z}^2$ onto itself. Thus, the point $(2n, 0)$ belongs to Γ. So does the vertex $(\mathcal{E}, \mathcal{E}_-)$. Therefore, the small $\mathbf{e}_1$-step of Γ divides the difference $2n-\mathcal{E}$, which is positive by (4.3). Consequently, we obtain

$$\mathcal{E} \leq 2n - 2. \tag{4.28}$$

Now we use the above inequalities to estimate N. Summing (4.20)–(4.23) and subsequently using (4.24)–(4.27) and (4.28), we obtain

$$\begin{aligned} 2N = \sum_{k=1}^{4} 2N_k + \sum_{k=1}^{4} 2M_k &\leq 2\mathcal{E} + (2M_1 - (\mathcal{S}_+ - \mathcal{S}_-)) \\ &+ (2M_3 - (\mathcal{N}_+ - \mathcal{N}_-)) + (2M_4 - (\mathcal{W}_+ - \mathcal{W}_-)) + 2(M_2 - 1) \\ &\leq 2\mathcal{E} \leq 2(2n-2), \end{aligned}$$

and the lemma follows. ∎

Lemma 4.5. *Let $n \geq 3$ be an integer and Q be a slope with N edges with respect to the basis $(-\mathbf{e}_1, \mathbf{e}_2)$. Suppose that the vertices of Q belong to a lattice Γ spanned by $\mathbf{e}_1 + a\mathbf{e}_2$ and $n\mathbf{e}_2$, where a is an integer. Suppose that the frame $((n, 0); -\mathbf{e}_1, \mathbf{e}_2)$ splits Q, the endpoints $\mathbf{v} = (v_1, v_2)$ and $\mathbf{w} = (w_1, w_2)$ of Q satisfy*

$$v_1 < n,\ v_2 < 0,\ n < w_1 \leq 2n-1,\ w_2 > 0, \tag{4.29}$$

the intersection of Q with the half-plane $x_1 \geq n$ lies in the slab

$$-n < x_2 - x_1 < 0, \tag{4.30}$$

and Q has a vertex $\mathbf{u} = (u_1, u_2)$ satisfying

$$u_2 - u_1 \leq -n - 1. \tag{4.31}$$

Then

$$2N \leq 2n - 1 - v_1. \tag{4.32}$$

Set $\mathbf{f}_1 = -\mathbf{e}_1$, $\mathbf{f}_2 = -\mathbf{e}_2$, so that Q_2 is a slope with respect to $(\mathbf{f}_1, \mathbf{f}_2)$. By hypothesis, the vectors $\mathbf{b}_1 = -\mathbf{f}_1 - a\mathbf{f}_2$, $\mathbf{b}_2 = -n\mathbf{f}_2$ form a basis of Γ. By Proposition 2.1, the vectors $\mathbf{f}_1 - (n-a)\mathbf{f}_2 = -\mathbf{b}_1 + \mathbf{b}_2$ and $n\mathbf{f}_2 = -\mathbf{b}_2$ form a basis of Γ as well, and as $1 \leq n - a \leq n - 1$, we apply Proposition 2.7 and conclude that there exists an integer s_2 such that

$$2N_2 \leq \mathcal{E} - \mathcal{N}_+ + s_2, \tag{4.49}$$
$$\mathcal{N} - \mathcal{E}_+ \geq \frac{2(n-a) + (s_2-1)n}{2} s_2. \tag{4.50}$$

As $n - a \geq 1$, (4.50) implies

$$\mathcal{N} - \mathcal{E}_+ \geq \frac{2 + (s_2-1)n}{2} s_2. \tag{4.51}$$

Applying Theorem 2.13 to the slope Q_3 and the frame $((n, n), \mathbf{e}_1, -\mathbf{e}_2)$ and to the slope Q_4 and the frame $(\mathbf{0}; \mathbf{e}_2, \mathbf{e}_1)$, we obtain

$$2N_3 \leq \mathcal{N}_- - \mathcal{W}_+ - 1, \tag{4.52}$$
$$2N_4 \leq \mathcal{S}_- + \mathcal{W}_- - 1. \tag{4.53}$$

By assertion (vi) of Lemma 4.2, the points $(\mathcal{E}, \mathcal{E}_-)$, $(\mathcal{E}, \mathcal{E}_+)$, and $(\mathcal{N}_+, \mathcal{N})$ lie in the half-plane $x_1 \geq n$, so by assertion (v) of the same lemma we have

$$-n + 1 \leq \mathcal{E}_- - \mathcal{E} \leq -1, \tag{4.54}$$
$$-n + 1 \leq \mathcal{E}_+ - \mathcal{E} \leq -1, \tag{4.55}$$
$$-n + 1 \leq \mathcal{N} - \mathcal{N}_+ \leq -1. \tag{4.56}$$

From (4.3) we also have

$$\mathcal{E} \leq 2n - 1. \tag{4.57}$$

It is clear that $\mathbf{e}_1 \notin \Gamma$, so the large $\mathbf{e}_1$-step of Γ cannot be less then 2. It is easily seen that the large $\mathbf{e}_2$-step of Γ equals n. By Proposition 2.17, we get

$$\mathcal{S}_+ - \mathcal{S}_- \geq 2M_1, \tag{4.58}$$
$$\mathcal{E}_+ - \mathcal{E}_- \geq nM_1, \tag{4.59}$$
$$\mathcal{N}_+ - \mathcal{N}_- \geq 2M_3, \tag{4.60}$$
$$\mathcal{W}_+ - \mathcal{W}_- \geq nM_4. \tag{4.61}$$

Now we deduce a few implications of the inequalities.

Inequalities (4.54) and (4.55) yield

$$\mathcal{E}_+ - \mathcal{E}_- \leq n - 2.$$

the above we have $a_{N2}/a_{N1} \le 2$ and the required estimate follows from (4.35). On the other hand, if $\varepsilon_N \in E^{(2)}$, then $i' = N$ and by virtue of (4.41) we have

$$v_{i'-1,1} = v_{N1} - 2 = 3,$$

i. e. (4.44) holds, and the required estimate follows from (4.45). Assertion (a) is proved.

Assertion (b) is a corollary of (a), given that by virtue of (4.35), the ratio a_{i2}/a_{i1} cannot attain its maximum at $i < N$.

By hypothesis, the slope Q has a vertex (u_1, u_2) satisfying $u_2 < u_1 - n = u_1 - 3$. Set

$$i_0 = \max\{i\colon v_{i2} < v_{i1} - 3\}.$$

Observe that *a priori* $i_0 < N$. Then we have $v_{i_0+1,2} \ge v_{i_0+1,1} - 3$ and therefore,

$$\frac{a_{i_0+1,2}}{a_{i_0+1,1}} = \frac{v_{i_0+1,2} - v_{i_0,2}}{a_{i_0+1,1}} > \frac{(v_{i_0+1,1} - 3) - (v_{i_0,1} - 3)}{a_{i_0+1,1}} = \frac{v_{i_0+1,1} - v_{i_0,1}}{a_{i_0+1,1}} = 1.$$

We must have $v_{i_0} \le 2$, because otherwise we would have $v_{i_0} \ge 3 = n$, and by hypothesis, $v_{i_0 2} < v_{i_0 1} - 3$, contrary to the definition of i_0. Thus,

$$v_{i_0+1,1} = v_{i_0 1} + a_{i_0+1,1} \le 4.$$

This and (4.41) imply $i_0 + 1 < N$. Then assertion (b) and the inequality $a_{i_0+1,2}/a_{i_0+1,1} > 1$ proved above yield $a_{i_0+1,2}/a_{i_0+1,1} = 3/2$, which is only possible if $\mathbf{a}_{i_0+1} = (2, 3)$.

Let $\varepsilon_{i_1} \in E^{(1)}$, then by assertion (a), we have either $\mathbf{a}_{i1} = (1, 1)$ or $\mathbf{a}_{i1} = (1, 2)$. By Proposition 2.1, in both cases the vectors $\mathbf{a}_{i_1}$ and $\mathbf{a}_{i_0+1}$ form a basis of $\mathbb{Z}^2$, which is impossible, since they belong to its proper sublattice Γ. The contradiction proves the lemma in the case $n = 3$. ∎

Lemma 4.6. *Suppose that P is a type IV_n N-gon, the segment $[(0, -n), (n, 0)]$ splits P, and the vertices of P belong to a lattice $\Gamma \subset \mathbb{Z}^2$ having the basis $(\mathbf{e}_1 + a\mathbf{e}_2,\ n\mathbf{e}_2)$, $1 \le a \le n - 1$. Then*

$$N \le 2n - 2. \tag{4.47}$$

Proof: Lemma 4.2 ensures that we can apply Lemma 4.5 to the slope Q_1 and obtain

$$2N_1 \le 2n - 1 - \mathcal{S}_+. \tag{4.48}$$

Proof: Let $\mathbf{v}_0 = \mathbf{v}$, $\mathbf{v}_1$, …, $\mathbf{v}_N = \mathbf{w}$ be consecutive vertices of Q, $\varepsilon_i = [\mathbf{v}_{i-1}, \mathbf{v}_i]$ and $\mathbf{a}_i = \mathbf{v}_i - \mathbf{v}_{i-1}$ be the edges and their associated vectors, where $i = 1, \dots, N$, and let $\mathbf{v}_i = (v_{i1}, v_{i2})$, $\mathbf{a}_i = (a_{i1}, a_{i2})$. Of course, v_{ij} and a_{ij} are integers. It follows from the definition of a slope that

$$a_{i1} \ge 1 \qquad (i = 1, \dots, N); \tag{4.33}$$

$$a_{i2} \ge 1 \qquad (i = 1, \dots, N); \tag{4.34}$$

$$\frac{a_{12}}{a_{11}} < \frac{a_{22}}{a_{21}} < \dots < \frac{a_{N2}}{a_{N1}}; \tag{4.35}$$

$$v_{01} < v_{11} < \dots < v_{N1}. \tag{4.36}$$

Set

$$E^{(j)} = \{\varepsilon_i\colon a_{i1} = j\},\ N^{(j)} = |E^{(j)}| \qquad (j = 1, 2, \dots).$$

Of course, $E^{(j)} \ne \varnothing$ and $N^{(j)} \ne 0$ for finitely many j. As the vectors $\mathbf{a}_i$ are distinct, we have

$$N = \sum_{j=1}^{\infty} N^{(j)}, \tag{4.37}$$

$$w_1 - v_1 = \sum_{i=1}^{N} a_{i1} = \sum_{j=1}^{\infty} jN^{(j)}. \tag{4.38}$$

We claim that

$$N^{(1)} \le 1. \tag{4.39}$$

It follows from (4.29) that $v_{N1} \ge n + 1$ and taking into account (4.33), we also obtain $v_{N-1,1} = v_{N1} - a_{N1} \le v_N - 1$. Therefore, there exists a point $\mathbf{y} = (y_1, y_2) \in \varepsilon_N$ such that $y_1 = v_{N1} - 1 \ge n$. Then

$$\frac{a_{N2}}{a_{N1}} = \frac{v_{N2} - y_2}{v_{N1} - y_1} = v_{N2} - y_2.$$

As the points $\mathbf{v}_N$ and $\mathbf{y}$ lie in the half-plane $x_1 \ge n$, they satisfy

$$-n < y_2 - y_1 < 0, \qquad -n < v_{N2} - v_{N1} < 0,$$

whence $v_{N2} \le v_{N1} - 1$ and

$$\frac{a_{N2}}{a_{N1}} < (v_{N1} - 1) - (y_1 - n) = n + (v_{N1} - y_1) - 1 = n.$$

Thus, according to (4.35),

$$a_{i2} = \frac{a_{i2}}{a_{i1}} < n \qquad (\varepsilon_i \in E^{(1)}). \tag{4.40}$$

As $\mathbf{a}_i \in \Gamma$, it is easily seen that possible values for $\mathbf{a}_i$ corresponding to $\varepsilon_i \in E^{(1)}$ belong to the set $\{\mathbf{e}_1 + (a + pn)\mathbf{e}_2 \colon p \in \mathbb{Z}\}$. Only one vector of this set satisfies both (4.34) and (4.40). As the vectors $\mathbf{a}_i$ are distinct, we conclude that $E^{(1)}$ contains at most one edge, and (4.33) follows.

Having established all these auxiliary facts, we start proving (4.32). Assuming the converse, we have

$$2n - v_1 - 2N \le 0,$$

or, equivalently,

$$(2n - 1 - w_1) + (1 - N^{(1)}) + \sum_{j=3}^{\infty} (j-2)N^{(j)} \le 0,$$

where we have used (4.37) and (4.38). In view of (4.29) and (4.40), the three summands on the left-hand side are nonnegative. Consequently, we obtain

$$w_1 = 2n - 1, \tag{4.41}$$

$$N^{(1)} = 1, \tag{4.42}$$

$$N^{(j)} = 0, \qquad (j = 3, 4, \ldots) \tag{4.43}$$

We must have $E^{(2)} \ne \varnothing$, for otherwise $N^{(2)} = 0$ and (4.42), (4.43), and (4.38) would give $w_1 - v_1 = 1$, which together with (4.41) implies

$$v_1 = w_1 - 1 = 2n - 2 > n,$$

in contradiction to (4.29). Set

$$i' = \max\{i \colon \varepsilon_i \in E^{(2)}\}.$$

Let us show that if

$$v_{i'-1,1} \ge n, \tag{4.44}$$

then

$$a_{i2} \le n \qquad (\varepsilon_i \in E^{(2)}). \tag{4.45}$$

Indeed, if (4.44) holds, according to (4.36) we have $n \le v_{i'-1,1} < v_{i'1}$, so the vertices $\mathbf{v}_{i'-1}$ and $\mathbf{v}_{i'}$ belong to the slab (4.30), whence

$$-n + 1 \le v_{i'-1,2} - v_{i'-1,1} \le -1,$$
$$-n + 1 \le v_{i'2} - v_{i'1} \le -1,$$

Then

$$\frac{a_{i'2}}{a_{i'1}} = \frac{v_{i'2} - v_{i'-1,2}}{2} \le \frac{(v_{i'1} - 1) - (v_{i'-1,1} - n + 1)}{2} = \frac{a_{i'1} - 2 + n}{2} = \frac{n}{2}.$$

This and (4.35) imply (4.45).

Assume that $n \ge 4$. Let us show that in this case

$$N^{(2)} = 1. \tag{4.46}$$

To this end let us estimate $v_{i'-1,1}$. If $\varepsilon_N \in E^{(2)}$, we have $i' = N$ and by virtue of (4.41) we get

$$v_{i'-1,1} = v_{N1} - a_{N1} = 2n - 3 \ge n.$$

Otherwise, $\varepsilon_N \in E^{(1)}$, then $\varepsilon_{N-1} \in E^{(2)}$, since $|E^{(1)}| = 1$. Moreover, $i' = N-1$ and

$$v_{N-1,1} = v_{N1} - a_{N1} = 2n - 2,$$
$$v_{i'-1,1} = v_{N-1,1} - a_{N-1,1} = 2n - 4 \ge n.$$

Thus, in any case we have (4.44), so (4.45) holds.

As the vectors $\mathbf{a}_i$ associated with edges from $E^{(2)}$ belong to Γ, it is easily seen that they have the form $\mathbf{a}_i = (2, 2a + pn)$, where p is an integer. Clearly, only one vector of this form satisfies $1 \le a_{i2} \le n$ and as the vectors $\mathbf{a}_i$ are distinct, we see that $E^{(2)}$ contains at most one edge. Thus, (4.46) is proved.

From (4.38), (4.42), (4.43), and (4.46) it follows that

$$w_1 - v_1 = 3,$$

and according to (4.41),

$$v_1 = w_1 - 3 = 2n - 4 \ge n,$$

which contradicts (4.29). The contradiction proves the lemma for $n \ge 4$.

Now assume that $n = 3$. Let us prove the following assertions:

(a) The numbers a_{i2}/a_{i1} $(i = 1, \ldots, N)$ are positive integers or half-integers not exceeding 2.

(b) The numbers a_{i2}/a_{i1} $(i = 1, \ldots, N-1)$ are positive integers or half-integers not exceeding 3/2.

Let us show (a). Fix $i \in \{1, \ldots, N\}$. If $\varepsilon_i \in E^{(1)}$, then $a_{i2}/a_{i1} = a_{i2}$ is a positive integer (according to (4.33)) not exceeding 2 according to (4.40). Assume that $\varepsilon_i \in E^{(2)}$, then $a_{i2}/a_{i1} = a_{i2}/2$ is positive and either an integer or a half-integer. Let us show that it cannot be greater then 2. If $\varepsilon_N \in E^{(1)}$, by