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PERFECTNESS IN LOCALES

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Dedicated to Professor Bernhard Banaschewski on the occasion of his 90th birthday

ABSTRACT: This paper makes a comparison between two notions of perfectness for locales which come as direct reformulations of the two equivalent topological definitions of perfectness. These reformulations are no longer equivalent. It will be documented that a locale may appropriately be called *perfect* if each of its open sublocales is a join of countably many closed sublocales. Certain circumstances are exhibited in which both reformulations coincide. This paper also studies perfectness in mildly normal locales. It is shown that perfect and mildly normal locales coincide with the Oz locales extensively studied in the last decade.

KEYWORDS: Locale, sublocale, F_σ-sublocale, G_δ-sublocale, normality, mild normality, perfectness, perfect normality, pm-normality, Oz locale.

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1. Introduction

In this paper we look for the extension to the pointfree setting of what in topology is called perfectness. We recall that a topological space is called *perfect* if each open set is a union of countably many closed sets, i.e. open sets are F_σ. This is equivalent to the statement that each closed set is an intersection of countably many open sets, i.e. closed sets are G_δ. The two equivalent formulations of perfectness for spaces have direct reformulations for locales in terms of open sublocales and closed sublocales. The two resulting concepts, which will be called *F_σ-perfectness* and *G_δ-perfectness*, are no longer equivalent, for G_δ-perfectness is generally stronger than F_σ-perfectness.

The first purpose of this paper is to compare those two non-equivalent concepts. One nice feature of F_σ-perfectness is that the locale $\mathcal{O}X$ of all open sets of an arbitrary perfect space X is *always* F_σ-perfect, but may fail to be G_δ-perfect (Section 3); another one is that it behaves nicely with respect to closed maps (Section 5). This shows that, with respect to the criterion of conservativeness, F_σ-perfectness behaves much better than G_δ-perfectness. Moreover, F_σ-perfectness will be shown to be conservative in a quite large class of spaces (containing all

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T_1-spaces). In the class of normal locales, F_σ-perfectness and G_δ-perfectness coincide, and are conservative concepts for the class of perfect T_0-spaces. Due to all those circumstances we eventually drop the prefix F_σ- and call a locale *perfect* if each its open sublocale is a join of countably many closed sublocales.

The second purpose of this paper is to study perfectness in mildly normal locales (Section 4). One interesting observation is that perfect and mildly normal locales coincide with the so-called Oz locales extensively studied in the last decade (cf. [1, 2, 4, 5]).

2. Preliminaries on locales

For general background regarding locales and frames we refer to [11] or [14]. Here, we present a brief outline of the facts specifically needed for the paper.

A *locale* or a *frame* is a complete lattice L in which

$$a \wedge \bigvee B = \bigvee\{a \wedge b \colon b \in B\}$$

for all $a \in L$ and $B \subseteq L$. The topology of a topological space X is a locale and is denoted by $\mathcal{O}(X)$. Being a Heyting algebra, each locale L has the implication operator

$$a \to b = \bigvee\{x \in L \mid x \wedge a \leq b\}$$

satisfying the standard equivalence $c \wedge a \leq b$ iff $c \leq a \to b$. The *pseudocomplement* of an $a \in L$ is the element $a^* = a \to 0$. An element a is *regular* if $a^{**} = a$ (equivalently, if $a = b^*$ for some $b \in L$). Note that the first De Morgan law $(a \vee b)^* = a^* \wedge b^*$ holds in any locale (actually, more generally, $(\bigvee A)^* = \bigwedge_{a \in A} a^*$).

For any elements a and b in L, $a \prec b$ (a is *well inside* b) means that $a \wedge x = 0$ and $b \vee x = 1$ for some $x \in L$ (equivalently, $a^* \vee b = 1$).

A *sublocale* S of a locale L is a subset $S \subseteq L$ such that:

(S1) for every $A \subseteq S$, $\bigwedge A$ is in S, and
(S2) for every $s \in S$ and every $x \in L$, $x \to s$ is in S.

The set $\mathcal{S}(L)$ of all sublocales of L forms a *co-frame* (i.e., the dual lattice is a frame) under inclusion, in which arbitrary infima coincide with intersections. Regarding suprema, there is the formula

$$\bigvee_{i \in I} S_i = \{\bigwedge A \colon A \subseteq \bigcup_{i \in I} S_i\}$$

for every $\{S_i \in \mathcal{S}(L) \colon i \in I\}$.

Since $\mathcal{S}(L)$ is the dual of a complete Heyting algebra, it has co-pseudocomplements, given by the formula

$$S^{\#} = L \smallsetminus S = \bigcap\{T \in \mathcal{S}(L) \mid S \vee T = L\}.$$

Note that

$$\Big(\bigcap_{i\in I} S_i\Big)^{\#} = \bigvee_{i\in I} S_i^{\#}, \qquad \{S_i\}_{i\in I} \subseteq \mathcal{S}(L). \tag{\#}$$

For each $a \in L$, the sublocales

$$\mathfrak{c}(a) = {\uparrow}a \quad \text{and} \quad \mathfrak{o}(a) = \{a \to b \mid b \in L\}$$

are the *closed* and *open* sublocales of L induced by a, respectively. We summarize here the basic properties of sublocales used throughout the paper:

(P1) *For every $a \in L$, $\mathfrak{c}(a)$ and $\mathfrak{o}(a)$ are complements of each other in $\mathcal{S}(L)$.*
(P2) *For every $a, b \in L$, $\mathfrak{c}(b) \subseteq \mathfrak{o}(a)$ if and only if $a \vee b = 1$ and $\mathfrak{c}(b) \supseteq \mathfrak{o}(a)$ iff $a \wedge b = 0$.*
(P3) *For every $A \subseteq L$, $\bigvee_{a\in A} \mathfrak{o}(a) = \mathfrak{o}(\bigvee A)$ and $\bigcap_{a\in A} \mathfrak{c}(a) = \mathfrak{c}(\bigvee A)$.*

3. A comparison between F_σ-perfectness and G_δ-perfectness

Definition 3.1. A locale L is said to be:

(1) *F_σ-perfect* whenever any open sublocale of L is an F_σ-sublocale, that is, for each $a \in L$ there exists a countable family $(a_n)_{n\in\mathbb{N}}$ in L such that

$$\mathfrak{o}(a) = \bigvee_{n\in\mathbb{N}} \mathfrak{c}(a_n).$$

(2) *G_δ-perfect* if any closed sublocale is a G_δ-sublocale, that is, for each $a \in L$ there exists a countable family $(a_n)_{n\in\mathbb{N}}$ in L such that

$$\mathfrak{c}(a) = \bigcap_{n\in\mathbb{N}} \mathfrak{o}(a_n).$$

By [14, Proposition V.1.4], each F_σ-perfect locale is subfit, and by [14, Proposition V.1.3.2], each G_δ-perfect locale is fit. We recall that a locale is subfit (resp. fit) if every open (resp. closed) sublocale is a join of closed sublocales (resp. meet of open sublocales).

Since $\mathcal{S}(L)$ is no longer a (complete) Boolean algebra, it is not surprising that these two concepts are not equivalent, in general. More specifically, by (#) and (P1) we have:

Remark 3.2. *Each G_δ-perfect locale is F_σ-perfect.*

However, the converse is far from being true. The following example shows that a F_σ-perfect locale need not even be fit.

Example 3.3. Let $\mathbb{N}$ be endowed with the cofinite topology

$$\mathcal{O}\mathbb{N} = \{\varnothing\} \cup \{U \subseteq \mathbb{N} \mid \mathbb{N} \smallsetminus U \text{ finite}\}.$$

For each $U, V \in \mathcal{O}\mathbb{N}$ we have

$$U \to V = \begin{cases} \mathbb{N}, & \text{if } U = \varnothing; \\ \operatorname{Int}(\mathbb{N} \smallsetminus U) = \varnothing, & \text{if } U \neq \varnothing = V; \\ \operatorname{Int}((\mathbb{N} \smallsetminus U) \cup V) = (\mathbb{N} \smallsetminus U) \cup V, & \text{if } U, V \neq \varnothing. \end{cases}$$

Hence

$$U \to V = V \iff (U = \varnothing \text{ and } V = \mathbb{N}) \text{ or } (U \neq \varnothing = V) \text{ or } (U \neq \varnothing \text{ and } \mathbb{N} \smallsetminus U \subseteq V).$$

Consequently, $\mathfrak{o}(\varnothing) = \{\mathbb{N}\}$ and, for each $\varnothing \neq U \in \mathcal{O}\mathbb{N}$,

$$\mathfrak{o}(U) = \{\varnothing\} \cup \{V \in \mathcal{O}\mathbb{N} \mid \mathbb{N} \smallsetminus U \subseteq V\} = \{\varnothing\} \cup \{V \in \mathcal{O}\mathbb{N} \mid U \cup V = \mathbb{N}\}.$$

We shall now prove that

$$\mathfrak{o}(U) = \bigvee_{n \in U} \mathfrak{c}(\mathbb{N} \smallsetminus \{n\}),$$

for every $U \in \mathcal{O}\mathbb{N}$. For $U = \varnothing$ this is trivial since $\mathfrak{o}(\varnothing) = \{\mathbb{N}\}$ is the bottom element of $\mathcal{S}(\mathcal{O}\mathbb{N})$. Further, let $\varnothing \neq U \in \mathcal{O}\mathbb{N}$ and $n \in U$. Then

$$\mathfrak{c}(\mathbb{N} \smallsetminus \{n\}) = \{\mathbb{N}, \mathbb{N} \smallsetminus \{n\}\} \subseteq \mathfrak{o}(U)$$

and $\bigvee_{n \in U} \mathfrak{c}(\mathbb{N} \smallsetminus \{n\}) \subseteq \mathfrak{o}(U)$. Conversely, we first notice that

$$\varnothing = \operatorname{Int}(\mathbb{N} \smallsetminus U) = \operatorname{Int}\Big(\bigcap_{n \in U} (\mathbb{N} \smallsetminus \{n\})\Big) = \bigwedge_{n \in U} (\mathbb{N} \smallsetminus \{n\}) \in \bigvee_{n \in U} \mathfrak{c}(\mathbb{N} \smallsetminus \{n\}).$$

Finally, for each $\varnothing \neq V \in \mathfrak{o}(U)$ we have

$$V = \operatorname{Int} V = \operatorname{Int}\Big(\bigcap_{n \in \mathbb{N} \smallsetminus V} (\mathbb{N} \smallsetminus \{n\})\Big) = \bigwedge_{n \in \mathbb{N} \smallsetminus V} (\mathbb{N} \smallsetminus \{n\}) \in \bigvee_{n \in U} \mathfrak{c}(\mathbb{N} \smallsetminus \{n\}).$$

We conclude that $\mathfrak{o}(U) \subseteq \bigvee_{n \in U} \mathfrak{c}(\mathbb{N} \smallsetminus \{n\})$. Hence $\mathfrak{o}(U) = \bigvee_{n \in U} \mathfrak{c}(\mathbb{N} \smallsetminus \{n\})$ is an F_σ-sublocale which shows that $\mathcal{O}\mathbb{N}$ is an F_σ-perfect locale.

On the other hand, the only closed sublocales of $\mathcal{O}\mathbb{N}$ which are meets of open sublocales are $\mathfrak{c}(\mathbb{N})$ and $\mathfrak{c}(\varnothing)$ and thus $\mathcal{O}\mathbb{N}$ is not fit, hence neither G_δ-perfect.

Let us recall that a localic property LP is a *conservative extension* of a topological property P if, given a topological space X, the locale $\mathcal{O}(X)$ has property LP if and only if X has property P.

Since the space $(\mathbb{N}, \mathcal{O}\mathbb{N})$ is perfect (as any countable T_1-space does), G_δ-perfectness is not a conservative extension of topological perfectness. Unlike G_δ-perfectness, the following holds:

Proposition 3.4. *If a space X is perfect, then $\mathcal{O}X$ is F_σ-perfect.*

Proof: Let $U \in \mathcal{O}X$. By hypothesis, there exists a countable family $(U_n)_{n\in\mathbb{N}}$ in $\mathcal{O}X$ such that $U = \bigcup_{n\in\mathbb{N}}(X \smallsetminus U_n)$. It follows that $U_n \cup U = X$ for each $n \in \mathbb{N}$ and thus $\bigvee_{n\in\mathbb{N}} \mathfrak{c}(U_n) \subseteq \mathfrak{o}(U)$ by (P2). On the other hand, let $V \in \mathfrak{o}(U)$ and $V_n = U_n \cup V \in \mathfrak{c}(U_n)$ for each $n \in \mathbb{N}$, then

$$V = U \to V = \operatorname{Int}((X \smallsetminus U) \cup V) = \operatorname{Int}((\bigcap_{n\in\mathbb{N}} U_n) \cup V) = \operatorname{Int}(\bigcap_{n\in\mathbb{N}} (U_n \cup V))$$
$$= \operatorname{Int}(\bigcap_{n\in\mathbb{N}} V_n) = \bigwedge_{n\in\mathbb{N}} V_n \in \bigvee_{n\in\mathbb{N}} \mathfrak{c}(U_n).$$

Hence $\mathfrak{o}(U) \subseteq \bigvee_{n\in\mathbb{N}} \mathfrak{c}(U_n)$ and we conclude that $\mathcal{O}X$ is an F_σ-perfect locale. ■

The converse implication is not true in general, as shown by the following example:

Example 3.5. Let X be a T_1 topological space, $\infty \notin X$, $Y = X \cup \{\infty\}$ and

$$\mathcal{O}Y = \{\varnothing\} \cup \{U \cup \{\infty\} \mid \varnothing \neq U \in \mathcal{O}X\}.$$

$\mathcal{O}Y$ is a topology in Y if and only if $\varnothing$ is meet-irreducible in $\mathcal{O}X$. In this case $\mathcal{O}Y$ and $\mathcal{O}X$ are clearly isomorphic locales and consequently $\mathcal{O}Y$ is an F_σ-perfect locale if and only if so is $\mathcal{O}X$. Moreover, the space Y is perfect if and only if X is indiscrete. Consequently, if X is a non-indiscrete and perfect topological space such that $\varnothing$ is meet-irreducible in $\mathcal{O}X$ (e.g. $\mathbb{N}$ endowed with the cofinite topology), it follows from Proposition 3.4 that $\mathcal{O}X$ is an F_σ-perfect locale and hence so is $\mathcal{O}Y$. However, Y fails to be perfect.

This construction of space Y is related with space Σ in [6, Problem 4M, page 64]. Note that Σ is completely normal and extremally disconnected. See also [6, Problem 6R, page 98] and [14, Example 3.1].

The point of this example is that Y fails to be T_D. Recall that a space X is T_D if $(X \setminus \overline{\{x\}}) \cup \{x\}$ is open for each $x \in X$. The space Y in the previous example is clearly T_0, but fails to be T_D since $(Y \setminus \overline{\{\infty\}}) \cup \{\infty\} = \{\infty\}$ is not open in Y. However, if we restrict ourselves to T_D-spaces we have the following result:

Proposition 3.6. *Let X be a T_D-space. Then X is perfect if and only if $\mathcal{O}X$ is F_σ-perfect.*

Proof: Let $U \in \mathcal{O}X$. By hypothesis, there exists a countable family $(U_n)_{n\in\mathbb{N}}$ in $\mathcal{O}X$ such that $\mathfrak{o}(U) = \bigvee_{n\in\mathbb{N}} \mathfrak{c}(U_n)$. Then $\mathfrak{c}(U_n) \subseteq \mathfrak{o}(U)$ for each $n \in \mathbb{N}$ and so it follows from property (P2) that $U \cup U_n = X$ for each $n \in \mathbb{N}$. Consequently, $\bigcup_{n\in\mathbb{N}}(X \smallsetminus U_n) \subseteq U$.
On the other hand, let $x \in U$. Since X is T_D it follows that there exists an open $V \ni x$ such that $W = V \smallsetminus \{x\}$ is open as well. We have that

$$U \to W \in \mathfrak{o}(U) = \bigvee_{n\in\mathbb{N}} \mathfrak{c}(U_n)$$

and so there exists a countable family $(V_n)_{n\in\mathbb{N}}$ in $\mathcal{O}X$ such that $U_n \subseteq V_n$ for each $n \in \mathbb{N}$ and

$$U \to W = \bigwedge_{n\in\mathbb{N}} V_n = \mathrm{Int}\big(\bigcap_{n\in\mathbb{N}} V_n\big).$$

Since $x \in U \cap V$ it follows that $U \cap V \not\subseteq W$ and thus $V \not\subseteq U \to W$, from which it follows that $x \notin U \to W$. Hence

$$x \in X \smallsetminus \mathrm{Int}\big(\bigcap_{n\in\mathbb{N}} V_n\big) = \overline{\bigcup_{n\in\mathbb{N}}(X \smallsetminus V_n)}.$$

Since V is an open neighborhood of x it follows that $V \cap \big(\bigcup_{n\in\mathbb{N}}(X \smallsetminus V_n)\big) \neq \varnothing$. But $U \to W \subseteq \bigcap_{n\in\mathbb{N}} V_n$. Hence

$$x \in \bigcup_{n\in\mathbb{N}}(X \smallsetminus V_n) \subseteq \bigcup_{n\in\mathbb{N}}(X \smallsetminus U_n). \qquad \blacksquare$$

In conclusion, F_σ-perfect locales model perfect spaces with the same proviso as in [14, III.7.2.1 (2) and III.7.3.1 (1)], that is, inside the class of T_D-spaces.

Recall that a locale L is *normal* if $a \vee b = 1$ implies that $a \vee u = 1 = b \vee v$ for some $u, v \in L$ satisfying $u \wedge v = 0$. It follows from [8, Proposition 3.5] that the classes of F_σ-perfect locales and G_δ-perfect locales coincide under normality. We include a direct proof here for the sake of completeness.

Proposition 3.7. *A normal locale is F_σ-perfect if and only if it is G_δ-perfect.*

Proof: We only need to prove necessity. Let L be a normal F_σ-perfect locale and $a \in L$. By hypothesis there exists a countable family $(a_n)_{n\in\mathbb{N}}$ in L such that $\mathfrak{o}(a) = \bigvee_{n\in\mathbb{N}} \mathfrak{c}(a_n)$. By (P2), $a \vee a_n = 1$ for each $n \in \mathbb{N}$. Now, the normality of L provides $u_n, v_n \in L$ such that

$$a \vee u_n = 1 = a_n \vee v_n \quad \text{and} \quad u_n \wedge v_n = 0, \qquad n \in \mathbb{N}.$$

It follows by (P2) that $\mathfrak{c}(a_n) \subseteq \mathfrak{o}(v_n)$. Moreover, $v_n \prec a$ and therefore, by (P3),

$$\mathfrak{o}(a) = \bigvee_{n\in\mathbb{N}} \mathfrak{c}(a_n) \subseteq \bigvee_{n\in\mathbb{N}} \mathfrak{o}(v_n) = \mathfrak{o}\big(\bigvee_{n\in\mathbb{N}} v_n\big) \subseteq \mathfrak{o}(a).$$

Hence $a = \bigvee_{n\in\mathbb{N}} v_n$ with $v_n \prec a$ for each $n \in \mathbb{N}$. Finally, by (P2) and (P3),

$$\mathfrak{c}(a) \subseteq \bigcap_{n\in\mathbb{N}} \mathfrak{o}(u_n) \subseteq \bigcap_{n\in\mathbb{N}} \mathfrak{c}(v_n) = \mathfrak{c}(\bigvee_{n\in\mathbb{N}} v_n) = \mathfrak{c}(a). \qquad \blacksquare$$

After all these considerations we drop the prefix F_σ and introduce the following:

Definition 3.8. We call a locale *perfect* if each open sublocale is a join of countable many closed sublocales.

Then, we have the following (cf. [8, Propositions 3.5 and 4.2]):

Proposition 3.9. *The following are equivalent for any locale L:*

(1) *L is perfectly normal.*
(2) *L is a normal and perfect locale.*
(3) *For each $a \in L$ there is a countable family $(b_n)_{n\in\mathbb{N}}$ in L such that $a = \bigvee_{n\in\mathbb{N}} b_n$ and $b_n \prec a$ for all $n \in \mathbb{N}$.*

Remarks 3.10. (1) Perfect normality in pointfree topology was first considered by Charalambous [3] in the context of σ-frames. In [7], Gilmour observed that in the class of σ-frames perfect normality and regularity are equivalent concepts.

(2) Condition (3) was taken as the definition of a perfectly normal locale in [8]. In the terminology of [9], it says that every element in the locale is *regular-F_σ* (i.e., a countable join of elements well inside it). Note that, for any topological space X, the regular-F_σ elements of the locale $\mathcal{O}X$ consist exactly of the regular-F_σ subsets of X (the complements of the usual regular-G_δ subsets of X [13]). It should be also noted that in the definition of a regular-F_σ one may assume that each b_n is regular. Indeed, $b_n \prec a$ implies ${b_n}^{**} \prec a$ and hence

$$a = \bigvee_{n\in\mathbb{N}} b_n \leq \bigvee_{n\in\mathbb{N}} {b_n}^{**} \leq a.$$

(3) For each regular-F_σ element a, the closed sublocale $\mathfrak{c}(a)$ is a G_δ-sublocale (and therefore the open sublocale $\mathfrak{o}(a)$ is an F_σ-sublocale). Indeed, if $a = \bigvee_{n\in\mathbb{N}} b_n$ with ${b_n}^* \vee a = 1$ for each $n \in \mathbb{N}$ then by (P2) and (P3) we get

$$\mathfrak{c}(a) \subseteq \bigcap_{n\in\mathbb{N}} \mathfrak{o}({b_n}^*) \subseteq \bigcap_{n\in\mathbb{N}} \mathfrak{c}(b_n) = \mathfrak{c}(\bigvee_{n\in\mathbb{N}} b_n) = \mathfrak{c}(a).$$

It is also easy to check that if we add normality to Proposition 3.6, then we can conclude (under T_0) that pointfree perfect normality, as normality, is a conservative extension of the classical notion, that is, a T_0 topological space X is perfectly normal if and only if $\mathcal{O}X$ is perfectly normal:

Proposition 3.11. *Let X be a topological space.*

(1) *If X is perfectly normal, then $\mathcal{O}X$ is perfectly normal.*

(2) *If X is T_0, then $\mathcal{O}X$ is perfectly normal if and only if X is perfectly normal.*

Proof: (1) follows from Proposition 3.4. Regarding (2), we first note that if $\mathcal{O}X$ is perfect and normal, then it is subfit and thus, by [10, Lemma 2.4], it is a T_1 space (hence T_D). Finally, it follows from Proposition 3.6 that X is perfectly normal. ■

4. Variants of normality and Oz locales

Now recall that a locale L is *almost normal* (resp. *mildly normal*) if for any $a, b \in L$ satisfying $a \vee b = 1$, with a regular (resp. a and b regular), there exist $u, v \in L$ such that $u \wedge v = 0$ and $a \vee u = b \vee v = 1$ (note that it is redundant to impose here u and v to be regular since $u \wedge v = 0$ iff $u^{**} \wedge v^{**} = 0$).

We can now prove the following result which is directly related to Proposition 3.7:

Proposition 4.1. *Let L be a locale and let a be a regular element in L.*

(1) *If L is almost normal, then $\mathfrak{c}(a)$ is a G_δ-sublocale if and only if it is an F_σ-sublocale.*

(2) *If L is mildly normal, then $\mathfrak{c}(a) = \bigcap_{n\in\mathbb{N}} \mathfrak{o}(a_n)$, with all a_n regular, if and only if $\mathfrak{o}(a) = \bigvee_{n\in\mathbb{N}} \mathfrak{c}(a_n)$.*

Proof: In both cases the proof of sufficiency follows the lines of that of Proposition 3.7 replacing normality by almost and mild normality, respectively. ■

By Proposition 3.9, a frame L is perfectly normal if and only if any element in L is regular-F_σ. We say now that a locale L is *perfectly mildly normal* (or *pm-normal* for short) if any regular element in L is regular-F_σ. Hence, pm-normal locales are to perfectly normal locales the same as mildly normal locales are to normal locales. Note that all the variants of normality we have considered are conservative extensions of their topological counterparts.

Lemma 4.2. *Suppose that $a, b \in L$ satisfy $a \vee b = 1$ and that there exist two countable families $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ of regular elements such that*

$$\bigvee_{n\in\mathbb{N}} a_n \vee b = 1 = a \vee \bigvee_{n\in\mathbb{N}} b_n,$$

with $a_n \prec a$ and $b_n \prec b$ for every $n \in \mathbb{N}$. Then there exist $u, v \in L$ such that $u \wedge v = 0$ and $a \vee u = 1 = b \vee v$.

Proof: Let

$$u = \bigvee_{n\in\mathbb{N}} \Big(b_n \wedge \bigwedge_{i=1}^{n} a_i{}^*\Big) \quad \text{and} \quad v = \bigvee_{n\in\mathbb{N}} \Big(a_n \wedge \bigwedge_{i=1}^{n} b_i{}^*\Big).$$

Then

$$a \vee u = \bigvee_{n\in\mathbb{N}} \Big(a \vee \Big(b_n \wedge \bigwedge_{i=1}^{n} a_i{}^*\Big)\Big) = \bigvee_{n\in\mathbb{N}} \Big((a \vee b_n) \wedge \Big(\bigwedge_{i=1}^{n}(a \vee a_i{}^*)\Big)\Big) = \bigvee_{n\in\mathbb{N}} (a \vee b_n) = 1.$$

Similarly $b \vee v = 1$. On the other hand,

$$u \wedge v = \bigvee_{n\in\mathbb{N}} \bigvee_{m\in\mathbb{N}} \Big(b_n \wedge \bigwedge_{i=1}^{n} a_i{}^* \wedge a_m \wedge \bigwedge_{i=1}^{m} b_i{}^*\Big) = 0$$

since, for each pair of naturals n, m,

$$b_n \wedge \bigwedge_{i=1}^{n} a_i{}^* \wedge a_m \wedge \bigwedge_{i=1}^{m} b_i{}^* \le b_n \wedge \bigwedge_{i=1}^{m} b_i{}^* \le b_n \wedge b_n{}^* = 0$$

in case $n \le m$ and

$$b_n \wedge \bigwedge_{i=1}^{n} a_i{}^* \wedge a_m \wedge \bigwedge_{i=1}^{m} b_i{}^* \le \bigwedge_{i=1}^{n} a_i{}^* \wedge a_m \le a_m{}^* \wedge a_m = 0$$

otherwise. ■

Lane proved in [12] that any pm-normal topological space is mildly normal. In our pointfree (and conservative!) setting we prove more with a much simpler proof.

Proposition 4.3. *The following are equivalent for any locale L:*

(1) *L is pm-normal.*
(2) *L is mildly normal and for each regular element a in L there exists a countable family $(a_n)_{n\in\mathbb{N}}$ of regular elements in L such that $\mathfrak{c}(a) = \bigcap_{n\in\mathbb{N}} \mathfrak{o}(a_n)$.*
(3) *L is mildly normal and for each regular element a in L there exists a countable family $(a_n)_{n\in\mathbb{N}}$ of regular elements in L such that $\mathfrak{o}(a) = \bigvee_{n\in\mathbb{N}} \mathfrak{c}(a_n)$.*

Proof: (1) $\Longrightarrow$ (2): Let a and b be regular elements in L such that $a \vee b = 1$. By pm-normality,

$$a = \bigvee_{n\in\mathbb{N}} x_n \quad \text{and} \quad b = \bigvee_{n\in\mathbb{N}} y_n \quad \text{with} \quad x_n \prec a \quad \text{and} \quad y_n \prec b \quad \text{for every } n \in \mathbb{N}.$$

Obviously the elements $a_n = x_n{}^*$ and $b_n = y_n{}^*$ satisfy the conditions of the Lemma 4.2 and thus there exist $u, v \in L$ such that $u \wedge v = 0$ and $a \vee u = 1 = b \vee v$.

Hence L is mildly normal. On the other hand, for each regular element $a \in L$, by pm-normality, $a = \bigvee_{n\in\mathbb{N}} x_n$ with $x_n \prec a$ for every $n \in \mathbb{N}$. Hence, by (P2) and (P3),

$$\mathfrak{c}(a) \subseteq \bigcap_{n\in\mathbb{N}} \mathfrak{o}(x_n{}^*) \subseteq \bigcap_{n\in\mathbb{N}} \mathfrak{c}(x_n) = \mathfrak{c}(\bigvee_{n\in\mathbb{N}} x_n) = \mathfrak{c}(a).$$

(2) $\Longleftrightarrow$ (3): This follows from Proposition 4.1 (2).

(3) $\Longrightarrow$ (1): Let a be a regular element in L. By hypothesis there exists a countable family $(a_n)_{n\in\mathbb{N}}$ of regular elements in L such that $\mathfrak{o}(a) = \bigvee_{n\in\mathbb{N}} \mathfrak{c}(a_n)$. Hence $a \vee a_n = 1$ for each $n \in \mathbb{N}$. Since L is mildly normal, it follows that there exist $u_n, v_n \in L$ such that $u_n \wedge v_n = 0$ and $a \vee u_n = 1 = a_n \vee v_n$ for each $n \in \mathbb{N}$. Consequently (by (P2) and (P3) again),

$$\mathfrak{o}(a) = \bigvee_{n\in\mathbb{N}} \mathfrak{c}(a_n) \subseteq \bigvee_{n\in\mathbb{N}} \mathfrak{o}(v_n) \subseteq \bigvee_{n\in\mathbb{N}} \mathfrak{c}(u_n) \subseteq \mathfrak{o}(a). \qquad \blacksquare$$

Locales where each regular element is a cozero element are called *Oz locales* and are the natural pointfree counterpart of Oz spaces. They were introduced in [2] and further studied in [1]. Recall that, by Proposition 2.3 of [1], a locale is Oz if and only if every element of the form $\bigvee_{n\in\mathbb{N}}(a_n \wedge b_n)$ with all a_n and b_n being regular is a countable union of elements well inside it.

The next result, which seems to have escaped to the authors of [1], shows that the class of Oz locales contains that of perfectly normal locales.

Proposition 4.4. *A locale is Oz if and only if it is pm-normal.*

Proof: Necessity is obvious. For sufficiency, let $a = \bigvee_{n\in\mathbb{N}}(a_n \wedge b_n)$ with all a_n and b_n being regular. By pm-normality, each regular element L is regular-F_σ and therefore

$$a_n = \bigvee\{x \in L^* \mid x \prec a_n\} \quad \text{and} \quad b_n = \bigvee\{y \in L^* \mid y \prec b_n\} \quad \text{for each } n \in \mathbb{N}.$$

Then

$$a = \bigvee_{n\in\mathbb{N}} \bigvee\{x \wedge y \mid x, y \in L^*,\ x \prec a_n,\ y \prec b_n\}.$$

For each such x and y, we have that $x\wedge y \leq (x\wedge y)^{**} \in L^*$ and $(x\wedge y)^{**} \prec (a_n\wedge b_n) \leq a$, since

$$(x \wedge y)^* \vee (a_n \wedge b_n) \geq (x^* \vee a_n) \wedge (y^* \vee b_n) = 1.$$

Hence $a = \bigvee\{z \in L^* \mid z \prec a\}$. $\blacksquare$

Remarks 4.5. (1) Cozero elements are regular-F_σ, since $a \in \operatorname{Coz} L$ if and only if $a = \bigvee_{n\in\mathbb{N}} a_n$ for some $a_n \lll a$ (where $\lll$ denotes the really inside relation [11]). The converse is obviously true in Oz locales.

(2) If $\prec$ is interpolative (e.g., if L is a normal locale), then regular-F_σ elements are cozero elements also. More generally, in any almost normal locale, each regular-F_σ element belongs to $\operatorname{Coz} L$. In fact, for $a = \bigvee_{n\in\mathbb{N}} a_n$ with $a_n \prec a$ and a_n regular, by almost normality there exist u_n and v_n such that $u_n \wedge v_n = 0$ and $a_n^* \vee u_n = 1 = v_n^* \vee a$, hence $a_n \prec u_n \prec a$ (since $u_n^* \vee a \geq v_n \vee a = 1$). Then $a_n \lll a$.

5. Images of perfect locales

In this final section, we show that, as happens with normality, perfectness is an invariant property under closed maps, providing more evidence for our choice in Definition 3.8.

We start by recalling from [14] that a *localic map* is a map $f\colon L \to M$ satisfying

(1) $f(\bigwedge S) = \bigwedge f(S)$ for any $S \subseteq L$,]
(2) $f(a) = 1$ implies that $a = 1$, and
(3) $f(f^*(b) \to a) = b \to f(a)$ for every $a \in L$ and $b \in M$,

where f^* denotes the left adjoint of f, that exists by condition (1). This left adjoint is a *frame homomorphism* (i.e., it preserves arbitrary joins and finite meets). A localic map f is *closed* whenever the image of each closed sublocale of the domain is closed. In that case, $f[\mathfrak{c}(a)] = \mathfrak{c}(f(a))$.

Each localic map $f\colon L \to M$ induces the *image map*

$$f[-]\colon \mathcal{S}(L) \to \mathcal{S}(M),$$

left adjoint to the *preimage map*

$$f^{-1}[-]\colon \mathcal{S}(M) \to \mathcal{S}(L).$$

Open sublocales are preserved by preimages. More specifically:

$$f^{-1}[\mathfrak{o}(b)] = \mathfrak{o}(f^*(b)) \quad \text{for every } b \in M.$$

Furthermore, if f is surjective then the composite ff^{-1} satisfies

$$ff^{-1}[\mathfrak{o}(b)] = \mathfrak{o}(b) \quad \text{for every } b \in M.$$

Indeed: the inclusion "$\subseteq$" follows from the adjunction $f[-] \dashv f^{-1}[-]$; moreover, for each $b \to y$ in $\mathfrak{o}(b)$, we have

$$b \to y = b \to f(a) = f(f^*(b) \to a)$$

for some $a \in L$ (by ontoness of f) where $f^*(b) \to a \in \mathfrak{o}(f^*(b)) = f^{-1}[\mathfrak{o}(b)]$.

We can now prove that perfectnes is invariant under closed localic maps.

Proposition 5.1. *Let $f\colon L \to M$ be a surjective localic map. If f is closed and L is perfect, then M is also perfect.*

Proof: Let $b \in M$. Since L is perfect it follows that $f^*(b) = \bigvee_{n\in\mathbb{N}} \mathfrak{c}(a_n)$ for some countable family $(a_n)_{n\in\mathbb{N}}$ in L. Then, since $f[-]$ preserves arbitrary joins, we have

$$\mathfrak{o}(b) = ff^{-1}[\mathfrak{o}(b)] = f[\mathfrak{o}(f^*(b))] = f[\bigvee_{n\in\mathbb{N}} \mathfrak{c}(a_n)] = \bigvee_{n\in\mathbb{N}} f[\mathfrak{c}(a_n)] = \bigvee_{n\in\mathbb{N}} \mathfrak{c}(f(a_n)). \quad ■$$

Remark 5.2. Note that in general $f[-]$ does not preserve countable meets, so that the previous argument does not work if we replace perfectness by G_δ-perfectness. This gives us one more argument for choosing F_σ-perfectness as the right way to extend the topological notion of perfectness to the pointfree setting.

Corollary 5.3. *Let $f\colon L \to M$ be a surjective localic map. If f is closed and L is perfectly normal, then M is also perfectly normal.*

Proof: Just combine the proposition above with the fact, proved in [9, Corollary 9.4], that normality is also invariant under closed localic maps. ■

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