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REGULARITY OF THE VANISHING IDEAL OVER A PARALLEL COMPOSITION OF PATHS

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Abstract: Let G be a graph obtained by taking $r \geq 2$ paths and identifying all first vertices and identifying all the last vertices. We compute the Castelnuovo–Mumford regularity of the quotient $S/I(X)$, where S is the polynomial ring on the edges of G and $I(X)$ is the vanishing ideal of the projective toric subset parameterized by G. The case we consider is the first case where the regularity was unknown, following earlier computations (by several authors) of the regularity when G is a tree, cycle, complete graph or complete bipartite graph, but specially in light of the reduction of the computation of the regularity in the bipartite case to the computation of the regularity of the blocks of G. We also prove new inequalities relating the Castelnuovo–Mumford regularity of $S/I(X)$ with the combinatorial structure of G, for a general graph.

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1. Introduction

Let K be a field and denote by S the polynomial ring $K[t_1, \ldots, t_s]$ with the standard grading. If M is a finitely generated graded S-module and

$$0 \to F_c \xrightarrow{\phi_c} \cdots \longrightarrow F_2 \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0 \longrightarrow M \tag{1}$$

is a minimal graded free resolution, the Castelnuovo–Mumford regularity of M is the integer:

$$\operatorname{reg} M = \max_{i,j} \{j - i \mid b_{ij} \neq 0\},$$

where b_{ij} are the graded Betti numbers of M, defined by $F_i \cong \oplus_{j\in\mathbb{Z}} S(-j)^{b_{ij}}$. The regularity of M reflects the size of the degrees of the entries of the matrices in (1), and therefore, in a certain sense, the complexity of M as

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a graded module. In the case when $M = S/I$, with I a Cohen–Macaulay homogeneous ideal, we know that (cf. [17, Proposition 4.2.3]):

$$\operatorname{reg} S/I = \max_j \{j - c \mid b_{cj} \neq 0\} = \deg F_{S/I}(t) + \dim S/I, \tag{2}$$

where $F_{S/I}(t)$ is the Hilbert Series of the module S/I in rational function form.

Recently, many authors have studied the Castelnuovo-Mumford regularity of ideals associated to some combinatorial structure. For square free monomial ideals generated in degree 2, so-called *edge ideals* as their generators correspond to the edges of a graph (cf. [17, Chapter 6]), the regularity can be bounded using the induced matching number of the associated graph (cf. [7], [8, Lemma 2.2] and [18]). For chordal graphs, it has been shown that the regularity actually coincides with this graph invariant (see [7, Corollary 6.9]). Several families of binomial ideals associated with a combinatorial structure have also been studied. The class of *toric ideals*, i.e., the ideal of relations of the edge subring of a graph, whose generators correspond to *even closed walks* on the graph (cf. [17, Chapter 8]), is one such example. For a complete graph $\mathcal{K}_n$, the regularity of its edge subring is equal to $\lfloor n/2 \rfloor$, while for a complete bipartite graph $\mathcal{K}_{a,b}$, this invariant coincides with $\min\{a, b\} - 1$ (cf. [17]). Lower and upper bounds for the regularity of toric ideals, in terms of the structure of the underlying graph, have recently been established (cf. [1]). Another class of binomial ideals which has been extensively studied in recent times is the class of *binomial edge ideals*. These ideals are generated by the maximal minors of a $2 \times s$ generic matrix, whose column indices correspond to the edges of a graph. The regularity of these ideals can also be expressed and bounded in terms of graph-theoretic invariants (cf. [3, 9, 10]).

For the purposes of this work, K will be a finite field of cardinality q. In the rest of the paper all *graphs* will be undirected and without loops; multiple edges are allowed. The vertex set of a graph G will be denoted by V_G and its edge set by E_G. We denote the number of edges by s and we fix an ordering of the set of edges given by an identification of E_G with the set of variables of $K[t_1, \ldots, t_s]$. If H is a subgraph of G we denote by $K[E_H]$ the polynomial subring on the variables of E_H, under the above identification. To G we associate a set X defined by

$$X = \left\{(\mathbf{x}^{t_1}, \mathbf{x}^{t_2}, \ldots, \mathbf{x}^{t_s}) \in \mathbb{P}^{s-1} \mid \mathbf{x} \in (K^*)^{V_G}\right\}, \tag{3}$$

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where, if $\mathbf{x} = \sum_{v \in V_G} x_v v$, with $x_v \in K^*$, for all $v \in V_G$, and t_i is the edge $\{v, w\}$ (with $v \neq w$), we set $\mathbf{x}^{t_i} = x_v x_w$. As $\mathbf{x}^{t_i} \neq 0$, for all i, X is a subset of the projective torus $\mathbb{T}^{s-1} \subset \mathbb{P}^{s-1}$. We refer to X as *the projective toric subset parameterized by* G. Denote by $I(X)$ the vanishing ideal of X. Observe that

$$I(\mathbb{T}^{s-1}) = (t_1^{q-1} - t_s^{q-1}, \ldots, t_{s-1}^{q-1} - t_s^{q-1}) \subset I(X).$$

The notion of parameterized projective toric subsets and the study of their vanishing ideals was introduced in [14]. Unlike in the case of the edge ideal of G, we know that $I(X)$ is always a Cohen–Macaulay homogeneous binomial ideal of height $s-1$ (Cf. [14, Theorem 2.1]).

In the original definition of a parameterized projective toric subset, G is assumed to be a simple graph. However, on the one hand, we note that multiple edges play no part in the invariants of $K[E_G]/I(X)$. More precisely, if G' is the simple graph obtained from G by removing all extra edges through any two given vertices and X' is the projective toric subset parameterized by G', then

$$K[E_{G'}]/I(X') \cong K[E_G]/I(X),$$

simply because $t_j - t_i \in I(X)$, for every extra edge t_j between the endpoints of t_i. On the other hand, allowing extra edges eases notation and simplifies statements and proofs.

As X is a finite set, the value of the Hilbert function of $K[E_G]/I(X)$ is eventually equal to $|X|$, the cardinality of X; therefore, $\deg K[E_G]/I(X) = |X|$. A formula for the degree was first given in [14] for connected graphs and then generalized to any graph in [12, Theorem 3.2]:

$$\deg K[E_G]/I(X) = \begin{cases} \left(\frac{1}{2}\right)^{\gamma-1}(q-1)^{n-m+\gamma-1}, & \text{if } \gamma \geq 1 \text{ and } q \text{ is odd,} \\ (q-1)^{n-m+\gamma-1}, & \text{if } \gamma \geq 1 \text{ and } q \text{ is even,} \\ (q-1)^{n-m-1}, & \text{if } \gamma = 0, \end{cases} \tag{4}$$

where (q is the cardinality of K), n is the cardinality of V_G, m is the number of connected components of G and γ the number of those that are non-bipartite.

Using the identity (2) and the fact that $\dim K[E_G]/I(X) = 1$, we deduce that the regularity of $K[E_G]/I(X)$ coincides with its *regularity index*, i.e., the minimum degree d for which the value of the Hilbert function at k is equal to the value of the Hilbert polynomial at k, for every $k \geq d$. (Cf. [17, Corollary 4.1.12].) Since the Hilbert function of $K[E_G]/I(X)$ is

strictly increasing for $0 \leq d \leq \operatorname{reg} K[E_G]/I(X)$ and the Hilbert polynomial is equal to $|X| = \deg K[E_G]/I(X)$ we conclude that $\operatorname{reg} K[E_G]/I(X)$ is the minimum d for which the value of the Hilbert function at d is equal to $|X| = \deg K[E_G]/I(X)$.

In Table 1 we list cases for which this invariant is known. When X coincides

	$\operatorname{reg} K[E_G]/I(X)$
$X = \mathbb{T}^{s-1}$	$(s-1)(q-2)$
$G = \mathcal{K}_n$	$\lceil (n-1)(q-2)/2 \rceil$
$G = \mathcal{K}_{a,b}$	$(\max\{a,b\}-1)(q-2)$
$G = C_{2k}$	$(k-1)(q-2)$
$G = \mathcal{K}_{\alpha_1,\ldots,\alpha_r}$	$\max\{\alpha_1(q-2),\ldots,\alpha_r(q-2),\lceil (n-1)(q-2)/2 \rceil\}$

TABLE 1. Known values of $\operatorname{reg} K[E_G]/I(X)$

with the projective torus $\mathbb{T}^{s-1}$ (which, from (4), is the case, for example, if G is a tree or an odd cycle),

$$I(X) = (t_1^{q-1} - t_s^{q-1}, \ldots, t_{s-1}^{q-1} - t_s^{q-1}).$$

Thus the regularity can be computed from (2), (see also [15]). The regularity in the case $G = \mathcal{K}_n$ is given in [6, Remark 3]. The case $G = \mathcal{K}_{a,b}$ is given in [4, Corollary 5.4] and the case of an even cycle, $G = C_{2k}$, in [12, Theorem 6.2]. In the case of a complete multipartite graph, $G = \mathcal{K}_{\alpha_1,\ldots,\alpha_r}$ this invariant was computed in [11, Theorem 4.3]. (Here $r \geq 3$ and the n in the formula is $|V_G| = \alpha_1 + \cdots + \alpha_r$.)

A graph G is said to be 2-connected if $|V_G| > 2$ and, for every vertex $v \in V_G$, the graph $G - v$ is connected. Any graph decomposes into *blocks*, which consist of either maximal 2-connected subgraphs, single edges or isolated vertices. When G is bipartite, we know that $\operatorname{reg}(E_G)/I(X)$ can be computed from its block decomposition. More precisely, if G is a simple bipartite graph with no isolated vertices and $H_1, \ldots, H_r$ are the blocks of G, then

$$\operatorname{reg} K[E_G]/I(X) = \sum_{k=1}^{r} \operatorname{reg} K[E_{H_k}]/I(X_k) + (r-1)(q-2), \qquad (5)$$

	$\operatorname{reg} K[E_G]/I(X)$
$l = 0$	$(k_1/2 + \cdots + k_r/2 - 1)(q-2)$
$l = r$	$(\lfloor k_1/2 \rfloor + \cdots + \lfloor k_r/2 \rfloor)(q-2)$
$l = 1, r = 2$	$(k_1 + k_2 - 1)(q-2)$
$l = 1, r > 2$	$(k_1 + k_2/2 + \cdots + k_r/2 - 1)(q-2)$
$l > 1, r = l+1$	$(\lfloor k_1/2 \rfloor + \cdots + \lfloor k_{r-1}/2 \rfloor + k_r)(q-2)$
$l > 1, r > l+1$	$(\lfloor k_1/2 \rfloor + \cdots + \lfloor k_l/2 \rfloor + k_{l+1}/2 + \cdots + k_r/2)(q-2)$

TABLE 2. Values of $\operatorname{reg} K[E_G]/I(X)$ for a parallel composition of paths.

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which coincides with $\operatorname{reg} K[E_{H_1}]/I(X_1) + \operatorname{reg} K[E_{H_2}]/I(X_2) + (q-2)$. Since, by Proposition 2.5, $\operatorname{reg} K[E_G]/I(X) \geq \operatorname{reg}[E_{G'}]/I(X')$ we obtain the desired inequality.

The case $l > 1$ and $r - l = 1$. In this case we construct a graph G' by identifying all vertices in $P_1, \ldots, P_l$ at an even number of edges away from v with the vertex v. This graph consists of an odd cycle of length $k_r + 1$ (given

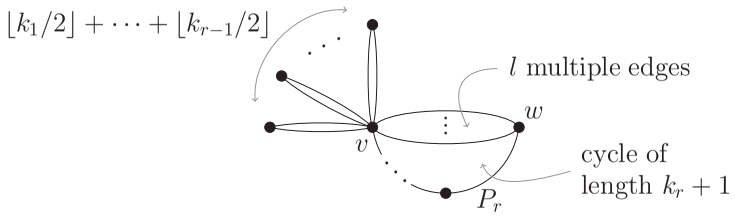

FIGURE 8. G', obtained by identifying with v every other vertex in $P_1, \ldots, P_{r-1}$.

by P_r and (a choice of) an edge $\{v, w\}$) that has l multiple edges between v and w and of a set of $\lfloor k_1/2 \rfloor + \cdots \lfloor k_{r-1}/2 \rfloor$ double edges incident at v (cf. Figure 8). Arguing as above, we get:

$$\begin{aligned} \operatorname{reg} K[E_G]/I(X) &\geq \operatorname{reg} K[E_{G'}]/I(X') \\ &= (\lfloor k_1/2 \rfloor + \cdots \lfloor k_{r-1}/2 \rfloor)(q-2) + k_r(q-2) \\ &= \operatorname{reg} K[E_{H_1}]/I(X_1) + \operatorname{reg} K[E_{H_2}]/I(X_2) + (q-2). \end{aligned}$$

The case $l > 1$ and $r - l > 1$. As in the previous case, let G' be the graph obtained by identifying the vertices in $P_1, \ldots, P_l$ at an even number of edges away from v with this vertex. We notice that the subgraph of G' consisting of the paths $P_{l+1}, \ldots, P_r$ and (a choice of) an edge $\{v, w\}$ belongs to the second case, above. Consequently,

$$\begin{aligned} &\operatorname{reg} K[E_G]/I(X) \geq (\lfloor k_1/2 \rfloor + \cdots + \lfloor k_l/2 \rfloor + k_{l+1}/2 + \cdots + k_r/2)(q-2) \\ &= \operatorname{reg} K[E_{H_1}]/I(X_1) + \operatorname{reg} K[E_{H_2}]/I(X_2) + (q-2). \quad \blacksquare \end{aligned}$$

The proof of Theorem 1.2 is obtained by combining Lemma 3.4 and Theorem 3.5. In Table 3 we give explicit formulas for the regularity of $K[E_G]/I(X)$ when G is a parallel composition of $r \geq 2$ paths of lengths $k_1, \ldots, k_r$, of which $k_1, \ldots, k_l$ are odd and $k_{l+1}, \ldots, k_r$ are even.

where X_k is the projective toric subset parameterized by the graph H_k, for each $k = 1, \ldots, r$ (cf. [13, Theorem 7.4]). This reduces the problem of computing $\operatorname{reg} K[E_G]/I(X)$ for a bipartite graph to the case of 2-connected graphs. Notice that (5), together with the formula for the regularity in the case of even cycles, gives the regularity for any bipartite *cactus graph* (a simple graph the blocks of which are edges or even cycles).

A 2-connected graph can be reconstructed from one of its cycles by adding a path by its endpoints (also known as an *ear*) to the cycle and successively repeating this operation (a finite number of times) to the graphs obtained (cf. [2, Proposition 3.1.1]). The simplest 2-connected graph is a cycle. The second simplest 2-connected graph is a cycle with an attached ear. This graph can also be obtained by identifying the endpoints of 3 paths, which, in turn, is also known as the *parallel composition* of 3 paths. Therefore the parallel composition of 3 paths is the first case of a 2-connected graph for which the regularity of $K[E_G]/I(X)$ was not known.

The aim of this work is to compute the Castelnuovo–Mumford regularity of $K[E_G]/I(X)$, when X belongs to the family of projective toric subsets parameterized by a graph given as the parallel composition of $r \geq 2$ paths, as illustrated in Figure 1. (Notice that this graph may well have multiple edges if more than one P_i has length equal to 1.)

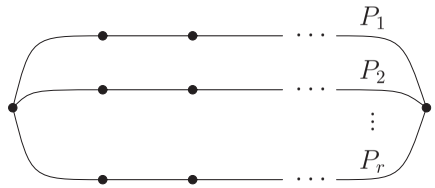

FIGURE 1. G, the parallel composition of paths $P_1, P_2, \ldots, P_r$.

Our first main result concerns the bipartite case.

Theorem 1.1. *Let X be the projective toric subset parameterized by the parallel composition of $r \geq 2$ paths, the lengths of which, $k_1, \ldots, k_r$, have the same parity. Then*

$$\operatorname{reg} K[E_G]/I(X) = \begin{cases} (\lfloor k_1/2 \rfloor + \cdots + \lfloor k_r/2 \rfloor)(q-2), & \text{if } k_i \text{ are odd,} \\ (k_1/2 + \cdots + k_r/2 - 1)(q-2), & \text{if } k_i \text{ are even.} \end{cases}$$

We prove this result in Section 3, by proving the two inequalities involved. The lower bound is a straightforward consequence of the fact that G is bipartite (cf. (7) and Lemma 3.1, below). For the upper bound we divide the proof into two cases. The case of k_i even is worked out by induction on r and arguing using suitable coverings of G (cf. Proposition 3.2). The case of k_i odd is harder and relies on a characterization of the homogeneous binomials in $I(X)$ (cf. Theorem 3.3).

With Theorem 1.1 we are able to study the non-bipartite case.

Theorem 1.2. *Let X be the projective toric subset parameterized by a graph G that is the parallel composition of $r \geq 2$ paths, the lengths of which have mixed parities. Then*

$$\operatorname{reg} K[E_G]/I(X) = \operatorname{reg} K[E_{H_1}]/I(X_1) + \operatorname{reg} K[E_{H_2}]/I(X_2) + (q-2),$$

where H_1 is the parallel composition of the paths of odd lengths, H_2 is the parallel composition of the paths of even lengths, and X_1, X_2, respectively, are the projective toric subsets they parameterize.

We point out that the formula of Theorem 1.2 includes the case when only one path has length of different parity. In this situation, the corresponding summand of the formula does not follow from Theorem 1.1, rather, it can be retrieved from the first formula of Table 1; more precisely, if H_i consists of a path of length k then $\operatorname{reg} K[E_{H_i}]/I(X_i) = (k-1)(q-2)$.

The proof of Theorem 1.2 occupies the second half of Section 3. As with our other main result we prove the two inequalities separately (cf. Lemma 3.4 and Theorem 3.5). This time, the easier inequality is the one giving the upper bound. For the lower bound inequality we need to use different techniques to those used in the proof of Theorem 1.1.

Section 2 provides the background theory and the results that are used in our proofs. We single out the new contributions of Proposition 2.5, Proposition 2.6 and Proposition 2.7, as we believe these results will prove useful in the study of the regularity for a general graph.

2. Preliminaries

Let K be a finite field of cardinality q. As in Section 1, G will denote a graph with edge set E_G of cardinality s (we always assume that G has no isolated vertices). We fix an identification of the variables of $K[t_1, \ldots, t_s]$ with E_G and denote the former by $K[E_G]$. Let X be the projective toric

$\operatorname{reg} K[E_G]/I(X) = (k_1+k_2-1)(q-2)$. On the other hand H_1 and H_2 are paths of lengths k_1 and k_2 and the projective toric subsets they parameterized are the tori $\mathbb{T}^{k_1-1}$ and $\mathbb{T}^{k_2-1}$ so that, again by the same formula, $\operatorname{reg} K[E_{H_1}]/I(X_1) = (k_1-1)(q-2)$ and $\operatorname{reg} K[E_{H_2}]/I(X_2) = (k_2-1)(q-2)$. We deduce that

$$\operatorname{reg} K[E_G]/I(X) = \operatorname{reg} K[E_{H_1}]/I(X_1) + \operatorname{reg} K[E_{H_2}]/I(X_2) + (q-2).$$

In the other cases, we will use vertex identifications and Proposition 2.5. For this purpose, let us denote the terminal vertices of the parallel composition yielding G by v and w.

The case $l = 1$, $k_1 = 1$ and $r - l > 1$. Consider the vertices of $P_2, \ldots, P_r$ at an odd number of edges away from v (or w). They form an independent set of vertices of cardinality $k_2/2 + \cdots + k_r/2$. Then, by Proposition 2.7, we get

$$\operatorname{reg} K[E_G]/I(X) \geq (k_2/2 + \cdots + k_r/2)(q-2). \tag{19}$$

Now, by Theorem 1.1, the right-hand of (19) is equal to

$$0 + \operatorname{reg} K[E_{H_2}]/I(X_2) + (q-2) = \operatorname{reg} K[E_{H_1}]/I(X_1) + \operatorname{reg} K[E_{H_2}]/I(X_2) + (q-2).$$

The case $l = 1$, $k_1 > 1$ and $r - l > 1$. Let G' be the graph obtained by identifying all the vertices in the paths $P_2, \ldots, P_r$ at an even number of edges away from v (or w) with the vertex v. The resulting graph G' consists of an odd cycle of length k_1 with a set of $k_2/2 + \cdots + k_r/2$ double edges incident to one of its vertices (cf. Figure 7). Let X' the projective toric

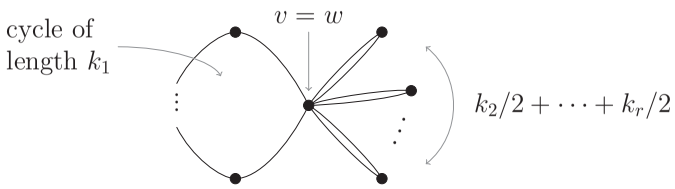

FIGURE 7. G', obtained by identifying every other vertex in $P_2, \ldots, P_r$.

subset parameterized by G'. The regularity of $K[E_{G'}]/I(X')$ is the same as if in G' all double edges were single edges. Hence by Proposition 2.4 and the formula for the odd cycle case we get:

$$K[E_{G'}]/I(X') = (k_1-1)(q-2) + (k_2/2 + \cdots + k_r/2)(q-2)$$

Let us now show that (16) holds. Since $t^a - \widehat{t_1^k} t_2^k g_1' g_2 \cdots g_r t^d$ is homogeneous, it will suffice to check the congruences at each vertex of G. Since for the binomial in (15), from which we obtain this binomial, the congruences are satisfied at all vertices of G', it will be enough to check the congruences for the vertices v_1, v_2 and v_3. At v_1, we have:

$$a_{\sigma_2+1} + \cdots + a_{\sigma_r+1} \equiv (q-1) - k + d_{\sigma_2+1} + \cdots + d_{\sigma_r+1} \pmod{q-1},$$

at v_2, $0 \equiv (q-1) - k + k \pmod{q-1}$ and at v_3, $a_3 \equiv k + d_3 \pmod{q-1}$, all of which hold, by virtue of (17). This completes the proof of the theorem. ∎

The proof of Theorem 1.1 follows from Lemma 3.1, Proposition 3.2 and Theorem 3.3.

We now turn to the proof of Theorem 1.2. In this case, G is the parallel composition of paths $P_1, \ldots, P_r$ the lengths of which have mixed parity. We assume, without loss of generality, that $P_1, \ldots, P_l$ have odd lengths and $P_{l+1}, \ldots, P_r$ have even lengths, for some $1 \leq l < r$. We will keep the notation for the edges of G as in the beginning of this section and recall that (as in the statement of Theorem 1.2) we will be denoting by H_1 the parallel composition of the paths of odd lengths, by H_2 the parallel composition of the paths of even lengths and by X_1, X_2, respectively, the projective toric subsets they parameterize.

Lemma 3.4.

$$\operatorname{reg} K[E_G]/I(X) \leq \operatorname{reg} K[E_{H_1}]/I(X_1) + \operatorname{reg} K[E_{H_2}]/I(X_2) + (q-2).$$

Proof: Consider the cover of G given by H_1 and H_2' where H_2' is given by $\{t_1\} \cup H_2$. Then $E_{H_1} \cap E_{H_2'} \neq \emptyset$ and therefore by Proposition 2.6,

$$\operatorname{reg} K[E_G]/I(X) \leq \operatorname{reg} K[E_{H_1}]/I(X_1) + \operatorname{reg} K[E_{H_2'}]/I(X_2'), \qquad (18)$$

where X_2' is the projective toric subset parameterized by H_2'. By Proposition 2.4, we know that $\operatorname{reg} K[E_{H_2'}]/I(X_2') = \operatorname{reg} K[E_{H_2}]/I(X_2) + (q-2)$. Combining this with (18) completes the proof of the lemma. ∎

Theorem 3.5.

$$\operatorname{reg} K[E_G]/I(X) \geq \operatorname{reg} K[E_{H_1}]/I(X_1) + \operatorname{reg} K[E_{H_2}]/I(X_2) + (q-2).$$

Proof: We divide the proof into cases.

The case $l = 1$ and $r = 2$. In this case G is a cycle of (odd) length $k_1 + k_2$. Accordingly, X coincides with $\mathbb{T}^{k_1+k_2-1}$ and, by the formula in Table 1,

subset parameterized by G, as defined in (3). If $a = (a_1, \ldots, a_s) \in \mathbb{N}^s$, t^a denotes the monomial $t_1^{a_1} \cdots t_s^{a_s} \in K[E_G]$.

We start by recalling a criterion for membership in $I(X)$ of a homogeneous binomial that only involves the combinatorics of G. It involves checking a linear congruence at every vertex of the graph. Let $v \in V_G$ and let $t_{i_1}, \ldots, t_{i_r}$

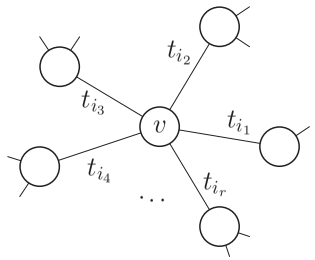

FIGURE 2. Congruence at vertex v

be the edges incident to v (cf. Figure 2). Then by [11, Lemma 2.3], a homogeneous binomial $t^a - t^b \in K[E_G]$ belongs to $I(X)$ if and only if, for every vertex $v \in V_G$, if $i_1, \ldots, i_r$ are the indices of the edges incident to it, the congruence

$$a_{i_1} + \cdots + a_{i_r} \equiv b_{i_1} + \cdots + b_{i_r} \pmod{q-1} \qquad (6)$$

is satisfied. It follows easily from this criterion, that if H is a subgraph of G and Y is the projective toric subset parameterized by H, then

$$I(Y) = I(X) \cap K[E_H].$$

The following lemma will be used in the proof of Theorem 3.3, below. Recall that an ear of G is a path which is maximal with respect to the condition that all of its interior vertices have degree 2 in G.

Lemma 2.1. *Let $t^a - t^b \in K[E_G]$ be a homogeneous binomial. Let t_i and t_j be edges along an ear of G in a same parity position along this path. Let $\sigma\colon K[E_G] \to K[E_G]$ be the automorphism defined by swapping the two edges t_i and t_j. Then*

$$t^a - t^b \in I(X) \iff \sigma(t^a) - \sigma(t^b) \in I(X).$$

Proof: It is clear we can reduce to the case illustrated in Figure 3. Since $\sigma(t^a) - \sigma(t^b)$ is homogeneous if and only if $t^a - t^b$ is, it suffices to check

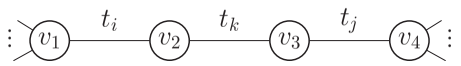

FIGURE 3. Swapping edges along an ear of G.

the equivalence of the system of 4 linear congruences given by the 4 vertices v_1, v_2, v_3 and v_4. Let $E(v_i)$ denote the set of edges incident to v_i and denote by E_1 the set $E(v_1) \setminus \{t_i\}$ and, likewise, $E_4 = E(v_4) \setminus \{t_j\}$. Let

$$A_1 = \sum_{t_\ell \in E_1} a_\ell, \quad A_4 = \sum_{t_\ell \in E_4} a_\ell, \quad B_1 = \sum_{t_\ell \in E_1} b_\ell, \quad \text{and} \quad B_4 = \sum_{t_\ell \in E_4} b_\ell.$$

Then, we need to show that the two systems of congruences modulo $q-1$

$$\begin{cases} A_1 + a_i \equiv B_1 + b_i \\ a_i + a_k \equiv b_i + b_k \\ a_k + a_j \equiv b_k + b_j \\ a_j + A_4 \equiv b_j + B_4 \end{cases} \quad \text{and} \quad \begin{cases} A_1 + a_j \equiv B_1 + b_j \\ a_j + a_k \equiv b_j + b_k \\ a_k + a_i \equiv b_k + b_i \\ a_i + A_4 \equiv b_i + B_4 \end{cases}$$

are equivalent, which is clearly true. ∎

Our approach to computing $\operatorname{reg} K[E_G]/I(X)$ is to consider an Artinian quotient $K[E_G]/I(X, g)$, where $g \in K[E_G]$ is a suitable monomial.

Proposition 2.2. *Let $g \in K[E_G]$ be a monomial.*

(i) *There exists a monomial order and a binomial Gröbner basis $\mathcal{B}$ of $I(X)$ such that $\mathcal{B} \cup \{g\}$ is a Gröbner basis for $(I(X), g) \subset K[E_G]$.*
(ii) *A monomial $t^a \in K[E_G]$ belongs to $(I(X), g)$ if and only if there exists a monomial $t^b \in K[E_G]$ such that $t^a - gt^b$ is homogeneous and belongs to $I(X)$.*

Proof: Since $I(X)$ is generated by homogeneous binomials, the Gröbner basis obtained from such a set, after fixing any monomial order, consists of homogeneous binomials, by Buchberger's Algorithm. Let $t_{i_1}, \ldots, t_{i_r}$ be the variables dividing g. Fix the graded reverse lexicographical order after reordering the variables in way such that $t_{i_1} \succ \cdots \succ t_{i_r}$ are the last variables of the ring. Let $\mathcal{B}$ be a binomial Gröbner basis of $I(X)$ with respect to such order. To prove (i) it suffices to show that $S(f, g)$ reduces to 0 modulo $\mathcal{B} \cup \{g\}$, for every $f \in \mathcal{B}$. Let $f = t^a - t^b \in \mathcal{B}$. Assume, without loss of generality, that $\operatorname{lt}(f) = t^a$. If t_{i_r} divides t^a, then t_{i_r} does not divide t^b (we may

with monomials $h_i' \in K[E_{P_i}]$, we get $\deg(h_1') = \deg(h_1) - 1$. Repeating the previous argument, we deduce that $t^{a'} \in (I(X), g)$, which, using (14) implies that $t^a \in (I(X), g)$.

We are left with the case of $a_1 = a_2 = 0$. We regard t^a as a monomial in $K[E_{G'}]$, where G' is the graph obtained as the parallel composition of $P_1 \setminus \{t_1, t_2\}$, $P_2, \ldots, P_r$. Let X' be the projective toric subset parameterized

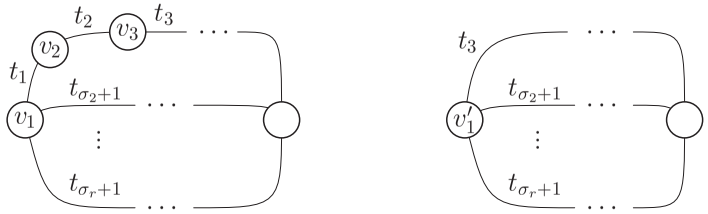

FIGURE 6. G (left) and G' (right).

by G'. By the induction hypothesis

$$\operatorname{reg} K[E_{G'}]/I(X') = (\lfloor k_1/2 \rfloor + \cdots + \lfloor k_r/2 \rfloor - 1)(q-2).$$

Hence, by Proposition 2.3 and Proposition 2.2, there exists $t^d \in K[E_{G'}]$, where $d \in \mathbb{N}^s$ is supported on the edges of G', such that

$$t^a - g_1' g_2 \cdots g_r t_3^{q-1} t^d \in I(X'), \tag{15}$$

where $g_1' = g_1/t_2 \in K[E_{G'}]$. We claim there exists $k \in \{1, 2, \ldots, q-1\}$ such that

$$t^a - t_1^{\widehat{k}} t_2^k g_1' g_2 \cdots g_r t^d \in I(X) \tag{16}$$

with $\widehat{k} = q - 1 - k$. We define k using the congruence at vertex v_1' of G' (see Figure 6) which, according to [11, Lemma 2.3], is satisfied for the binomial in (15). This congruence is:

$$\begin{aligned} &a_3 + a_{\sigma_2+1} + \cdots + a_{\sigma_r+1} \equiv q - 1 + d_3 + d_{\sigma_2+1} + \cdots + d_{\sigma_r+1} \pmod{q-1} \\ \iff &a_3 - d_3 \equiv d_{\sigma_2+1} + \cdots + d_{\sigma_r+1} - (a_{\sigma_2+1} + \cdots + a_{\sigma_r+1}) \pmod{q-1}. \end{aligned}$$

Let $k \in \{1, 2, \ldots, q-1\}$ to be such that:

$$k \equiv a_3 - d_3 \equiv d_{\sigma_2+1} + \cdots + d_{\sigma_r+1} - (a_{\sigma_2+1} + \cdots + a_{\sigma_r+1}) \pmod{q-1}. \tag{17}$$

Thus, we may assume $k_i \geq 3$, for all $i = 1, \ldots, r$. According to Proposition 2.3, to show that

$$\operatorname{reg} K[E_G]/I(X) \leq (\lfloor k_1/2 \rfloor + \cdots + \lfloor k_r/2 \rfloor)(q-2),$$

it suffices to show that any monomial $t^a \in K[E_G]$ of degree

$$(\lfloor k_1/2 \rfloor + \cdots + \lfloor k_r/2 \rfloor)(q-2) + \lfloor k_1/2 \rfloor + \cdots + \lfloor k_r/2 \rfloor \tag{10}$$

belongs to the ideal $(I(X), g) \subset K[E_G]$, where $g = g_1 \cdots g_r$ and g_i were defined in (8). Let t^a be one such monomial and write it as the product of monomials, $h_1 \cdots h_r$, where $h_i \in K[E_{P_i}]$. By (10), we have $\deg(h_i) \leq \lfloor k_i/2 \rfloor (q-1)$, for some $i \in \{1, \ldots, r\}$. Without loss of generality we assume $i = 1$. In particular,

$$\deg(h_2 \cdots h_r) \geq (\lfloor k_2/2 \rfloor + \cdots + \lfloor k_r/2 \rfloor)(q-2) + \lfloor k_2/2 \rfloor + \cdots + \lfloor k_r/2 \rfloor. \tag{11}$$

Since g is invariant under the swapping of variables corresponding to edges of P_1 in a same parity position, using Lemma 2.1, we may assume that

$$a_1 \leq a_3 \leq \cdots \leq a_{2\lceil k_1/2 \rceil - 1} \text{ and } a_2 \leq a_4 \leq \cdots \leq a_{2\lfloor k_1/2 \rfloor}. \tag{12}$$

Let H be the subgraph of G given by $P_2 \cup \cdots \cup P_r$ and denote by Y the projective toric subset parameterized by it. By induction,

$$\operatorname{reg} K[E_H]/I(Y) = (\lfloor k_2/2 \rfloor + \cdots + \lfloor k_r/2 \rfloor)(q-2).$$

Then, by (11), Proposition 2.3 and Proposition 2.2, there exists a monomial $t^b \in K[E_H]$, for some $b \in \mathbb{N}^s$ supported on the edges of H, such that $h_2 \cdots h_r - g_2 \cdots g_r t^b \in I(Y) \subset I(X)$ and hence

$$t^a - h_1 g_2 \cdots g_r t^b \in I(X). \tag{13}$$

If $a_2 \neq 0$, then from (12) we deduce that g_1 divides h_1 and we are done. If $a_1 \neq 0$, then there exists $c \in \mathbb{N}^s$ such that $h_1 = f_1 t^c$. Accordingly, $h_1 g_2 \cdots g_r t^b = f_1 g_2 \cdots g_r t^{b+c}$. Since $f_1 g_2 - f_2 g_1 \in I(X)$, we deduce that

$$f_1 g_2 \cdots g_r t^{b+c} - f_2 g_1 g_3 \cdots g_r t^{b+c} \in I(X),$$

which, together with (13), implies that

$$t^a - f_2 g_1 g_3 \cdots g_r t^{b+c} \in I(X). \tag{14}$$

Consider $a' \in \mathbb{N}^s$ such that $t^{a'} = f_2 g_1 g_3 \cdots g_r t^{b+c}$. Since $h_1 = f_1 t^c$ and the monomials $f_2, g_3, \ldots, g_r, t^b$ are supported away from the edges of P_1, we see that, if $1 \leq i \leq k_1$, $a'_i = a_i - 1$, when i is odd, and $a'_i = a_i + 1$, when i is even. In particular, $a'_2 \neq 0$ and, in the corresponding decomposition $t^{a'} = h'_1 \cdots h'_r$

assume the generating set we start with consists of irreducible binomials). This implies that $t^b \succ t^a$, hence t_{i_r} does not divide t^a. Arguing in the same way, by induction, we conclude that none of $t_{i_1}, \ldots, t_{i_r}$ divides t^a and thus $\gcd(g, t^a) = 1$. Accordingly,

$$S(f, g) = g(t^a - t^b) - t^a g = -g t^b$$

which reduces to zero modulo $\mathcal{B} \cup \{g\}$. This completes the proof of (i).

Let t^a be a monomial. One direction of the equivalence in (ii) is clear. Assume that $t^a \in (I(X), g)$. Then, considering the Gröbner basis $\mathcal{B} \cup \{g\}$ obtained in (i), t^a has zero remainder after division with $\mathcal{B} \cup \{g\}$. Since the division of a monomial by a binomial is still a monomial, the division algorithm stops the first time g is used. Thus, the partial quotients of division are monomials $t^a = h_0, h_1, \ldots, h_k$ such that $h_i - h_{i-1} \in I(X)$, for all $i = 1, \ldots, k$ and such that g divides h_k. Writing $h_k = g t^b$, we get a homogeneous binomial $t^a - g t^b$ which belongs to $I(X)$, as required. ∎

Proposition 2.3. *Let $g \in K[E_G]$ be a monomial. Then $K[E_G]/(I(X), g)$ is zero in degree d if and only if $d \geq \operatorname{reg} K[E_G]/I(X) + \deg(g)$.*

Proof: We denote $K[E_G]/I(X)$ by R and $K[E_G]/(I(X), g)$ by R/g, by abuse of notation. Since g is an R-regular element and R is Cohen–Macaulay,

$$\dim R/g = \dim R - 1 = 0.$$

Moreover, since R/g is a quotient of a polynomial ring with the standard grading by a homogeneous ideal, its regularity index is the minimum degree d for which $(R/g)_d = 0$. (It is easy to see that $(R/g)_d = 0$, for some d, implies $(R/g)_{d+k} = 0$, for all $k \geq 0$.) Hence we need to show that the regularity index of R/g is equal to $\operatorname{reg} K[E_G]/I(X) + \deg(g)$. Consider the following exact sequence of graded $K[E_G]$-modules:

$$0 \to R[-\deg(g)] \xrightarrow{\cdot g} R \to R/g \to 0.$$

Comparing the degree of the Hilbert series of the three terms and using the identity (2), we get $\deg F_{R/g} + 1 = \operatorname{reg} R + \deg(g)$, where $F_{R/g}$ is the Hilbert Series of the $K[E_G]$-module R/g in rational function form. As $\deg F_{R/g} + 1$ is the regularity index (cf. [17, Corollary 4.1.12]), we have proved the claim. ∎

We note that the following proposition can be easily derived from [13, Theorem 7.4] in the bipartite case, and from [15, Corollary 3.10] and [5, Lemma 1] in the non-bipartite case, when G is a unicyclic connected graph

and the only cycle of G is odd. Here, we do not assume G is bipartite nor a unicyclic connected graph with an odd cycle.

Proposition 2.4. *Let $v \in V_G$ be a vertex of degree 1. Assume that $|E_G| > 1$. Consider the graph $G' = G - v$ and denote by X' the projective toric subset parameterized by it. Then*

$$\operatorname{reg} K[E_G]/I(X) = \operatorname{reg} K[E_{G'}]/I(X') + (q-2).$$

Proof: Let $t_i \in E_G$ be incident to v and let $t_j \in E_G \setminus t_i$. According to Proposition 2.3, to show that $\operatorname{reg} K[E_G]/I(X) \leq \operatorname{reg} K[E_{G'}]/I(X') + (q-2)$ it suffices to show that for any monomial $t^a \in K[E_G]$ of degree

$$\operatorname{reg} K[E_{G'}]/I(X') + (q-2) + 1$$

we have $t^a \in (I(X), t_j)$. Let t^a be such a monomial. If $a_i \geq q-1$ then writing $t^a = t^{a'} t_i^{q-1}$ for some $a' \in \mathbb{N}^s$, we get:

$$t^a = t^{a'}(t_i^{q-1} - t_j^{q-1}) + t^{a'} t_j^{q-1} \in (I(X), t_j).$$

Assume now that $a_i < q-1$. Consider $a' \in \mathbb{N}^s$, with $a'_i = 0$, such that $t^a = t^{a'} t_i^{a_i}$. Then $\deg t^{a'} = \deg t^a - a_i \geq \operatorname{reg} K[E_{G'}]/I(X') + 1$, by our assumptions. As $t^{a'}$ belongs to $K[E_{G'}]$, using Proposition 2.3 we get

$$t^{a'} \in (I(X'), t_j) \subset K[E_{G'}].$$

As G' is a subgraph of G we have $I(X') \subset I(X)$ and therefore $t^{a'} \in (I(X), t_j)$.

Using the same idea, let us now show that

$$\operatorname{reg} K[E_{G'}]/I(X') \leq \operatorname{reg} K[E_G]/I(X) - (q-2).$$

Let $t^a \in K[E_{G'}]$ be a monomial of degree $\operatorname{reg} K[E_G]/I(X) - (q-2) + 1$. Then $t^a t_i^{q-2}$ belongs to $K[E_G]$ and has degree $\operatorname{reg} K[E_G]/I(X) + 1$. We deduce that $t^a t_i^{q-2} \in (I(X), t_j)$. By Proposition 2.2, there exists a monomial $t^b \in K[E_G]$ such that $t^a t_i^{q-2} - t_j t^b \in I(X)$. However the congruence at vertex v gives $b_i = q - 2 + k(q-1)$, for some $k \geq 0$. Let $b' \in \mathbb{N}^s$ be such that $b'_i = 0$ and $t^b = t^{b'} t_i^{b_i}$. Then:

$$t^a t_i^{q-2} - t_j t^b \in I(X) \implies t^a - t_j t_i^{k(q-1)} t^{b'} \in I(X) \implies t^a - t_j^{1+k(q-1)} t^{b'} \in I(X).$$

Since $t^a - t_j^{1+k(q-1)} t^{b'} \in K[E_{G'}]$ and $I(X') = I(X) \cap K[E_{G'}]$, we deduce that $t^a \in (I(X'), t_j)$. ∎

induction hypothesis,

$$\operatorname{reg} K[E_{H_1}]/I(X_1) = (k_1/2 + k_2/2 - 1)(q-2) \quad \text{and}$$
$$\operatorname{reg} K[E_{H_2}]/I(X_2) = (k_1/2 + k_3/2 + \cdots + k_r/2 - 1)(q-2).$$

Hence, if $\deg(t^b t^d) \geq (k_1/2 + k_2/2 - 1)(q-2) + 1$, we get

$$t^b t^d \in (I(X_1), t_1) \subset (I(X), t_1)$$

which implies that $t^a \in (I(X), t_1)$. Similarly, if

$$\deg(t^b t^e) \geq (k_1/2 + k_3/2 + \cdots + k_r/2 - 1)(q-2) + 1.$$

Suppose that

$$\deg(t^b t^d) \leq (k_1/2 + k_2/2 - 1)(q-2) \quad \text{and}$$
$$\deg(t^b t^e) \leq (k_1/2 + k_3/2 + \cdots + k_r/2 - 1)(q-2).$$

Since $\deg(t^a) = \deg(t^b t^d) + \deg(t^b t^e) - \deg(t^b)$, we deduce that

$$\deg(t^a) \leq (k_1/2 + \cdots + k_r/2 - 1)(q-2) - 1,$$

which is a contradiction. ∎

Theorem 3.3. *If k_i are odd, then*

$$\operatorname{reg} K[E_G]/I(X) \leq (\lfloor k_1/2 \rfloor + \cdots + \lfloor k_r/2 \rfloor)(q-2).$$

Proof: We will use induction on $k_1 + \cdots + k_r$. In the base case, $r = 2$ and $k_1 = k_2 = 1$. As we mentioned earlier, $\operatorname{reg} K[E_G]/I(X) = 0$.

Assume that $k_1 + \cdots + k_r > 3$ and, as induction hypothesis, that the statement of the theorem holds for any $k'_1, \ldots, k'_{r'}$ and $r' \geq 2$ such that $k'_1 + \cdots + k'_{r'} < k_1 + \cdots + k_r$. If $r = 2$, then, as observed in the beginning of this section, G is an even cycle of length $k_1 + k_2$ and accordingly

$$\operatorname{reg} K[E_G]/I(X) = ((k_1 + k_2)/2 - 1)(q-2) = (\lfloor k_1/2 \rfloor + \lfloor k_2/2 \rfloor)(q-2).$$

Hence, we may assume $r \geq 3$. If, for some i, $k_i = 1$, let G' be the subgraph of G given as the parallel composition of all $P_1, \ldots, P_r$ but P_i. We note that G' is a connected spanning subgraph of G and hence, if X' is the projective toric subset parameterized by G', by the induction hypothesis, since $\lfloor k_i/2 \rfloor = 0$, we get

$$\operatorname{reg} K[E_G]/I(X) \leq \operatorname{reg} K[E_{G'}]/I(X') \leq (\lfloor k_1/2 \rfloor + \cdots + \lfloor k_r/2 \rfloor)(q-2).$$

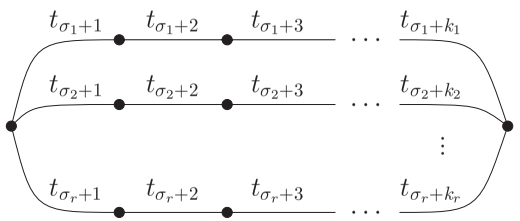

FIGURE 5. Labeling of the edges of G.

that, for all $i \neq j$,

$$f_i g_j - f_j g_i \in I(X). \tag{9}$$

Proposition 3.2. *If k_i are even, then*

$$\operatorname{reg} K[E_G]/I(X) \leq (k_1/2 + \cdots + k_r/2 - 1)(q-2).$$

Proof: According to Proposition 2.3, it suffices to show that any monomial $t^a \in K[E_G]$ of degree

$$(k_1/2 + \cdots + k_r/2 - 1)(q-2) + 1$$

belongs to $(I(X), t_1)$. We may assume t_1 does not divide t^a. We will argue by induction on r. For $r = 2$, as observed earlier, the result holds true. Assume now that $r \geq 3$. Let H be the subgraph of G given by $\{t_1\} \cup P_2 \cup \cdots \cup P_r$ and Y be the projective toric subset parameterized by G. By induction and [13, Theorem 7.4],

$$\operatorname{reg} K[E_H]/I(Y) = (k_2/2 + \cdots + k_r/2)(q-2).$$

Set $t^a = t^b t^c$, with $t^b \in K[E_{P_1}]$ and $t^c \in K[E_H]$. If

$$\deg(t^c) \geq (k_2/2 + \cdots + k_r/2)(q-2) + 1,$$

then, by Proposition 2.3, $t^c \in (I(Y), t_1) \subset (I(X), t_1)$, which implies that $t^a \in (I(X), t_1)$. Assume that $\deg(t^c) \leq (k_2/2 + \cdots + k_r/2)(q-2)$. Then $\deg(t^b) \geq (k_1/2 - 1)(q-2) + 1$. Consider now the subgraphs of G given by

$$H_1 = P_1 \cup P_2 \quad \text{and} \quad H_2 = P_1 \cup P_3 \cup \cdots \cup P_r$$

and denote by X_1 and X_2, respectively, the projective toric subsets parameterized by them. Set $t^c = t^d t^e$ with $t^b t^d \in K[E_{H_1}]$ and $t^b t^e \in K[E_{H_2}]$. By the

Let G be a connected graph and a spanning subgraph of a bipartite graph H. Let Y be the projective toric subset parameterized by H. Then, by [16, Lemma 2.13], if $|X| = |Y|$, it follows that

$$\operatorname{reg} K[E_G]/I(X) \geq \operatorname{reg} K[E_H]/I(Y).$$

Hence if G is a connected bipartite spanning subgraph of $\mathcal{K}_{a,b}$, by (4) the assumption on the cardinality of the associated parameterized projective toric subsets holds and we obtain:

$$\operatorname{reg} K[E_G]/I(X) \geq (\max\{a,b\} - 1)(q-2). \tag{7}$$

In the remainder of this section we introduce three new inequalities involving $\operatorname{reg} K[E_G]/I(X)$. They will play an important role in the proofs of Theorem 1.1 and Theorem 1.2.

Proposition 2.5. *Let v_1 and v_2 be two vertices of G such that $\{v_1, v_2\}$ is a non-edge of G. Let G' be the graph obtained by identifying v_1 with v_2 and denote by X' the projective toric subset parameterized by it. Then* $\operatorname{reg} K[E_G]/I(X) \geq \operatorname{reg} K[E_{G'}]/I(X')$.

Proof: The edge sets of G and G' have the same cardinality. Morevoer, there is an induced identification of the edges of G' with the variables of the polynomial ring $K[t_1, \ldots, t_s]$ under which $K[E_G] = K[E_{G'}]$. Since the parameterization of X' is obtained by adding the restriction that the coefficient of v_1 in the formal sum $\sum_{v \in V_G} x_v v$ be equal to the coefficient of v_2 we obtain $X' \subset X$ (cf. (3)), and thus, $I(X) \subset I(X')$. Let t_1 be an edge. According to Proposition 2.3, to show that

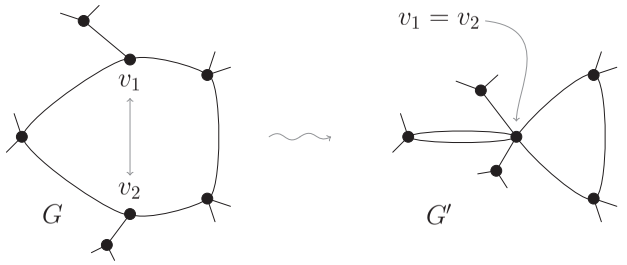

FIGURE 4. The graph obtained by identifying two vertices of G.

$$\operatorname{reg} K[E_{G'}]/I(X') \leq \operatorname{reg} K[E_G]/I(X)$$

it suffices to prove that for any monomial t^a of degree $\operatorname{reg} K[E_G]/I(X)+1$ we have $t^a \in (I(X'), t_1)$. Let t^a be such a monomial. Then, using again Proposition 2.3, we deduce that $(t^a \in I(X), t_1)$. Since $I(X) \subset I(X')$ we get $(t^a \in I(X'), t_1)$. ∎

Proposition 2.6. *Let $H_1, H_2 \subset G$ be subgraphs such that $E_G = E_{H_1} \cup E_{H_2}$ and $E_{H_1} \cap E_{H_2} \neq \emptyset$. Let X_1 and X_2 be the projective toric subsets parameterized by H_1 and H_2 and $I(X_1) \subset K[E_{H_1}]$, $I(X_2) \subset K[E_{H_2}]$ their corresponding vanishing ideals. Then*

$$\operatorname{reg} K[E_G]/I(X) \leq \operatorname{reg} K[E_{H_1}]/I(X_1) + \operatorname{reg} K[E_{H_2}]/I(X_2).$$

Proof: Let $t_i \in E_{H_1} \cap E_{H_2}$. According to Proposition 2.3, it suffices to show that any monomial $t^a \in K[E_G]$, of degree

$$\operatorname{reg} K[E_{H_1}]/I(X_1) + \operatorname{reg} K[E_{H_2}]/I(X_2) + 1,$$

belongs to $(I(X), t_i)$. Let us write $t^a = t^b t^c$ for some $t^b \in K[E_{H_1}]$ and $t^c \in K[E_{H_2}]$. Since $\deg(t^a) = \deg(t^b) + \deg(t^c)$, we have

$$\deg(t^b) \geq \operatorname{reg} K[E_{H_1}]/I(X_1) + 1 \quad \text{or} \quad \deg(t^c) \geq \operatorname{reg} K[E_{H_2}]/I(X_2) + 1.$$

By Proposition 2.3 it follows that

$$t^b \in (I(X_1), t_i) \subset K[E_{H_1}] \quad \text{or} \quad t^c \in (I(X_2), t_i) \subset K[E_{H_2}],$$

respectively. In both cases we conclude that $t^a \in (I(X), t_i)$. ∎

Proposition 2.7. *Let $\{v_1, \ldots, v_r\}$ be an independent set of vertices of G. Assume that there is an edge in $G - \{v_1, \ldots, v_r\}$. Then*

$$\operatorname{reg} K[E_G]/I(X) \geq r(q-2).$$

Proof: By Proposition 2.3, to show that $\operatorname{reg} K[E_G]/I(X) \geq r(q-2)$ it suffices to show that there exists and edge t_i and a monomial $t^a \in K[E_G]$ of degree $r(q-2)$ that does not belong to $(I(X), t_i)$. Let t_i be an edge of $G - \{v_1, \ldots, v_r\}$ and, for every $i = 1, \ldots, r$, let t_{j_i} be an edge incident to v_i. Such edges exist since we assume that G has no isolated vertices. Notice also that since $\{v_1, \ldots, v_r\}$ is an independent set the edges $t_{j_1}, \ldots, t_{j_r}$ are distinct. Consider the monomial:

$$t^a = (t_{j_1} \cdots t_{j_r})^{q-2}$$

and let us show that $t^a \notin (I(X), t_i)$. Suppose the contrary holds. Then, by Proposition 2.2, there exists a monomial t^b such that $t^a - t_i t^b$ is homogeneous and belongs to $I(X)$. Since t_i is not incident to any of the vertices of $\{v_1, \ldots, v_r\}$, evaluating the congruence at a particular vertex of this set, we conclude that the degree of t^b in the edges incident to it is $\geq q-2$. Since, by assumption, these vertices possess no common incident edges we deduce that the degree of t^b in edges incident to the vertices of $\{v_1, \ldots, v_r\}$ is $\geq r(q-2)$. In particular, $\deg(t^b) \geq r(q-2)$. But this implies that $t^a - t_i t^b$ is not homogeneous, which is a contradiction. ∎

We note that Proposition 2.7 implies (7).

3. Proof of the main results

The aim of this section is to prove Theorem 1.1 and Theorem 1.2. In what follows G is the parallel composition of $r \geq 2$ paths $P_1, \ldots, P_r$ of lengths $k_1, \ldots, k_r$. In a first instance, we assume that these integers have the same parity, so that G is bipartite. If $r = 2$ and one of k_1, k_2 is > 1, then G is an even cycle of length $k_1 + k_2$. In this case, by [12, Theorem 6.2], we know that $\operatorname{reg} K[E_G]/I(X) = ((k_1+k_2)/2 - 1)(q-2)$. If $r = 2$ and $k_1 = k_2 = 1$, then G is a graph on 2 vertices with 2 multiple edges. Hence the value of the regularity is the same as in the case of a tree with a single edge, which is $(s-1)(q-2) = 0$ (cf. Table 1). Both cases agree with the formula in Theorem 1.1.

Lemma 3.1.

$$\operatorname{reg} K[E_G]/I(X) \geq \begin{cases} (\lfloor k_1/2 \rfloor + \cdots + \lfloor k_r/2 \rfloor)(q-2), & \text{if } k_i \text{ are odd,} \\ (k_1/2 + \cdots + k_r/2 - 1)(q-2), & \text{if } k_i \text{ are even.} \end{cases}$$

Proof: If k_i are odd, then G is a connected spanning subgraph of $\mathcal{K}_{\rho,\rho}$, where ρ is the integer $1 + \lfloor k_1/2 \rfloor + \cdots + \lfloor k_r/2 \rfloor$. If k_i are even, then G is a connected spanning subgraph of $\mathcal{K}_{(\rho-r+2),\rho}$ where ρ is the integer $k_1/2 + \cdots + k_r/2$. Hence the claim follows from (7). ∎

In the next two results we prove the opposite inequalities in each case. We need to fix some notation. For each $i \in \{1, \ldots, r\}$, let $\sigma_i = k_1 + \cdots + k_{i-1}$, so that, in particular, $\sigma_1 = 0$.
Let us label the edges of G as in Figure 5. For each $i \in \{1, \ldots, r\}$, let $f_i, g_i \in K[E_G]$ be:

$$f_i = t_{\sigma_i+1} \cdot t_{\sigma_i+3} \cdots t_{\sigma_i+2\lceil k_i/2 \rceil - 1} \quad \text{and} \quad g_i = t_{\sigma_i+2} \cdot t_{\sigma_i+4} \cdots t_{\sigma_i+2\lfloor k_i/2 \rfloor}. \tag{8}$$

(In other words, f_i is the product of every other edge in P_i starting with t_{σ_i+1} and g_i is the product of every other edge in P_i starting with t_{σ_i+2}.) We notice