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JOINS OF CLOSED SUBLOCALES

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ABSTRACT: Sublocales that are joins of closed ones constitute a frame $\mathsf{S}_{\bigvee\mathfrak{c}}(L)$ embedded as a join-sublattice into the coframe $\mathsf{S}(L)$ of sublocales of L. We prove that in the case of subfit L it is a subcolocale of $\mathsf{S}(L)$, that it is then a Boolean algebra and in fact precisely the Booleanization of $\mathsf{S}(L)$. In case of a T_1-space X, $\mathsf{S}_{\bigvee\mathfrak{c}}(\Omega(X))$ picks precisely the sublocales corresponding to induced subspaces. In linear L and more generally if L is also a coframe, $\mathsf{S}_{\bigvee\mathfrak{c}}(L)$ is both a frame and a coframe, but with trivial exceptions not Boolean and not a subcolocale.

KEYWORDS: Frame, locale, sublocale, nucleus, sublocale lattice, open sublocale, closed sublocale, T_1-space, subfit frame, fit frame, Booleanization.

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Introduction

This paper is about sublocales, the natural subobjects in the category of locales (which one may think about as generalized topological spaces), that is, in the dual category of the category of frames. Sublocales of a frame L are well defined subsets of L, and constitute, in the natural inclusion order, a coframe $\mathsf{S}(L)$. One has open and closed sublocales (precisely corresponding to classical open and closed subspaces, see 1.3 – 1.6 below), complementing each other.

A separation axiom called subfitness (making sense for classical spaces as well, slightly weaker than T_1) is characterized by the property that *every open sublocale is a join of closed ones*, and another, stronger, called fitness (akin to regularity) is characterized by the fact that *every closed sublocale is an intersection of open ones*. These properties sound dual to each other, but is not quite so: in fact *in a fit frame every sublocale whatsoever is an intersection of open ones* which has no counterpart in the subfit case. Now

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what does the property that *every sublocale whatsoever is a join of closed ones* mean? In a previous paper [2] it was shown that it characterizes the so called scattered frames (quite analogous to scattered topological spaces), formally the L with Boolean $\mathsf{S}(L)$ (for more about scatteredness see e.g. [7, 12, 2]).

This paper is devoted to the study of the system $\mathsf{S}_{\bigvee\mathfrak{c}}(L)$ of all the sublocales that are joins of closed ones, in the setting of a general frame L. A closer scrutiny of one of the proofs in [2] shows that $\mathsf{S}_{\bigvee\mathfrak{c}}(L)$ is always a frame, even if it does not coincide with the whole of $\mathsf{S}(L)$. Now the question naturally arises: since it is a join-sublattice, is it not also a coframe, or even a subcolocale of $\mathsf{S}(L)$?

In Section 2 we prove the basics about $\mathsf{S}_{\bigvee\mathfrak{c}}(L)$ and in Section 3 we give a complete answer for subfit frames L. There, indeed, $\mathsf{S}_{\bigvee\mathfrak{c}}(L)$ is a subcolocale (and in fact this is another characterization of subfitness). Moreover, it is a Boolean algebra and in fact precisely the Booleanization of $\mathsf{S}(L)$. Further, we have here a frame extension $L \to \mathsf{S}_{\bigvee\mathfrak{c}}(L)$ by *open* sublocales; this is compared with the well known frame extension $L \to \mathsf{S}(L)^{\mathrm{op}}$ by *closed* sublocales (the embedding into the frame of congruences), and the relation is analysed.

In Section 4 we turn to the spatial case, namely to the case of T_1-spaces (in localic representation). Subspaces of a space can be viewed as sublocales (more precisely, sublocales of the associated frame $\Omega(X)$). But in general there are more sublocales than subspaces ("a space has typically generalized subspaces that are not classical induced ones"). Now it turns out that the classical ones constitute precisely the $\mathsf{S}_{\bigvee\mathfrak{c}}(L)$, and hence the Booleanization of $\mathsf{S}(L)$.

In the ultimate Section 5 we discuss the L that are sort of opposite to the subfit ones (linear frames and, more generally, frames the duals of which are frames as well). Here, the $\mathsf{S}_{\bigvee\mathfrak{c}}(L)$ are again both frames and coframes, but with trivial exceptions not subcolocales of $\mathsf{S}(L)$, and not Boolean.

1. Preliminaries

1.1. Notation. As usual, for a subset A of a poset $(X, \leq)$ we write

$$\uparrow A = \{x \in X \mid x \geq a,\ a \in A\} \text{ and abbreviate } \uparrow\{x\} \text{ to } \uparrow x.$$

If $\uparrow A = A$ we speak of an *up-set*.

A join (supremum) of a subset $A \subseteq (X, \leq)$ – if it exists – will be denoted by $\bigvee A$, and we write $a \vee b$ for $\bigvee\{a, b\}$; similarly we write $\bigwedge A$ and $a \wedge b$ for infima.

The smallest element in a poset, if it exists (the supremum $\bigvee \emptyset$), will be denoted by 0, and the largest one, the infimum $\bigwedge \emptyset$, will be denoted by 1.

The dual of a poset $(X, \leq)$, that is, the poset with the order on X reversed, will be denoted by $(X, \leq)^{\mathrm{op}}$.

1.2. Adjoint maps. If X, Y are posets we say that monotone maps $f\colon X \to Y$, $g\colon Y \to X$ are *adjoint, f to the left* and *g to the right*, and write $f \dashv g$, if

$$f(x) \leq y \quad \text{iff} \quad x \leq g(y).$$

Recall that this is characterized by $fg(y) \leq y$ and $x \leq gf(x)$, that $fgf = f$ and $gfg = g$, and in particular that

1.2.1. *if $f \dashv g$ then f (resp. g) preserves all the existing suprema (resp. infima)*, and

1.2.2. *if X, Y are complete lattices then an $f\colon X \to Y$ preserving all suprema (resp. a $g\colon Y \to X$ preserving all infima) has a right (resp. left) adjoint.*

1.3. Frames and coframes. A *frame*, resp. *coframe*, is a complete lattice L satisfying the distributivity law

$$(\bigvee A) \wedge b = \bigvee\{a \wedge b \mid a \in A\}, \tag{frm}$$

$$\text{resp.} \quad (\bigwedge A) \vee b = \bigwedge\{a \vee b \mid a \in A\}, \tag{cofrm}$$

for all $A \subseteq L$ and $b \in L$; a *frame* (resp. *coframe*) *homomorphism* preserves all joins and all finite meets (resp. all meets and all finite joins). The lattice $\Omega(L)$ of all open subsets of a topological space is a typical frame, and a typical frame homomorphism is obtained from a continuous $f\colon X \to Y$ by setting $\Omega(f)(U) = f^{-1}[U]$.

1.3.1. Note. In a frame, (cofrm) generally does not hold, similarly (frm) does not hold in a coframe. But

if b is complemented, then both (frm) *and* (cofrm) *hold in any frame and any coframe*

(see [4, 9]; for a complemented element see 1.5 below).

1.4. The (co)Heyting structure. The equality (frm) states that the maps $(x \mapsto x \wedge b)\colon L \to L$ preserve all joins. Hence, by 1.2.2, every frame is a Heyting algebra with the Heyting operation $\to$ satisfying

$$a \wedge b \le c \quad \text{iff} \quad a \le b \to c.$$

Similarly, every coframe is a coHeyting algebra with the coHeyting operation $\smallsetminus$ (which will be referred to as the *difference*) satisfying

$$a \smallsetminus b \le c \quad \text{iff} \quad a \le b \vee c.$$

From 1.2.1 we immediately obtain the following rules.

1.4.1. (1) $b \to (\bigwedge_i c_i) = \bigwedge_i (b \to c_i)$ and $(\bigvee a_i) \smallsetminus b = \bigvee_i (a_i \smallsetminus b)$,
(2) $(\bigvee_i b_i) \to c = \bigwedge_i (b_i \to c)$ and $a \smallsetminus (\bigwedge_i b_i) = \bigvee_i (a \smallsetminus b_i)$.

1.5. Pseudocomplements, supplements and complements. The Heyting resp. coHeyting structure yields in a frame the pseudocomplement $a^* = a \to 0$ $(= \bigvee\{x \mid x \wedge a = 0\})$ and in a coframe the supplement $a^\# = 1 \smallsetminus a$ $(= \bigwedge\{x \mid x \vee a = 1\})$.

An element a is said to be *complemented* if there exists a b such that $a \wedge b = 0$ and $a \vee b = 1$; this b will be referred to as the *complement* of a. In a distributive lattice (in particular, in a frame or a coframe) the complement, if it exists, is uniquely determined, and is simultaneously the pseudocomplement and the supplement of a. Therefore we will also denote it by a^* (it will be always clear we have in mind a complement which in the case in question happens to exist). The following is standard.

1.5.1. Fact. *Let a be complemented in a frame resp. coframe. Then $a \to b = a^* \vee b$ resp. $b \smallsetminus a = b \wedge a^*$.*

1.6. Sublocales and nuclei. A meet-subset in a complete lattice L is a subset $S \subseteq L$ such that for every $M \subseteq S$ the meet $\bigwedge M$ is in S. The set of all meet-sets of L will be denoted by

$$\mathcal{M}(L).$$

Obviously, for every subset $A \subseteq L$ there is the least meet-subset containing it, namely $\mathfrak{m}(A) = \{\bigwedge M \mid M \subseteq A\}$. It is easy to check that $\mathcal{M}(L)$ is a complete lattice with intersections for meets, and the joins defined by

$$\bigvee S_i = \{\bigwedge M \mid M = \bigcup_i S_i\} = \mathfrak{m}(\bigcup_i S_i).$$

Now let L be a frame. A *sublocale* of L is a meet-subset S such that

$$\forall s \in S\ \forall x \in L, \quad x \to s \in S. \tag{$*$}$$

Every sublocale S of L is a frame; the joins in S typically differ from those in L, but the meets, and hence by $(*)$ also the Heyting operation, obviously coincide. Sublocales are the natural subobjects in the category of locales, the dual of the category of frames; see e.g. [8, 9]).

Sublocales of L constitute a complete sublattice of $\mathcal{M}(L)$. It will be denoted by

$$\mathsf{S}(L)$$

(as for the join, recall 1.4.1(2)). An important fact is that *it is a coframe* (see, e.g., [5, 9]). Note that the zero of $\mathsf{S}(L)$, the least sublocale, is $\mathsf{O} = \{1\}$.

In particular we have the *open* sublocales $\mathfrak{o}(a) = \{a \to x \mid x \in L\} = \{x \mid x = a \to x\}$ and the *closed* sublocales $\mathfrak{c}(a) = {\uparrow}a$ (corresponding to the Isbell's open and *closed parts* from [4]). We will also speak of the ${\uparrow}a$ in a general complemented lattice as of closed meet-sets. The following holds.

1.6.1. Facts. (1) *$\mathfrak{o}(a)$ and $\mathfrak{c}(a)$ are complements of each other.*
(2) $\mathfrak{o}(0) = \mathsf{O}$, $\mathfrak{o}(1) = L$, $\mathfrak{o}(a \wedge b) = \mathfrak{o}(a) \cap \mathfrak{o}(b)$, $\mathfrak{o}(\bigvee_i a_i) = \bigvee_i \mathfrak{o}(a_i)$, $\mathfrak{c}(0) = L$, $\mathfrak{c}(1) = \mathsf{O}$, $\mathfrak{c}(a \wedge b) = \mathfrak{c}(a) \cap \mathfrak{c}(b)$, $\mathfrak{c}(\bigvee_i a_i) = \bigcap_i \mathfrak{o}(a_i)$.
(3) *Each sublocale S can be represented as $S = \bigcap_i(\mathfrak{o}(a_i) \vee \mathfrak{c}(b_i))$.*

1.6.2. Nuclei. A *nucleus* on a frame L is a monotone map $\nu\colon L \to L$ such that $a \le \nu(a)$, $\nu\nu(a) = \nu(a)$ and $\nu(a \wedge b) = \nu(a) \wedge \nu(b)$. Nuclei are in a one-one correspondence with sublocales, given by

$$S \mapsto \nu_S,\ \nu_S(x) = \bigwedge\{s \mid x \le s, s \in S\} \quad \text{and} \quad \nu \mapsto S_\nu = \nu[S].$$

The restriction of ν to a map $L \to S_\nu$ is a frame homomorphism, and it is the left adjoint to the embedding $\mathsf{j}\colon S_\nu \subseteq L$.

1.6.3. Booleanization. The Booleanization of L is the sublocale $\mathfrak{B}(L) = \{x \in L \mid x = x^{**}\} = \{x^* = 0 \to x \mid x \in L\}$; it is associated with the nucleus $(x \mapsto x^{**})$. Note that each sublocale containing 0 has to contain the whole of $\mathfrak{B}(L)$.

1.6.4. Dually. In the context of a coframe L we speak about a *subcolocale* $S \subseteq L$ if it is closed under joins and if for all $s \in S$ and $x \in L$, $s \smallsetminus x \in S$, and of a *conucleus* $\kappa\colon L \to L$ if $a \ge \kappa(a)$, $\kappa\kappa(a) = \kappa(a)$ and $\kappa(a \vee b) = \kappa(a) \vee \kappa(b)$.

1.7. Regularity, fitness and subfitness. Write $x \prec y$ if $x^* \vee y = 1$. A frame L is *regular* if

$$\forall a \in L,\ a = \bigvee\{x \mid x \prec a\}, \tag{reg}$$

subfit if

$$\forall a, b\ a \not\leq b\ \Rightarrow\ \exists c,\ a \vee c = 1 \neq b \vee c, \tag{sfit}$$

and *fit* if every sublocale S of L is subfit.

If X is a space then $\Omega(X)$ is regular in the sense above iff X is regular.

A frame is subfit iff every open sublocale is a join of closed ones, and it is fit iff every closed sublocale is a meet of open ones (these characteristics were in fact the original definitions of subfitness and fitness in [4]). We have the implications

$$\text{(reg)}\ \Rightarrow\ \text{(fit)}\ \Rightarrow\ \text{(sfit)}.$$

(For more about fitness and subfitness see e.g. [10, 11, 13, 14].)

A frame L is *zero-dimensional* if each $a \in L$ is a join of complemented elements. Thus, for instance, because of 1.6.1(3),

the frame $\mathsf{S}(L)^{op}$ *is zero-dimensional.*

Obviously every zero-dimensional frame is regular (and hence subfit).

1.7.1. It is a standard fact that *every sublocale of a regular frame is regular.*

1.7.2. A rather surprising formula. (See, e.g., [2, 11].) *If L is fit then for every $a \in L$,*

$$a^* = \bigwedge\{x \mid a \vee x = 1\}.$$

Note that it formally coincides with the formula for supplement; for it being really the supplement we would need the coframe distributivity. Anyway, from this observation we obtain

1.7.3. Corollary. *If L is subfit* (*in particular, regular*) *frame which is also a coframe then it is a Boolean algebra.* ■

For more about frames see e.g. [5, 9].

2. The frame $\mathsf{S}_{\bigvee\mathfrak{c}}(L)$

2.1. Let L be a complete lattice. Consider the frame

$$\mathfrak{U}(L) = \{A \subseteq L \mid \emptyset \neq A = {\uparrow}A\}$$

(ordered by inclusion) and the adjunction

$$\begin{tikzcd} \mathfrak{U}(L) \arrow[r, shift left, "\mathfrak{m}"] & \mathcal{M}(L) \arrow[l, shift left, "\mathfrak{u}"] \end{tikzcd}$$

with $\mathfrak{m}(A) = \{\bigwedge M \mid M \subseteq A\}$ and $\mathfrak{u}(A) = \{x \mid {\uparrow}x \subseteq A\}$. Checking that

$$\mathfrak{m}(A) \subseteq B \quad \text{iff} \quad A \subseteq \mathfrak{u}(B)$$

is straightforward.

2.2. Proposition. $\mathfrak{um}$ *is a nucleus on* $\mathfrak{U}(L)$ *and hence* $\mathfrak{um}[\mathfrak{U}(L)]$ *is a frame.*

Proof: By the adjunction we have $A \subseteq \mathfrak{um}(A)$ and $\mathfrak{um}(A)(\mathfrak{um}(A)) = \mathfrak{um}(A)$, hence we have to prove that $\mathfrak{um}$ preserves finite intersections. Since $\mathfrak{u}$ is a right adjoint the point is in proving that $\mathfrak{u}(\mathfrak{m}(A \cap B)) \supseteq \mathfrak{um}(A) \cap \mathfrak{um}(B) = \mathfrak{u}(\mathfrak{m}(A) \cap \mathfrak{m}(B))$.

Let $x \in \mathfrak{u}(\mathfrak{m}(A) \cap \mathfrak{m}(B))$, that is, ${\uparrow}x \subseteq \mathfrak{m}(A) \cap \mathfrak{m}(B)$. In particular, $x \in \mathfrak{m}(A)$ and $x = \bigwedge_{i \in I} a_i$ for some $a_i \in A$. As $a_i \in {\uparrow}x \subseteq B$ we have $a_i = \bigwedge_{j \in J_j} b_{ij}$ for some $b_{ij} \in B$. Since A, B are up-sets we obtain that $x = \bigwedge_{i \in I, j \in J_i} b_{ij}$ in $\mathfrak{m}(A \cap B)$. Now we can conclude that ${\uparrow}x \subseteq \mathfrak{m}(A \cap B)$ applying this reasoning to an arbitrary $y \in {\uparrow}x$. ■

2.3. Now we have the situation as in the following diagram

$$\begin{tikzcd}[column sep=large, row sep=huge] \mathfrak{U}(L) \arrow[r, shift left, "\mathfrak{m}"] \arrow[d, "\sigma"'] & \mathcal{M}(L) \arrow[l, shift left, "\mathfrak{u}"] \arrow[d, shift right, "\kappa'"'] \\ \mathfrak{um}[\mathfrak{U}(L)] \arrow[r, shift left, "\widetilde{\mathfrak{m}}"] & \mathfrak{mu}[\mathcal{M}(L)] \arrow[l, shift left, "\widetilde{\mathfrak{u}}"] \arrow[u, shift right, "\mathsf{j}'=\subseteq"'] \end{tikzcd}$$

with σ resp. κ' restrictions of $\mathfrak{um}$ resp. $\mathfrak{mu}$, and $\widetilde{\mathfrak{m}}$, $\widetilde{\mathfrak{u}}$ restrictions of $\mathfrak{m}$, $\mathfrak{u}$.

$\widetilde{\mathfrak{m}}$ and $\widetilde{\mathfrak{u}}$ are obviously mutually inverse isomorphisms, and hence

$$\mathsf{S}_{\bigvee\mathfrak{c}}(L) = \mathfrak{mu}[\mathcal{M}(L)]$$

is also a frame (the notation comes from "joins of closeds" and will be explained shortly). The mapping σ is a frame homomorphism, and the nature of κ' and in particular its restriction κ (see 2.4 below) is one of the main objectives of this paper.

Recall the formula for supremum in $\mathcal{M}(L)$ in 1.6. We have, for a meet-set S, $\mathfrak{mu}(S) = \mathfrak{m}(\bigcup\{\uparrow a \mid \uparrow a \subseteq A\}) = \bigvee\{\uparrow a \mid \uparrow a \subseteq A\}$. Thus,

$$\mathfrak{mu}(S) \textit{ is the largest join of closed meet-set contained in } S \tag{2.3.1}$$

and

$$\mathsf{S}_{\bigvee\mathfrak{c}}(L) \textit{ is the system of all joins of closed meet-sets.} \tag{2.3.2}$$

Now $\mathsf{j}'\kappa'(S) = \mathfrak{mu}(S) \subseteq S$ and for $T \in \mathsf{S}_{\bigvee\mathfrak{c}}(L)$, and hence $T = \mathfrak{mu}(T)$, $\mathsf{j}'\kappa'\mathsf{j}'(T) = \mathsf{j}'\mathfrak{mu}(T) = \mathsf{j}'(T)$. Since j' is one- one, we have $\kappa'\mathsf{j}'(T) = T$ and conclude that κ' is adjoint to the right to j' and hence

$$\kappa' \textit{ preserves meets.} \tag{2.3.3}$$

(preserving joins by j' is obvious anyway).

2.4. Now let us restrict ourselves to frames L and consider the coframe $\mathsf{S}(L)$, a complete sublattice of $\mathcal{M}(L)$. Using the notation from [2], that is, defining $U\colon \mathsf{S}(L) \to \mathfrak{U}(L)$ and $J\colon \mathfrak{U}(L) \to \mathsf{S}(L)$ by

$$\begin{aligned} U(S) &= \{a \mid \uparrow a \subseteq S\} = \bigcap\{\uparrow a \mid \uparrow a \subseteq S\} \quad \text{and} \\ J(A) &= \bigvee\{a \mid \uparrow a \subseteq S\} = \{\bigwedge B \mid B \subseteq A\}. \end{aligned}$$

we obtain an adjunction $J \dashv U$ restricting the $\mathfrak{m} \dashv \mathfrak{u}$ (by straightforward checking we see that $UJ = \mathfrak{um}$) and we get (recall 2.3) that

$\kappa(S) = JU(S)$ *is the largest join of closed sublocales contained in* S and
$\mathsf{S}_{\bigvee\mathfrak{c}}(L)$ *is the system of all joins of closed sublocales.*

The situation is depicted in the following diagram:

$$
\begin{tikzcd}
 & & \mathcal{M}(L) \arrow[dll, dotted, bend left=5, "\mathfrak{u}"] \arrow[dl, dotted] \arrow[dl, dotted, shift left=6, "\kappa'"'] \\
\mathfrak{U}(L) \arrow[urr, dotted, bend left=10, "\mathfrak{m}"] \arrow[r, shift left, "J"] \arrow[d, "\sigma"'] & \mathsf{S}(L) \arrow[l, shift left, "U"] \arrow[d, shift right, "\kappa"'] & \\
\sigma[\mathfrak{U}(L)] \arrow[r, shift left, "\widetilde{J}=\widetilde{\mathfrak{m}}"] & \mathsf{S}_{\bigvee\mathfrak{c}}(L) \arrow[l, shift left, "\widetilde{U}=\widetilde{\mathfrak{u}}"] \arrow[u, shift right, "\mathsf{j}=\subseteq"'] \arrow[uur, dotted, "\mathsf{j}'=\subseteq"'] &
\end{tikzcd}
$$

(the adjoint situation between $\mathsf{S}(L)$ and $\mathcal{M}(L)$ is not of interest here and is indicated just for completeness).

The main objective now is the nature of the mapping κ. Similarly as in 2.3 (and obviously) it is the right adjoint to the embedding j and hence

κ preserves meets.

In the sequel we will discuss the question when we have more, in particular, when $\mathsf{S}_{\bigvee\mathfrak{c}}(L)$ is a coframe and κ preserves finite joins (in other words, when j is an embedding of a subcolocale, and $JU = \mathsf{j}\kappa$ the corresponding conucleus).

2.4.1. Note. The question merits explanation. Of course, once we have more on the left hand side (σ preserves finite meets) one would like to have, by symmetry, a dual property on the right hand side. But the point is deeper. Recall the category of join-lattices from [6]. We can think of it as a relaxed category of frames (or, rather, think of frames as natural geometric specialization of join-lattices). The morphisms are join-preserving maps (right adjoints), and $\mathcal{M}(L)$ is the lattice of natural subobjects (objects embedded by meet-preserving maps). Then $\mathsf{S}_{\bigvee\mathfrak{c}}(L)$ is embedded into $\mathcal{M}(L)$ by a *join-preserving map* is a generalization of a natural embedding of a coframe into $\mathcal{M}(L)$. Thus, our question amounts to whether j is such an embedding in the frame setting – and if not, when it is.

3. The case of subfit L

3.1. Subfit frames are characterized by the fact that each open sublocale is a join of closed ones. Further, the standard facts from 1.6.1 make the mapping

$$\mathfrak{o} = (a \mapsto \mathfrak{o}(a)) \colon L \to \mathsf{S}_{\bigvee\mathfrak{c}}(L)$$

a frame embedding (and $\mathfrak{o}$ is a frame embedding only for subfit L).

3.2. Lemma. *Let L be subfit. Then for any $T \in \mathsf{S}(L)$ and $x \in L$, we have $\uparrow x \smallsetminus T \in \mathsf{S}_{\bigvee\mathfrak{c}}(L)$.*

Proof: By 1.6.1(3)

$$T = \bigcap_i (\mathfrak{o}(a_i) \vee \mathfrak{c}(b_i))$$

and therefore, by 1.4.1(1)(2),

$$\begin{aligned}\uparrow x \smallsetminus T = \mathfrak{c}(x) \smallsetminus \bigcap_i (\mathfrak{o}(a_i) \vee \mathfrak{c}(b_i)) &= \bigvee_i (\mathfrak{c}(x) \smallsetminus (\mathfrak{o}(a_i) \vee \mathfrak{c}(b_i))) \\ = \bigvee_i (\mathfrak{c}(x) \wedge \mathfrak{c}(a_i) \wedge \mathfrak{o}(b_i)) &= \bigvee_i (\mathfrak{c}(x \vee a_i) \wedge \mathfrak{o}(b_i))\end{aligned}$$

Now let L be subfit. Then, for each i, $\mathfrak{o}(b_i) = \bigvee_j \mathfrak{c}(d^i_j)$ and hence

$$\uparrow x \smallsetminus T = \bigvee_i (\mathfrak{c}(x \vee a_i) \wedge \bigvee_j \mathfrak{c}(d^i_j)) = \bigvee_{i,j} \mathfrak{c}(x \vee a_i \vee d^i_j). \quad \blacksquare$$

3.3. Theorem. *Let L be subfit (in particular, regular). Then $\mathsf{S}_{\bigvee\mathfrak{c}}(L)$ is a subcolocale of $S(L)$ (with JU from 2.4 the associated conucleus), and it is a Boolean algebra.*

Proof: Let $S = \bigvee_i \uparrow x_i$ be in $\mathsf{S}_{\bigvee\mathfrak{c}}(L)$. Since $\mathsf{S}_{\bigvee\mathfrak{c}}(L)$ is closed under joins, we need to prove that

$$\forall T \in \mathsf{S}(L),\ S \smallsetminus T \in \mathsf{S}_{\bigvee\mathfrak{c}}(L).$$

By 1.4.1(1)

$$S \smallsetminus T = \bigvee (\uparrow x_i \smallsetminus T).$$

By 3.2, all the $\uparrow x_i \smallsetminus T$ are in $\mathsf{S}_{\bigvee\mathfrak{c}}(L)$, and we see that $\mathsf{S}_{\bigvee\mathfrak{c}}(L)$ is a subcolocale. We have already observed in 2.4 that $\kappa \colon \mathsf{S}(L) \to \mathsf{S}_{\bigvee\mathfrak{c}}(L)$ is the left adjoint to the embedding, hence it is a coframe homomorphism, and $JU = \mathsf{j}\kappa$ is the associated conucleus.

Now by 1.7, $\mathsf{S}(L)^{\mathrm{op}}$ is regular and hence (recall 1.7.1) its sublocale $\mathsf{S}_{\bigvee\mathfrak{c}}(L)^{\mathrm{op}}$ is a regular frame which is also a coframe, hence by 1.7.3 a Boolean algebra. $\blacksquare$

3.4. Theorem. *Let L be subfit. Then $\kappa \colon \mathsf{S}(L) \to \mathsf{S}_{\bigvee\mathfrak{c}}(L)$ is the Booleanization of $\mathsf{S}(L)$. In particular, $\kappa = (S \mapsto S^{\#\#})$.*

Proof: By 3.3, for every sublocale $S \in \mathsf{S}(L)$, $S^{\#} = L \smallsetminus S \in \mathsf{S}_{\bigvee\mathfrak{c}}(L)$.

Now $\mathsf{S}_{\bigvee\mathfrak{c}}(L)$, as a subcolocale of $\mathsf{S}(L)$, is closed under the difference, and hence in particular under the supplement. By 3.3 again, we know that $\mathsf{S}_{\bigvee\mathfrak{c}}(L)$ is a Boolean algebra and hence the restriction of the supplement from $\mathsf{S}(L)$ is the complement in $\mathsf{S}_{\bigvee\mathfrak{c}}(L)$. Thus, for $S \in \mathsf{S}_{\bigvee\mathfrak{c}}(L)$, $S = (S^{\#})^{\#}$ and we conclude that

$$\mathsf{S}_{\bigvee\mathfrak{c}}(L) = \{S^{\#} \mid s \in \mathsf{S}(L)\} = \beta\mathsf{S}(L),$$

the Booleanization.

$(S \mapsto S^{\#\#})\colon \mathsf{S}(L) \to \mathsf{S}_{\bigvee\mathfrak{c}}(L)$ is (as always) the left adjoint to the embedding; but from 2.4 we know that the left adjoint to j is here κ and hence $\kappa = (S \mapsto S^{\#\#})$. ■

3.5. Theorem. *The following statements about a frame L are equivalent.*

(1) *L is subfit.*

(2) *$\kappa : \mathsf{S}(L) \to \mathsf{S}_{\bigvee\mathfrak{c}}(L)$ is the Booleanization of L.*

(3) *$\mathsf{S}_{\bigvee\mathfrak{c}}(L)$ is a subcolocale of $\mathsf{S}(L)$.*

(4) *$\mathsf{S}_{\bigvee\mathfrak{c}}(L)$ is Boolean.*

Proof: (1)⇒(2) is in 3.4 and (2)⇒(3)&(4) is trivial.

(3)⇒(1) immediately follows from preserving the co-Heyting operation in subcolocales. We have $L \in \mathsf{S}_{\bigvee\mathfrak{c}}(L)$ and hence $L \smallsetminus S$ in $\mathsf{S}_{\bigvee\mathfrak{c}}(L)$ for all S; in particular every $\mathfrak{o}(a) = L \smallsetminus \mathfrak{c}(a)$ is a join of closed sublocales, that is, L is subfit.

(4)⇒(1): The joins in $\mathsf{S}_{\bigvee\mathfrak{c}}(L)$ and $\mathsf{S}(L)$ coincide. Now denote by $S \sqcap T$ the meet in $\mathsf{S}_{\bigvee\mathfrak{c}}(L)$. For closed sublocales we have $\mathfrak{c}(a) \sqcap \mathfrak{c}(b) = \mathfrak{c}(a) \cap \mathfrak{c}(b) = {\uparrow}(a \vee b)$, that is, the same meet as in $\mathsf{S}(L)$. Consequently, if $S = \bigvee \mathfrak{c}(b_i)$ and if $\mathfrak{c}(a) \sqcap S = \mathsf{O}$ we have by distributivity

$$\mathsf{O} = \bigvee(\mathfrak{c}(a) \sqcap \mathfrak{c}(b_i)) = \bigvee(\mathfrak{c}(a) \cap \mathfrak{c}(b_i)) = \mathfrak{c}(a) \cap S.$$

Hence the complement S of $\mathfrak{c}(a)$ in $\mathsf{S}_{\bigvee\mathfrak{c}}(L)$ has to be also the complement of $\mathfrak{c}(a)$ in $\mathsf{S}(L)$, that is, it coincides with $\mathfrak{o}(a)$. So, again, every open sublocale is a join of closed ones. ■

3.6. $\mathsf{S}_{\bigvee\mathfrak{c}}(L)$ and the extension of L to the congruence frame. It is a well-known fact $\mathsf{S}(L)^{\mathrm{op}}$ is isomorphic to the frame of congruences on L and that the mapping $\mathfrak{c} = (a \mapsto \mathfrak{c}(a))\colon L \to \mathsf{S}(L)^{\mathrm{op}}$ is universal for frame homomorphisms $f\colon L \to M$ such that each element of $f[L]$ is complemented in M.

Hence, if L is subfit, because each $\mathfrak{o}(a)$ is in $\mathsf{S}_{\bigvee\mathfrak{c}}(L)$ and $\mathfrak{o}(a)^* = L \smallsetminus \mathfrak{o}(a) = \mathfrak{c}(a) \in \mathsf{S}_{\bigvee\mathfrak{c}}(L)$, we have a frame homomorphism $h\colon \mathsf{S}^{\mathrm{op}}(L) \to \mathsf{S}_{\bigvee\mathfrak{c}}(L)$ such that $h \cdot \mathfrak{c} = \mathfrak{o}$. Since we have $h(\mathfrak{c}(a)) = \mathfrak{o}(a)$, we obtain by complementation that $h(\mathfrak{o}(a)) = \mathfrak{c}(a)$.

The joins in $\mathsf{S}_{\bigvee\mathfrak{c}}(L)$ coincide with those in $\mathsf{S}(L)$ and hence we have for a general element $\bigvee_i \mathfrak{c}(a_i)$ of $\mathsf{S}_{\bigvee\mathfrak{c}}(L)$,

$$\bigvee_i \mathfrak{c}(a_i) = h(\bigvee_i \mathfrak{o}(a_i)) = h(\bigcap_i \mathfrak{o}(a_i)))$$

(the last intersection understood in the coframe $\mathsf{S}(L)$). Thus, h is an onto homomorphism, and hence a representation of a sublocale in $\mathsf{S}^{\mathrm{op}}(L)$.

3.6.1. Consider the mappings

$$\kappa^{\mathrm{op}}\colon \mathsf{S}(L)^{\mathrm{op}} \to \mathsf{S}_{\bigvee\mathfrak{c}}(L)^{\mathrm{op}}, \quad \mathsf{j}^{\mathrm{op}}\colon \mathsf{S}_{\bigvee\mathfrak{c}}(L)^{\mathrm{op}} \to \mathsf{S}(L)^{\mathrm{op}}$$

carried by the same mappings as κ, j, the anti-isomorphism

$$\delta\colon \mathsf{S}(L)^{\mathrm{op}} \to \mathsf{S}(L)$$

carried by the identity, and the anti-isomorphism

$$*\colon \mathsf{S}_{\bigvee\mathfrak{c}}(L)^{\mathrm{op}} \to \mathsf{S}_{\bigvee\mathfrak{c}}(L)$$

defined by $*(S) = S^*$ (we already know that $\mathsf{S}_{\bigvee\mathfrak{c}}(L)$ is a Boolean algebra). They are indicated in the following diagram.

$$\begin{tikzcd}
L \arrow[r, "\mathfrak{c}"] \arrow[drr, bend right, "\mathfrak{o}"'] & \mathsf{S}(L)^{\mathrm{op}} \arrow[r, "\delta"] \arrow[d, shift right, "\kappa^{\mathrm{op}}"'] \arrow[dr, "h"] & \mathsf{S}(L) \arrow[d, shift right, "\kappa"'] \\
& \mathsf{S}_{\bigvee\mathfrak{c}}(L)^{\mathrm{op}} \arrow[r, "*"] \arrow[u, shift right, "\mathsf{j}^{\mathrm{op}}"'] & \mathsf{S}_{\bigvee\mathfrak{c}}(L) \arrow[u, shift right, "\mathsf{j}=\subseteq"']
\end{tikzcd}$$

Proposition. *We have $h = * \cdot \kappa^{op}$ and hence $h \cdot \mathsf{j} = *$.*

Proof: $* \cdot \kappa^{\mathrm{op}}$ is a frame homomorphism (it should not confuse us that also κ^{op} is one). Indeed, we have

$$*\kappa^{\mathrm{op}}(\bigcap S_i) = *\kappa(\bigcap S_i) = * \bigwedge \kappa(S_i) = \bigvee *\kappa(S_i)$$

(note that $\bigwedge S_i$ is the *join* in $\mathsf{S}(L)^{\mathrm{op}}$ and $*$ is the complement in $\mathsf{S}_{\bigvee\mathfrak{c}}(L)$), and

$$*\kappa^{\mathrm{op}}(S \vee T) = *\kappa(S \vee T) = *\kappa(s) \wedge *\kappa(T).$$

We have

$$*\kappa^{\mathrm{op}}\mathfrak{c}(a) = *\mathfrak{c}(a) = \mathfrak{o}(a) = h\mathfrak{c}(a) \quad \text{that is,} \quad * \cdot \kappa^{\mathrm{op}} \cdot \mathfrak{c} = h \cdot \mathfrak{c}$$

and since $\mathfrak{c}$ is an epimorphism in the category of frames (see e.g. [5, 9]) we obtain that $* \cdot \kappa^{\mathrm{op}} = h$. Since $\kappa^{\mathrm{op}} \cdot \mathsf{j}^{\mathrm{op}}$ is the identity, the second equality follows. ■

3.6.2. Proposition. *The sublocale of* $\mathsf{S}(L)^{op}$ *associated with the onto homomorphism* h *is*

$$\{T^* \mid T \in \mathsf{S}_{\bigvee\mathfrak{c}}(L)\}.$$

Consequently we also have a subcolocale $\{T^* \mid T \in \mathsf{S}_{\bigvee\mathfrak{c}}(L)\}$ *of* $\mathsf{S}(L)$ *isomorphic with* $\mathsf{S}_{\bigvee\mathfrak{c}}(L)$.

Proof: We have $h(S) = *\kappa^{\mathrm{op}}(S) \le T$ in $\mathsf{S}_{\bigvee\mathfrak{c}}(L)$ iff $\kappa^{\mathrm{op}}(S) \ge T^*$ in $\mathsf{S}_{\bigvee\mathfrak{c}}(L)$, that is, $\kappa^{\mathrm{op}}(S) \le T^*$ in $\mathsf{S}_{\bigvee\mathfrak{c}}(L)^{\mathrm{op}}$, iff $S \le \mathsf{j}^{\mathrm{op}}(T^*)$ in $\mathsf{S}(L)^{\mathrm{op}}$. Thus, the right adjoint to h is $(T \mapsto \mathsf{j}(T^*) = T^*)$. ■

4. T_1-spatial frames

If X is a T_1-space then it (resp. the frame $\Omega(X)$) is subfit. Hence the subcolocale and Booleanization facts of 4.3 and 4.5 below follow from the results in Section 3. The point of this section is in providing some concrete formulas and explaining the mechanisms in this particular spatial case.

4.1. Recall that a frame is *spatial* if it is isomorphic to $\Omega(X)$ for some topological space X. Not every spatial L can be represented using a T_1-space X. If this can be done we speak of a T_1-*spatial* frame.

A spatial frame L is characterized by the fact that every element $a \in L$ is a meet of *prime* elements ($p \neq 1$ is prime if $p \le x \wedge y$ implies that either $p \le x$ or $p \le y$; the set of primes of L will be denoted by $\mathsf{Prime}(L)$) while in a T_1-spatial frame each a is a meet of *maximal* elements (that is, $p < 1$ such that $p < x$ only for $x = 1$; the set of such p will be denoted by $\mathsf{Max}(L)$). It should be noted that in his famous theorem on spatiality of subfit compact frames ([4], see also [10]), Isbell in fact proved that these frames are T_1-spatial.

A spatial frame typically has sublocales that are not spatial. But even a spatial sublocale of $\Omega(X)$ is not necessarily $\Omega(Y)$ for a subspace Y. Of those that are we speak as of *induced subspaces*. Induced subspaces of T_1-spatial frames are obviously T_1-spatial. The spaces such that all sublocales of $\Omega(X)$ are (induced) subspaces are precisely the scattered ones ([7, 9]).

4.2. Recall that a subset $\{p, 1\}$ of a frame is a sublocale iff p is prime [9]. Hence, induced spatial sublocales of a spatial frame L are precisely the sublocales of the form

$$S = \bigvee\{P = \{p, 1\} \mid p \in X\} \quad \text{with} \quad X \subseteq \mathsf{Prime}(L),$$

and since $\{p, 1\}$ with $p < 1$ is an up-set iff p is maximal, the induced sublocales *of the T_1-representation* are then precisely the

$$S = \bigvee\{P = \{p, 1\} \mid p \in X\} \quad \text{with} \quad X \subseteq \mathsf{Max}(L), \tag{4.2.1}$$

the T_1*-subspaces* of L in the terminology of 4.1.

4.3. Theorem. *Let L be T_1-spatial. Then every S in $\mathsf{S}_{\bigvee\mathfrak{c}}(L)$ is of the form $S = \bigvee\{\uparrow m \mid m \in M\}$ for some $M \subseteq \mathsf{Max}(L)$. Consequently, $\mathsf{S}_{\bigvee\mathfrak{c}}(L)$ is precisely the set of all T_1-subspaces of L (and hence, it is a Boolean algebra).*

Proof: Let S be in $\mathsf{S}_{\bigvee\mathfrak{c}}(L)$. Then we have

$$S = \bigvee\{\uparrow a \mid a \in A\}$$

where the $A \subseteq L$ can be assumed an up-set. For $x \in L$ set

$$M_x = \uparrow x \cap \mathsf{Max}(L).$$

Since L is T_1-spatial we have $x = \bigwedge M_x$. Since, obviously, $\uparrow a = \bigvee\{\uparrow x \mid x \geq a\}$ we have $\uparrow a = \bigvee\{\uparrow m \mid m \in M_x, x \geq a\}$ so that, taking into account that A is an up-set,

$$S = \bigvee\{\uparrow m \mid m \in \bigcup\{M_a \mid a \in A\}\} \tag{4.3.1}$$

(if $b = \bigwedge N$ with $N \subseteq A$ then $M_n \subseteq A$ for all $n \in N$ and we have

$$b = \bigwedge_{n \in N} \bigwedge M_n = \bigwedge(\bigcup\{M_n \mid n \in N\})$$

with $\bigcup\{M_n \mid n \in N\} \subseteq \bigcup\{M_a \mid a \in A\}$; obviously $\bigcup\{M_a \mid a \in A\} \subseteq A$).

On the other hand, since the points $\{p, 1\}$ with $p \in \mathsf{Max}(L)$ are closed, by (4.2.1) each T_1-subspace is in $\mathsf{S}_{\bigvee\mathfrak{c}}(L)$. ■

4.3.1. Observation. *If $S = \bigvee\{\uparrow m \mid m \in M\}$ for some $M \subseteq \mathsf{Max}(L)$ then $M = \mathsf{Max}(L) \cap S$.*

Proof: Each $m \in M$ is in S, hence $M \subseteq \mathsf{Max}(L) \cap S$. On the other hand, let $s \in \mathsf{Max}(L) \cap S$. Then $s = \bigwedge_i m_i$ for some $m_i \in M$. Since $s \leq m_i$ and since it is maximal and $m_i \neq 1$, we have $s = m_i \in M$. ■

4.4. Proposition. *Let L be T_1-spatial. Let S be in $\mathsf{S}_{\bigvee\mathfrak{c}}(L)$ and let T be in $\mathsf{S}(L)$. Then we have for the difference (co-Heyting operation in $\mathsf{S}(L)$)*

$$S \smallsetminus T = \bigvee\{\uparrow x \mid x \in \mathsf{Max}(L),\ x \in S,\ x \notin T\}.$$

Proof: Set $V = \bigvee\{\uparrow x \mid x \in \mathsf{Max}(L),\ x \in S,\ x \notin T\}$. Suppose $U \vee T \supseteq S$. Let $x \in S$ be maximal and $x \notin T$. Then $x = u \wedge t$ with $u \in U$ and $t \in T$, hence $x \leq u, t$ and by maximality and since $x \neq t \in T$, $t = 1$ and $x = u \in U$, and finally $\uparrow x \subseteq U$. Thus, $U \supseteq W$.

On the other hand, let $U \supseteq W$. Since $S \in \mathsf{S}_{\bigvee\mathfrak{c}}(L)$ it suffices to prove that for every $s \in S$ such that $\uparrow s \subseteq S$, $s \subseteq U \vee T$. Take such an s; like every element of L it is $\bigwedge\{x \mid x \in \mathsf{Max}(L),\ x \geq s\}$. Write it as

$$s = \bigwedge\{x \mid x \in \mathsf{Max}(L),\ x \geq s, x \notin T\} \wedge \bigwedge\{x \mid x \in \mathsf{Max}(L),\ x \geq s,\ x \in T\}.$$

Since $\uparrow s \subseteq S$ the first factor is in W and hence in U, and the second one is in T, hence $s \in U \vee T$. ■

Now we have a very explicit formula for $S \smallsetminus T$ and obtain, like in 3.3,

4.4.1. Theorem. *Let L be T_1-spatial. Then $\mathsf{S}_{\bigvee\mathfrak{c}}(L)$ is a subcolocale of $S(L)$ (with $\kappa\colon \mathsf{S}(L) \to \mathsf{S}_{\bigvee\mathfrak{c}}(L)$ and JU from 2.4 the associated frame homomorphism and conucleus respectively).*

4.5. Theorem. *Let X be a T_1-space. Then the Booleanization of $\mathsf{S}(\Omega(X))$ consists precisely of the classical subspaces of X. The homomorphism κ from 2.4 is again the Booleanization map $S \mapsto S^{\#\#}$ and can be written as $\kappa(S) = \bigvee\{\uparrow x \mid x \in \mathsf{Max}(L) \cap S\}$.*

Proof: By 4.4, we have the formula for the supplement

$$S^{\#} = L \smallsetminus T = \bigvee\{\uparrow x \mid x \in \mathsf{Max}(L) - S\}$$

("$-$" stands for the standard difference of subsets). Further,

$$S^{\#\#} = L \smallsetminus S^{\#} = \bigvee\{\uparrow x \mid x \in \mathsf{Max}(L) - S^{\#}\}.$$

By 4.3.1, $\mathsf{Max}(L) \cap S^{\#} = \mathsf{Max}(L) - S$ and hence

$$\begin{aligned}\mathsf{Max}(L) - S^{\#} &= \mathsf{Max}(L) - (\mathsf{Max}(L) \cap S^{\#})\\ &= \mathsf{Max}(L) - (\mathsf{Max}(L) - S) = \mathsf{Max}(L) \cap S.\end{aligned}$$

Now it is also straightforward to see directly that

$$\bigvee\{\uparrow x \mid x \in \mathsf{Max}(L) \cap S\} = \bigvee\{\uparrow x \mid \uparrow x \subseteq S\}. \quad \blacksquare$$

4.6. Remark. The question naturally arises what about the spaces X that are subfit but not T_1. In this case, of course, 3.4 still holds and we have that $\mathsf{S}_{\bigvee\mathfrak{c}}(L)$ is the Booleanization od $\mathsf{S}(L)$. But it is not any more the system of all subspaces. Representing X as a part of the spectrum (see [9, 11]) we cannot do with taking just the maximum elements p for representing the points. For a p that is prime but not maximum, that has to appear in the representation of one of the one-point sets, the associated sublocale $\{p, 1\}$ is not closed, and for trivial reasons cannot be written as a join of closed ones.

But there is a deeper reason. If the space is not T_D ([1]), subspaces are not perfectly represented by special sublocales anyway (see, e.g., [3, 11]), while if a space is T_D and subfit then it is automatically T_1.

5. Some very simple, and different, cases

5.1. Proposition. *If L is linear then $\mathsf{S}_{\bigvee\mathfrak{c}}(L) \cong L^{op}$. Then $\mathsf{S}_{\bigvee\mathfrak{c}}(L)$ is the set of all closed sublocales ordered by inclusion and hence it is both a frame and a coframe (but with the exception of $L = \{0\}$ and $L = \{0, 1\}$ not a Boolean algebra).*

Proof: We have $\bigvee\{\uparrow a \mid a \in A\} = \uparrow\bigwedge A$ so that $\mathsf{S}_{\bigvee\mathfrak{c}}(L)$ is the lattice of all the closed sublocales ordered by inclusion. $\blacksquare$

5.2. The fact in the previous proposition is a part of a more general one.

Theorem. *Let L be both a frame and a coframe. Then $\mathsf{S}_{\bigvee\mathfrak{c}}(L)$ is the set of all closed sublocales ordered by inclusion, hence $\mathsf{S}_{\bigvee\mathfrak{c}}(L) \cong L^{op}$ (and it is both a frame and coframe).*

Proof: We have

$$\bigvee\uparrow a_i = \uparrow(\bigwedge a_i).$$

Indeed, obviously $\bigwedge a_i$ is the least element of $\bigvee\uparrow a_i$ and hence $\bigvee\uparrow a_i \subseteq \uparrow(\bigwedge a_i)$. Now let $x \geq \bigwedge a_i$. Then $x = x \vee \bigwedge a_i = \bigwedge(x \vee a_i)$ (we are in a coframe) and $x \vee a_i \in \uparrow a_i$ so that $x \in \bigvee\uparrow a_i$. $\blacksquare$

5.3. Another, this time non-linear, case included in 5.2 is the following.

Corollary. *If L is quasidiscrete (Alexandroff), that is, the topology of up-sets of a poset, then* $\mathsf{S}_{\bigvee\mathfrak{c}}(L) \cong L^{op}$.

5.4. In all the cases in this section, $\mathsf{S}_{\bigvee\mathfrak{c}}(L)$ are, as in the previous ones, not only frames but also coframes. But there is a fundamental difference: they are typically not Boolean, and, what is more important, not subcolocales.

5.4.1. Observations. (1) *If L is linear then for any two sublocales, $S \vee T = S \cup T$.*
(2) *If L is linear then S is a sublocale iff it is a meet-set.*
(3) *If L is linear then the difference of two closed sublocales is $\uparrow a \smallsetminus \uparrow b = \langle a, b) \cup \{1\}$.*

Proof: (1) $S \vee T = \{s \wedge t \mid s \in S,\ t \in T\}$ and $s \wedge t \in \{s, t\}$.
(2) $x \to s$ is either 1 or s.
(3) $\langle a, b) \cup \{1\} = (\uparrow a - \uparrow b) \cup \{1\}$ where "$-$" designates, again, the set-theoretical difference (note that the right hand side is really a meet-set and hence a sublocale). Now for any subset $U \subseteq L$ we have $\uparrow a - \uparrow b \subseteq U$ iff $\uparrow a \subseteq U \cup \uparrow b$ and hence by Observations (1) and (3), for any sublocale U, $\uparrow a \smallsetminus \uparrow b \subseteq U$ iff $\uparrow a \subseteq U \vee \uparrow b$. ∎

5.4.2. Proposition. *Let L be linear and let it contain at least four elements. Then $\mathsf{S}_{\bigvee\mathfrak{c}}(L)$ is not a subcolocale of $\mathsf{S}(L)$.*

Proof: Take a chain $a < b < x < 1$ in L. By Observation (3), the difference $\uparrow a \smallsetminus \uparrow b$ in $\mathsf{S}(L)$ does not contain x and hence it is not an up-set, unlike the difference in $\mathsf{S}_{\bigvee\mathfrak{c}}(L)$. ∎

5.5. Note. The $\mathsf{S}_{\bigvee\mathfrak{c}}(L)$'s in this section were not subcolocales of $\mathsf{S}(L)$ (and not Boolean, either) but they were still both frames and coframes. An example of an $\mathsf{S}_{\bigvee\mathfrak{c}}(L)$ that is not a coframe (or proving that it is always one) remains an open problem.

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