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A SEIFERT–VAN KAMPEN THEOREM IN NON-ABELIAN ALGEBRA

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Abstract: We prove a Seifert–van Kampen theorem in a non-additive setting, providing sufficient conditions on a functor $F\colon \mathscr{C} \to \mathscr{X}$ from an algebraically coherent semi-abelian category with enough projectives $\mathscr{C}$ to an abelian category $\mathscr{X}$ for its first left derived functor $\mathscr{L}_1 F$ to preserve pushouts of split monomorphisms.

Keywords: Derived functor, fundamental group, homology coproduct theorem, categorical Galois theory, algebraically coherent semi-abelian category.
Math. Subject Classification (2010): 18G10, 18G50, 18A40, 20J, 55N.

1. Introduction

Seifert and Van Kampen's famous theorem on the fundamental group of a union of two spaces [66, 71] has been sharpened and extended to other contexts in many ways [17, 40, 56, 20, 67, 19, 74, 21, 68]. Let us here recall the following elementary version. Consider a commutative square of pointed topological spaces and base-point-preserving continuous maps as on the left,

$$\begin{array}{ccc} O & \xrightarrow{j} & B \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \iota_B} \\ A & \xrightarrow[\iota_A]{} & X \end{array} \qquad\qquad \begin{array}{ccc} \pi_1(O) & \xrightarrow{\pi_1(j)} & \pi_1(B) \\ {\scriptstyle \pi_1(i)}\downarrow & & \downarrow{\scriptstyle \pi_1(\iota_B)} \\ \pi_1(A) & \xrightarrow[\pi_1(\iota_A)]{} & \pi_1(X) \end{array}$$

where A, B and O are open, path-connected subspaces of X, and

· the square is a pushout, $X = A +_O B$ is the amalgamated sum of A and B over O;

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· the square is a pullback, $O = A \times_X B = A \cap B$ is the intersection of A and B in X.

Then the square on the right obtained by applying the fundamental group functor is a pushout in the category of groups.

The aim of our article is to present a variation on this result, valid in a general algebraic context. We prove a Seifert–van Kampen theorem in a non-additive setting, giving sufficient conditions on a functor $F\colon \mathscr{C} \to \mathscr{X}$ from an *algebraically coherent* [24] *semi-abelian* [49] category $\mathscr{C}$ to an *almost abelian* [65] category $\mathscr{X}$ for its fundamental group functor π_1^F to preserve pushouts of split monomorphisms. Here $\mathscr{C}$ may, for instance, be the category of groups, rings, crossed modules, or Lie algebras over a field; while $\mathscr{X}$ may for example be a category of modules, Banach spaces or locally compact abelian groups. When $\mathscr{X}$ is abelian, this fundamental group functor π_1^F is precisely the classical left derived functor $\mathscr{L}_1 F$ of F.

Motivation. This work is part of a long-term project aiming to understand group (co)homology from a categorical-algebraic perspective. Within this larger context, our interest in a general Seifert–van Kampen theorem is twofold. On the one hand, through an analysis of this theorem we are studying which aspects of group (co)homology are typical for groups, and which arise for purely formal reasons, so that a categorical argument suffices to comprehend and apply these in other settings. On the other hand, such an analysis helps us see what the needs of homological algebra tell us about categories of non-abelian algebraic structures: we are asking ourselves what are the right conditions on a category for a given result in homological algebra to hold. These questions may have unexpected answers—for instance, the one given here which involves a seemingly unrelated technical condition called *algebraic coherence* [24].

A special case: preservation of binary sums. In the special case where the pushout under consideration is a coproduct, our Seifert–van Kampen theorem may be seen as a non-abelian version of a fact which is well known in the abelian case. Indeed, for any additive functor $F\colon \mathscr{C} \to \mathscr{X}$ between abelian categories $\mathscr{C}$ and $\mathscr{X}$ where $\mathscr{C}$ has enough projectives, the left derived functors $\mathscr{L}_n F\colon \mathscr{C} \to \mathscr{X}$ are additive [23]. In other words, if F preserves binary coproducts, then each derived functor of F does. One goal of the article is to explore how this extends to a general *homology coproduct theorem* in the sense of [3], valid in a non-abelian algebraic context.

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Known non-abelian results. Of course several instances of a non-abelian homology coproduct theorem can already be found in the literature. For example, given any two groups X, Y and any $n \geqslant 0$, there is the isomorphism

$$H_{n+1}(X+Y,\mathbb{Z}) \cong H_{n+1}(X,\mathbb{Z}) \oplus H_{n+1}(Y,\mathbb{Z}) \qquad (\mathbf{A})$$

describing the integral homology of the free product $X+Y$ as the direct sum of the homologies of X and Y. See [2, 3, 73] for an algebraic proof of this result, which may also be obtained by using the corresponding isomorphism for topological spaces (via the classical Seifert–van Kampen theorem). This may be reinterpreted as an isomorphism

$$\mathscr{L}_n F(X+Y) \cong \mathscr{L}_n F(X) \oplus \mathscr{L}_n F(Y)$$

where F is the abelianisation functor $\mathrm{ab}\colon \mathsf{Gp} \to \mathsf{Ab}$, which sends a group X to its abelianisation $\mathrm{ab}(X) = X/[X,X]$.

Similar results hold, for instance, for rational homotopy of topological spaces [26] and for homology of graded associative algebras [5, 44, 42]; see also [54]. Barr and Beck give several examples of homology coproduct theorems in the context of comonadic homology [3].

A negative result. Attempting to find a general non-additive homology coproduct theorem for $\mathscr{L}_n F$, one is immediately confronted with the following counterexample, which was obtained in [51] as a test of the homology theory for varieties of groups established in [53]. If $F\colon \mathsf{Nil}_2 \to \mathsf{Ab}$ is the reflector from the category of groups of nilpotency class at most two to the category of abelian groups, so the restriction to Nil_2 of the abelianisation functor considered above, then

$$\mathscr{L}_2 F(C_2 + C_2) = C_2 \oplus C_2 \oplus C_2 \ncong C_2 \oplus C_2 = \mathscr{L}_2 F(C_2) \oplus \mathscr{L}_2 F(C_2),$$

where C_2 is the cyclic group of order two.

We have to conclude that a general homology coproduct theorem applicable to the functor $\mathrm{ab}\colon \mathsf{Nil}_2 \to \mathsf{Ab}$ cannot exist. This leaves open essentially only two courses of investigation:

(1) to study the case $n = 1$ in a setting wide enough to include the variety of groups of nilpotency class at most two;
(2) to study the situation for general n, but in a much more restricted context which includes groups while ruling out the category Nil_2.

In the present paper we focus on (1) in order to prove a general homology coproduct theorem for $\mathscr{L}_1 F$ as a particular case of an algebraic Seifert–van

Kampen theorem. We hope to come back to (2) in future work; for now, in Section 5 we make some further remarks about this problem.

Our setting. The situation we consider in this paper is that of a regular epi–reflection

$$\mathscr{C} \underset{\supseteq}{\overset{F}{\underset{\perp}{\rightleftarrows}}} \mathscr{X}$$

where $\mathscr{C}$ is an algebraically coherent [24] semi-abelian [49] category with enough regular projectives and $\mathscr{X}$ is an abelian category. It is this functor F which we are going to derive—using simplicial resolutions [69] in the abstract case, or comonadic resolutions [3] when $\mathscr{X}$ is a variety of algebras. Alternatively, the standard Quillen model structure on the category of simplicial objects in $\mathscr{C}$ may be used [61, 70].

Some examples. In Section 2 we shall recall the definitions of the above-mentioned categorical-algebraic conditions. For the time being, let us just mention that any *Orzech category of interest* [60] is semi-abelian algebraically coherent, so that we might consider any reflector from such a variety $\mathscr{C}$ to an abelian subvariety $\mathscr{X}$ of $\mathscr{C}$. For instance, the reflector from groups to abelian groups, from groups of a certain nilpotency class to abelian groups, from crossed modules to abelian crossed modules, from Lie algebras over a field $\mathbb{K}$ to $\mathbb{K}$-vector spaces, from G-actions to $\mathbb{Z}[G]$-modules for a given group G all fit into this setting. More examples will be given in Section 2 and Section 3.

The fundamental group functor. The proof of our result is based on categorical Galois theory [46, 47, 48, 9], with in particular the interpretation [36, 32, 28] of the derived functor $\mathscr{L}_1 F$ as a fundamental group functor π_1^F. This leads to a slightly more general result, which remains valid even when $\mathscr{X}$ is not abelian, but only *almost abelian* in the sense of [65].

In the given context, there is the concept of an *extension* (= a regular epimorphism in $\mathscr{C}$) and that of a *normal extension* (defined with respect to the functor $F\colon \mathscr{C} \to \mathscr{X}$). It turns out that the inclusion $\mathsf{NExt}(\mathscr{C}) \to \mathsf{Ext}(\mathscr{C})$ of the category of normal extensions into the category of extensions admits a left adjoint. This left adjoint functor is called the **normalisation functor** relative to F and is denoted $F_1\colon \mathsf{Ext}(\mathscr{C}) \to \mathsf{NExt}(\mathscr{C})$.

For instance, an extension in the category of groups is just a surjective group homomorphism. If now F is the abelianisation functor $\mathrm{ab}\colon \mathsf{Gp} \to \mathsf{Ab}$, then a

Proof: The pushout result is Theorem 5.1 via the interpretation of left derived functors as Galois groups, taking the discussion in Subsection 3.6 into account. Recall that $\mathscr{L}_0 F = F$ preserves pushouts since it is a left adjoint functor. The short exact sequences follow from the construction of pushouts in abelian categories [35, Proposition 2.53], and the fact that the composite

$$\mathscr{L}_n F(O) \xrightarrow{\Delta_{\mathscr{L}_n F(O)}} \mathscr{L}_n F(O) \oplus \mathscr{L}_n F(O) \xrightarrow{i_* \oplus j_*} \mathscr{L}_n F(A) \oplus \mathscr{L}_n F(B)$$

is a (split) monomorphism. ∎

Remark 5.3. One technique used in [3] by Barr and Beck in order to obtain a homology coproduct theorem such as (**A**) is to assume that a sum of two resolutions is again a resolution. This hardly ever happens, so this technique is of limited use when trying to prove general results. However, a trace of their idea occurs in the proof of Theorem 5.1, when Proposition 2.7 is used to obtain a presentation of a pushout square.

Following the interpretation in [31] of n-fold extensions as finite-dimensional resolutions, part of Barr and Beck's argument may be recoverable when the sum of two n-fold extensions is still an n-fold extension. We do not know how this condition relates to other conditions studied in categorical algebra.

Remark 5.4. Another categorical-algebraic condition which may be related to the problem of extending Theorem 5.2 to higher degrees is the following. In [39, 14] a semi-abelian variety is called **locally algebraically cartesian closed (LACC)** when the change of base functors considered in 2.8 preserve binary coproducts. It is easy to see that this condition implies algebraic coherence. Examples include the categories of groups, Lie algebras, crossed modules, cocommutative Hopf algebras over a field of characteristic zero. On the other hand, the category Nil_2 of groups of nilpotency class at most 2 is not an example. Incidentally, Nil_2 is the counterexample from [51] which appeared on page 3. Since (LACC) is one of the rare categorical-algebraic conditions satisfied by Gp but not by Nil_2, perhaps a higher-order version of Theorem 5.1 is valid for (LACC) semi-abelian categories with enough projectives.

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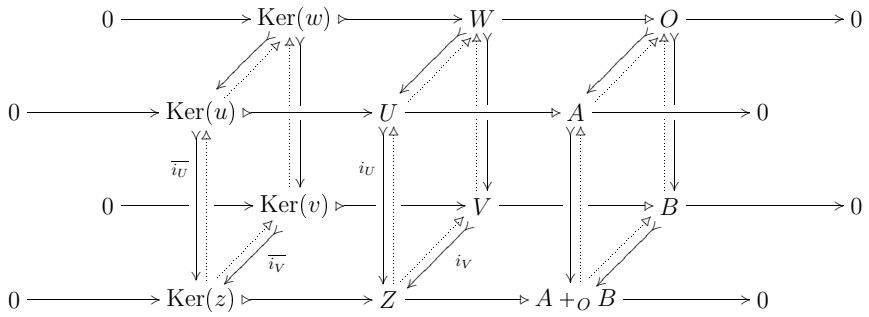
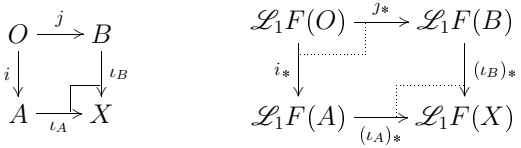


FIGURE 5. Taking kernels in the right hand side cube of Figure 3 yields a short exact sequence in $\mathsf{Pt}^2(\mathscr{C})$.

is a pushout diagram in $\mathscr{X}$. Consequently, by Proposition 2.12 the dotted square in the same diagram is a pullback, which is preserved by the right adjoint inclusion H.

Let us now prove that also the central dotted square in Figure 4 is a pullback. Again by Proposition 2.12, for this it suffices that the pair $(\overline{i_U}, \overline{i_V})$ is jointly (strongly) epimorphic in $\mathscr{X}$ or, equivalently, in $\mathscr{C}$. To see this, we consider the diagram in Figure 5 and apply Proposition 4.5. ∎

Theorem 5.2. *Let $F\colon \mathscr{C} \to \mathscr{X}$ be the Birkhoff reflector from a semi-abelian algebraically coherent category with enough projectives $\mathscr{C}$ to an abelian category $\mathscr{X}$. The functor F may, for instance, be the reflector from an* Orzech category of interest *to an abelian subvariety. Consider in $\mathscr{C}$ a pushout of split monomorphisms as on the left.*



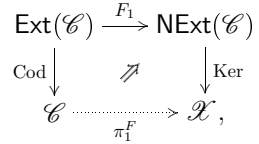
Then the square on the right is a pushout square in $\mathscr{X}$. Moreover, we have "Mayer–Vietoris" short exact sequences [58, 72]

$$0 \longrightarrow \mathscr{L}_nF(O) \xrightarrow{\langle i_*, j_*\rangle} \mathscr{L}_nF(A) \oplus \mathscr{L}_nF(B) \xrightarrow{[(\iota_A)_* \; -(\iota_B)_*]} \mathscr{L}_nF(X) \longrightarrow 0$$

for $n \in \{0, 1\}$. As a consequence, the pushout on the right is also a pullback.

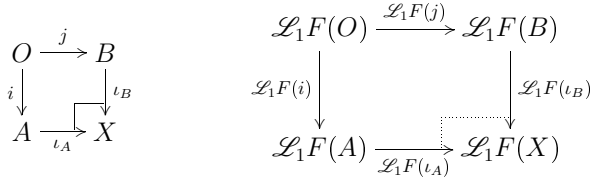
normal extension is the same thing as a central extension: a surjective group homomorphism $f\colon X \to Y$ for which the elements of the kernel K commute with all the elements of X. The functor F_1 takes an extension $f\colon X \to Y$ and sends it to the quotient $F_1(f)\colon X/[K, X] \to Y$. Here the commutator $[K, X]$ is generated by the elements of the form $kxk^{-1}x^{-1}$ where $k \in K$ and $x \in X$.

When it exists, the **fundamental group functor** $\pi_1^F\colon \mathscr{C} \to \mathscr{X}$ relative to the functor F is defined as the pointwise right Kan extension [55]



where Cod sends an extension $f\colon X \to Y$ to its codomain Y and Ker sends a (normal) extension g to its kernel $\mathrm{Ker}(g)$. If $\mathscr{C}$ has enough projectives and $\mathscr{X}$ is abelian, then the functor π_1^F corresponds to the first left derived functor of F obtained via simplicial resolutions: we have an isomorphism $\pi_1^F \cong \mathscr{L}_1F$, where the derived functor on the right is as in [3, 69]. In the case of abelianisation of groups, for instance, we see that $\pi_1^F(X) \cong H_2(X, \mathbb{Z})$, the second integral homology group.

The result. We prove that the first fundamental group functor $\pi_1^F\colon \mathscr{C} \to \mathscr{X}$ preserves pushouts of split monomorphisms, so that—when the derived functor in question is a fundamental group functor—for any pushout of split monomorphisms i and j in $\mathscr{C}$ as on the left,



the resulting square on the right is a pushout in $\mathscr{X}$. In particular, by taking $O = 0$ it follows that for any A and B in $\mathscr{C}$,

$$\mathscr{L}_1F(A + B) \cong \mathscr{L}_1F(A) \oplus \mathscr{L}_1F(B).$$

Structure of the text. In Section 2 we recall some basic terminology and sketch the context in which we shall be working. Section 3 recalls Galois-theoretic notions and results concerning the fundamental group functor. Most of the real work is done in Section 4 where some technical lemmas having to do with jointly strongly epimorphic pairs of arrows are proved. Section 5 contains our main result, the Seifert–van Kampen theorem. We obtain two versions: first, in Theorem 5.1, we consider the case where $\mathscr{X}$ is an almost abelian category [65], and the left adjoint reflector F preserves pullbacks of split epimorphisms along regular epimorphisms; then we find that π_1^F preserves pushouts of split monomorphisms. We next restrict the setting to the case where $\mathscr{X}$ is abelian and F is a Birkhoff reflector. Then π_1^F is a genuine left derived functor of F, and we obtain Theorem 5.2.

2. Terminology

Semi-abelian categories were introduced in [49] in order to unify several historically important approaches to a categorical framework for the study of homological algebra of non-abelian algebraic objects (e.g., [41, 45, 1, 10]). Next to all abelian categories, any *variety of* Ω*-groups* in the sense of [41] is an example. So for a variety of algebras to be semi-abelian, it suffices that amongst the operations and identities of the theory, there is a unique constant and a group operation. This easily provides us with many examples such as groups, (non-unitary) rings, associative algebras, Lie algebras, crossed modules, etc. Further examples of a different kind include loops [7], Heyting semilattices [52], cocommutative Hopf algebras over a field of characteristic zero [38], and the dual of the category of pointed sets [13].

2.1. Formal definition. A category is **semi-abelian** [49] when it is pointed, exact and protomodular with binary coproducts. In presence of the other axioms, the existence of binary coproducts suffices for the category to be finitely complete and finitely cocomplete.

Pointed means that there is a **zero object**, written 0: an initial object which is also terminal. Often, the zero object is a one-element algebra. A morphism which factors through the zero object is called a **zero map**; given any two objects A and B, a unique such $0\colon A \to B$ exists. For any morphism $f\colon A \to B$, the coequaliser of f and 0 is called the **cokernel** of f, and written $\operatorname{coker}(f)\colon B \to \operatorname{Coker}(f)$. Dually, we write $\ker(f)\colon \operatorname{Ker}(f) \to A$ for the **kernel** of f, the equaliser of f and 0. A **short exact sequence** is a pair of

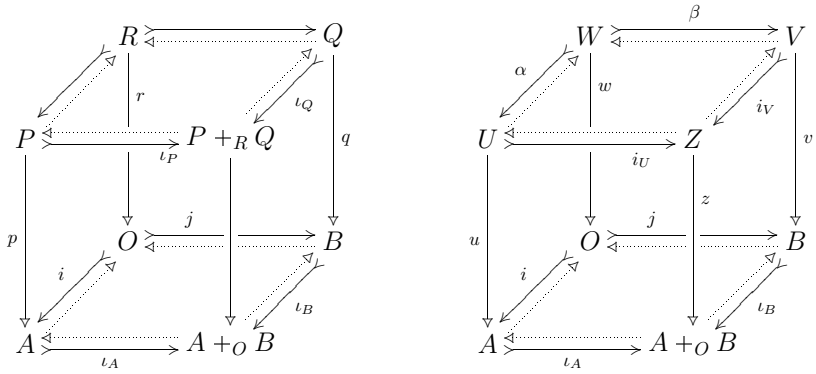


FIGURE 3. A projective cover of a pushout square, and its normalisation.

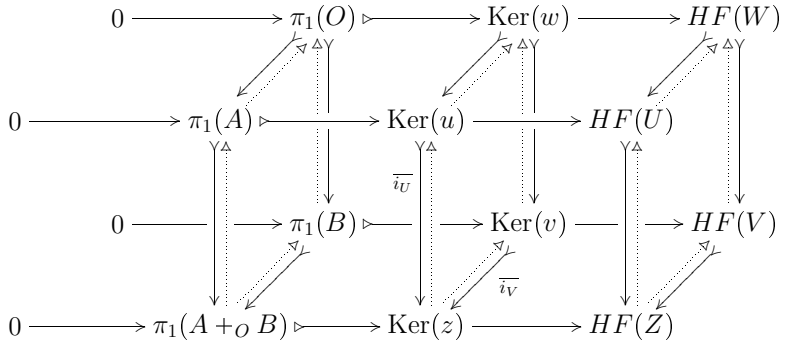
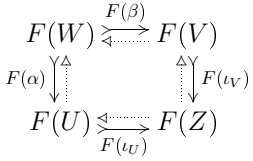


FIGURE 4. The exact sequence (**G**) induced by the right hand side cube of normal extensions in Figure 3.

find that the solid square in



it a trivial extension—see 3.1. Furthermore, the kernel of $f^*(f)$ is K. The conclusion now follows from Lemma 4.3. ∎

5. The Seifert–van Kampen theorem

We are now ready to prove our main result. We start with a version of the Seifert–van Kampen theorem for fundamental group functors, Theorem 5.1. We then restrict our attention to the situation where those fundamental group functors are the derived functors of a Birkhoff reflector and obtain Theorem 5.2. After that we finish with some final questions and comments.

Theorem 5.1. *Let Γ be a Galois structure as in* (**E**) *and let us consider in the category $\mathscr{C}$ a pushout of split monomorphisms as on the left.*

$$\begin{array}{ccc} O & \overset{j}{\rightleftarrows} & B \\ {\scriptstyle i}\downarrow\uparrow & & \uparrow\downarrow{\scriptstyle \iota_B} \\ A & \underset{\iota_A}{\rightleftarrows} & A +_O B \end{array} \qquad \begin{array}{ccc} \pi_1^F(O) & \overset{\pi_1^F(j)}{\rightleftarrows} & \pi_1^F(B) \\ {\scriptstyle \pi_1^F(i)}\downarrow\uparrow & & \uparrow\downarrow{\scriptstyle \pi_1^F(\iota_B)} \\ \pi_1^F(A) & \underset{\pi_1^F(\iota_A)}{\rightleftarrows} & \pi_1^F(A +_O B) \end{array}$$

Then the square on the right is a pushout square in $\mathscr{X}$.

Proof: Proposition 2.7 gives us a way of covering the given pushout square by a pushout of projective presentations as in the left hand side cube of Figure 3. Normalising, we obtain a square in $\mathsf{NExt}(\mathscr{C})$ represented by the right hand side cube. The top diagram in this cube is an object of $\mathsf{Pt}^2(\mathscr{C})$, in which the solid square is jointly strongly epimorphic: (i_U, i_V) is a jointly strongly epimorphic pair, as a regular quotient of the pair (ι_P, ι_Q).

In Figure 4, all rows are exact as in (**G**). In order to prove our claim, we only need to show that the central and the right hand side upward directed dotted squares of this figure are pullbacks. Indeed, then the dotted square on the left is a pullback as well, and the result follows from Proposition 2.12.

We first show that the right hand side dotted square in Figure 4 is a pullback. Recalling from Subsection 3.2 that $F(P) \cong F(U)$, $F(Q) \cong F(V)$, $F(R) \cong F(W)$ and $F(P +_R Q) \cong F(Z)$, we see that the two cubes in Figure 3 have the same image through F. Since the reflector F preserves colimits, we

morphisms (k, f) where $k = \ker(f)$ and $f = \operatorname{coker}(k)$; this situation is usually pictured as a sequence

$$0 \longrightarrow K \overset{k}{\rightarrowtriangle} A \overset{f}{\twoheadrightarrow} B \longrightarrow 0.$$

A morphism which is the cokernel of some morphism is called a **normal epimorphism**, and a morphism which is the kernel of some morphism is called a **normal monomorphism**. In many varieties of algebras, normal monomorphisms correspond to inclusions of (suitably defined) ideals. The cokernel of a homomorphism $f\colon A \to B$ may then be obtained as the quotient of B by the normal closure of the image of f. It is important to notice that not all monomorphisms need to be normal: indeed, non-normal subgroups exist.

The concept of an image itself is formalised as follows. A morphism is called a **regular epimorphism** when it is the coequaliser of some parallel pair of arrows. In any variety of algebras these are precisely the surjective homomorphisms—unlike the epimorphisms, which in this context turn out to play a lesser role. Obviously, by definition, any normal epimorphism is a regular epimorphism; conversely, in a semi-abelian category, the two concepts may be seen to coincide. A category is **regular** [1] if it is finitely complete, every morphism can be factorised as a regular epimorphism followed by a monomorphism, and regular epimorphisms are stable under pullbacks. The monomorphism in this (essentially unique) factorisation of a morphism f is called the **image** of f.

A **relation** (R, d_1, d_2) from A to B is a subobject of the product $A \times B$, represented by a monomorphism $\langle d_1, d_2\rangle\colon R \to A \times B$. Relations are naturally ordered by inclusion. In the context of a regular category, it is possible to consider the composite of two relations, and then an **internal equivalence relation** on an object A may be defined as a relation R which is reflexive ($\Delta_A \leqslant R$, where $\Delta_A = \langle 1_A, 1_A\rangle\colon A \to A \times A$), symmetric ($R^{\mathrm{op}} \leqslant R$, where $(R^{\mathrm{op}}, d_1^{\mathrm{op}}, d_2^{\mathrm{op}}) = (R, d_2, d_1)$) and transitive ($R \circ R \leqslant R$). The **kernel relation** or **kernel pair** $(\mathrm{Eq}(f), \mathrm{pr}_1^f, \mathrm{pr}_2^f)$ of a morphism $f\colon A \to B$, which is the pullback of f along itself, considered as a subobject of $A \times A$, is always an equivalence relation on A. The converse need not hold in general. Whence the definition of a **(Barr) exact** [1] category, which is regular and such that every internal equivalence relation is the kernel relation of some morphism. It is well known that all varieties of algebras are Barr exact.

A pointed category with finite limits is **protomodular** [10] when the **Split Short Five Lemma** holds. This means that whenever we have a diagram

$$\begin{array}{ccccc} \mathrm{Ker}(p) & \rightarrowtail & A & \underset{s}{\overset{p}{\rightleftarrows}} & C \\ {\scriptstyle k}\downarrow & & {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle c} \\ \mathrm{Ker}(p') & \rightarrowtail & A' & \underset{s'}{\overset{p'}{\rightleftarrows}} & C' \end{array}$$

in which the three squares commute while $p \circ s = 1_C$ and $p' \circ s' = 1_{C'}$, if the morphisms k and c are isomorphisms then a is an isomorphism.

A category which is pointed, regular and protomodular is said to be **homological** [7]. In a homological category, the basic lemmas of homological algebra, such as the *Short Five Lemma*, the 3×3 *Lemma* and the *Snake Lemma* hold. Such a category is Barr exact if and only if the image of a composite $p \circ k$, where p is a normal epimorphism and k is a normal monomorphism, is again a normal monomorphism. Then it suffices that binary coproducts exist for the category to be semi-abelian.

Examples are given below in 2.10. For now, let us just recall the following

Lemma 2.2. [22, Theorem 5.7] *In a semi-abelian category, let us consider a square of regular epimorphisms*

$$\begin{array}{ccc} X & \overset{p_B}{\twoheadrightarrow} & B \\ {\scriptstyle p_A}\downarrow & & \downarrow{\scriptstyle b} \\ A & \underset{a}{\twoheadrightarrow} & O. \end{array}$$

The comparison morphism $\langle p_A, p_B \rangle \colon X \to A \times_O B$ to the pullback of a and b is a regular epimorphism if and only if the square is a pushout. ∎

2.3. The category of points. Protomodularity may be reformulated in more abstract terms as follows.

Let $\mathscr{C}$ be a category with pullbacks of split epimorphisms along any morphism. A **point in** $\mathscr{C}$ is a pair $(p \colon A \to C, s \colon C \to A)$ of morphisms in $\mathscr{C}$ such that $p \circ s = 1_C$: it is a **split epimorphism** p with a chosen splitting s. (Equivalently, of course, a point (p, s) may be considered as a **split monomorphism** s with a chosen splitting p.) The **category of points** $\mathsf{Pt}(\mathscr{C})$ has as objects points in $\mathscr{C}$. Morphisms are natural transformations between those.

where B and D are jointly strongly epimorphic squares and C is a pushout square. Then A is also a jointly strongly epimorphic square.

Proof: Let us consider the commutative diagram of solid arrows in Figure 2, where the objects and arrows in $\mathscr{C}$ are numbered as in (**H**). Lemma 2.6 tells us that the morphisms $\overline{k}_3 \coloneqq k_3 \circ \langle a_1 \ a_2 \rangle \colon A_1 +_{A_0} A_2 \to D_3$ and $\overline{f}_3 \coloneqq f_3 \circ \langle a_1 \ a_2 \rangle \colon A_1 +_{A_0} A_2 \to B_3$ are regular epimorphisms. By Lemma 2.2, it remains to show that the square $g_3 \circ \overline{f}_3 = h_3 \circ \overline{k}_3$ is a pushout. This is easily checked using the definition of pushouts and the fact that C is a pushout square. ∎

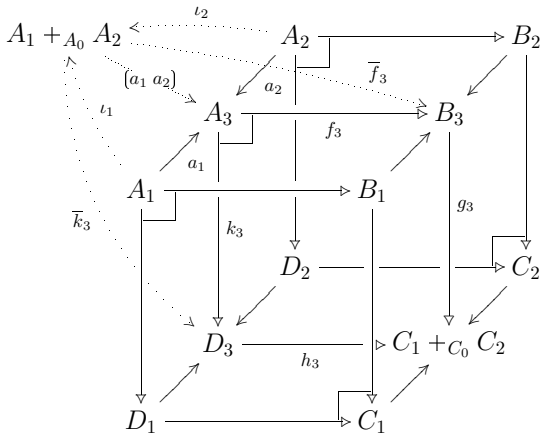


FIGURE 2. Situation sketch for Lemma 4.4.

Proposition 4.5. *Let*

$$0 \longrightarrow K \overset{k}{\rightarrowtail} A \overset{f}{\twoheadrightarrow} B \longrightarrow 0$$

be a short exact sequence in $\mathsf{Pt}^2(\mathscr{C})$ where f is component-wise a normal extension. If A, considered as a square of split monomorphisms, is jointly strongly epimorphic, and B is a pushout square, then K is also a jointly strongly epimorphic square.

Proof: Lemma 4.4 allows us to reduce the situation to the special case where f is a both a split epimorphism and a trivial extension. Indeed, the pullback $f^*(f)$ of f along itself is a split epimorphic normal extension, which makes

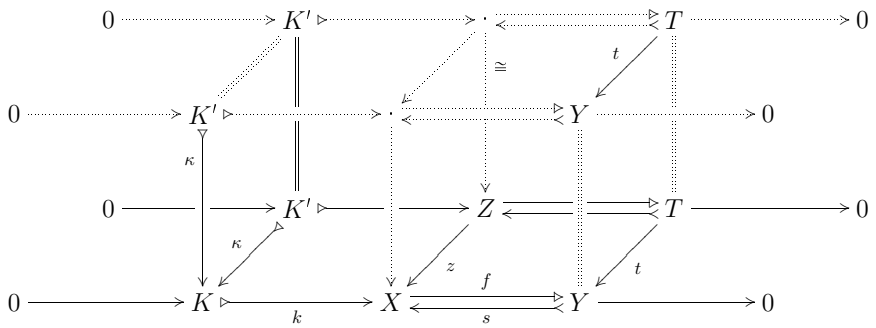


FIGURE 1. A square of split extensions induced by Lemma 4.1.

If f is a trivial extension, κ and κ' are split monomorphisms and (z, z') is a jointly strongly epimorphic pair, then also the pair (κ, κ') is jointly strongly epimorphic.

Proof: Lemma 3.4 tells us that the composites $k \circ \kappa$ and $k \circ \kappa'$ are normal monomorphisms. Hence we can apply Lemma 4.2 twice to reduce the situation to the special case where t and t' are identities. The result then follows from algebraic coherence via Proposition 2.9. ∎

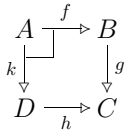
When the cospans under consideration are induced by squares of split monomorphisms, this result may be refined as in Proposition 4.5.

We say that a commutative square

$$A = \begin{pmatrix} A_0 & \longrightarrow & A_2 \\ \downarrow & & \downarrow{\scriptstyle a_2} \\ A_1 & \xrightarrow[a_1]{} & A_3 \end{pmatrix} \in \mathsf{Arr}^2(\mathscr{C}) \qquad (\mathbf{H})$$

is **jointly strongly epimorphic** when such is its underlying cospan (a_1, a_2). Note that a pushout square is always jointly strongly epimorphic.

Lemma 4.4. *Suppose that $\mathscr{C}$ is semi-abelian. Consider in $\mathsf{Arr}^2(\mathscr{C})$ a pullback of regular epimorphisms*



More precisely, an arrow

$$f\colon (p\colon A \to C, s\colon C \to A) \to (p'\colon A' \to C', s'\colon C' \to A')$$

is a pair $f = (f_1\colon A \to A', f_0\colon C \to C')$ such that $p' \circ f_1 = f_0 \circ p$ and $f_1 \circ s = s' \circ f_0$.

We write $\mathsf{Pt}^2(\mathscr{C})$ for the category $\mathsf{Pt}(\mathsf{Pt}(\mathscr{C}))$. Its objects are split epimorphisms of split epimorphisms with chosen splittings: squares such as (**D**) below.

The forgetful codomain functor

$$\mathrm{Cod}\colon \mathsf{Pt}(\mathscr{C}) \to \mathscr{C}\colon \quad (p\colon A \to C, s\colon C \to A) \mapsto C, \quad f = (f_1, f_0) \mapsto f_0$$

is a fibration (see [6] for instance) which is called the **fibration of points**. We write $\mathsf{Pt}_X(\mathscr{C})$ for the fibre over X, which is the subcategory of $\mathsf{Pt}(\mathscr{C})$ determined by the morphisms f such that $\mathrm{Cod}(f) = 1_X$. The objects of this category are called **points over X in $\mathscr{C}$**.

For any morphism $f\colon A \to B$ in $\mathscr{C}$, pulling back a point (p, s) along the morphism f induces a change-of-base functor

$$f^*\colon \mathsf{Pt}_B(\mathscr{C}) \to \mathsf{Pt}_A(\mathscr{C}).$$

It is not very difficult to see that the category $\mathscr{C}$ is protomodular if, and only if, all change-of-base functors f^* are **conservative**, which means that they reflect isomorphisms.

When $\mathscr{C}$ is homological, there is [15] a concept of *internal action* which induces an equivalence between "split extensions" and "actions"—formally, this is the monadicity of the functor $\mathrm{Cod}\colon \mathsf{Pt}(\mathscr{C}) \to \mathscr{C}$. For us right now, it suffices to understand the following interpretation of this fact. Any point (p, s) induces a split extension

$$0 \longrightarrow K \overset{\ker(p)}{\rightarrowtail} A \underset{s}{\overset{p}{\rightleftarrows}} C \longrightarrow 0.$$

This split extension corresponds to an internal action ξ of C on K in the sense of [15] via a general semi-direct product construction: there is a split extension

$$0 \longrightarrow K \overset{\ker(p_\xi)}{\rightarrowtail} K \rtimes_\xi C \underset{s_\xi}{\overset{p_\xi}{\rightleftarrows}} C \longrightarrow 0$$

which turns out to be isomorphic to the given split extension.

In other words, a point (p, s) over C in $\mathscr{C}$ may be seen as some kind of a "non-abelian Beck module" [4] via the internal C-action it induces on the kernel K of p. Protomodularity thus amounts to the condition that any morphism of C-actions which induces an isomorphism $K \to K'$ on the underlying objects

(= kernels) K and K' necessarily is an isomorphism itself. This point of view may help better understanding the definition in 2.8 below.

2.4. Epimorphisms. A morphism $f\colon A \to B$ in a finitely complete category $\mathscr{C}$ is

- an **extremal epimorphism** if, for every commutative triangle as on the left with m a monomorphism, m is an isomorphism;

- a **strong epimorphism** if, for every commutative square as on the right with m a monomorphism, there exists a (unique) morphism $t\colon B \to D$ such that $v = m \circ t$ (and $t \circ f = w$).

It turns out that the two concepts are equivalent and that, when $\mathscr{C}$ is semi-abelian, both concepts coincide with the notions of regular and of normal epimorphism, so with the surjective homomorphisms in case $\mathscr{C}$ is a variety of algebras. However, to be a split epimorphism is a stronger condition, and to be an epimorphism is weaker.

2.5. Cospans. Let $\mathscr{C}$ be a finitely complete category. A **cospan** in $\mathscr{C}$ is a pair (f, g)

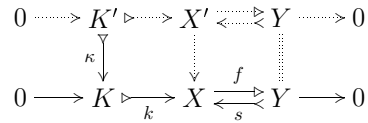
$$A \xrightarrow{f} B \xleftarrow{g} C \tag{B}$$

of morphisms in $\mathscr{C}$. We will make use of the category $\mathsf{Cospan}(\mathscr{C})$ of cospans. A morphism $(a, b, c)\colon (f', g') \to (f, g)$ is a commutative diagram

$$\begin{array}{ccccc} A' & \xrightarrow{f'} & B' & \xleftarrow{g'} & C' \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} \\ A & \xrightarrow[f]{} & B & \xleftarrow[g]{} & C. \end{array} \tag{C}$$

A cospan such as (f, g) in (**B**) is said to be

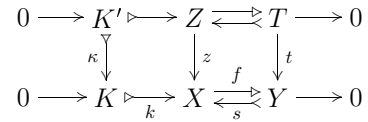
Lemma 4.1. [25, Lemma 2.6] *In a semi-abelian category, consider a point with chosen kernel as in the bottom row of the diagram*



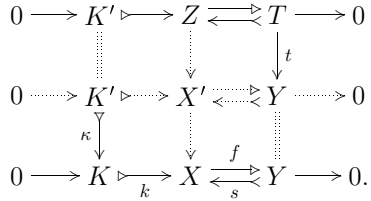
and a monomorphism $\kappa\colon K' \to K$ such that $k \circ \kappa$ is a normal monomorphism. Then this point lifts along κ to yield a morphism of points with chosen kernels. Moreover, this process is functorial in the data given. ∎

The functoriality in the above lemma implies the following.

Lemma 4.2. *In a semi-abelian category, given a morphism of split extensions such as*

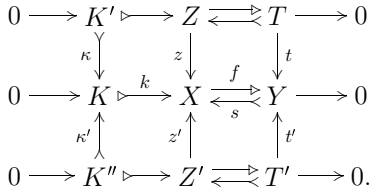


where $k \circ \kappa$ is a normal monomorphism, this morphism factors as a composite



Proof: The needed factorisation is induced by Lemma 4.1 as in the square of split extensions of Figure 1. ∎

Proposition 4.3. *In an algebraically coherent semi-abelian category, consider a cospan of split extensions*



and

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ker}(\eta_P) & \rightarrowtail & P & \xrightarrow{\eta_P} & HF(P) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \rho_p^1} & & \| & & \\ 0 & \longrightarrow & \mathrm{Ker}(\eta_U) & \rightarrowtail & U & \xrightarrow[\eta_U]{} & HF(U) & \longrightarrow & 0 \end{array}$$

so that by using some standard compositions / cancelations of pullbacks, we may see that (**F**) holds.

3.6. Birkhoff subcategories. When the subcategory $\mathscr{X}$ is closed under regular quotients, so that $\mathscr{X}$ is a **Birkhoff subcategory** [48] of $\mathscr{C}$, results of [30] show that the first fundamental group functor is the left derived functor of the reflector F: we have an isomorphism $\pi_1^F \cong \mathscr{L}_1(F)$. An important class of examples occurs when F is the reflector from a semi-abelian algebraically coherent variety $\mathscr{C}$ to an abelian subvariety $\mathscr{X}$ of $\mathscr{C}$. For instance, the reflector from groups to abelian groups, from groups of a certain nilpotency class to abelian groups, from crossed modules to abelian crossed modules, from Lie algebras over a field $\mathbb{K}$ to $\mathbb{K}$-vector spaces, from G-actions to $\mathbb{Z}[G]$-modules for a given group G.

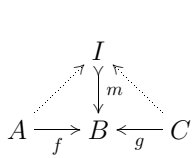
By Corollary 9 and Example 13 in [8] combined, the condition that $\mathscr{X}$ is a Birkhoff subcategory of $\mathscr{C}$ is equivalent to its exactness. That is to say, $\mathscr{X}$ is abelian (= additive + exact, by Tierney's result [1]) if and only if it is Birkhoff. Note, furthermore, that any Birkhoff reflector preserves pullbacks of regular epimorphisms along split epimorphisms [37, 33].

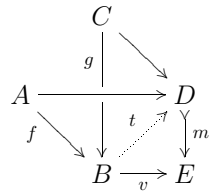
Other (non-Birkhoff) examples are, for instance, the reflection of Gp to the category of torsion-free abelian groups, or the "reflection" from the category of cocommutative Hopf algebras over a field of characteristic zero to the category of abelian groups (considered as group Hopf algebras).

4. Some results on jointly strongly epimorphic pairs

Here we prove some technical results having to do with the preservation of jointly strongly epimorphic pairs by kernel functors in algebraically coherent categories. We start with Proposition 4.3, a variation on Proposition 2.9.

· **jointly extremally epimorphic** if, for every commutative diagram as on the left with m a monomorphism, m is an isomorphism;





· **jointly strongly epimorphic** if, for every commutative diagram of solid arrows as on the right with m a monomorphism, there exists a (unique) morphism $t\colon B \to D$ such that $v = m \circ t$.

As above in 2.4, the two concepts are actually equivalent. We may view such a cospan as a way of surjectively covering the object B with the objects A and C.

It is easily seen that a jointly strongly epimorphic pair (f, g) in a regular category with binary coproducts is the same thing as a regular epimorphism $\begin{pmatrix} f & g \end{pmatrix}\colon A+C \to B$ with a coproduct $A+C$ as domain. Consequently, in such a category, the full subcategory of $\mathsf{Cospan}(\mathscr{C})$ determined by the jointly strongly epimorphic pairs is closed under regular quotients. There is also the following, closely related result, which follows easily from the definitions:

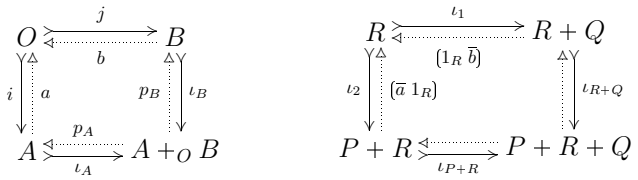
Lemma 2.6. *In a finitely complete category, consider a commutative diagram such as* (**C**)*. If a and c are strong epimorphisms and the pair (f, g) is jointly strongly epimorphic, then b is a strong epimorphism.*

Proof: Pulling any monomorphism m through which the cospan $(f \circ a, g \circ c)$ would factor back along the morphisms f and g, then using that the morphisms a, c and the pair (f, g) are extremally epimorphic, it is easily shown that b is an extremal epimorphism as well. ∎

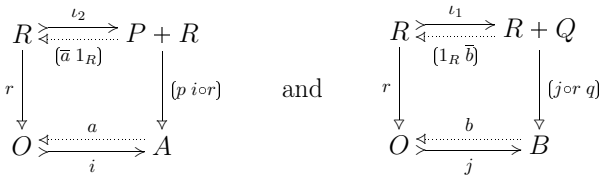
A **projective presentation** $p\colon P \to A$ of an object A is a regular epimorphism where the object P is projective (with respect to the class of regular epimorphisms). A category $\mathscr{C}$ **has enough projectives** when for every object a projective presentation exists. For instance, in a variety of algebras we may always take the canonical homomorphism $LU(A) \to A$, from the free algebra $LU(A)$ on the underlying set $U(A)$ of A, to the given algebra A.

Proposition 2.7. *Let $\mathscr{C}$ be finitely (co)complete with enough projectives. Any pushout of split monomorphisms in $\mathscr{C}$ may be covered by a pushout in $\mathsf{Arr}(\mathscr{C})$ of split monomorphisms of presentations.*

Proof: Given a pushout of split monomorphisms as on the left

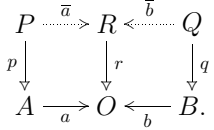


we consider projective presentations $p\colon P \to A$, $q\colon Q \to B$ and $r\colon R \to O$, and construct the pushout on the right. Note that all objects in it are projective. Since the squares of solid arrows



commute, Lemma 2.6 tells us that the induced arrow between the pushouts $(p\ i{\circ}r) +_r (j{\circ}r\ q)\colon P + R + Q \to A +_O B$ is a projective presentation.

We let $\overline{a}$ and $\overline{b}$ be liftings in the diagram



These make the two above squares commute to the left, so that the original pushout square of split monomorphisms is covered by a pushout of projectives in a way which is compatible with the given splittings. ∎

2.8. Algebraically coherent categories. A category $\mathscr{C}$ is **algebraically coherent** [24] when it is finitely complete and for every morphism $f\colon A \to B$ in $\mathscr{C}$, the change of base functor

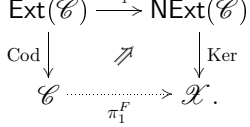
$$f^*\colon \mathsf{Pt}_B(\mathscr{C}) \to \mathsf{Pt}_A(\mathscr{C})$$

is **coherent**, which means that it preserves finite limits and jointly strongly epimorphic pairs. In the semi-abelian context, a simple argument shows that

As a consequence, $\eta_A{\circ}k{\circ}n$ is the kernel of $HF(q)$. It follows that $k{\circ}n$ is the kernel of q, since q is a trivial extension, so that the square $\eta_Q{\circ}q = HF(q){\circ}\eta_A$ is a pullback.

This latter claim still remains to be shown. On the one hand, it is easily verified by hand that the square $\eta_Q{\circ}q = HF(q){\circ}\eta_A$ is a pushout. Hence Lemma 2.2 tells us that the comparison morphism $\langle \eta_A, q\rangle\colon A \to HF(A) \times_{HF(Q)} Q$ is a regular epimorphism. On the other hand, since the square $\eta_B{\circ}f = HF(f){\circ}\eta_A$ is a pullback, the pair (η_A, f) is jointly monomorphic. Hence also η_A and q are jointly monomorphic, so that $\langle \eta_A, q\rangle$ is an isomorphism, and the claim follows. ∎

3.5. First fundamental group functor and Hopf formula. As explained in the introduction, the first fundamental group functor is defined as a pointwise right Kan extension: $\pi_1^F = \mathrm{Ran}_{\mathrm{Cod}}(\mathrm{Ker}\circ F_1)$,



Its value in some object A can be described by a so-called *Hopf formula*—first obtained in [43, 18, 34, 27, 33, 30], then extended to the present context in [28].

Given a projective presentation $p\colon P \to A$ of A, we have

$$\pi_1^F(A) \cong \frac{\mathrm{Ker}(p) \cap \mathrm{Ker}(\eta_P)}{\mathrm{Ker}(\rho_p^1)}.$$

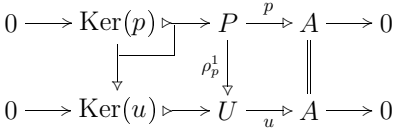
Using this, we easily find that

$$\pi_1^F(A) \cong \mathrm{Ker}(u) \cap \mathrm{Ker}(\eta_U) \qquad (\mathbf{F})$$

for $u \coloneqq F_1(p)\colon U \to A$, so that there is an exact sequence

$$0 \longrightarrow \pi_1^F(A) \rightarrowtail \mathrm{Ker}(u) \longrightarrow HF(U). \qquad (\mathbf{G})$$

Actually, there exist morphisms of short exact sequences



an extension $f\colon A \to B$ is of the form $(\rho_f^1, 1_B)$

$$\begin{array}{ccc} A & \xrightarrow{\rho_f^1} & F_1[f] \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle F_1(f)} \\ B & = & B \end{array}$$

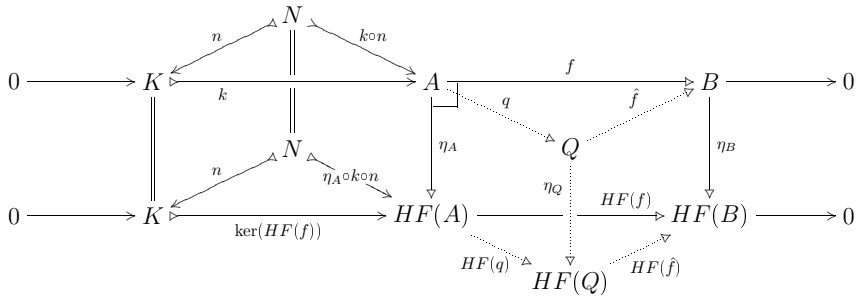
where ρ_f^1 is a regular epimorphism. Furthermore, $F(\rho_f^1)$ is an isomorphism. As a consequence:

Proposition 3.3. *The full subcategory* $\mathsf{NExt}(\mathscr{C})$ *determined by the normal extensions is a regular epi–reflective subcategory of the category* $\mathsf{Ext}(\mathscr{C})$. ∎

See [28] for a more complete account. This allows us to prove the following lemma, needed in Section 4.

Lemma 3.4. *If* $n\colon N \to K$ *is a split monomorphism and* $k\colon K \to A$ *is the kernel of a trivial extension, then the composite monomorphism* $k{\circ}n\colon N \to A$ *is also the kernel of a trivial extension.*

Proof: Let $f\colon A \to B$ be a trivial extension of which k is the kernel, and let us consider the commutative diagram



where q is the cokernel of the composite $k{\circ}n$. Since f is a trivial extension, the bottom right square is a pullback, which implies that the kernel of $HF(f)$ is $\eta_A{\circ}k\colon K \to HF(A)$. In particular, the object K lies in $\mathscr{X}$. Since $\mathscr{X}$ is closed under subobjects in $\mathscr{C}$, also N is in $\mathscr{X}$. Hence $\eta_N = 1_N$, so that $F(k{\circ}n) = \eta_A{\circ}k{\circ}n$. This latter composite $\eta_A{\circ}k{\circ}n$ is a normal monomorphism as a composite of two regular monomorphisms ($\ker(HF(f))$ and n) in the cohomological category $\mathscr{X}$. Since the left adjoint F preserves cokernels,

$$F(q) = \operatorname{coker}(F(k{\circ}n)) = \operatorname{coker}(\eta_A{\circ}k{\circ}n).$$

this condition is equivalent to the coherence of the "kernel functors"

$$\mathrm{Ker} = !_B^*\colon \mathsf{Pt}_B(\mathscr{C}) \to \mathsf{Pt}_0(\mathscr{C}) \cong \mathscr{C}$$

forgetting the B-action, which can be expressed quite easily as follows.

Proposition 2.9. *A semi-abelian category* $\mathscr{C}$ *is algebraically coherent if and only if given any cospan of points over any object* B

$$\begin{array}{ccccc} K' & \xrightarrow{h} & K & \xleftarrow{k} & K'' \\ {\scriptstyle \ker(p')}\downarrow & & \downarrow{\scriptstyle \ker(p)} & & \downarrow{\scriptstyle \ker(p'')} \\ A' & \xrightarrow{f} & A & \xleftarrow{g} & A'' \\ {\scriptstyle s'}\uparrow\downarrow{\scriptstyle p'} & & {\scriptstyle s}\uparrow\downarrow{\scriptstyle p} & & {\scriptstyle s''}\uparrow\downarrow{\scriptstyle p''} \\ B & = & B & = & B \end{array}$$

where the pair (f, g) *is jointly strongly epimorphic in* $\mathscr{C}$*, the pair* (h, k) *is also jointly strongly epimorphic in* $\mathscr{C}$. ∎

In other words, if a B acts on K, K' and K'', and the objects K' and K'' cover K in the category of B-actions in $\mathscr{C}$, then K' and K'' cover K in the underlying category $\mathscr{C}$.

Algebraically coherent semi-abelian categories have good stability properties. For instance, categories of diagrams and categories of regular epimorphisms in an algebraically coherent category are also algebraically coherent, as is any regular epi–reflective subcategory. Furthermore, as explained in [24], algebraic coherence has important categorical-algebraic consequences, such as the *Smith is Huq* property [57], *normality of Higgins commutators* [25], or *strong protomodularity* [12, 62].

2.10. Examples. Any additive category, in particular any almost abelian category, is semi-abelian and algebraically coherent. On the other hand, not every semi-abelian category is: for instance, the categories of loops and of Jordan algebras are not. Examples of algebraically coherent semi-abelian categories are the categories of groups, associative algebras, Lie algebras, Leibniz algebras, Poisson algebras over a commutative ring with unit, all *Neumann varieties of groups* [59], all *Orzech categories of interest* [60], next to the categories of rings, crossed modules, and cocommutative Hopf algebras over a field of characteristic zero [38].

2.11. Almost abelian categories. A category $\mathscr{C}$ is **almost abelian** [65] when both $\mathscr{C}$ and $\mathscr{C}^{\mathrm{op}}$ are homological [64]. The original definition is the following: $\mathscr{C}$ is almost abelian if it is additive, every morphism has a kernel and a cokernel, normal epimorphisms are stable under pullbacks and normal monomorphisms are stable under pushouts. Examples of almost abelian categories are the categories of locally compact abelian groups, normed vector spaces, Banach spaces (with morphisms the bounded linear maps) and Fréchet spaces.

In such a category, every **regular monomorphism** (equaliser of some parallel pair of morphisms) is a normal monomorphism (a kernel).

We will use the following proposition (see [11] and [63, Lemma 1.8]):

Proposition 2.12. *Let $\mathscr{C}$ be an almost abelian category and*

$$\begin{array}{ccc} A & \underset{f_1}{\overset{s_1}{\rightleftarrows}} & A' \\ {\scriptstyle p}\downarrow\uparrow{\scriptstyle s} & & {\scriptstyle p'}\downarrow\uparrow{\scriptstyle s'} \\ C & \underset{f_0}{\overset{s_0}{\rightleftarrows}} & C' \end{array} \tag{D}$$

be an object of $\mathsf{Pt}^2(\mathscr{C})$, so that every square in (**D**) *commutes and*

$$f_1 \circ s_1 = 1_{A'}, \quad f_0 \circ s_0 = 1_{C'}, \quad p \circ s = 1_C, \quad p' \circ s' = 1_{C'}.$$

Then the square $f_0 \circ p = p' \circ f_1$ is a pullback if and only if $s \circ s_0 = s_1 \circ s'$ is a pushout. Consequently, in a diagram as above, the comparison morphism $\langle p, f_1 \rangle \colon A \to C \times_{C'} A'$ always has a section $\nu \colon C \times_{C'} A' \to A$ such that

$$s = \nu \circ \langle 1_C, s' \circ f_0 \rangle, \quad s_1 = \nu \circ \langle s_0 \circ p', 1_{A'} \rangle.$$

Then the square $f_0 \circ p = p' \circ f_1$ is a pullback precisely when the pair (s, s_1) is jointly epimorphic. ∎

3. Fundamental group functors and derived functors

Our proof of the Seifert–van Kampen theorem is based on an interpretation of the first left derived functor $\mathscr{L}_1 F$ of a given functor $F \colon \mathscr{C} \to \mathscr{X}$ as a fundamental group functor relative to F, defined via categorical Galois theory. Thus the context where our result applies is, first of all, limited by the scope of the latter general framework. We consider a fixed adjunction

$$\mathscr{C} \underset{H}{\overset{F}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathscr{X} \tag{E}$$

with unit $\eta \colon 1_{\mathscr{C}} \Rightarrow H \circ F$ and counit $\epsilon \colon F \circ H \Rightarrow 1_{\mathscr{X}}$ such that

- $\mathscr{C}$ is semi-abelian and algebraically coherent with enough projectives;
- $\mathscr{X}$ is almost-abelian;
- $\eta_C \colon C \to HF(C)$ is a regular epimorphism for all C in $\mathscr{C}$;
- $\epsilon_X \colon FH(X) \to X$ is an isomorphism for all X in $\mathscr{X}$;
- F preserves pullbacks of split epimorphisms along regular epimorphisms.

Such an adjunction naturally gives rise to a **closed Galois structure** Γ; in the sense of [47]. We call the regular epimorphisms in $\mathscr{C}$ **extensions** and we denote by $\mathsf{Ext}(\mathscr{C})$ the full subcategory of the category $\mathsf{Arr}(\mathscr{C})$ of arrows in $\mathscr{C}$ determined by the extensions; morphisms are commutative squares between them. Since the condition on ϵ makes $\mathscr{X}$ a reflective subcategory of $\mathscr{C}$, we shall sometimes omit the right adjoint inclusion H. It is well known that the condition on the η_C being regular epimorphisms is equivalent to $\mathscr{X}$ being closed under subobjects in $\mathscr{C}$.

3.1. Trivial and normal extensions. With respect to a closed Galois structure Γ as above, an extension $f \colon A \to B$ is said to be **trivial** if the naturality square

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & HF(A) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle HF(f)} \\ B & \xrightarrow[\eta_B]{} & HF(B) \end{array}$$

is a pullback, and **normal** if the first projection $\mathrm{pr}_1^f = f^*(f) \colon \mathrm{Eq}(f) \to A$ of the kernel pair of f is trivial. We write $\mathsf{NExt}(\mathscr{C})$ for the category of normal extensions, considered as a full subcategory of $\mathsf{Ext}(\mathscr{C})$. Any trivial extension is a normal extension, and any split epimorphism which is a normal extension is automatically a trivial extension (see [29, Lemma 11] or [16, Corollary 2.11]). Normal *extensions* should not be confused with normal *epimorphisms* (cokernels), since by definition extensions and normal epimorphisms are precisely the same thing, whereas normal extensions are introduced in order to model concepts such as central extensions of groups.

3.2. Normalisation. The inclusion functor $H_1 \colon \mathsf{NExt}(\mathscr{C}) \to \mathsf{Ext}(\mathscr{C})$ has a left adjoint F_1, called the **normalisation functor**. The component of the unit at