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ON SOME CATEGORICAL-ALGEBRAIC CONDITIONS IN S-PROTOMODULAR CATEGORIES

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Dedicated to Jiří Adámek

Abstract: In the context of semi-abelian categories, several additional conditions have been considered in order to obtain a closer group-like behavior. Among them are locally algebraic cartesian closedness and algebraic coherence. The recent notion of S-protomodular categories, whose main examples are the category of monoids and, more generally, categories of monoids with operations and Jónsson-Tarski varieties, raises a similar question: how to get a description of S-protomodular categories with a strong monoid-like behavior. In this paper we consider relative versions of the conditions mentioned above, in order to exhibit the parallelism with the "absolute" semi-abelian context and to obtain a hierarchy among S-protomodular categories.

Keywords: S-protomodular categories, monoids, monoids with operations, Jónsson-Tarski varieties, algebraic cartesian closedness, algebraic coherence.
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1. Introduction

Semi-abelian categories [16] have been introduced in order to give a categorical description of group-like algebraic structures, as well as abelian categories describe abelian groups and modules over commutative rings. However, the family of semi-abelian categories revealed to be too large for this purpose, since, together with algebraic structures like groups, rings, Lie algebras, Ω-groups in the sense of [14], it contains many other examples, like the duals of the categories of pointed objects in a topos [4] (in particular, the dual of the category of pointed sets is semi-abelian).

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In order to get closer to group-like structures, several conditions have been asked for a category, in addition to the condition of being semi-abelian. A well studied one, which has several important consequences in commutator theory and in the description of internal structures, is the so-called "Smith is Huq" condition [22]: two internal equivalence relations on the same object centralize each other in the sense of Smith-Pedicchio [27, 25] if and only if their normalizations commute in the sense of Huq [15]. An example, due to G. Janelidze, of a semi-abelian category (actually, a semi-abelian variety in the sense of universal algebra) which doesn't satisfy this condition is the category of digroups. As shown in [18], every category of *groups with operations* in the sense of Porter [26] (see also [23], where the axioms for groups with operations have been first considered, without giving a name to such structures) is a semi-abelian category which satisfies the "Smith is Huq" condition. A characterization of the "Smith is Huq" condition in terms of the so-called *fibration of points* (see Section 2 for a description of this fibration) has been given in [11]: the change-of-base functors of the fibration of points reflect the commutation of normal subobjects.

Other conditions have been introduced more recently, by constructing a parallelism with topos theory. We recall, among them, locally algebraic cartesian closedness [13, 5], fibrewise algebraic cartesian closedness [5] and algebraic coherence [12]. These conditions are obtained from the classical ones of locally cartesian closedness, fibrewise cartesian closedness and coherence by replacing, in their definition, the basic fibration with the fibration of points. See Section 2 for more details. Every *category of interest* in the sense of Orzech [23] is algebraically coherent [12, Theorem 4.15] and fibrewise algebraically cartesian closed (this is a consequence of [12, Theorem 6.27]). Much less are the examples of locally algebraically cartesian closed categories: the main ones are the categories of groups and of Lie algebras over a commutative ring. Hence these conditions create a hierarchy among semi-abelian categories: to a stronger condition corresponds a smaller family of examples, closer to the main example, the category of groups.

An important property of semi-abelian categories is the fact that internal actions (defined as in [2]) are equivalent to split extensions [6, Theorem 3.4]. In the category of groups, internal actions correspond to classical group actions: an action of a group B on a group X is a group homomorphism

$B \to \mathrm{Aut}(X)$. In the category of monoids, which is not semi-abelian, the equivalence mentioned above does not hold. Classical monoid actions are defined as monoid homomorphisms $B \to \mathrm{End}(X)$, where $\mathrm{End}(X)$ is the monoid of endomorphisms of the monoid X. Looking for a class of split extensions that are equivalent to such actions in the category of monoids, in [20] we identified a particular kind of split epimorphisms, that we called *Schreier split epimorphisms* (the name is inspired by the work of Patchkoria on Schreier internal categories [24]). A similar equivalence between actions and Schreier split epimorphisms holds also for semirings, and, more generally, for every category of monoids with operations [20].

Further investigations on the class of Schreier split epimorphisms in monoids, semirings and monoids with operations [9, 8] allowed to discover that this class satisfies several properties that are typically satisfied by the class of all split epimorphisms in a semi-abelian category, like for example the Split Short Five Lemma, which is a key ingredient in the definition of semi-abelian categories (in a pointed finitely complete context, it is equivalent to Bourn-protomodularity [3]). In order to describe this situation categorically, the notion of pointed *S-protomodular category*, relatively to a suitable class S of points (i.e. split epimorphisms with a fixed section) has been introduced in [10] (see Section 3 for more details). In [10] it is shown that S-protomodular categories satisfy, relatively to the class S, many properties of semi-abelian categories. In [19] it is proved that every Jónsson-Tarski variety [17] is S-protomodular w.r.t. the class of Schreier split epimorphisms. This is the case, in particular, for monoids with operations, as already observed in [10].

The aim of this paper is to study additional conditions on an S-protomodular category, similarly to what has been done for semi-abelian categories, in order to create a hierarchy among them which allows to get closer to a categorical description of the category of monoids, which is the central example of an S-protomodular category. A relative version of the "Smith is Huq" condition was already studied in [19]: two S-equivalence relations (i.e. equivalence relations such that the two projections, with the reflexivity morphism, form points belonging to S, see [10]) on the same object centralize each other if and only if their normalizations commute. In [19] it was shown that every category of monoids with operations satisfies this relative version of the "Smith is Huq" condition. This already permits to distinguish monoids with operations among Jónsson-Tarski varieties.

Now our aim is to consider relative versions of the other additional conditions we mentioned, namely locally algebraic cartesian closedness, fibrewise algebraic cartesian closedness and algebraic coherence. In order to do that, we replace the fibration of points with its subfibration of points belonging to the class S, which is supposed to be stable under pullbacks (this assumption is necessary to get a subfibration). We show that the category of monoids is locally algebraically cartesian closed w.r.t. the class S of Schreier points, while this property fails for semirings. Furthermore, the categories of monoids and semirings are fibrewise algebraically cartesian closed and algebraically coherent, relatively to S, while these two properties fail, in general, for monoids with operations. Hence we get the hierarchy we were looking for.

2. Semi-abelian categories and additional conditions on them

A semi-abelian category [16] is a pointed, Barr-exact [1], Bourn-protomodular [3] category with finite coproducts. Every variety of Ω-groups in the sense of Higgins [14] is a semi-abelian category. More generally, semi-abelian varieties of universal algebras have been characterized in [7]. Non-varietal examples of semi-abelian categories are the category of compact Hausdorff groups and the dual of every category of pointed objects in a topos [4].

The condition of Bourn-protomodularity can be expressed in terms of a property of the so-called *fibration of points*. A *point* in a category $\mathcal{C}$ is a 4-tuple (A, B, f, s), where $f\colon A \to B$, $s\colon B \to A$ and $fs = 1_B$. In other terms, a point is a split epimorphism with a chosen splitting. A morphism between a point (A, B, f, s) and a point (A', B', f', s') is a pair (g, h) of morphisms such that the two reasonable squares in the following diagram commute:

$$\begin{tikzcd} A \arrow[r, shift left, "f"'] \arrow[d, "g"'] & B \arrow[l, shift left, "s"'] \arrow[d, "h"] \\ A' \arrow[r, shift left, "f'"'] & B', \arrow[l, shift left, "s'"'] \end{tikzcd}$$

i.e. $hf = f'g$ and $gs = s'h$. There is a functor

$$\mathrm{cod}\colon Pt(\mathcal{C}) \to \mathcal{C}$$

which associates its codomain with every point: $\mathrm{cod}(A, B, f, s) = B$. If $\mathcal{C}$ has pullbacks, this functor is a fibration, called the *fibration of points*. A finitely complete category $\mathcal{C}$ is Bourn-protomodular if every change-of-base functor of the fibration of points is conservative. If $\mathcal{C}$ is pointed, Bourn-protomodularity is equivalent to the fact that the Split Short Five Lemma holds.

We now recall the definitions of the additional conditions we are interested in.

Definition 2.1 ([13, 5])**.** *A semi-abelian category* $\mathcal{C}$ *is* locally algebraically cartesian closed*, or* LACC*, if, for every morphism* $h\colon E \to B$*, the corresponding change-of-base functor of the fibration of points:*

$$h^*\colon Pt_B(\mathcal{C}) \to Pt_E(\mathcal{C})$$

has a right adjoint.

As shown in [13], the categories of groups and of Lie algebras over a commutative ring are LACC.

Definition 2.2 ([5])**.** *A semi-abelian category* $\mathcal{C}$ *is* fibrewise algebraically cartesian closed*, or* FWACC*, if, for every split epimorphism* $h\colon E \to B$*, the corresponding change-of-base functor of the fibration of points has a right adjoint.*

Obviously every LACC category is FWACC. As shown in [13, Propositions 6.7 and 6.8], the category of (non-unitary) rings is a FWACC category which is not LACC.

We recall that, in a finitely complete category, a pair (f, g) of morphisms with the same codomain is *jointly strongly epimorphic* if, whenever f and g factor through a monomorphism m, m is an isomorphism.

Definition 2.3 ([12])**.** *A semi-abelian category* $\mathcal{C}$ *is* algebraically coherent *if, for every morphism* $h\colon E \to B$*, the corresponding change-of-base functor of the fibration of points preserves jointly strongly epimorphic pairs.*

It is immediate to see that a LACC category is algebraically coherent [12, Theorem 4.5]. Any semi-abelian algebraically coherent variety is FWACC [12, Theorem 6.27]. Every *category of interest* in the sense of Orzech [23] is algebraically coherent [12, Theorem 4.15], and hence FWACC. This is not the case for every group with operations in the sense of Porter: for example, the category of non-associative rings is not algebraically coherent [12, Examples 4.10].

3. S-protomodular categories

Given a finitely complete category $\mathcal{C}$, let S be a class of points in $\mathcal{C}$ which is stable under pullbacks along any morphism. Denoting by $SPt(\mathcal{C})$ the full subcategory of $Pt(\mathcal{C})$ whose objects are the points in S, we obtain a subfibration of the fibration of points:

$$S\text{-cod}\colon SPt(\mathcal{C}) \to \mathcal{C}.$$

A point (A, B, f, s) is called a *strong point* [10] (or *regular point*, as in [21], in a regular context) if the morphisms k and s, where k is a kernel of f, are jointly strongly epimorphic.

Definition 3.1 ([10]). *A pointed, finitely complete category* $\mathcal{C}$ *is said to be* S-protomodular *if:*

(1) *every point in S is a strong point;*
(2) *$SPt(\mathcal{C})$ is closed under finite limits in $Pt(\mathcal{C})$.*

The following result implies that the Split Short Five Lemma holds for points in S in an S-protomodular category:

Theorem 3.2 ([10], Theorem 3.2). *In an S-protomodular category, every change-of-base functor of the subfibration S-cod is conservative.*

Every protomodular category $\mathcal{C}$ is S-protomodular for the class S of all points in $\mathcal{C}$. In order to give other, more meaningful examples, we recall the following:

Definition 3.3 ([17]). *A variety in the sense of universal algebra is a* Jónsson-Tarski variety *if the corresponding theory contains a unique constant* 0 *and a binary operation* $+$ *satisfying the equalities* $0 + x = x + 0 = x$ *for any* x.

Definition 3.4 ([20, 19]). *A split epimorphism* $A \underset{f}{\overset{s}{\leftrightarrows}} B$ *in a Jónsson-Tarski variety is said to be a* Schreier split epimorphism *when, for any* $a \in A$, *there exists a unique* α *in the kernel* $Ker(f)$ *of* f *such that* $a = \alpha + sf(a)$.

Equivalently, a split epimorphism (A, B, f, s) as above is a Schreier split epimorphism if there exists a unique map $q_f\colon A \to Ker(f)$ (which is not a homomorphism, in general) such that $a = q_f(a) + sf(a)$ for all $a \in A$. This map q_f is called the *Schreier retraction* of the split epimorphism (A, B, f, s).

Proposition 3.5 ([19], Proposition 2.5)**.** *If* $\mathbb{C}$ *is a Jónsson-Tarski variety and* S *is the class of Schreier split epimorphisms, then* $\mathbb{C}$ *is an* S*-protomodular category.*

As shown in [20] (see also Chapter 5 in [8]), in the category of monoids Schreier split epimorphisms are equivalent to monoid actions: an action of a monoid B on a monoid X is a monoid homomorphism $\varphi\colon B \to \mathrm{End}(X)$, where $\mathrm{End}(X)$ is the monoid of endomorphisms of X. Given a Schreier split epimorphism
$X \underset{k}{\overset{q_f}{\leftrightarrows}} A \underset{f}{\overset{s}{\leftrightarrows}} B$ of monoids, the corresponding action φ is given by

$$\varphi(b)(x) = q_f(s(b) + k(x)),$$

where we use the additive notation for the monoid operation; conversely, given an action of B on X, the corresponding Schreier split epimorphism is obtained via a semidirect product construction. Actually, Schreier split epimorphisms have been identified in order to get such an equivalence. A similar equivalence with a suitable notion of action holds in any category of *monoids with operations* [20], a family of varieties which includes monoids, commutative monoids, semirings (i.e. rings where the additive structure is not necessarily a group, but just a commutative monoid), join-semilattices with a bottom element, distributive lattices with a bottom element (or a top one).

The equivalence between actions and Schreier split epimorphisms does not hold in any Jónsson-Tarski variety, that's why originally in [20] Schreier split epimorphisms were only considered in monoids with operations. A conceptual explanation of this phenomenon was given in [19]: monoids with operations satisfy, with respect to the class S of Schreier split epimorphisms, the relative version of the "Smith is Huq" condition: two S-equivalence relations on the same object centralize each other if and only if their normalizations commute. S-equivalence relations (see [10]) are equivalence relations such that the two projections, with the reflexivity morphism, form points belonging to S. So, this condition allows to distinguish monoids with operations from general Jónsson-Tarski varieties. In order to get a more refined classification, in the next section we consider relative versions of the conditions on semi-abelian categories we recalled in Section 2.

4. Relative conditions on the fibration of points

Throughout this section, let $\mathcal{C}$ be an S-protomodular category, for a fixed class S of points. By replacing the fibration of points $\mathrm{cod}\colon Pt(\mathcal{C}) \to \mathcal{C}$ with its subfibration $S\text{-cod}\colon SPt(\mathcal{C}) \to \mathcal{C}$ of points in S, we can formulate relative versions of the conditions considered in Section 2:

Definition 4.1. *An S-protomodular category $\mathcal{C}$ is* S-locally algebraically cartesian closed*, or* S-LACC*, if, for every morphism $h\colon E \to B$, the corresponding change-of-base functor of the fibration S-cod:*

$$h^*\colon SPt_B(\mathcal{C}) \to SPt_E(\mathcal{C})$$

has a right adjoint.

Definition 4.2. *An S-protomodular category $\mathcal{C}$ is* S-fibrewise algebraically cartesian closed*, or* S-FWACC*, if, for every split epimorphism $h\colon E \to B$, the corresponding change-of-base functor of the fibration S-cod has a right adjoint.*

Definition 4.3. *An S-protomodular category $\mathcal{C}$ is* S-algebraically coherent *if, for every morphism $h\colon E \to B$, the corresponding change-of-base functor of the fibration S-cod preserves jointly strongly epimorphic pairs.*

Exactly as in the "absolute" case (i.e. when S is the class of all points), it is clear that every S-LACC category is S-FWACC and S-algebraically coherent.

Our aim is to describe how these relative conditions allow to distinguish "poor" S-protomodular categories, like general Jónsson-Tarski varieties, from "richer" ones, closer to the category of monoids. We start by showing that our key example, the category of monoids, is S-LACC:

Proposition 4.4. *The category* Mon *of monoids (and monoid homomorphisms) is S-LACC, where S is the class of Schreier split epimorphisms.*

Proof: We already observed that a Schreier split epimorphism $A \underset{f}{\overset{s}{\leftrightarrows}} B$ with kernel X corresponds to an action of B on X, i.e. to a monoid homomorphism $\varphi\colon B \to \mathrm{End}(X)$. Moreover, such actions can also be described as functors $F\colon B \to \mathsf{Mon}$, where the monoid B is seen as a category with only one object $*_B$: given φ as above, we define F by putting $F(*_B) = X$ and, for any $b \in B$, $F(b) = \varphi(b)$. For every monoid B, there is then an equivalence of categories

$$SPt_B(\mathsf{Mon}) \cong \mathsf{Mon}^B.$$

Given a monoid homomorphism $h\colon E \to B$, we have to prove that the change-of-base functor

$$h^*\colon SPt_B(\mathsf{Mon}) \to SPt_E(\mathsf{Mon})$$

has a right adjoint. From the remarks above it follows that h^* is naturally isomorphic to the functor

$$\mathsf{Mon}^h\colon \mathsf{Mon}^B \to \mathsf{Mon}^E.$$

The category Mon is complete, hence the right adjoint R_h of Mon^h can be constructed by means of the right Kan extension. Let us give a concrete description of R_h.

Given $F\colon E \to \mathsf{Mon}$ with $F(*_E) = M$, we have that $R_h(F)\colon B \to \mathsf{Mon}$ is defined by

$$R_h(*_B) = L(B, M) = \{u\colon B \to M \mid e{\cdot}u(b) = u(h(e){+}b), \text{ for all } b \in B, e \in E\},$$

where the maps u are not required to be monoid homomorphisms and $e \cdot u(b)$ denotes the action of $e \in E$ on $u(b) \in M$; the definition of R_h on morphisms is given by

$$(R_h(b_0)(u))(b) = u(b + b_0).$$

This last equality describes the action of B on $L(B, M)$. The counit ε of the adjunction $\mathsf{Mon}^h \dashv R_h$ has components $\varepsilon_F\colon L(B, M) \to M$ given by evaluation of $u \in L(B, M)$ at the neutral element of B: $\varepsilon_F(u) = u(0_B)$. Let us check the universality of this counit. Given a functor $G\colon B \to \mathsf{Mon}$, with $G(*_B) = S$, and a natural transformation $\beta\colon Gh \to F$, there is a unique natural transformation $\gamma\colon G \to R_h(F)$ such that $\varepsilon\gamma h = \beta$. The component of γ at the unique object $*_B$ of B is the map $\gamma_{*_B}\colon S \to L(B, M)$ defined by

$$\gamma_{*_B}(s)(b) = \beta(b \cdot s), \text{ for all } s \in S, b \in B. \qquad \blacksquare$$

We observe that, with the notation of the previous proof, if $h\colon E \to B$ is a surjective monoid homomorphism, and if we choose a set-theoretical section

$s\colon B \to E$, then the description of $L(B, M)$ can be simplified. In fact, every $u \in L(B, M)$ is completely determined by its value at 0_B:

$$u(b) = u(b + 0_B) = u(hs(b) + 0_B) = s(b) \cdot u(0_B).$$

Hence $L(B, M)$ is isomorphic to the following submonoid of M:

$$\{m \in M \mid es(b) \cdot m = s(h(e) + b) \cdot m, \text{ for every } b \in B, e \in E\}.$$

Being S-LACC, the category of monoids is also S-FWACC and S-algebraically coherent. The category SRng of semirings is not S-LACC with respect to the class S of Schreier split epimorphisms: if B is a ring, $B \neq 0$, then every split epimorphism with codomain B is a Schreier one [8, Proposition 6.1.6]: $SPt_B(\mathsf{SRng}) = Pt_B(\mathsf{SRng})$. Then Proposition 6.7 in [13] implies that the functor

$$\mathrm{Ker}_B\colon SPt_B(\mathsf{SRng}) \to \mathsf{SRng} = SPt_0(\mathsf{SRng}),$$

which is the change-of-base functor determined by the unique morphism $0 \to B$, does not have a right adjoint. However, SRng is S-FWACC. Actually, we can prove something slightly stronger:

Proposition 4.5. *If $h\colon E \to B$ is a regular epimorphism (i.e. a surjective homomorphism) in the category SRng of semirings (and semiring homomorphisms), then the change-of-base functor*

$$h^*\colon SPt_B(\mathsf{SRng}) \to SPt_E(\mathsf{SRng})$$

of the fibration of Schreier points has a right adjoint.

Proof: Similarly to what happens for monoids, Schreier split epimorphisms of semirings correspond to actions [20]. An action of a semiring B on a semiring X is a pair $\varphi = (\varphi_l, \varphi_r)$ of functions

$$\varphi_l\colon B \times X \to X, \qquad \varphi_r\colon X \times B \to X,$$

whose images are simply denoted by $b \cdot x$ and $x \cdot b$, respectively, such that the following conditions are satisfied for all $b, b_1, b_2 \in B$ and all $x, x_1, x_2 \in X$:

(1) $0 \cdot x = x \cdot 0 = 0 \cdot b = b \cdot 0 = 0$;
(2) $b \cdot (x_1 + x_2) = b \cdot x_1 + b \cdot x_2, \quad (x_1 + x_2) \cdot b = x_1 \cdot b + x_2 \cdot b$;
(3) $(b_1 + b_2) \cdot x = b_1 \cdot x + b_2 \cdot x, \quad x \cdot (b_1 + b_2) = x \cdot b_1 + x \cdot b_2$;
(4) $b \cdot (x_1 x_2) = (b \cdot x_1) x_2, \quad (x_1 x_2) \cdot b = x_1 (x_2 \cdot b)$;
(5) $(b_1 b_2) \cdot x = b_1 \cdot (b_2 \cdot x), \quad x \cdot (b_1 b_2) = (x \cdot b_1) \cdot b_2$;
(6) $x_1 (b \cdot x_2) = (x_1 \cdot b) x_2, \quad (b_1 \cdot x) \cdot b_2 = b_1 \cdot (x \cdot b_2)$.

Given a Schreier split epimorphism $X \underset{k}{\overset{q_f}{\leftrightarrows}} A \underset{f}{\overset{s}{\leftrightarrows}} B$ of semirings, the corresponding action is given by

$$b \cdot x = q_f(s(b)k(x)), \qquad x \cdot b = q_f(k(x)s(b));$$

conversely, given an action of B on X, the corresponding Schreier split epimorphism is obtained via a semidirect product construction (see [20] for more details). If we denote by B-Act the category whose objects are pairs (X, φ), with $X \in \mathsf{SRng}$ and φ an action of B on X, and whose morphisms are equivariant homomorphisms, we get an equivalence of categories $SPt_B(\mathsf{SRng}) \cong B\text{-}Act$. Unlike the case of monoids, a B-action can't be represented as a functor into SRng. Still, given a surjective morphism $h \colon E \to B$ in SRng, in order to prove that the change-of-base functor

$$h^* \colon SPt_B(\mathsf{SRng}) \to SPt_E(\mathsf{SRng})$$

has a right adjoint, we can consider the equivalent functor

$$h^* \colon B\text{-}Act \to E\text{-}Act$$

which sends (X, φ) to (X, ψ), where the action ψ is defined by

$$e \cdot x = h(e) \cdot x, \qquad x \cdot e = x \cdot h(e).$$

The right adjoint $R_h \colon E\text{-}Act \to B\text{-}Act$ of h^* is defined as follows. Given $(X, \psi) \in E\text{-}Act$, we define

$$R_h(X) = \{x \in X \mid e_1 \cdot x = e_2 \cdot x,\ x \cdot e_1 = x \cdot e_2 \text{ for all } e_1, e_2 \in E \text{ s.t. } h(e_1) = h(e_2)\}.$$

It is immediate to see that $R_h(X)$ is a submonoid of X. The action φ of B on $R_h(X)$ is given by

$$b \cdot x = e \cdot x, \quad x \cdot b = x \cdot e \quad \text{for all } e \in E \text{ such that } h(e) = b.$$

Of course this is well-defined thanks to the definition of $R_h(X)$. Given a morphism $g \colon X \to Y$ in E-Act, $R_h(g)$ is just its restriction to $R_h(X)$: it takes values in $R_h(Y)$ because of the equivariance of g. It is straightforward to check that R_h is the right adjoint to h^*. ■

Corollary 4.6. *The category* SRng *is* S*-FWACC, where* S *is the class of Schreier split epimorphisms.*

The category of semirings is not only S-FWACC, but also S-algebraically coherent:

Proposition 4.7. *The category* SRng *is* S*-algebraically coherent, where* S *is the class of Schreier split epimorphisms.*

Proof: Since all the change-of-base functors of the fibration of Schreier points in SRng are conservative, as we recalled in Section 3, and they obviously preserve monomorphisms, we can use Lemmas 3.9 and 3.11 in [12] to conclude that, in order to prove that SRng is S-algebraically coherent, it suffices to show that, for every semiring B, the kernel functor

$$\mathrm{Ker}_B \colon SPt_B(\mathsf{SRng}) \to \mathsf{SRng}$$

preserves jointly strongly epimorphic pairs. Accordingly, consider the following diagram, where f and g are morphisms in $SPt_B(\mathsf{SRng})$:

$$\begin{tikzcd}
H \arrow[r, "f_{|H}"] \arrow[d, tail] & K \arrow[d, tail] & L \arrow[l, "g_{|L}"'] \arrow[d, tail] \\
A \arrow[r, "f"] \arrow[d, shift right, "p'"'] & D \arrow[d, shift right, "p"'] & C \arrow[l, "g"'] \arrow[d, shift right, "p''"'] \\
B \arrow[u, shift right, "s'"'] \arrow[r, equal] & B \arrow[u, shift right, "s"'] \arrow[r, equal] & B, \arrow[u, shift right, "s''"']
\end{tikzcd}$$

and suppose that f and g are jointly strongly epimorphic (in $SPt_B(\mathsf{SRng})$); we have to prove that their restrictions to H and L, respectively, are jointly strongly epimorphic in SRng. It is not difficult to see that f and g are jointly strongly epimorphic in $SPt_B(\mathsf{SRng})$ if and only if they have the same property in SRng. This happens if and only if D, as a semiring, is generated by the images $f(A)$ and $g(C)$. This means that every $d \in D$ may be written as sums of elements of the following 4 forms:

$$f(a_1)g(c_1)f(a_2)g(c_2)\dots f(a_n)g(c_n), \qquad f(a_1)g(c_1)f(a_2)g(c_2)\dots f(a_n),$$

$$g(c_1)f(a_2)g(c_2)\dots f(a_n)g(c_n), \qquad g(c_1)f(a_2)g(c_2)\dots f(a_n).$$

In particular, every element $k \in K$ has this property. We have to show that k can be written as a sum of products as above, but only using elements of $f(H)$ and $g(L)$.

We start by considering the simplest case: suppose that $k = f(a)g(c)$ for some $a \in A$, $c \in C$. Since (p', s') and (p'', s'') are Schreier split epimorphisms, there are (unique) $h \in H$, $l \in L$ such that $a = h + s'p'(a)$ and $c = l + s''p''(c)$.

Then

$$\begin{aligned} k = f(a)g(c) = f(h + s'p'(a))g(l + s''p''(c)) = \\ = f(h)g(l) + f(h)gs''p''(c) + fs'p'(a)g(l) + fs'p'(a)gs''p''(c) = \\ = f(h)g(l) + f(h)sp''(c) + sp'(a)g(l) + sp'(a)sp''(c). \end{aligned} \tag{1}$$

We have that

$$f(h)sp''(c) = f(h)fs'p''(c) = f(h\,s'p''(c))$$

and

$$p'(h\,s'p''(c)) = pf(h\,s'p''(c)) = pf(h)psp''(c) = 0p''(c) = 0,$$

hence $h\,s'p''(c) \in H$. Similarly, we show that $sp'(a)g(l) = g(s''p'(a)\,l)$, with $s''p'(a)\,l \in L$. From (1) we get

$$\begin{aligned} 0 = p(k) = p(f(h)g(l) + f(h)sp''(c) + sp'(a)g(l) + sp'(a)sp''(c)) = \\ = p(sp'(a)sp''(c)) = ps(p'(a)p''(c)); \end{aligned}$$

since $ps = 1_B$, we get that $p'(a)p''(c) = 0$ and hence

$$s(p'(a)p''(c)) = sp'(a)sp''(c) = 0.$$

This means that k can be written as a sum of products of elements in $f(H)$ and $g(L)$. The case in which $k = g(c)f(a)$, for some $c \in C$ and $a \in A$, is similar.

If $k = f(a_1)g(c)f(a_2)$ for some $a_1, a_2 \in A$ and $c \in C$, then

$$\begin{aligned} k = f(h_1 + s'p'(a_1))g(l + s''p''(c))f(h_2 + s'p'(a_2)) = \\ = (f(h_1) + sp'(a_1))(g(l) + sp''(c))(f(h_2) + sp'(a_2)) \end{aligned}$$

for suitable $h_1, h_2 \in H$, $l \in L$. Making the calculations in the last expression using the distributivity law, we get a sum in which every summand, except $sp'(a_1)sp''(c)sp'(a_2)$, contains an element of $f(H)$ or of $g(L)$, and then all these summands belong either to $f(H)$, to $g(L)$ or are made of products of elements of $f(H)$ and $g(L)$; let's consider, for instance, one of them:

$$f(h_1)sp''(c)sp'(a_2) = f(h_1\,s'p''(c)s'p'(a_2)),$$

and $h_1\,s'p''(c)s'p'(a_2) \in H$, since $p'(h_1\,s'p''(c)s'p'(a_2)) = 0$. So, it only remains to consider the summand $sp'(a_1)sp''(c)sp'(a_2)$; but, for the same reasons as in the case $k = f(a)g(c)$, this is equal to 0. So, if $k = f(a_1)g(c)f(a_2)$, then k is a sum of products of elements of $f(H)$ and $g(L)$. All the other cases are dealt analogously. ∎

It is not true that every category of monoids with operations is S-algebraically coherent w.r.t. the class S of Schreier split epimorphisms. This is not the case even for groups with operations, as shown in Examples 4.10 in [12]. Hence the additional conditions we considered in this section gave us the hierarchy among S-protomodular categories we were looking for. When S is the class of Schreier split epimorphisms, this hierarchy is summarized by the following table:

Property	**True in**
S-LACC	Mon
S-algebraically coherent	Mon, SRng
S-FWACC	Mon, SRng
S-"Smith is Huq"	categories of monoids with operations
S-protomodularity	Jónsson-Tarski varieties

References

[1] M. Barr, *Exact categories*, in Lecture Notes in Mathematics, vol. 236 (1971), Springer-Verlag, 1-120.

[2] F. Borceux, G. Janelidze, G.M. Kelly, *Internal object actions*, Commentationes Mathematicae Universitatis Carolinae 46 n.2 (2005), 235-255.

[3] D. Bourn, *Normalization equivalence, kernel equivalence and affine categories*, in Lecture Notes in Mathematics, vol. 1488 (1991), Springer-Verlag 43-62.

[4] D. Bourn, *Protomodular aspect of the dual of a topos*, Adv. Math. 187 (2004), no. 1, 240255.

[5] D. Bourn, J.R.A. Gray, *Aspects of algebraic exponentiation*, Bull. Belg. Math. Soc. Simon Stevin 19 (2012), 823-846.

[6] D. Bourn, G. Janelidze, *Protomodularity, descent, and semidirect products*, Theory Appl. Categ. 4 (1998), No. 2, 37-46.

[7] D. Bourn, G. Janelidze, *Characterization of protomodular varieties of universal algebras*, Theory Appl. Categ. 11 (2003), No. 6, 143147.

[8] D. Bourn, N. Martins-Ferreira, A. Montoli, M. Sobral, *Schreier split epimorphisms in monoids and in semirings*, Textos de Matemática (Série B), Departamento de Matemática da Universidade de Coimbra, vol. 45 (2013).

[9] D. Bourn, N. Martins-Ferreira, A. Montoli, M. Sobral, *Schreier split epimorphisms between monoids*, Semigroup Forum 88 (2014), 739-752.

[10] D. Bourn, N. Martins-Ferreira, A. Montoli, M. Sobral, *Monoids and pointed S-protomodular categories*, Homology, Homotopy and Applications 18 n.1 (2016), 151-172.

[11] D. Bourn, N. Martins-Ferreira, T. Van der Linden, *A characterisation of the "Smith is Huq" condition in the pointed Mal'tsev setting*, Cah. Topol. Géom. Différ. Catégor. 54 (2013), n. 4, 243-263.

[12] A.S. Cigoli, J.R.A. Gray, T. Van der Linden, *Algebraically coherent categories*, Theory Appl. Categories 30 (2015), 1864-1905.

[13] J.R.A. Gray, *Algebraic exponentiation in general categories*, Appl. Categ. Structures 20 (2012), 543-567.

[14] P.J. Higgins, *Groups with multiple operators*, Proc. Lond. Math. Soc. (3) 6 (1956), no. 3, 366-416.

[15] S.A. Huq, *Commutator, nilpotency and solvability in categories*, Q. J. Math. 19 (1968) n.2, 363-389.

[16] G. Janelidze, L. Márki, W. Tholen, *Semi-abelian categories*, J. Pure Appl. Algebra 168 (2002), no. 2-3, 367-386.

[17] B. Jónsson, A. Tarski, *Direct decompositions of finite algebraic systems*, Notre Dame Mathematical Lectures, Notre Dame, Indiana (1947).

[18] S. Mantovani, G. Metere, *Internal crossed modules and Peiffer condition*, Theory Appl. Categories 23 (2010), 113-135.

[19] N. Martins-Ferreira, A. Montoli, *On the "Smith is Huq" condition in S-protomodular categories*, Appl. Categ. Structures, published online, DOI 10.1007/s10485-015-9411-1 (2015).

[20] N. Martins-Ferreira, A. Montoli, M. Sobral, *Semidirect products and crossed modules in monoids with operations*, J. Pure Appl. Algebra 217 (2013), 334-347.

[21] N. Martins-Ferreira, A. Montoli, M. Sobral, *Semidirect products and Split Short Five Lemma in normal categories*, Appl. Categorical Structures 22 (2014), 687-697.

[22] N. Martins-Ferreira, T. Van der Linden, *A note on the "Smith is Huq" condition*, Appl. Categorical Structures 20 (2012), 175-187.

[23] G. Orzech, *Obstruction theory in algebraic categories I*, J. Pure Appl. Algebra 2 (1972), 287-314.

[24] A. Patchkoria, *Crossed semimodules and Schreier internal categories in the category of monoids*, Georgian Math. Journal 5 (1998), n.6, 575-581.

[25] M.C. Pedicchio, *A categorical approach to commutator theory*, J. Algebra 177 (1995), 647-657.

[26] T. Porter, *Extensions, crossed modules and internal categories in categories of groups with operations*, Proceedings of the Edinburgh Math. Society 30 (1987), 373-381.

[27] J.D.H. Smith, *Mal'cev varieties*, Lecture Notes in Mathematics, vol. 554 (1976), Springer.

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