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ASPECTS OF ALGEBRAIC ALGEBRAS

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We dedicate this paper to Jirka Adámek whose mathematics has enchanted us since the first seminars in Bremen and in Coimbra.

Abstract: In this paper we investigate important categories lying strictly between the Kleisli category and the Eilenberg–Moore category, for a Kock-Zöberlein monad on an order-enriched category. Firstly, we give a characterisation of free algebras in the spirit of domain theory. Secondly, we study the existence of weighted (co)limits, both on the abstract level and for specific categories of domain theory like the category of algebraic lattices. Finally, we apply these results to give a description of the idempotent split completion of the Kleisli category of the filter monad on the category of topological spaces.

1. Introduction

The Eilenberg-Moore categories of idempotent monads are precisely the full reflective isomorphism-closed subcategories of the base category. A substantial study in category theory has been dedicated to full reflective subcategories since the 1970's, and this is one of the many subjects to which Jiří Adámek has given a remarkable contribution (see [4, 6, 5], just to name a few). The notion of Kock-Zöberlein monad ([21, 35]), also named lax-idempotent monad, is a fruitful generalisation of idempotent monads to the more general setting of 2-categories. As discovered by Martin Escardó and others in a series of papers in the late 1990's, Kock-Zöberlein monads provide a new insight into important examples of domain theory and topology, when our 2-categories are just order-enriched categories (e.g., [13, 15, 14]). In this case, the Eilenberg-Moore categories are reflective subcategories of the base category as well; however, in general they are not anymore full. In [9] and other related papers, this kind of subcategories were called *KZ-monadic subcategories*. As demonstrated in a series of recent papers [10, 9, 32, 8],

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several important well-known properties and notions on full reflective subcategories of ordinary category theory have an order-enriched counterpart when we replace full reflectivity by KZ-monadicity.

Associated with each monad $\mathbb{T} = (T, m, e)$ on a category X, we have a faithfully full functor $E : \mathsf{X}_{\mathbb{T}} \hookrightarrow \mathsf{X}^{\mathbb{T}}$ between the Kleisli category $\mathsf{X}_{\mathbb{T}}$ and the Eilenberg-Moore category $\mathsf{X}^{\mathbb{T}}$. Moreover, we have adjunctions $F_{\mathbb{T}} \dashv U_{\mathbb{T}} : \mathsf{X}_{\mathbb{T}} \longrightarrow \mathsf{X}$ and $F^{\mathbb{T}} \dashv U^{\mathbb{T}} : \mathsf{X}^{\mathbb{T}} \longrightarrow \mathsf{X}$ with $U^{\mathbb{T}}E = U_{\mathbb{T}}$ and $EF_{\mathbb{T}} = F^{\mathbb{T}}$. In fact, these two adjunctions are the initial and terminal objects of the obvious category of all adjunctions which induce the monad $\mathbb{T}$.

When $\mathbb{T}$ is an idempotent monad (i.e., the multiplication m is a natural isomorphism), $\mathsf{X}^{\mathbb{T}}$ can be identified with a full subcategory of X, and $\mathsf{X}_{\mathbb{T}} \simeq \mathsf{X}^{\mathbb{T}}$. Thus, when the functor T is injective on objects, as it happens in most significant examples, $\mathsf{X}^{\mathbb{T}}$ is just the closure under isomorphisms of $\mathsf{X}_{\mathbb{T}}$ in X. Hence, there are no interesting subcategories strictly between $\mathsf{X}_{\mathbb{T}}$ and $\mathsf{X}^{\mathbb{T}}$ to be considered.

The situation is dramatically different when we work with Kock-Zöberlein monads in order-enriched categories. In this case we have yet $\mathsf{X}^{\mathbb{T}}$ as a (usually non-full) subcategory of X. And $\mathsf{X}^{\mathbb{T}}$ is now, on objects and morphisms, the closure of $\mathsf{X}_{\mathbb{T}}$ under left adjoint retractions on X ([10]). But, between $\mathsf{X}_{\mathbb{T}}$ and $\mathsf{X}^{\mathbb{T}}$ there are interesting subcategories which are quite distinct. As an example, take the open filter monad $\mathbb{F}$ over $\mathsf{X} = \mathsf{Top}_0$. Then $\mathsf{X}^{\mathbb{F}}$ is precisely the category of continuous lattices and maps preserving directed suprema and arbitrary infima. And between $\mathsf{X}_{\mathbb{F}}$ and $\mathsf{X}^{\mathbb{F}}$ we have at least two remarkable subcategories: the category ALat of algebraic lattices is properly contained in $\mathsf{X}^{\mathbb{F}}$, and between $\mathsf{X}_{\mathbb{F}}$ and ALat we have the idempotent completion of $\mathsf{X}_{\mathbb{F}}$ which we characterise here as consisting of all algebraic lattices whose compact elements form the dual of a frame (Section 6). We also prove that, for that monad $\mathbb{F}$, ALat is precisely the closure under weighted limits of $\mathsf{X}_{\mathbb{F}}$ in $\mathsf{X}^{\mathbb{F}}$ (then, also in Top_0).

In this paper we embark on a study of important categories lying strictly between the Kleisli category and the Eilenberg–Moore category, for a Kock-Zöberlein monad on an order-enriched category; with particular focus on various filter monads on the category Top_0 of T_0 topological spaces and continuous maps. After recalling the necessary background material in Section 2, the aim of Section 3 is to give a general treatment of the notion of algebraic lattice. In continuation of [29], where the authors observe that "these theorems characterizing completely distributive lattices are not really about lattices" but rather "about a mere monad D on a mere category", in Theorem 3.17 we give a characterisation of free algebras for a general Kock-Zöberlein monad, the *algebraic algebras*, which resembles the classical notion of (totally) algebraic lattice.

Taking seriously the fact that Top_0 is order-enriched forces us to not just consider ordinary completeness but rather study *weighted* limits and colimits. In this spirit, in Section 4 we prove an interesting general result which has an important application in Section 5: every order-enriched category with weighted limits and a regular cogenerator has also weighted colimits.

In Section 5 we consider the full subcategory of $\mathsf{X}^{\mathbb{T}}$ defined by those algebras which are in a suitable sense *cogenerated* by the Sierpiński space. For the various filter monads we show that these algebras coincide with well-known objects in domain theory: algebraic lattices and spectral spaces. In particular, we conclude that the corresponding categories have weighted limits and weighted colimits.

Finally, in Section 6, we consider the filter monad $\mathbb{F}$ on Top_0. By the results of the previous section, its Kleisli category is a full subcategory of the category ALat of algebraic lattices with maps preserving directed suprema and all infima. As the latter one is complete, it contains in particular the idempotent split completion of $(\mathsf{Top}_0)_{\mathbb{F}}$; and we identify its objects as precisely those algebraic lattices where the compact elements form the dual of a frame.

2. Background material on Kock-Zöberlein monads

In this section we recall the main facts about Kock-Zöberlein monads on order-enriched categories needed in this paper. For general 2-categories, this type of monads were introduced independently by Volker Zöberlein [35] and Anders Kock (see [21]). We also refer to [13] and [15] for a detailed study of Kock-Zöberlein monads in the context of domain theory, the one treated in this paper. In particular, the three theorems of this section are presented there. In this case, we work with a special type of 2-categories, the ***order-enriched*** categories, that is, categories enriched in the category Pos of partially ordered sets and monotone maps. This means that the hom-sets are posets and the composition of morphisms preserves the order on the left and on the right. An order-enriched functor between order-enriched categories is one which preserves the order of each hom-poset.

Definition 2.1. A monad $\mathbb{T} = (T, m, e)$ on an order-enriched category X is called ***order-enriched*** whenever $T : \mathsf{X} \to \mathsf{X}$ is so. An order-enriched monad $\mathbb{T} = (T, m, e)$ is ***of Kock-Zöberlein type*** whenever $Te_X \le e_{TX}$, for all object X in X.

We note that, for an order-enriched monad $\mathbb{T} = (T, m, e)$, the full and faithful functor $\mathsf{X}_{\mathbb{T}} \to \mathsf{X}^{\mathbb{T}}$ of the Kleisli category into the Eilenberg-Moore category is also an order-isomorphism on hom-sets. The condition "$Te_X \le e_{TX}$" in the definition of Kock-Zöberlein monad is somehow arbitrarily chosen, the following theorem presents alternative descriptions.

Theorem 2.2. *Let $\mathbb{T} = (T, m, e)$ be an order-enriched monad on an order-enriched category X. For every object X in X, the following assertions are equivalent.*

(i) $Te_X \le e_{TX}$.
(ii) $m_X \dashv e_{TX}$.
(iii) $Te_X \dashv m_X$.

We turn now our attention to Eilenberg–Moore algebras.

Theorem 2.3. *Let $\mathbb{T} = (T, m, e)$ be a Kock-Zöberlein monad on an order-enriched category X and let $\alpha : TX \to X$ in X. Then the following assertions are equivalent.*

(i) $\alpha : TX \to X$ *is a* $\mathbb{T}$*-algebra structure on* X.
(ii) $\alpha \cdot e_X = \mathrm{id}_X$.
(iii) $\alpha \dashv e_X$.

As a consequence of the equivalence (i) $\iff$ (iii) of the above theorem, the Eilenberg-Moore category $\mathsf{X}^{\mathbb{T}}$ is a subcategory of X (up to isomorphism of categories). Moreover, $\mathsf{X}^{\mathbb{T}}$ is also an order-enriched category with the order inherited from X.

Before presenting examples, we recall some standard notions from order theory and topology.

Definition 2.4. In (1)-(6) we follow the terminology of [17].

(1) A subset $D \subseteq X$ of a partially ordered set X is called ***directed*** whenever $D \neq \varnothing$ and, for all $x, y \in D$, there is some $z \in D$ with $x \le z$ and $y \le z$. We are going to use the notation $\bigvee^{\uparrow} D$ to express the supremum of a set D and, at the same time, indicate that D is directed.

(2) The ***way below*** relation $\ll$ is defined as follows: $x \ll y$ provided that, for every directed subset $D \subseteq X$, if $y \leq \bigvee^{\uparrow} D$, then $x \leq d$ for some $d \in D$. An element $x \in X$ is called ***compact*** whenever $x \ll x$.
(3) The ***totally below*** relation $\lll$ is defined in a similar way: $x \lll y$ whenever, for every subset $S \subseteq X$, if $y \leq \bigvee S$, then $x \leq d$ for some $d \in S$. An element $x \in X$ is called ***totally compact*** whenever $x \lll x$.
(4) A partially ordered set X is called ***directed complete*** whenever every directed subset of X has a supremum. Furthermore, X is said to be ***bounded complete*** if every subset with an upper bound has a least one; equivalently, it has all non-empty infima.
(5) A partially ordered set X is ***continuous*** if each one of its elements x is the directed supremum of all elements y with $y \ll x$. A ***domain*** is a continuous poset with directed suprema. Furthermore, a complete partially ordered set X is called ***completely distributive*** whenever every $x \in X$ is the supremum of all elements y with $y \lll x$.
(6) A domain X with each $x \in X$ satisfying the equality $x = \bigvee^{\uparrow}\{y \in X \mid y \leq x, \, y \ll y\}$ is an ***algebraic domain***. The designation of ***continuous lattice*** [30] is used for a domain which is also a lattice; hence, a continuous lattice is a complete and continuous partially ordered set. Analogously, an ***algebraic lattice*** is an algebraic domain which is also a lattice. A completely distributive partially ordered set where $x = \bigvee\{y \in X \mid y \leq x, \, y \lll y\}$, for every $x \in X$, is called ***totally algebraic***.
(7) A topological space X is called ***stably compact*** whenever X is sober, locally compact and every finite intersection of compact saturated subsets is compact (see [19]). A continuous map $f : X \to Y$ between stably compact spaces is called ***spectral*** whenever $f^{-1}(K)$ is compact, for every compact saturated subset $K \subseteq Y$. We denote by $\mathsf{StablyComp}$ the category of stably compact spaces and spectral maps. A stably compact space X is called ***spectral*** whenever the compact open subsets form a basis for the topology of X; equivalently, if the cone $(f : X \to \mathrm{S})_f$ of all spectral maps into the the Sierpiński space is initial with respect to the forgetful functor $\mathsf{Top} \to \mathsf{Set}$; and this in turn is equivalent to $(f : X \to \mathrm{S})_f$ being initial with respect to the canonical forgetful functor $\mathsf{StablyComp} \to \mathsf{Set}$. It is also well-known that a continuous map $f : X \to Y$ between spectral spaces is spectral if and

only if $f^{-1}(K)$ is compact, for every compact open subset $K \subseteq Y$. The full subcategory of $\mathsf{StablyComp}$ defined by all spectral spaces we denote by Spec; it is a reflective subcategory since by definition it is closed under initial cones (see [3, Theorem 16.8]). Finally, we note that $\mathsf{StablyComp}$ is equivalent to the category of Nachbin's partially ordered compact Hausdorff spaces and monotone continuous maps. Here a stably compact space X corresponds to the partially ordered compact Hausdorff space with the same underlying set, the order relation is the specialisation order, and the compact Hausdorff topology is given by the so-called patch topology (see [25, 19] for details).

Examples 2.5. The following monads are of Kock-Zöberlein type.

(1) The category Pos of partially ordered sets and monotone maps is order-enriched, with the pointwise order of monotone maps. The downset monad $\mathbb{D} = (D, m, e)$ on Pos is given by
- the downset functor $D : \mathsf{Pos} \to \mathsf{Pos}$ which sends an ordered set X to the set DX of downclosed subsets of X ordered by inclusion, and, for $f : X \to Y$ monotone, $Df : DX \to DY$ sends a downclosed subset A of X to the downclosure of $f(A)$;
- the unit $e_X : X \to DX$ sends $x \in X$ to the downclosure $\downarrow x$ of x; and
- the multiplication $m_X : DDX \to DX$ sends a downset of downsets to its union.

The category $\mathsf{Pos}^{\mathbb{D}}$ of Eilenberg–Moore algebras and homomorphisms is equivalent to the category Sup of complete partially ordered sets and sup-preserving maps.

(2) An interesting submonad of $\mathbb{D} = (D, m, e)$ is given by the monad $\mathbb{I} = (I, m, e)$ where IX is the set of directed downclosed subset of X, ordered by inclusion. Furthermore, $\mathsf{Pos}^{\mathbb{I}}$ is equivalent to the category DSup of partially ordered sets with directed suprema and maps preserving directed suprema.

(3) We denote the category of topological T_0-spaces and continuous maps by Top_0. The topology of a T_0-space X induces the specialisation order on the set X: for $x, x' \in X$, $x \leq x' \iff \Omega(x) \subseteq \Omega(x')$, where $\Omega(x)$ denotes the set of open sets. Every continuous map preserves this order, and, thus, also its dual. We consider Top_0 as an order-enriched

category by taking the dual of the specialisation order pointwisely on hom-sets.

The filter functor $F : \mathsf{Top}_0 \to \mathsf{Top}_0$ sends a topological space X to the space FX of all filters on the lattice ΩX of open subsets of X. The topology on FX is generated by the sets

$$A^{\#} = \{\mathfrak{f} \in FX \mid A \in \mathfrak{f}\}$$

where $A \subseteq X$ is open. For a continuous map $f : X \to Y$, the map $Ff : FX \to FY$ is defined by

$$\mathfrak{f} \mapsto \{B \subseteq Y \mid f^{-1}(B) \in \mathfrak{f}\},$$

for $\mathfrak{f} \in FX$. Since $(Ff)^{-1}(B^{\#}) = (f^{-1}(B))^{\#}$ for every $B \subseteq Y$ open, Ff is continuous. The filter functor is part of the filter monad $\mathbb{F} = (F, m, e)$ on Top_0, here the unit $e_X : X \to FX$ sends $x \in X$ to its neighbourhood filter $\Omega(x)$, and the multiplication $m_X : FFX \to FX$ sends $\mathfrak{F} \in FFX$ to the filter $\{A \subseteq X \mid A^{\#} \in \mathfrak{F}\}$. The category $\mathsf{Top}_0^{\mathbb{F}}$ of Eilenberg–Moore algebras for the filter monad is equivalent to the category $\mathsf{ContLat}$ of continuous lattices and maps preserving directed suprema and arbitrary infima (see [12, 33]). Here a continuous lattice is viewed as a topological space with the Scott topology, and the algebra structure $\alpha : FX \to X$ picks for every $\mathfrak{f} \in FX$ the largest convergence point with respect to the specialisation order.

(4) In this paper we will consider several submonads of the filter monad $\mathbb{F}$ on Top_0; in particular, the proper filter monad $\mathbb{F}_1 = (F_1, m, e)$ where F_1X is the subspace of FX consisting of all proper filters, and the prime filter monad $\mathbb{F}_2 = (F_2, m, e)$ where F_2X is the subspace of FX consisting of all prime filters. Indeed, we have a chain of Kock-Zöberlein submonads $\mathbb{F}_n$ of $\mathbb{F}$, for n a regular cardinal, where F_nX is the subspace of FX of all n-prime filters; and the union of this chain is the completely prime filter Kock-Zöberlein monad $\mathbb{F}_c = (F_c, m, e)$ where F_cX is the subspace of FX consisting of all completely prime filters (see [11]). In the latter case, the category $\mathsf{Top}_0^{\mathbb{F}_c}$ is equivalent to the category of sober spaces and continuous maps (see [15]). It is shown in [31] that the category $\mathsf{Top}_0^{\mathbb{F}_2}$ is equivalent to the category $\mathsf{StablyComp}$ of stably compact spaces and spectral maps. Moreover, $\mathsf{Top}_0^{\mathbb{F}_1}$ is equivalent to the category of bounded complete domains (also

known as continuous Scott domains) and maps preserving directed suprema and non-empty infima (see [34, 15]).

The notion of Kock-Zöberlein monad generalises the one of idempotent monad; we recall that a monad $\mathbb{T} = (T, m, e)$ on a category X is idempotent whenever $m : TT \to T$ is an isomorphism. By Theorem 2.2, $\mathbb{T}$ is idempotent if and only if $\mathbb{T}$ is of Kock-Zöberlein type with respect to the discrete order on the hom-sets of X; i.e. if $Te_X = e_{TX}$. This observation motivates the designation *lax idempotent monad* for this type of monads, which is also used in the literature. Furthermore, we recall that

(1) For every adjunction $\begin{tikzcd} \mathsf{A} \arrow[r, bend left, "G"] \arrow[r, phantom, "\top"] & \mathsf{X} \arrow[l, bend left, "F"] \end{tikzcd}$, G is fully faithful if and only if the counit $\varepsilon : FG \to \mathrm{Id}$ is an isomorphism.
(2) Every fully faithful and right adjoint functor $G : \mathsf{A} \to \mathsf{X}$ is monadic, and the induced monad is idempotent.
(3) For every monad $\mathbb{T}$ on X, $G^{\mathbb{T}} : \mathsf{X}^{\mathbb{T}} \to \mathsf{X}$ is full if and only if $\mathbb{T}$ is idempotent.

We also remark that the completely prime filter monad $\mathbb{F}_c = (F_c, m, e)$ on Top_0 is actually idempotent.

For an order-enriched monad $\mathbb{T} = (T, m, e)$ on an order-enriched category X, we put

$$\mathcal{M}_{\mathbb{T}} = \{h : X \to Y \text{ in } \mathsf{X} \mid Th \text{ has a right adjoint } g \text{ with } g \cdot Th = \mathrm{id}_{TX}\}.$$

Clearly, if X is locally discrete, then $\mathcal{M}_{\mathbb{T}}$ is the class of all morphisms $h : X \to Y$ where Th is an isomorphism. Equivalently, $\mathcal{M}_{\mathbb{T}}$ is the largest class of morphisms of X with respect to which the subcategory $\mathsf{X}^{\mathbb{T}}$ is orthogonal. The concept of orthogonality is the particularisation, to the locally discrete case, of the concept of *Kan-injectivity*. We recall that an object A is left Kan-injective with respect to a morphism $h : X \to Y$, if and only if the hom-map $\mathsf{X}(h, A) : \mathsf{X}(Y, A) \to \mathsf{X}(X, A)$ is a right adjoint retraction in the category Pos. And a morphism $f : A \to B$ is Kan-injective with respect to h if A and B are so and the left adjoint maps $(\mathsf{X}(h, A))^\star$ and $(\mathsf{X}(h, B))^\star$ satisfy the equality $\mathsf{X}(X, f) \cdot (\mathsf{X}(h, A))^\star = (\mathsf{X}(h, B))^\star \cdot \mathsf{X}(Y, f)$.

Next we recall a characterisation of Eilenberg–Moore algebras of Kock-Zöberlein monads in terms of injetivity.

Theorem 2.6. *Let A be in X and $\mathbb{T}$ be a Kock-Zöberlein monad on X. Then the following assertions are equivalent.*

(i) A is injective with respect to $\{e_X : X \to TX \mid X \text{ in } \mathsf{X}\}$.
(ii) A is a $\mathbb{T}$-algebra.
(iii) A is injective with respect to $\mathcal{M}_{\mathbb{T}}$.
(iv) A is Kan-injective with respect to $\mathcal{M}_{\mathbb{T}}$.

Moreover, as shown in [10], $\mathcal{M}_{\mathbb{T}}$ is the largest class of morphisms of X with respect to which the subcategory $\mathsf{X}^{\mathbb{T}}$ is Kan-injective. For a detailed study on Kan-injectivity, see also [9].

If $\mathbb{T}$ is of Kock-Zöberlein type, then, by Theorem 2.2, $e_X : X \to TX$ belongs to $\mathcal{M}_{\mathbb{T}}$, for all objects X in X. However, in contrast to the idempotent case, the following example shows that this property does not characterise Kock-Zöberlein monads.

Example 2.7. Let X be the order-enriched category of all complete partially ordered sets and all monotone maps, ordered pointwise; and let A be the subcategory of X with the same objects, and as morphisms those morphisms of X which preserve the top and the bottom element. The inclusion functor $\mathsf{A} \hookrightarrow \mathsf{X}$ is right adjoint: for each object X of X, the reflection map

$$\eta_X : X \to FX = \{\perp\} + X + \{\top\}$$

is given by freely adjoining a largest and a smallest element to X. Furthermore, $F\eta_X : FX \to FFX$ sends the bottom element of FX to the bottom element of FFX, the top element of FX to the top element of FFX, and $x \in X$ to itself. Since X has a largest element, every supremum in FX of elements of X is in X, therefore $F\eta_X$ preserves all suprema and consequently has a right adjoint $g : FFX \to FX$ in X. Moreover, since $F\eta_X : FX \to FFX$ is an order-embedding, we obtain $g \cdot F\eta_X = \mathrm{id}_{FX}$. Since η_{FX} neither preserves the top nor the bottom element, we get $F\eta_X \not\leq \eta_{FX}$ and $\eta_{FX} \not\leq F\eta_X$; in particular, the induced monad is not of Kock-Zöberlein type, neither for the order $\leq$ nor for its dual.

Remark 2.8. For every $h : X \to Y$ in X, a right adjoint $g : TY \to TX$ of Th is necessarily a $\mathbb{T}$-algebra homomorphism. To see this, just observe that the diagram

$$\begin{tikzcd} TTX \arrow[r, "TTh"] & TTY \\ TX \arrow[u, "Te_X"] \arrow[r, "Th"'] & TY \arrow[u, "Te_Y"'] \end{tikzcd}$$

commutes, therefore the diagram of the corresponding right adjoints $Th \dashv g$, $TTh \dashv Tg$, $Te_X \dashv m_X$ and $Te_Y \dashv m_Y$ commutes as well. We also recall that, for an adjunction $f \dashv g$ in an order-enriched category, the inequalities $\mathrm{id} \leq gf$ and $fg \leq \mathrm{id}$ imply $fgf = f$; hence, if f is a monomorphism, then $gf = \mathrm{id}$. Consequently,

$$\begin{aligned} \mathcal{M}_{\mathbb{T}} &= \{h : X \to Y \text{ in } \mathsf{X} \mid Th \text{ is a left adjoint monomorphism in } \mathsf{X}\} \\ &= \{h : X \to Y \text{ in } \mathsf{X} \mid Th \text{ is a left adjoint monomorphism in } \mathsf{X}^{\mathbb{T}}\}. \end{aligned}$$

In the sequel we call

- a morphism $h : X \to Y$ in X ***order-mono*** whenever, for all $f, g : A \to X$ in X, $h \cdot f \leq h \cdot g$ implies $f \leq g$.
- a morphism $h : X \to Y$ in X ***order-epi*** whenever, for all $f, g : Y \to B$ in X, $f \cdot h \leq g \cdot h$ implies $f \leq g$.
- a functor $T : \mathsf{X} \to \mathsf{X}$ ***order-faithful*** whenever, for all $f, g : A \to X$ in X, $Tf \leq Tg$ implies $f \leq g$.

We denote the class of all order-monos of X by ord-mono(X), and the class of all order-epis by ord-epi(X). Clearly, if X is order-enriched with the discrete order, then the notions above coincide with mono, epi and faithful, respectively. Furthermore, order-mono implies mono, order-epi implies epi and order-faithful implies faithful.

Proposition 2.9. *Let $\mathbb{T} = (T, m, e)$ be an order-enriched monad on X. Then the following assertions are equivalent.*

(i) For every object X in X, e_X is order-mono.
(ii) T is order-faithful.

Moreover, for a Kock-Zöberlein monad $\mathbb{T}$, the two assertions above are also equivalent to $\mathcal{M}_{\mathbb{T}} \subseteq$ ord-mono(X).

Proof: It is immediate, taking into account that, for an order-enriched monad $\mathbb{T} = (T, m, e)$, the inequality $Tf \cdot e_X \leq Tg \cdot e_X$ implies $f \leq g$, and that the maps e_X belong to $\mathcal{M}_{\mathbb{T}}$. ■

Remark 2.10. For all monads $\mathbb{T} = (T, m, e)$ of Examples 2.5, the functor T is order-faithful.

3. Abstract algebraic objects

The role model of this section is the theory of completely distributive and of totally algebraic lattices in the spirit of [28, 29]. We recall that, for $\mathbb{T} = \mathbb{D}$ being the downset monad on Pos, a partially ordered set Y is isomorphic to some DX if and only if Y is totally algebraic (see Definition 2.4 (6)). Analogously, trading the downset monad for the directed downset monad $\mathbb{T} = \mathbb{I}$, a partially ordered set Y is isomorphic to some IX if and only if Y is an algebraic domain. The principal observation of this section is that these results are not particularly about order theory but hold in more general for a Kock-Zöberlein monad on an order-enriched category. To achieve this, an important tool is the equivalence [29] between category of split algebras and the idempotent split completion of the Kleisli monad which allows us to move back and forth between these categories.

Hence, in this section we consider a Kock-Zöberlein monad $\mathbb{T} = (T, m, e)$ on an order-enriched category X. Then the Kleisli category $\mathsf{X}_\mathbb{T}$ is order-enriched as well, and the canonical functor

$$\mathsf{X} \longrightarrow \mathsf{X}_\mathbb{T},\ (f : X \to Y) \longmapsto (f_* = e_Y \cdot f : X \mathrel{\circ\!\!\!\rightarrow} Y)$$

is order-enriched; it is even locally an order embedding provided that T is order-faithful. We note that, for arrows $r : A \mathrel{\circ\!\!\!\rightarrow} X$ and $s : Y \mathrel{\circ\!\!\!\rightarrow} B$ in $\mathsf{X}_\mathbb{T}$ and $f : X \to Y$ in X,

$$f_* \circ r = Tf \cdot r \qquad \text{and} \qquad s \circ f_* = s \cdot f.$$

The following definition is motivated by [23].

Definition 3.1. An object Y in X is called ***Cauchy complete*** whenever every left adjoint morphism $r : X \mathrel{\circ\!\!\!\rightarrow} Y$ in $\mathsf{X}_\mathbb{T}$ is of the form $r = f_*$, for some $f : X \to Y$ in X.

Examples 3.2. For the downset monad $\mathbb{D}$ on Pos, every partially ordered set X is Cauchy complete. For each of the filter monads on Top_0, a T_0-space X is Cauchy complete if and only if X is sober.

Theorem 3.3. *Every $\mathbb{T}$-algebra Y is Cauchy complete. Moreover, if T is order-faithful, an object Y of X is a $\mathbb{T}$-algebra if and only if Y is Cauchy-complete and Te_Y has a left adjoint in X.*

Proof: Assume first that Y is a $\mathbb{T}$-algebra, with left adjoint $\beta : TY \to Y$ of $e_Y : Y \to TY$. Since T is order-enriched, also $T\beta \dashv Te_Y$, hence Te_Y has a left adjoint. Let $s : Y \nrightarrow X$ be the right adjoint of $r : X \to Y$ in $\mathsf{X}_{\mathbb{T}}$. Hence,

$$e_X \le s \circ r = m_X \cdot Ts \cdot r \qquad \text{and} \qquad e_Y \ge r \circ s = m_Y \cdot Tr \cdot s.$$

We put $f = \beta \cdot r$, then $f_* = e_Y \cdot \beta \cdot r \ge r$. In fact, $f_* \dashv s$ in $\mathsf{X}_{\mathbb{T}}$ since $s \circ f_* \ge s \circ r \ge e_X$ and

$$f_* \circ s = T\beta \cdot Tr \cdot s \le T\beta \cdot e_{TY} \cdot m_Y \cdot Tr \cdot s \le T\beta \cdot e_{TY} \cdot e_Y = T\beta \cdot Te_Y \cdot e_Y = e_Y;$$

and therefore $r = f_*$.

Assume now that T is order-faithful and let Y be a Cauchy-complete X-object so that Te_Y has a left adjoint in X. Then, since $Te_Y : TY \to TTY$ corresponds to $(e_Y)_* : Y \nrightarrow TY$ in $\mathsf{X}_{\mathbb{T}}$, $(e_Y)_*$ has a left adjoint $r : TY \nrightarrow Y$ in $\mathsf{X}_{\mathbb{T}}$ (see also Remark 2.8). Since Y is Cauchy complete, $r = \beta_*$ for some $\beta : TY \to Y$. Finally, $(-)_* : \mathsf{X} \to \mathsf{X}_{\mathbb{T}}$ is locally an order embedding by hypothesis, therefore $\beta \dashv e_Y$. ∎

Corollary 3.4. *Let $\mathbb{T} = (T, m, e)$ be an idempotent monad on a category X where T is faithful. Then an object Y of X is a $\mathbb{T}$-algebra if and only if Y is Cauchy complete.*

We recall now the general notion of a split algebra for a monad as introduced in [29].

Definition 3.5. A $\mathbb{T}$-algebra X is called ***split*** whenever the left adjoint $\alpha : TX \to X$ of $e_X : X \to TX$ has a left adjoint $t : X \to TX$ in X; and X is called ***algebraic*** whenever X is isomorphic to a free algebra in $\mathsf{X}^{\mathbb{T}}$.

Examples 3.6. A partially ordered set X is a split algebra for the downset monad if and only if X is completely distributive, in this case the map $t : X \to DX$ sends $x \in X$ to the set $\{y \in X \mid y \lll x\}$ of all elements $y \in Y$ which are totally below x (see [27, 16]). Similarly, a directed cocomplete partially ordered set X is a split algebra for $\mathbb{I} = (I, m, e)$ if and only if X is a domain (see Definition 2.4(4)); in this case the splitting $t : X \to IX$ is given by $x \mapsto \{y \in X \mid y \ll x\}$. Regarding the filter monad $\mathbb{F} = (F, m, e)$ on Top_0, a continuous lattice X (equipped with the Scott topology) is a split algebra for $\mathbb{F}$ if and only if X is $\mathbb{F}$-disconnected in the sense of [18]. Here, with $\alpha : FX \to X$ denoting the algebra structure of X, for an open subset $A \subseteq X$ we put $\mu(A) = \{x \in X \mid \alpha(\mathfrak{f}) = x$ for some $\mathfrak{f} \in FX$ with $A \in \mathfrak{f}\}$. Then X is $\mathbb{F}$-disconnected precisely when $\mu(A)$ is open, for every open subset

$A \subseteq X$; and in this case the map $t : X \to FX$ sends $x \in X$ to the filter $t(x) = \{A \subseteq X \mid A \text{ open}, x \in \mu(A)\}$. The case of the prime filter monad is similar, with $\mu(A)$ now defined using only prime filters. In terms of partially ordered compact Hausdorff spaces, every split algebra for $\mathbb{F}_2$ is a Priestley space, more precise, a Priestley space is a split algebra for $\mathbb{F}_2$ if and only if it is an f-space in the sense of [26]. In Section 6 we give a different characterisation of the split algebras for the filter monad, by means of the way below relation. In Examples 3.18 we describe algebraic $\mathbb{T}$-algebras.

We denote the full subcategory of $\mathsf{X}^{\mathbb{T}}$ of all split $\mathbb{T}$-algebras by $\mathrm{Spl}(\mathsf{X}^{\mathbb{T}})$. Since $\mathbb{T}$ is of Kock-Zöberlein type, every free $\mathbb{T}$-algebra TY (with algebra structure $m_Y : TTY \to TY$) is split since $Te_Y \dashv m_Y \dashv e_{TY}$. Hence, every algebraic $\mathbb{T}$-algebra is split. Next we recall that the split $\mathbb{T}$-algebras are precisely those algebras where the algebra structure has a homomorphic splitting.

Proposition 3.7. *Let X be a $\mathbb{T}$-algebra with $\alpha \dashv e_X$ in X and let $t : X \to TX$ in X. Then $t \dashv \alpha$ in X if and only if t is a $\mathbb{T}$-homomorphism with $\alpha \cdot t = \mathrm{id}_X$.*

The following two results exhibit the connection with idempotents in $\mathsf{X}_{\mathbb{T}}$ as shown in [29].

Proposition 3.8. *For every split $\mathbb{T}$-algebra X with $t \dashv \alpha \dashv e_X$, $t \leq e_X$ and $t \circ t = t$.*

Recall that X is *idempotent split complete*, or just *idempotent complete*, whenever every idempotent morphism $e : X \to X$ in X is of the form $s \cdot r$, for some $r : X \to Y$ and $s : Y \to X$ in X with $r \cdot s = \mathrm{id}_Y$. (see [7], for instance). Every category with equalisers or with coequalisers is idempotent split complete.

Theorem 3.9. *Assume that X is idempotent split complete. Then $\mathrm{Spl}(\mathsf{X}^{\mathbb{T}})$ is equivalent to the idempotent split completion $\mathrm{kar}(\mathsf{X}_{\mathbb{T}})$ of $\mathsf{X}_{\mathbb{T}}$.*

In the remainder of this section we aim for a characterisation of algebraic $\mathbb{T}$-algebras in an intrinsic way, for idempotent split complete order-enriched categories X. Under the equivalence $\mathrm{Spl}(\mathsf{X}^{\mathbb{T}}) \simeq \mathrm{kar}(\mathsf{X}_{\mathbb{T}})$, a split algebra X with $t \dashv \alpha \dashv e_X$ corresponds to (X, t) in $\mathrm{kar}(\mathsf{X}_{\mathbb{T}})$; in particular, the free algebra TY corresponds to (TY, Te_Y). Moreover, for every Y in X, $(Y, e_Y) \simeq (TY, Te_Y)$ in $\mathrm{kar}(\mathsf{X}_{\mathbb{T}})$. Hence:

Corollary 3.10. *Assume that* X *is idempotent split complete. A split* $\mathbb{T}$-*algebra* X *with* $t \dashv \alpha \dashv e_X$ *is algebraic if and only if* $(X,t) \simeq (Y, e_Y)$ *in* $\mathrm{kar}(\mathsf{X}_{\mathbb{T}})$, *for some* Y *in* X.

To describe this condition, we introduce the following notion.

Definition 3.11. A morphism $f : X \to Y$ in X is called $\mathbb{T}$-***dense*** whenever $f_* : X \nrightarrow Y$ has a right adjoint $f^* : Y \nrightarrow X$ in $\mathsf{X}_{\mathbb{T}}$.

Remark 3.12. Clearly, $f_* : X \nrightarrow Y$ has a right ajoint in $\mathsf{X}_{\mathbb{T}}$ if and only if the corresponding algebra homomorphism $Tf : TX \to TY$ has a right adjoint in $\mathsf{X}^{\mathbb{T}}$. By Remark 2.8, this is equivalent to Tf being left adjoint in X. Thus, if T is order-faithful,

$$\mathcal{M}_{\mathbb{T}} = \mathbb{T}\text{-dense} \cap \text{ord-mono}(\mathsf{X}).$$

Examples 3.13. (1) For $\mathbb{T} = \mathbb{D}$ being the downset monad on Pos , every monotone map $f : X \to Y$ is $\mathbb{D}$-dense. In fact, for a monotone map $f : X \to Y$, the right adjoint $f^* : Y \nrightarrow X$ of f_* in $\mathsf{Pos}_{\mathbb{D}}$ is given by $f^*(y) = \{x \in X \mid f(x) \leq y\}$, for all $y \in Y$.

(2) If we consider the monad $\mathbb{I} = (I, m, e)$ instead, then f_* has a right adjoint if and only if "f^* lives in $\mathsf{Pos}_{\mathbb{I}}$", that is, if and only if $\{x \in X \mid f(x) \leq y\}$ is directed, for all $y \in Y$.

(3) For $\mathbb{T} = \mathbb{F}$ being the filter monad on Top_0, every continuous map $f : X \to Y$ is $\mathbb{F}$-dense. Here, for a continuous map $f : X \to Y$, the right adjoint $f^* : Y \nrightarrow X$ of $f : X \nrightarrow Y$ is given by $f^*(y) = \langle\{f^{-1}(B) \mid B \in \Omega(y)\}\rangle \in FX$, for all $y \in Y$.

(4) For the proper filter monad $\mathbb{F}_1$ on Top_0, a continuous map $f : X \to Y$ is $\mathbb{F}_1$-dense if and only if the filter $\langle\{f^{-1}(B) \mid B \in \Omega(y)\}\rangle$ is proper, for each $y \in Y$; and this in turn is equivalent to f being dense in the usual topological sense.

(5) Similarly, for the prime filter monad $\mathbb{F}_2$ on Top_0, a continuous map $f : X \to Y$ is $\mathbb{F}_2$-dense if and only if the filter $\langle\{f^{-1}(B) \mid B \in \Omega(Y)\}\rangle$ is prime. By [15, Lemma 6.5], this condition is equivalent to f being flat. More generally, for the n-prime filter monads $\mathbb{F}_n$, to be $\mathbb{F}_n$-dense is equivalent to be n-flat [11].

Assumption 3.14. From now on we also assume that

- X *has equalisers* and
- T *sends regular monomorphisms to monomorphisms.*

Since X has equalisers, X is also idempotent split complete. We remark that these conditions are satisfied in all Examples 2.5.

Lemma 3.15. *If $i : A \to X$ is a regular monomorphism in X, then i_* is a monomorphism in $\mathsf{X}_{\mathbb{T}}$.*

Proof: Just observe that $i_* \circ r = i_* \circ s$ in $\mathsf{X}_{\mathbb{T}}$ translates to $Ti \cdot r = Ti \cdot s$ in X. ■

Proposition 3.16. *Let X be a split $\mathbb{T}$-algebra with $t \dashv \alpha \dashv e_X$ and let*

$$A \xrightarrow{i} X \underset{t}{\overset{e_X}{\rightrightarrows}} TX$$

be an equaliser diagram. Then the following assertions hold.

(1) $i_* : A \,\circ\!\!\!\rightarrow X$ is a morphism of type $i_* : (A, e_A) \,\circ\!\!\!\rightarrow (X, t)$ in $\mathrm{kar}(\mathsf{X}_{\mathbb{T}})$.

(2) X is algebraic if and only if $i : A \to X$ is $\mathbb{T}$-dense and $i_* \circ i^* = t$.

Proof: To show the first assertion, we calculate

$$t \circ i_* = t \cdot i = e_X \cdot i = Ti \cdot e_A = i_* \circ e_A.$$

Regarding the second assertion, assume first that X is algebraic, that is, there are arrows $r : (Y, e_Y) \,\circ\!\!\!\rightarrow (X, t)$ and $s : (X, t) \,\circ\!\!\!\rightarrow (Y, e_Y)$ in $\mathrm{kar}(\mathsf{X}_{\mathbb{T}})$ with $s \circ r = e_Y$ and $r \circ s = t$. Since $t \leq e_X$, we conclude that $r \dashv s$ in $\mathsf{X}_{\mathbb{T}}$ and, since the $\mathbb{T}$-algebra X is Cauchy complete (see Theorem 3.3), $r = f_*$ for $f = \alpha \cdot r : Y \to X$. Furthermore,

$$t \cdot f = t \cdot \alpha \cdot r = m_X \cdot Tt \cdot r = t \circ r = r = f_* = e_X \cdot f,$$

hence there is an arrow $h : Y \to A$ in X with $i \cdot h = f$. Then

$$i_* \circ h_* \circ s = f_* \circ s = r \circ s = t \leq e_Y$$

and

$$i_* \circ h_* \circ s \circ i_* = t \circ i_* = i_* \circ e_A,$$

hence $h_* \circ s \circ i_* = e_A$, by Lemma 3.15. Putting $i^* = h_* \circ s$, we have seen that $i_* \dashv i^*$ in $\mathsf{X}_{\mathbb{T}}$ and $i_* \circ i^* = t$.

Conversely, assume now that i_* has a right adjoint i^* with $i_* \circ i^* = t$. Since

$$i^* \circ t = i^* \circ i_* \circ i^* = i^* = e_A \circ i^*,$$

$i^* : (X, t) \,\circ\!\!\!\rightarrow (A, e_Y)$ is a morphism in $\mathrm{kar}(\mathsf{X}_{\mathbb{T}})$; it is indeed an isomorphism since $i^* \circ i_* = e_A$ and $i_* \circ i^* = t$. ■

Finally, we can simplify the condition $i_* \circ i^* = t$ and obtain:

Theorem 3.17. *With the same assumption as in Proposition 3.16, X is algebraic if and only if $i : A \to X$ is $\mathbb{T}$-dense and $\alpha \cdot Ti$ is an epimorphism in* X.

Proof: For $r : X \to\!\!\!\!\!\circ\; Y$ in $\mathsf{X}_{\mathbb{T}}$, we write $\widehat{r} : TX \to TY$ for the corresponding $\mathbb{T}$-algebra homomorphism. With the notation of Proposition 3.16, if $i : A \to X$ is $\mathbb{T}$-dense with right adjoint i^*, then $i_* \circ i^* = t$ if and only if $\widehat{t} = \widehat{i_*} \cdot \widehat{i^*} = Ti \cdot \widehat{i^*}$ if and only if the $\mathbb{T}$-algebra homomorphisms $Ti : TA \to TX$ and $\widehat{i^*} : TX \to TA$ split the idempotent $\widehat{t} : TX \to TX$. But since $\widehat{t} : TX \to TX$ is also split by $\alpha : TX \to X$ and $t : X \to TX$, $i_* \circ i^* = t$ if and only if

$$\widehat{i^*} \cdot t \cdot \alpha \cdot Ti = \mathrm{id}_{TA} \qquad \text{and} \qquad \alpha \cdot Ti \cdot \widehat{i^*} \cdot t = \mathrm{id}_X .$$

Furthermore, the first equality is always true:

$$\widehat{i^*} \cdot t \cdot \alpha \cdot Ti = \widehat{i^*} \cdot m_X \cdot Tt \cdot Ti = \widehat{i^*} \cdot m_X \cdot Te_X \cdot Ti = \widehat{i^*} \cdot Ti = \mathrm{id}_{TA};$$

therefore the second one holds precisely when $\alpha \cdot Ti$ is an epimorphism in X. ∎

Examples 3.18. We continue here Examples 3.6.

(1) For $\mathbb{T} = \mathbb{D}$ being the downset monad on Pos, Theorem 3.17 tells us that a completely distributive lattice L is algebraic for $\mathbb{D}$ if and only if L is totally algebraic, that is, if every element is the supremum of all the elements totally below it.

(2) We consider now the directed downset monad $\mathbb{I} = (I, m, e)$ on Pos. In this case, a directed cocomplete partially ordered set X is a split algebra if and only if it is a domain; in this case the splitting $t : X \to IX$ is given by $x \mapsto \{y \in X \mid y \ll x\}$. Moreover, X is algebraic if and only if, for every $x \in X$, the set $\{y \in X \mid y \ll y \ll x\}$ is directed and has x as supremum; that ist, if X is algebraic in the sense of domain theory (see [1]).

(3) Let now X be a $\mathbb{F}$-disconnected continuous lattice. Then the elements of A are precisely those elements $x \in X$ where, for all open subsets $B \subseteq X$, $x \in \mu(B)$ implies that $x \in B$. Then X is algebraic if and only if every $x \in X$ is the largest convergence point (with respect to the specialisation order) of a filter $\mathfrak{f}$ with $A \in \mathfrak{f}$.

(4) Similarly, an f-space X is a algebraic for the prime filter monad $\mathbb{F}_2$ if and only if every $x \in X$ is the largest convergence point (with respect to the specialisation order) of a prime filter $\mathfrak{f}$ with $A \in \mathfrak{f}$.

4. Weighted (co)limits and cogenerators

"Cocompleteness almost implies completenes" is the title of the paper [2] of Jiří Adámek, Horst Herrlich and Jiří Reiterman, as well as the main theme of section 12 of the book [3]. The title announces several results giving conditions under which completeness and cocompleteness are equivalent. In particular, it is proved (the dual of) that a complete and wellpowered category with a cogenerator is cocomplete (and co-wellpowered).

In the setting of order-enriched categories, it is natural to consider "order-enriched" limits and colimits, the so-called ***weighted (co)limits***, or ***indexed (co)limits***. Thus, the question of knowing when weighted completeness does imply weighted cocompleteness arises. Here we show that it happens in the presence of a regular cogenerator.

Remark 4.1. 1. We start by recalling the notion of weighted limit ([20]) in the order-enriched setting. Let $D : \mathsf{D} \to \mathsf{X}$ and $W : \mathsf{D} \to \mathsf{Pos}$ be order-enriched functors, with D small. They give rise to the functor $\mathsf{Pos}^{\mathsf{D}}(W, \mathsf{X}(-, D))$ from X^{op} to Pos, where, for every $X \in \mathsf{X}$, $\mathsf{X}(-, D)(X)$ stands for the functor $\mathsf{X}(X, -) \cdot D : \mathsf{D} \to \mathsf{Pos}$. The ***limit of*** D ***weighted by*** W, in case it exists, is an object L of X which represents that functor, that is, there is a natural isomorphism

$$\mathsf{X}(-, L) \cong \mathsf{Pos}^{\mathsf{D}}(W, \mathsf{X}(L, D)). \tag{1}$$

This is equivalent to say that we have a family of morphisms

$$L \xrightarrow{l_d^x} Dd,\ d \in \mathsf{D},\ x \in Wd$$

satisfying the following conditions:

(a) $l_d^x \le l_d^y$ whenever $x \le y$, and $Dn \cdot l_d^x = l_{d'}^{Wn(x)}$, for all morphisms $n : d \to d'$ in D and all $x \in Wd$. (This gives the natural transformation from W to $\mathsf{X}(L, D)$ which is the image of id_L by the component of the natural transformation indexed by L.)

(b) The family $(l_d^x)_{d \in \mathsf{D}, x \in Wd}$ is universal, i.e., the natural transformation (1) is a natural isomorphism. This means that every family of morphisms $A \xrightarrow{a_d^x} Dd$, $d \in \mathsf{D}$, $x \in Wd$, satisfying (a) – with A and a in the place of L and l – is of the form $a_d^x = l_d^x t$ for a unique $t : A \to L$; and, moreover, for $t, t' : A \to L$, the inequality $l_d^x t \le l_d^x t'$, for all d and x, imply $t \le t'$.

When W is just the constant functor into a singleton, we speak of ***conical limits***. Thus, a conical limit is a limit in the ordinary sense whose projections are jointly order-monic.

Inserters and cotensor products are special types of weighted limts. The ***inserter*** of a pair of morphisms $f, g : X \to Y$ is just a morphism $i : I \to X$ with $fi \leq gi$ and universal with respect to that property (in the sense of (b) above). Given a poset I and an object X of X, the ***cotensor product*** of I and X, denoted by $\pitchfork(I, X)$, is a weighted limit with the domain D of the functors D and W the unit category, i.e., the category with just an object and the corresponding identity morphism. Thus, the projections of the cotensor product are of the form

$$\pitchfork(I, X) \xrightarrow{\quad l^i \quad} X \ , \ i \in I,$$

with $l_i \leq l_j$ for $i \leq j$.

In an order-enriched category, the existence of conical products and inserters guarantees the existence of all weighted limits.

The dual notions for weighted limits, inserters and cotensor products are, respectively, weighted colimits, coinserters and tensor products.

2. For every Kock-Zöberlein monad $\mathbb{T}$ over a category X with weighted limits, the subcategory $\mathsf{X}^{\mathbb{T}}$ is closed under them (since the forgetful functor from $\mathsf{X}^{\mathbb{T}}$ to X creates weighted limits). Indeed, as shown in [9], more than being closed under weighted limits, the subcategory $\mathsf{X}^{\mathbb{T}}$ is also an inserter-ideal. This means that, for every diagram

$$\begin{tikzcd} I \arrow[r, "i"] & A \arrow[r, shift left, "g"] \arrow[r, shift right, "f"'] & B \end{tikzcd}$$

with i the inserter of the pair (f, g) in X, if f is a morphism of $\mathsf{X}^{\mathbb{T}}$, then $i : I \to A$ lays in $\mathsf{X}^{\mathbb{T}}$ too.

Remark 4.2. We also use the "order-enriched" version of the notion of cogenerator. Along this paper an object S of an order-enriched category is said to be a ***cogenerator*** if it detects the order, in the sense that, for every pair of morphisms $f, g : X \to Y$, $f \leq g$ iff $tf \leq tg$ for all morphisms $t : Y \to S$. Next we give the notion of *regular cogenerator*. (Co)generators in this sense were considered for instance in [22].

Remark 4.3. 1. We recall that an order-enriched adjunction between order-enriched categories is an adjunction $F \dashv U : \mathsf{A} \to \mathsf{B}$ with U and F order-enriched, and for which there exists a natural isomorphism between the functors $\mathsf{B}(-, U-)$ and $\mathsf{A}(F-, -)$ from $\mathsf{B} \times \mathsf{A}$ to Pos. This is equivalent to say that we have an adjunction $F \dashv U : \mathsf{A} \to \mathsf{B}$ with U order-enriched, and the unit η satisfies the property that any inequality of the form $Uf \cdot \eta_X \leq Ug \cdot \eta X$, for f e g with common domain and codomain, implies $f \leq g$ ([11]). Clearly, an order-enriched adjunction induces an order-enriched monad; and, for an order-enriched monad $\mathbb{T}$, the adjunctions $F^{\mathbb{T}} \dashv U^{\mathbb{T}}$ and $F_{\mathbb{T}} \dashv U_{\mathbb{T}}$ are order-enriched.

2. In an order-enriched category A with weighted limits, given an object S, the cotensor product yields a functor

$$\pitchfork(-, S) : \mathsf{Pos} \longrightarrow \mathsf{A}^{\mathrm{op}} \tag{2}$$

which is an order-enriched left adjoint of $\mathsf{A}(-, S)$. For every $X \in \mathsf{A}$, the counit map is given by (the dual of) the morphism n_X determined by the universality of the cotensor product:

$$\begin{tikzcd} X \arrow[r, "n_X"] \arrow[dr, "f"'] & \pitchfork(\mathsf{A}(X, S), \mathrm{S}) \arrow[d, "\pi_f"] \\ & \mathrm{S} \end{tikzcd} \quad , \ f \in \mathsf{A}(X, S) \tag{3}$$

Given $X \in \mathsf{A}$, put

$$\hat{X} = \pitchfork(\mathsf{A}(X, S), S)$$

and consider the cotensor product

$$\pitchfork(\mathsf{A}(\hat{X}, S), S) \xrightarrow{\hat{\pi}_g} \mathrm{S}, \quad g \in \mathsf{A}(\hat{X}, S).$$

Let $\beta : \pitchfork(\mathsf{A}(X, \mathrm{S}), S) \longrightarrow \pitchfork(\mathsf{A}(\hat{X}, S), S)$ be the unique morphism of A which makes the following diagrams commutative:

$$\begin{tikzcd}
\hat{X} \arrow[rr, "\beta"] \arrow[dr, "\pi_{g\cdot n_X}"'] & & \pitchfork(\mathsf{A}(\hat{X}, S), S) \arrow[dl, "\hat{\pi}_g"] \\
& \mathrm{S} &
\end{tikzcd}, \quad g \in \mathsf{A}(\hat{X}, S).$$

Thus, putting $\alpha = n_{\hat{X}}$, we have the diagram

$$X \xrightarrow{n_X} \hat{X} \underset{\alpha}{\overset{\beta}{\rightrightarrows}} \pitchfork(\mathsf{A}(\hat{X}, S), S). \tag{4}$$

Definition 4.4. Let A be an order-enriched category with weighted limits. An object S of A is said to be a ***regular cogenerator*** if the the diagram (4) is an equaliser.

If A has weighted limits, every equaliser of A is conical; hence, it is immediate that every regular cogenerator detects the order, so, in particular, it is a cogenerator.

Theorem 4.5. *Every order-enriched category with weighted limits and a regular cogenerator has weighted colimits.*

Proof: Let A be an order-enriched category with weighted limits and a regular cogenerator S. Then, as seen in Remark 4.3, the functors

$$\mathsf{A}^{\mathrm{op}} \underset{\pitchfork(-,S)}{\overset{\mathsf{A}(-,S)}{\rightleftarrows}} \mathsf{Pos}$$

form an order-enriched adjunction. Let $\mathbb{T}$ be the corresponding monad and let $K : \mathsf{A}^{\mathrm{op}} \to \mathsf{Pos}^{\mathbb{T}}$ be the comparison functor:

$$\begin{tikzcd}
\mathsf{A}^{\mathrm{op}} \arrow[rr, "K"] \arrow[dr, "\mathsf{A}(-,S)"'] & & \mathsf{Pos}^{\mathbb{T}} \arrow[dl] \\
& \mathsf{Pos} &
\end{tikzcd}$$

Since S is a regular cogenerator, the morphism n_X^{op}, which is, pointwisely, the counit of the adjunction $\pitchfork(-, S) \dashv \mathsf{A}(-, S)$, is a regular epimorphism, and, consequently, K is a full and faithful right adjoint. Moreover, this adjunction is order-enriched, as it is explained in the next paragraph.

Let $F \dashv U : \mathsf{C} \to \mathsf{B}$ be an order-enriched adjunction with the counit being pointwisely a conical coequaliser, and C having conical coequalisers. It is well-known that, under these conditions, the comparison functor K is a full and faithful right adjoint [24]. It is clear that K is order-enriched. Then, in order to conclude that the adjunction $K : \mathsf{C} \to \mathsf{B}^{\mathbb{T}}$ is order-enriched, it suffices to show that, for every universal map $\eta^{\mathbb{T}}_{(X,\xi)} : (X,\xi) \to KA$ of the adjunction, and every pair $f, g : A \to B$ of morphisms in C with $Kf \cdot \eta^{\mathbb{T}}_{(X,\xi)} \leq Kg.\eta^{\mathbb{T}}_{(X,\xi)}$, we have $f \leq g$ (see Remark 4.3.1). Recall that, given $(X,\xi) \in \mathsf{B}^{\mathbb{T}}$, the universal map from (X,ξ) to K is obtained as follows: take the coequaliser $c : FX \to A$ of the pair

$$\begin{tikzcd} FUFX \arrow[r, shift left, "\varepsilon_{FX}"] \arrow[r, shift right, "F\xi"'] & FX \end{tikzcd}$$

where ε is the counit of the adjunction $F \dashv U$. Then $Uc \cdot UF\xi = Uc \cdot U\varepsilon_{FX}$. But $\xi = \mathrm{coeq}(U\varepsilon_{FX}, UF\xi)$; hence, there is a unique $\theta : X \to UA$ making the following triangle commutative:

$$\begin{tikzcd} UFX \arrow[r, "\xi"] \arrow[dr, "Uc"'] & X \arrow[d, "\theta"] \\ & UA \end{tikzcd}$$

and it holds $\theta = Uc.\eta_X$. It is known that

$$\eta^{\mathbb{T}}_{(X,\xi)} = \theta$$

and, for every $g : (X,\xi) \to KB$ in $\mathsf{B}^{\mathbb{T}}$, the unique $\bar{g} : A \to B$ in C making the triangle

$$\begin{tikzcd} (X,\xi) \arrow[r, "\theta"] \arrow[d, "g"'] & KA \arrow[dl, "K\bar{g}"] \\ KB & \end{tikzcd}$$

commutative is characterised by the equality

$$\bar{g} \cdot c = \varepsilon_B \cdot Fg.$$

We show that, given two morphisms $g, h : (X,\xi) \to KB$ with $g \leq h$ then $\bar{g} \leq \bar{h}$. Since F is order-enriched, the inequality $g \leq h$ implies $\varepsilon_B \cdot Fg \leq \varepsilon_B \cdot Fh$.

But then

$$\bar{g} \cdot c = \varepsilon_B \cdot Fg \leq \varepsilon_B \cdot Fh = \bar{h} \cdot c$$

and, since c is order-epic (because it is a conical coequaliser), $\bar{g} \leq \bar{h}$.

Now we have that, for our comparison functor $K : \mathsf{A}^{\mathrm{op}} \to \mathsf{Pos}^{\mathbb{T}}$,

- K is the right adjoint of an order-enriched adjunction;
- K is full and faithful, and it is full with respect to the order, that is, given a pair of morphisms $\underset{f}{\overset{g}{\rightrightarrows}}$ in A^{op}, $f \leq g$ in A^{op} iff $Kf \leq Kg$ in $\mathsf{Pos}^{\mathbb{T}}$.

Consequently, since $\mathsf{Pos}^{\mathbb{T}}$ has weighted limits, also A^{op} has weighted limits, and the weighted limits in A^{op} are constructed, up to isomorphism, as in $\mathsf{Pos}^{\mathbb{T}}$. (This can be easily proved in a way analogous to the one of the ordinary case.) That is, A has weighted colimits. ■

In the next section we apply this theorem to the categories ALat of algebraic lattices with maps which preserve directed suprema and all infima, the category ADom of bounded complete algebraic domains with maps which preserve directed suprema and all non-empty infima, and the category Spec of spectral topological spaces and spectral maps.

5. (Co)completeness of subcategories of $\mathsf{X}^{\mathbb{T}}$

In this and the next section we work within the category Top_0 of T_0 topological spaces and continuous maps. We consider the relation $\leq$ in a space to be the specialisation order, and we use also the symbol $\leq$ to refer to the corresponding order induced in the hom-sets of Top_0. We do this in order to fit our terminology on continuous domains and lattices with [17]. Thus, as mentioned already in Examples 2.5, the open filter monads are KZ with respect to $\geq$.

The category Top_0 has weighted limits, since its ordinary limits are conical, and the inserter of a pair (f, g) of morphisms with domain in X is just the subspace of all $x \in X$ with $f(x) \leq g(x)$. Therefore, for every Kock-Zöberlein monad $\mathbb{T}$ over $\mathsf{X} = \mathsf{Top}_0$ the corresponding Eilenberg-Moore category $\mathsf{X}^{\mathbb{T}}$ is closed under weighted limits in Top_0 (since the forgetful functor from $\mathsf{X}^{\mathbb{T}}$ to Top_0 creates limits). Hence, the cotensor product yields the functor

$$\pitchfork(-, \mathrm{S}) : \mathsf{Pos} \longrightarrow \left(\mathsf{X}^{\mathbb{T}}\right)^{\mathrm{op}}.$$

This functor is defined as in (2) of the previos section. Thus, we can consider the diagram defined as in (4):

$$X \xrightarrow{n_X} \hat{X} \underset{\alpha}{\overset{\beta}{\rightrightarrows}} \pitchfork(\mathbb{H}\mathrm{om}(\hat{X}, \mathrm{S}), \mathrm{S}) \tag{5}$$

where $\mathbb{H}\mathrm{om}$ refers to hom-posets of $\mathsf{X}^{\mathbb{T}}$. Let

$$\mathsf{X}_{\mathrm{alg}}$$

denote the full subcategory of $\mathsf{X}^{\mathbb{T}}$ for which the diagram (5) is an equaliser in Top_0, then also in $\mathsf{X}^{\mathbb{T}}$.

We are going to show that, concerning the filter, the proper filter and the prime filter monads, the subcategories $\mathsf{X}_{\mathrm{alg}}$ are well-known categories, namely: the category ALat of algebraic lattices with maps which preserve directed suprema and all infima, the category ADom of bounded complete algebraic domains with maps which preserve directed suprema and all non-empty infima, and the category Spec of spectral topological spaces and spectral maps (see Definition 2.4). We show that all of them are closed under weighted limits. Hence, the equaliser diagram (5) tells us that the Sierpiński space is a regular cogenerator of $\mathsf{X}_{\mathrm{alg}}$. Moreover, it allows us to conclude that:

(1) $\mathsf{X}_{\mathrm{alg}}$ is the closure under weighted limits of $\mathsf{X}_{\mathbb{T}}$ in $\mathsf{X}^{\mathbb{T}}$, and in Top_0 (Corollary 5.4);
(2) $\mathsf{X}_{\mathrm{alg}}$ has weighted colimits (Corollary 5.5).

We start by establishing the closedness under weighted limits:

Proposition 5.1. *Every one of the three categories,* ALat*,* ADom *and* Spec*, is closed under weighted limits in* Top_0*.*

Proof: Let $\mathbb{T}$ be a Kock-Zöberlein monad over Top_0; then $\mathsf{X}^{\mathbb{T}}$ is closed under weighted limits in Top_0. Inserters in Top_0 are topological embeddings, then also order embeddings. Thus, the same happens in $\mathsf{X}^{\mathbb{T}}$.

Let now $\mathcal{K}$ be a full subcategory of $\mathsf{X}^{\mathbb{T}}$. Then, in order to ensure that $\mathcal{K}$ is closed under weighted limits in Top_0, it suffices to show that $\mathcal{K}$ is closed in $\mathsf{X}^{\mathbb{T}}$ under

- (conical) products, and
- topological embedding subobjects, i.e., for every topological embedding
  $m : X \hookrightarrow Y$ in $\mathsf{X}^{\mathbb{T}}$ with Y in $\mathcal{K}$, also X belongs to $\mathcal{K}$.

Since for the filter and the proper filter monads the morphisms of $\mathsf{X}^{\mathbb{T}}$ are the maps preserving directed suprema and infima (respectively, non-empty infima), the closedness under products and topological embedding subobjects of ALat and ADom in the corresponding category $\mathsf{X}^{\mathbb{T}}$ is just Proposition I-4.12 and Corollary I-4.14 of [17].

Concerning the category Spec, we observed already in Definition 2.4 (7) and Example 2.5(4) that Spec is a reflecive full subcategory of $\mathsf{X}^{\mathbb{T}} \simeq \mathsf{StablyComp}$ since it is closed in it under initial cones. In particular, it is closed under products and embeddings. ■

Next we show that the Sierpiński space S is a regular cogenerator for each one of the three categories, ALat, ADom and Spec. For that, we first prove Lemma 5.2 below, where we present a common feature of the three categories, which gives the means for the proof of Theorem 5.3.

Before stating that lemma, we describe the morphism $n_X : X \longrightarrow \hat{X}$, defined in (3), in any full subcategory A of $\mathsf{X}^{\mathbb{T}}$ closed under weighted limits and containing the Sierpinski space S. Given $X \in \mathsf{A}$, let

$$\Lambda X = \{U \in \Omega X \mid \mathcal{X}_U : X \longrightarrow \mathrm{S} \text{ is a morphism of } \mathsf{A}\}.$$

Then $\hat{X} = \pitchfork(\mathbb{H}\mathrm{om}(X,\mathrm{S}),\mathrm{S})$ consists of all families $(z_U)_{U\in\Lambda X}$ in the product $\mathrm{S}^{\Lambda X}$ with the property $U \subseteq V \Rightarrow z_U \le z_V$, and $n_X(x) = (\chi_U(x))_{U\in\Lambda X}$. The topology of $\hat{X}$ is just the one induced by the product topology. Thus, it is generated by the sub-base of all sets

$$\Diamond U = \pi_{\chi_U}^{-1}(\{1\}) = \{(z_U)_{U\in\Lambda X} \mid z_U = 1\},\ U \in \Lambda X,$$

and we have $U = n_X^{-1}(\Diamond U)$. Moreover, since the projections π_{χ_U} belong to $\mathbb{H}\mathrm{om}(X,\mathrm{S})$, the sets $\Diamond U$ belong to $\Lambda\hat{X}$.

Lemma 5.2. *Let* A *be one of the categories* ALat*,* ADom *or* Spec*. Then* A *satisfies the following conditions:*

(i) The spaces of A *are sober and* $\mathrm{S} \in \mathsf{A}$*.*

(ii) A *is closed under weighted limits in* Top_0*.*

(iii) For every $X \in \mathsf{A}$*, the set* ΛX *is closed under finite intersections (in particular, contains* X*) and forms a base of the topology* ΩX*.*

(iv) For every $X \in \mathsf{A}$*, the morphism* $n_X : X \to \hat{X}$ *has the following property, for every family* V_i*,* $i \in I$*, of sets of* ΛX*:*

If $H = \bigcup_{i\in I} V_i \in \Lambda X$*, then* $H = n_X^{-1}(H')$*, for some* $H' \in \Lambda\hat{X}$ *with* $H' \subseteq \bigcup_{i\in I} \Diamond V_i$*.*

Proof: Condition (i) is well-known for the three categories.

Condition (ii) is Proposition 5.1.

We show condition (iii) for ALat. Given $X \in \mathsf{ALat}$ and $U \in \Omega X$, the characteristic function $\chi_U : X \to \mathrm{S}$ is a morphism of ALat iff it preserves arbitrary infima, and this is equivalent to U being closed under arbitrary infima. We show that it forms a base of ΩX. If U closed under infima, it is of the form $U = \uparrow c$ where $c = \bigwedge U$. But then the open sets of X closed under infima are precisely all of the form $\uparrow c$ with c a compact element of X, and these sets are known to be a base for the topology of the algebraic lattice X. Moreover, they are closed under finite intersections.

Condition (iii) for ADom is shown in an analogous way and we have, in this case,

$$\Lambda X = \{U \in \Omega X \mid U \text{ is closed under non-empty infima}\}.$$

Concerning (iii) for Spec, it is obvious that a continuous map $f : X \to \mathrm{S}$ is spectral iff $f^{-1}(\{1\})$ is compact. Thus

$$\Lambda X = \{U \in \Omega X \mid U \text{ is compact}\}$$

which is, by definition of spectral space, a base of ΩX.

Now we verify condition (iv) for the three categories.

$\mathsf{A} = \mathsf{ALat}$. Let $H = \bigcup_{i \in I} V_i$ belong to ΛX with all V_i in ΛX. Then, $\bigcup_{i \in I} V_i = \uparrow a$, with a a compact element of X; hence, $a \in V_{i_0}$ for some $i_0 \in I$; but $V_{i_0} = \uparrow V_{i_0}$, thus we have $\bigcup_{i \in I} V_i = V_{i_0}$. Consequently,

$$H = V_{i_0} = n^{-1}(\Diamond V_{i_0}) \text{ with } \Diamond V_{i_0} \subseteq \bigcup_{i \in I} \Diamond V_i.$$

$\mathsf{A} = \mathsf{ADom}$. The same proof as for ALat, in case $\bigcup_{i \in I} V_i \neq \varnothing$. The case $\bigcup_{i \in I} V_i = \varnothing$ is trivial.

$\mathsf{A} = \mathsf{Spec}$. Consider $H = \bigcup_{i \in I} V_i$ in ΛX, with $V_i \in \Lambda X, \quad i \in I$. Then, since $\bigcup_{i \in I} V_i$ is compact, it can be written as $\bigcup_{i \in I} V_i = \bigcup_{j \in J} V_j$, with $J \subseteq I$ finite. Hence, we obtain

$$H = \bigcup_{i \in I} V_i = \bigcup_{j \in J} V_j = n^{-1}(\bigcup_{j \in J} \Diamond V_j)$$

wth $\bigcup_{j \in J} \Diamond V_j \subseteq \bigcup_{i \in I} \Diamond V_i$, and $\bigcup_{j \in J} \Diamond V_j \in \Lambda \hat{X}$, because it is a finite union of compact open sets of $\hat{X}$. ■

Theorem 5.3. *For a subcategory* A *of* Top_0 *fulfilling conditions (i)-(iv) of Lemma 5.2, the diagram* (5) *is an equaliser in* Top_0. *As a consequence, the Sierpiński space is a regular cogenerator in* A*, and, in particular, in each one of the categories* ALat*,* ADom *and* Spec.

Proof: We prove that if A is a subcategory of Top_0 fulfilling conditions (i)-(iv) of Lemma 5.2, then (5) is an equaliser in Top_0. Since A contains S and is closed under weighted limits in Top_0, it immediatly follows that (5) is also an equaliser in A.

Put $n = n_X$. In order to conclude that n is indeed the equaliser of α and β, let

$$Y \xrightarrow{h} \hat{X} = \pitchfork (\mathbb{H}\mathrm{om}(X, \mathrm{S}), \mathrm{S})$$

be a morphism in Top_0 such that

$$\alpha h = \beta h.$$

For $y \in Y$, put

$$h(y) = (y_U)_{U \in \Lambda X}.$$

We show that:

(A) For every $y \in Y$, the set

$$F_y = \{U \in \Lambda X \mid y_U = 1\}$$

is a filter of the poset $(\Lambda X, \subseteq)$, and has the following property:

($\Diamond$) If $\bigcup_{i \in I} V_i \in F_y$ with all $V_i \in \Lambda X$, then $V_j \in F_y$ for some $j \in I$.

(B) Every filter F of the poset $(\Lambda X, \subseteq)$ satisfying property ($\Diamond$) is of the form

$$F = B(x) = \{U \in \Lambda X \mid x \in U\}$$

for a unique $x \in X$.

After proving (A) and (B), it is then clear that we can define $\bar{h} : Y \to X$ by putting

$$\bar{h}(y) = x \quad \text{ with } F_y = B(x),$$

and this is the unique map making the triangle

$$\begin{tikzcd} X \arrow[rr, "n"] & & \hat{X} \\ & Y \arrow[ul, dashed, "\bar{h}"] \arrow[ur, "h"'] & \end{tikzcd}$$

commutative. The fact that $\bar{h}$ is continuous follows, since n is a topological embedding.

Proof of (A). We observe that the equality $\alpha h(y) = \beta h(y)$ means that

$$\chi_H((y_U)_{U\in\Lambda X}) = y_{n^{-1}(H)}\ ,\quad H \in \Lambda\hat{X}.$$

Thus $F_y \neq \varnothing$, because $y_X = y_{n^{-1}(\hat{X})} = \chi_{\hat{X}}((y_U)_{U\in\Lambda X}) = 1$.

It is also clear that if U and V are two open sets of ΛX with $U \subseteq V$ and $U \in F_y$ then $V \in F_y$, by definition of $\hat{X}$. Moreover, F_y is closed under binary intersections: V and W laying in F_y means that $y_V = 1$ and $y_W = 1$, that is, $(y_U)_{U\in\Lambda X} \in (\Diamond V)\bigcap(\Diamond W)$. But then $\chi_{(\Diamond V)\bigcap(\Diamond W)}((y_U)_{U\in\Lambda X}) = 1$. Now, $(\Diamond V)\bigcap(\Diamond W) \in \Lambda\hat{X}$, because $\hat{X} \in \mathsf{A}$ (since $\mathsf{S} \in \mathsf{A}$ and A is closed under weighted limits), thus $\hat{X}$ satisfies (iii). Then, we have $y_{V\bigcap W} = y_{n^{-1}(\Diamond V\bigcap\Diamond W)} = 1$, that is, $V\bigcap W \in F_y$.

We show now that F_y satisfies $(\Diamond)$. Let V_i, $i \in I$, be a family of sets of ΛX with $\bigcup_{i\in I} V_i \in F_y$, that is, $\bigcup_{i\in I} V_i \in \Lambda X$ and $y_{\bigcup_{i\in I} V_i} = 1$. Then, by (iv), there is some $H' \in \Lambda\hat{X}$, with $n^{-1}(H') = \bigcup_{i\in I} V_i$ and $H' \subseteq \bigcup_{i\in I}\Diamond V_i$. Now, using the equality $\alpha h(y) = \beta h(y)$, we have:

$$1 = y_{\bigcup_{i\in I} V_i} = y_{n^{-1}(H')} = \chi_{H'}((y_U)_{U\in\Lambda X}).$$

Consequently,

$$(y_U)_{U\in\Lambda X} \in H' \subseteq \bigcup_{i\in I}\Diamond V_i.$$

Thus, for some $j \in I$, $(y_U)_{U\in\Lambda X} \in \Diamond V_j$ that is, $y_{V_j} = 1$, hence $V_j \in F_y$.

Proof of (B). It is clear that $B(x)$ is a filter of $(\Lambda X, \subseteq)$ with property $(\Diamond)$. Conversely, let F be a filter of $(\Lambda X, \subseteq)$ with property $(\Diamond)$, and put

$$A = \{z \in X \mid B(z) \subseteq F\}.$$

We show that A is a non-empty irreducible closed set.

Indeed, given $t \in X \setminus A$, there is some $V \in \Lambda X$ with $t \in V$ and $V \notin F$. But then all elements of V belong to $X \setminus A$, thus $t \in V \subseteq X \setminus A$; hence, A is closed. A is also non-empty, because, if for every $x \in X$, we have some $U_x \in \Lambda X$ with $U_x \notin F$ then, by $(\Diamond)$, we obtain that $\bigcup_{x\in X} U_x = X \notin F$, which contradicts the fact that F is a filter.

To show that A is irreducible, let $A = F_1 \bigcup F_2$ with F_1 and F_2 closed. If $A \neq F_1$ and $A \neq F_2$ then there is $x \in X\setminus F_1$ and $y \in X\setminus F_2$ with $x, y \in A$. But then we can find $U, V \in \Lambda X$ with $x \in U \subseteq X \setminus F_1$ and $y \in V \subseteq X \setminus F_2$, and

$U \bigcap V \in F$. Taking into account that $U \bigcap V \subseteq (X \setminus F_1) \bigcap (X \setminus F_2) = X \setminus A$, then, for every $z \in U \bigcap V$, there is some $V_z \in B(z)$ with $V_z \subseteq U \bigcap V$ and $V_z \notin F$. But then $U \bigcap V = \bigcup_{z \in U \bigcap V} V_z$ belongs to F with all $V_z \notin F$, which contradicts ($\Diamond$).

Since X is sober and $A \subseteq X$ is a non-empty irreducible closed set, we know that $A = \overline{\{x\}}$ for a unique $x \in X$. We show that $F = B(x)$. Clearly $B(x) \subseteq F$. Concerning the converse inclusion, condition ($\Diamond$) ensures that, for every $U \in F$, there is some $z \in U \bigcap A = U \bigcap \overline{\{x\}}$ – otherwise, we would find $V_z \in \Lambda X$, with $z \in V_z \notin F$ and $U = \bigcup_{z \in U} V_z$, a contradiction to ($\Diamond$); but then $x \in U$, i.e., $U \in B(x)$. ■

Corollary 5.4. *For the filter, the proper filter and the prime filter monads, the category* $\mathsf{X}_{\mathrm{alg}}$ *is, respectively,* ALat*,* ADom *and* Spec*. Moreover,* $\mathsf{X}_{\mathrm{alg}}$ *is the closure under weighted limits of* $\mathsf{X}_{\mathbb{T}}$ *in* $\mathsf{X}^{\mathbb{T}}$*, thus, also in* Top_0*.*

Proof: The above theorem shows that, in all the three cases $\mathsf{A} = \mathsf{ALat}$, ADom, Spec, A is indeed contained in $\mathsf{X}_{\mathrm{alg}}$. On the other hand, since $\mathrm{S} \in \mathsf{A}$, and A is closed under weighted limits in $\mathsf{X}^{\mathbb{T}}$, the diagram (5) is contained in A whenever it is an equaliser diagram. Hence A coincides with $\mathsf{X}_{\mathrm{alg}}$. Moreover, every X of $\mathsf{X}^{\mathbb{T}}$ making diagram (5) an equaliser belongs to the closure under weighted limits of $\mathsf{X}_{\mathbb{T}}$ in $\mathsf{X}^{\mathbb{T}}$, because S belongs to $\mathsf{X}_{\mathbb{T}}$. Indeed, for the open filter monad $\mathbb{T}$, S is homeomorphic to $T\mathrm{S}$, and, for the proper and the prime filter monads, S is homeomorphic to TX with X a singleton space. Therefore, in the three cases, $\mathsf{X}_{\mathrm{alg}}$ is precisely the closure under weighted limits in $\mathsf{X}^{\mathbb{T}}$ of $\mathsf{X}_{\mathbb{T}}$; and also in Top_0, since $\mathsf{X}^{\mathbb{T}}$ is closed under weighted limits in Top_0. ■

Corollary 5.5. *The categories* ALat*,* ADom *and* Spec *have weighted colimits.*

Proof: It is a consequence of Theorem 4.5, Proposition 5.1 and Theorem 5.3. ■

6. The idempotent split completion for the filter monad

Let $\mathbb{F} = (F, m, e)$ be the open filter monad on $\mathsf{X} = \mathsf{Top}_0$. As in the previous section, we use $\leq$ to refer to the order induced in the hom-sets of Top_0 by the *specialisation order*, thus the open filter monad is of Kock-Zöberlein type with respect to $\geq$. Accordingly, in all notions and results of Sections 2 and 3 on Kock-Zöberlein monads, regarding adjunctions between morphisms, "left adjoint" interchanges with "right adjoint".

As seen in Section 3, the idempotent split completion of $\mathsf{X}_{\mathbb{F}}$, denoted by $\mathrm{kar}(\mathsf{X}_{\mathbb{F}})$, is equivalent to the full subcategory $\mathrm{Spl}(\mathsf{X}^{\mathbb{F}})$ of $\mathsf{X}^{\mathbb{F}}$. And $\mathrm{Spl}(\mathsf{X}^{\mathbb{F}})$ consists of all $\mathbb{F}$-algebras (X, α) for which there is a morphism $t : X \to FX$ (in Top_0) such that $\alpha \dashv t$. Moreover, it is known that the subcategory $\mathsf{X}_{\mathbb{F}}$ is contained in ALat [13], and the latter is closed under weighted limits in $\mathsf{ContLat}$. Thus, we have the following full embeddings:

$$\mathsf{X}_{\mathbb{F}} \hookrightarrow \mathrm{kar}(\mathsf{X}_{\mathbb{F}}) \hookrightarrow \mathsf{ALat} \hookrightarrow \mathsf{X}^{\mathbb{F}} = \mathsf{ContLat}.$$

In this section we show that the idempotent split completion of $\mathsf{X}_{\mathbb{F}}$ consists precisely of all algebraic lattices whose set of compact elements forms the dual of a frame.

Notation 6.1. Along this section we use the sybmol $K(X)$ to denote the set of compact elements of a directed complete poset (see Definition 2.4).

Remark 6.2. Let X and Y be continuous lattices and let $Y \underset{e}{\overset{\alpha}{\rightleftarrows}} X$ be in Top_0 with $\alpha e = \mathrm{id}_X$ and $e\alpha \leq \mathrm{id}_Y$. Then α is defined by

$$\alpha(y) = \bigvee\{z \in X \mid e(z) \leq y\}.$$

This follows from Freyd Adjoint Theorem.

Lemma 6.3. *Let X, Y be directed complete posets with Y continuous, and let* $\begin{tikzcd} Y \arrow[r, bend left, "\alpha"] \arrow[r, phantom, "\perp"] & X \arrow[l, bend left, "t"] \end{tikzcd}$ *be in* Pos *with α a surjective map and t preserving directed suprema. Then α preserves the way-below relation $\ll$, and, as a consequence, X is also continuous and the set of compact elements of X is given by*

$$K(X) = \{\alpha(y) \mid y \in K(Y)\}.$$

Proof: Let $y_0, y_1 \in Y$ with $y_0 \ll y_1$. Assume that $\alpha(y_1) \leq \bigvee^{\uparrow}_{i\in I} z_i$. Then, since $\alpha \dashv t$, $y_1 \leq t(\bigvee^{\uparrow}_{i\in I} z_i) = \bigvee^{\uparrow}_{i\in I} t(z_i)$. By hypothesis, there is some $i \in I$ with $y_0 \leq t(z_i)$. Hence $\alpha(y_0) \leq \alpha t(z_i) \leq z_i$. Consequently, $\alpha(y_0) \ll \alpha(y_1)$. Thus α prserves the relation $\ll$, in particular it preserves compact elements.

Let now $x \in K(X)$. First we show that $x = \bigvee\{\alpha(y) \mid y \in K(Y), \alpha(y) \leq x\}$. Indeed, for every $y \in Y$, we have that the inequalities $\alpha(y) \leq x$ and $y \leq t(x)$ are equivalent, because $\alpha \dashv t$. Now, using also the fact that α is surjective and Y is continuous, we have that

$$x = \alpha t(x) = \alpha\left(\bigvee\{y \in K(Y) \mid y \leq t(x)\}\right) = \bigvee\{\alpha(y) \mid y \in K(Y),\, y \leq t(x)\}$$
$$= \bigvee\{\alpha(y) \mid y \in K(Y),\, \alpha(y) \leq x\}.$$

Let now $x \in K(X)$. The set $\{\alpha(y) \mid y \in K(Y),\, \alpha(y) \leq x\}$ is directed in X (because it is the image under α of a directed set). Then, as x is compact, it must be of the form $\alpha(y)$ for some $y \in K(Y)$. ∎

Lemma 6.4. *Let A be an algebraic lattice such that there is $t : A \to FA$ in Top_0 which is right adjoint to the $\mathbb{F}$-structure $\alpha : FA \to A$ (thus, $\alpha t = \mathrm{id}_A$ and $\mathrm{id}_{FA} \leq t\alpha$). Then the set $K(A)$ of compact elements of A is closed under arbitrary infima, and, in $K(A)$, finite suprema distribute over arbitrary infima.*

Proof: It is easy to see that in FA, the compact elements are closed under arbitrary infima. Indeed $K(FA) = \{\uparrow U \mid U \in \Omega A\}$, and we have that $\bigcap_{i\in I} \uparrow U_i = \uparrow \left(\bigcup_{i\in I} U_i\right)$.

Moreover, in $K(FA)$ finite suprema are distributive with respect to arbitrary infima. Indeed, it is easy to see that, for V_i, U and V in ΩA, we have in FA:

(i) $\bigwedge_{i\in I} \uparrow V_i = \bigcap_{i\in I} \uparrow lV_i = \uparrow \left(\bigcup_{i\in I} V_i\right)$; and
(ii) $(\uparrow U) \bigvee (\uparrow V) = \uparrow (U \bigcap V)$.

Hence,

$$\begin{aligned}
(\uparrow U) \vee \left(\bigwedge_{i\in I} \uparrow V_i\right) &= \uparrow \left(U \cap \left(\bigcup_{i\in I} V_i\right)\right) \\
&= \uparrow \left(\bigcup_{i\in I} (U \cap V_i)\right) \\
&= \bigcap_{i\in I} \uparrow (U \cap V_i) \\
&= \bigwedge_{i\in I} \left((\uparrow U) \vee (\uparrow V_i)\right).
\end{aligned}$$

Now, being simultaneously a right and a left adjoint, α preserves infima and suprema. Consequently, by Lemma 6.3, as in FA, compacts in A are closed under infima. Moreover, A also inherits the distribuivity of finite suprema over arbitrary infima for compact elements: putting $c = \alpha(d)$ and $c_i = \alpha(d_i)$ with d and all d_i compacts of A, we have:

$$\begin{aligned} c \vee \left(\bigwedge_{i \in I} c_i \right) &= \alpha\left(d\right) \vee \left(\bigwedge_{i \in I} \alpha\left(d_i\right) \right) = \alpha \left(d \vee \left(\bigwedge_{i \in I} d_i \right) \right) \\ &= \alpha \left(\bigwedge_{i \in I} \left(d \vee d_i\right) \right) = \bigwedge_{i \in I} \left(c \vee c_i\right). \end{aligned}$$

■

Theorem 6.5. *The idempotent split completion of the category $\mathsf{X}_{\mathbb{F}}$ of algebraic algebras is precisely the full subcategory of* $\mathsf{ContLat}$ *of all algebraic lattices whose subposet of compacts is the dual of a frame.*

Proof: We know that $\mathrm{kar}\,(\mathsf{X}_{\mathbb{F}})$ consists of all algebraic lattices A such that the $\mathbb{F}$-structure of A, $\alpha : FA \to A$, has a right adjoint $t : A \to FA$. In particular, since α is a retraction, also $\alpha t = \mathrm{id}_A$. Consequently, by Lemma 6.4, for every $A \in \mathrm{kar}\,(\mathsf{X}_{\mathbb{F}})$, the poset dual to $K(A)$ is a frame.

Conversely, let A be an algebraic lattice such that in its subposet $K(A)$ there are all infima and finite suprema are distributive with respect to arbitrary infima.

By Remark 6.2, the $\mathbb{F}$-structure map of A is given by

$$\alpha\left(\phi\right) = \bigvee \{x \in A \mid e_A(x) \subseteq \phi\}, \quad \phi \in FA.$$

We show that α has a right adjoint $t : A \to FA$.

For every $G \in \Omega A$, let $k(G)$ denote the compact elements of A which belong to G. Given $a \in A$, consider the subset of FA

$$S_a = \{\phi \in FA \mid \alpha(\phi) \leq a\} \tag{6}$$

and the subset of ΩA

$$\psi_a = \{G \in \Omega A \mid \bigwedge k(G) \leq a\}. \tag{7}$$

We show that ψ_a is a filter and $\psi_a = \bigvee S_a$.

First, we show that the union of all filters of S_a is precisely ψ_a. Let then $\phi \in FA$ with $\alpha(\phi) \leq a$, and let $G \in \phi$. Put $c = \bigwedge k(G)$. By hypothesis, $c \in K(A)$, then $\uparrow c$ is an open set containing G, hence belongs to ϕ. Consequently, $e_A(c) \subseteq \phi$, thus, $c \leq \alpha(\phi)$. Since, $\alpha(\phi) \leq a$, it follows that $c \leq a$, as desired. Conversely, let G be an open set of A with $\bigwedge k(G) \leq a$. Put $\phi = \uparrow G$. Then, for every $x \in A$, $e_A(x) \subseteq \phi$ means that every open set of which x is an element contains G, and, in particular, contans $k(G)$. But this implies that $x \leq c$ for all $c \in k(G)$, that is, $x \leq \bigwedge k(G)$, and, thus, $x \leq a$. Since this happens to all x with $e_A(x) \subseteq \phi$, we have $\alpha(\phi) \leq a$. Hence, $G \in \phi$ with ϕ a filter of S_a.

Now, we show that ψ_a is indeed a filter, then $\psi_a = \bigvee S_a$. First, observe that, every open G is the union of all sets $\uparrow c$ with $c \in k(G)$, and, moreover, if $\{c_i,\, i \in I\} \subseteq K(A)$ with $G = \bigcup_{i\in I} \uparrow c_i$, then $\bigwedge k(G) = \bigwedge_{i\in I} c_i$. Now, let G and H belong to ψ_a with $k(G) = \{c_i,\, i \in I\}$ and $k(H) = \{d_j,\, j \in J\}$. Then

$$G \bigcap H = \left(\bigcup_{i\in I} \uparrow c_i\right) \bigcap \left(\bigcup_{j\in J} \uparrow d_j\right) = \bigcup_{i\in I,\, j\in J} (\uparrow c_i \bigcap \uparrow d_j) = \bigcup_{i\in I,\, j\in J} \uparrow (c_i \vee d_j)$$

with all $c_i \vee d_j$ compact, because the supremum of two compacts is compact. Moreover, using the existing distributivity in $K(A)$,

$$\bigwedge_{i\in I,\, j\in J} (c_i \vee d_j) = \left(\bigwedge_{i\in I} c_i\right) \vee \left(\bigwedge_{j\in j} d_j\right) \leq a \wedge a = a.$$

Then $G \bigcap H$ belongs to ψ_a.

Now, put, for every $a \in A$,

$$t(a) = \bigvee S_a = \psi_a.$$

By the defnition of S_a, $t : A \to FA$ is indeed a right adjoint of α in Pos. It remains to show that the map t is continuous (equivalently, it preserves directed suprema). We know that the sets $U^{\#} = \{\phi \in FA \mid U \in \phi\}$, $U \in \Omega A$, form a base of the topology of FA (see Examples 2.5(3)). And we have that

$$t^{-1}(U^{\#}) = \{a \in A \mid U \in t(a)\} = \{a \in A \mid \bigwedge k(U) \leq a\} = \Big\uparrow (\bigwedge k(U))\,;$$

thus $t^{-1}(U^{\#})$ is open because, by hypothesis, $\bigwedge k(U)$ is compact. ■

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