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LAX ORTHOGONAL FACTORISATIONS IN MONAD-QUANTALE-ENRICHED CATEGORIES

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Dedicated to Jiří Adámek

ABSTRACT: We show that, for a quantale V and a Set-monad $\mathbb{T}$ laxly extended to V-Rel, the presheaf monad on the category of $(\mathbb{T}, V)$-categories is simple, giving rise to a lax orthogonal factorisation system (LOFS) whose corresponding weak factorisation system has embeddings as left part. In addition, we present presheaf submonads and study the LOFSs they define.

KEYWORDS: Quantale, monad, enriched category, $(\mathbb{T}, V)$-category, presheaf monad, injective morphism.
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1. Introduction

In 1985 Cassidy-Hébert-Kelly [6] studied orthogonal factorisations systems induced by reflective subcategories, with particular emphasis in the case when the reflection is simple. Among the lax orthogonal factorisation systems, that generalise the orthogonal ones in 2-categories, those arising from simple monads – as defined by the authors of this paper in [12, 13] – have particular relevance. This paper intends to give a systematic way of producing simple monads in (some) topological categories over Set using the presheaf monads of $(\mathbb{T}, V)$-Cat studied in [20, 9]. Given a quantale V and a well-behaved Set-monad $\mathbb{T}$, the category $(\mathbb{T}, V)$-Cat, of generalised V-enriched categories and their functors, is topological and locally preordered (see [8, 14]). As crucial examples we mention the categories Ord of (pre)ordered sets and monotone maps, Top of topological spaces and continuous maps, Met of Lawvere generalised metric spaces and non-expansive maps [25], and App of Lowen approach spaces and non-expansive maps [26]. Equipping the quantale V with a canonical $(\mathbb{T}, V)$-category structure, one gets naturally a Yoneda Lemma

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and a well-behaved presheaf monad that was shown to be lax idempotent in [20]. Here we show that it is simple, inducing a lax orthogonal factorisation system which underlies a weak factorisation system having embeddings as left part. (In order to avoid technicalities we restrict ourselves to separated, or skeletal, $(\mathbb{T}, V)$-categories, so that their hom-sets have an anti-symmetric order.) This encompasses the weak factorisation system in Ord studied by Adámek-Herrlich-Rosický-Tholen in [1].

These presheaf monads have interesting simple submonads, namely the one that has as algebras the Lawvere complete $(\mathbb{T}, V)$-categories (see [10]), and that gives – as shown by Lawvere in [25] – Cauchy-complete spaces when one takes $\mathbb{T} = \mathbb{I}\mathrm{d}$ the identity monad and V the complete half-real line. These also cover, following techniques developed in [9], the weak factorisation systems of Top_0 studied in [4], having as left parts embeddings, dense embeddings, flat embeddings and completely flat embeddings.

This paper does not intend to be self-contained. In Section 2 and 3 we present the basic definitions and results on lax orthogonal factorisation systems and on $(\mathbb{T}, V)$-categories that are needed for this work. For a better understanding of these topics we refer to the papers mentioned there and to the monograph [21]. In Section 4 we study the presheaf monads on $(\mathbb{T}, V)$-categories and their simplicity. In Section 5 we explore the examples of lax orthogonal factorisation systems induced by these presheaf monads.

2. Lax orthogonal factorisation systems

Throughout we will be working on a category C enriched in posets, or Ord-*enriched category*, so that each hom-set $\mathsf{C}(X, Y)$ is equipped with an order structure $\leq$ that is preserved by composition: if $f, f' : X \to Y$, with $f \leq f'$, $g : Y \to Z$ and $h : W \to X$, then $g \cdot f \leq g \cdot f'$ and $f \cdot h \leq f' \cdot h$.

2.1. Weak factorisation systems. Given morphisms f, g, we say that f *has the left lifting property with respect to* g, and that g *has the right lifting property with respect to* f, if every commutative square as shown has (a not necessarily unique) diagonal filler.

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, "f"'] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r] \arrow[ur, dotted] & \cdot \end{tikzcd}$$

A *weak factorisation system* (WFS) in a category is a pair $(\mathcal{L}, \mathcal{R})$ of families of morphisms such that:

- $\mathcal{L}$ consists of those morphisms with the left lifting property with respect to each morphism of $\mathcal{R}$.
- $\mathcal{R}$ consists of those morphisms with the right lifting property with respect to each morphism of $\mathcal{L}$.
- Each morphism in the category factors through an element of $\mathcal{L}$ followed by one of $\mathcal{R}$.

2.2. Algebraic weak factorisation systems. An Ord-functorial factorisation on an Ord-category C consists of a factorisation $\mathrm{dom} \overset{\lambda}{\Rightarrow} E \overset{\rho}{\Rightarrow} \mathrm{cod}$ of the natural transformation $\mathrm{dom} \Rightarrow \mathrm{cod}$ with component at $f \in \mathsf{C}^2$ equal to $f\colon \mathrm{dom}(f) \to \mathrm{cod}(f)$, in the category of locally monotone functors $\mathsf{C}^2 \to \mathsf{C}$. As in the case of functorial factorisations on ordinary categories, an Ord-functorial factorisation can be equivalently described as:

- A copointed endo-Ord-functor $\Phi\colon L \Rightarrow 1_{\mathsf{C}^2}$ on C^2 with $\mathrm{dom}(\Phi) = 1$.
- A pointed endo-Ord-functor $\Lambda\colon 1_{\mathsf{C}^2} \Rightarrow R$ on C^2 with $\mathrm{cod}(\Lambda) = 1$.

The three descriptions of an Ord-functorial factorisation are related by:

$$\mathrm{dom}(\Lambda_f) = Lf = \lambda_f \qquad \mathrm{cod}(\Phi_f) = Rf = \rho_f.$$

An *algebraic weak factorisation system*, abbreviated AWFS, on an Ord-category C consists of a pair $(\mathbb{L}, \mathbb{R})$, where $\mathbb{L} = (L, \Phi, \Sigma)$ is an Ord-comonad and $\mathbb{R} = (R, \Lambda, \Pi)$ is an Ord-monad on C^2, such that (L, Φ) and (R, Λ) represent the same Ord-functorial factorisation on C (i.e., the equalities above hold), fulfilling a distributivity condition we explain next.

Note that the components of Σ and Π are as follows

$$\Sigma_f = \begin{tikzcd} \cdot \arrow[r, equal] \arrow[d, "Lf"'] & \cdot \arrow[d, "L^2 f"] \\ \cdot \arrow[r, "\sigma_f"] & \cdot \end{tikzcd} \quad \text{and} \quad \Pi_f = \begin{tikzcd} \cdot \arrow[r, "\pi_f"] \arrow[d, "R^2 f"'] & \cdot \arrow[d, "Rf"] \\ \cdot \arrow[r, equal] & \cdot \end{tikzcd}$$

which together form a transformation $\Delta : LR \Rightarrow RL$, with $\Delta_f = (\sigma_f, \pi_f)$ as below.

$$\Delta_f = \begin{tikzcd} Ef \arrow[r, "\sigma_f"] \arrow[d, "LRf"'] \arrow[dr, dotted, "1"] & ELf \arrow[d, "RLf"] \\ ERf \arrow[r, "\pi_f"'] & Ef \end{tikzcd}$$

The distributivity axiom requires Δ to be a mixed distributive law between the comonad $\mathbb{L}$ and the monad $\mathbb{R}$, that reduces to the commutativity of the following diagrams.

$$\begin{tikzcd} LR^2 \arrow[r, "\Delta R"] \arrow[d, "L\Pi"'] & RLR \arrow[r, "R\Delta"] & R^2L \arrow[d, "\Pi L"] \\ LR \arrow[rr, "\Delta"] & & RL \end{tikzcd} \qquad \begin{tikzcd} LR \arrow[rr, "\Delta"] \arrow[d, "\Sigma R"'] & & RL \arrow[d, "R\Sigma"] \\ L^2R \arrow[r, "L\Delta"] & LRL \arrow[r, "\Delta L"] & RL^2 \end{tikzcd} \tag{a}$$

Algebraic weak factorisation systems were introduced by Grandis-Tholen in [18] under the name *natural factorisation system*; later, in [17], Garner added to this definition the distributivity conditions we described above.

Each AWFS has an underlying WFS $(\mathcal{L}, \mathcal{R})$, with $\mathcal{L} = \{f \mid f$ has an (L, Φ)-coalgebra structure$\}$ and $\mathcal{R} = \{g \mid g$ has an (R, Λ)-algebra structure$\}$. A coalgebra structure $(1_X, s : Y \to Ef)$ for $f \in \mathcal{L}$, so that $s \cdot f = Lf$ and $Rf \cdot s = 1_{Ef}$, and an (R, Λ)-algebra structure $(p : Eg \to Z, 1_W)$ for $g \in \mathcal{R}$, so that $g \cdot p = Rg$ and $p \cdot Lg = 1_Z$, give a natural lifting $d = p \cdot E(u, v) \cdot s$ for a commutative square $v \cdot f = g \cdot u$:

$$\begin{tikzcd} X \arrow[rr, "u"] \arrow[dr, "Lf"] \arrow[dd, "f"'] & & Z \arrow[r, equal] & Z \arrow[dd, "g"] \\ & Ef \arrow[r, "E(u,v)"] & Eg \arrow[ur, "p"] \arrow[dr, "Rg"] & \\ Y \arrow[r, equal] & Y \arrow[u, "s"] \arrow[rr, "v"] & & W. \end{tikzcd} \tag{b}$$

This lifting is unique – so that $(\mathcal{L}, \mathcal{R})$ is an *orthogonal factorisation system* – if, and only if, $\mathbb{L}$ and $\mathbb{R}$ are idempotent. In fact idempotency of $\mathbb{L}$ implies idempotency of $\mathbb{R}$ and vice-versa, as shown in [3].

2.3. Lax algebraic factorisation systems. Informally, a lax orthogonal factorisation system is an AWFS whose liftings as in (b) have a universal property, as we explain next. First we recall that:

Definition ([24])**.** An Ord-enriched monad $\mathbb{S} = (S, \eta, \mu)$ is *lax idempotent*, or *Kock-Zöberlein*, if it satisfies any of the following equivalent conditions:

(i) $S\eta \leq \eta S$;
(ii) $S\eta \dashv \mu$ (or, equivalently, $S\eta \cdot \mu \leq 1$);
(iii) $\mu \dashv \eta S$ (or, equivalently, $1 \leq \eta S \cdot \mu$);
(iv) a morphism $f : SX \to X$ is an $\mathbb{S}$-algebra structure for X if, and only if, $f \dashv \eta_X$ with $f \cdot \eta_X = 1_X$.

A lax idempotent Ord-comonad is defined dually.

Lemma. *If $\mathbb{S} = (S, \eta, \mu)$ is a lax idempotent monad on an* Ord*-category, the following conditions are equivalent, for an object X of* C*:*

(i) *X admits an $\mathbb{S}$-algebra structure;*
(ii) *X admits a unique $\mathbb{S}$-algebra structure;*
(iii) *$\eta_X : X \to SX$ has a right inverse, i.e. X admits an (S, η)-algebra structure;*
(iv) *X is a retract of SX;*
(v) *X is a retract of an $\mathbb{S}$-algebra.*

An AWFS $(\mathbb{L}, \mathbb{R})$ is a *lax orthogonal factorisation system*, abbreviated LOFS, if $\mathbb{L}$ and $\mathbb{R}$ are lax idempotent. These factorisations were introduced by the authors in [12] and further studied in the Ord-enriched categories setting, as used here, in [13].

Corollary. *If $(\mathbb{L}, \mathbb{R})$ is a* LOFS*, then its underlying weak factorisation system $(\mathcal{L}, \mathcal{R})$ consists of the class $\mathcal{L}$ of the morphisms admitting a (unique) $\mathbb{L}$-coalgebra structure and the class $\mathcal{R}$ consists of the morphisms admitting a (unique) $\mathbb{R}$-algebra structure.*

As for orthogonal factorisation systems, lax idempotency of $\mathbb{L}$ implies lax idempotency of $\mathbb{R}$ and vice-versa. In fact:

Theorem ([13])**.** (1) *Given an* AWFS *$(\mathbb{L}, \mathbb{R})$ on an* Ord*-category* C*, the following conditions are equivalent:*
  (i) *$(\mathbb{L}, \mathbb{R})$ is a* LOFS*;*
  (ii) *$\mathbb{L}$ is lax idempotent;*
  (iii) *$\mathbb{R}$ is lax idempotent.*

(2) *Given a domain-preserving* Ord*-comonad $\mathbb{L}$ and a codomain-preserving* Ord*-monad $\mathbb{R}$ inducing the same* Ord*-functorial factorisation $f = Rf \cdot Lf$, the following conditions are equivalent:*
  (i) *$(\mathbb{L}, \mathbb{R})$ is a* LOFS*.*
  (ii) *Both $\mathbb{L}$ and $\mathbb{R}$ are lax idempotent.*
  (iii) *One of $\mathbb{L}$ and $\mathbb{R}$ is lax idempotent and the distributive law axioms* (a) *hold.*

2.4. Lifting operations. Diagram (b) shows that every functorial factorisation system induces a canonical lifting operation from the forgetful Ord-functor $U : (L,\Phi)\text{-Coalg} \to \mathsf{C}^2$ to the forgetful Ord-functor $V : (R,\Lambda)\text{-Alg} \to \mathsf{C}^2$, meaning that every commutative diagram

$$\begin{tikzcd} \cdot \arrow[r, "h"] \arrow[d, "Ua"'] & \cdot \arrow[d, "Vb"] \\ \cdot \arrow[r, "k"] & \cdot \end{tikzcd} \tag{c}$$

has a canonical diagonal filler $\phi_{a,b}(h,k)$ so that $Vb \cdot \phi_{a,b}(h,k) = k$, $\phi_{a,b}(h,k) \cdot Ua = h$. Those fillers respect both composition and order in a natural way (see [13] for details).

A lifting operation from $U : \mathsf{A} \to \mathsf{C}^2$ to $V : \mathsf{B} \to \mathsf{C}^2$ is said to be KZ if, for every commutative diagram (c) and every diagonal filler d, one has $\phi_{a,b}(h,k) \le d$.

Theorem ([13])**.** *For an* AWFS $(\mathbb{L},\mathbb{R})$ *on an* Ord*-category* C*, the following conditions are equivalent:*

(i) $(\mathbb{L},\mathbb{R})$ *is a* LOFS*.*
(ii) *The lifting operation from* $\mathbb{L}$-Coalg $\to \mathsf{C}^2$ *to* $\mathbb{R}$-Alg $\to \mathsf{C}^2$ *is* KZ*.*

2.5. Simple monads and their LOFS**s.** The notion of simple monad we present here, studied in [12, 13], is the Ord-enriched version of simple reflection of [6]. Given an Ord-monad $\mathbb{S} = (S,\eta,\mu)$, we construct a monad $\mathbb{R}$ on C^2 by considering the comma-object $Kf = Sf \downarrow \eta_Y$ and defining $Rf : Kf \to Y$ as the second projection. Then Lf is the unique morphism making the following diagram commute.

$$\begin{tikzcd} X \arrow[dr, "Lf"] \arrow[drr, "\eta_X"] \arrow[ddr, "f"'] & & \\ & Kf \arrow[r, "q_f"] \arrow[d, "Rf"] \arrow[dr, phantom, "\ge"] & SX \arrow[d, "Sf"] \\ & Y \arrow[r, "\eta_Y"'] & SY \end{tikzcd}$$

The Ord-functorial factorisation $f = Rf \cdot Lf$ defines a copointed endo-Ord-functor $(L, \Phi : L \Rightarrow 1)$, with $\Phi_f = (1_X, Rf)$, and a pointed endo-Ord-functor (R,Λ), with $\Lambda_f = (Lf, 1_Y)$. Moreover, (R,Λ) underlies a monad $\mathbb{R}$ on C^2

whose multiplication $\Pi_f = (\pi_f, 1_Y)$ is defined by the unique morphism π_f given by the universal property of the comma-object:

$$\begin{tikzcd}
KRf \arrow[rr, "q_{Rf}"] \arrow[dr, "\pi_f"] \arrow[ddr, "RRf"'] & & SKf \arrow[r, "Sq_f"] & SSX \arrow[d, "\mu_X"] \\
& Kf \arrow[rr, "q_f"] \arrow[d, "Rf"] \arrow[drr, phantom, "\geq"] & & SX \arrow[d, "Sf"] \\
& Y \arrow[rr, "\eta_Y"'] & & SY
\end{tikzcd}$$

(See [13] for details.)

Lemma. *Given a monad $\mathbb{S}$ on C, the following conditions are equivalent for a morphism $f : X \to Y$ in C:*

(i) *f has an (L, Φ)-coalgebra structure.*
(ii) *Sf is a* LARI *(=left adjoint right inverse), that is, it has a right adjoint S^*f such that $S^*f \cdot Sf = 1$.*

Proof: (i)⇒(ii): If $(1_X, s : Y \to Kf)$ is an (L, Φ)-coalgebra structure for $f : X \to Y$, then $S^*f := \mu_X \cdot Sq_f \cdot Ss$ is a left inverse of Sf:

$$S^*f \cdot Sf = \mu_X \cdot Sq_f \cdot Ss \cdot Sf = \mu_X \cdot Sq_f \cdot SLf = \mu_X \cdot S\eta_X = 1;$$

and, moreover, it is right adjoint to Sf:

$$Sf \cdot S^*f = Sf \cdot \mu_X \cdot Sq_f \cdot Ss = \mu_Y \cdot SSf \cdot Sq_f \cdot Ss \leq \mu_Y \cdot S\eta_Y \cdot SRf \cdot Ss = 1.$$

(ii)⇒(i): Let S^*f be a right adjoint left inverse of Sf. By definition of comma-object, from $Sf \cdot (S^*f \cdot \eta_Y) \leq \eta_Y$ there exists a unique $s : Y \to Kf$ such that $Rf \cdot s = 1_Y$ and $q_f \cdot s = S^*f \cdot \eta_Y$. To conclude that $s \cdot f = Lf$, compose $s \cdot f$ with Rf and q_f:

$$Rf \cdot s \cdot f = f \text{ and } q_f \cdot s \cdot f = S^*f \cdot \eta_Y \cdot f = S^*f \cdot Sf \cdot \eta_X = \eta_X.$$

∎

A morphism f in C such that Sf is a LARI is called an *S-embedding.* Denote by $\mathbb{S}$-Emb the category that has as objects pairs (f, r) of morphisms of C such that $Sf \dashv S^*f$ with $S^*f \cdot Sf = 1$, and as morphisms $(h, k) : (f, S^*f) \to (g, S^*g)$ morphisms $(h, k) : f \to g$ in C^2 such that $S^*g \cdot Sk = Sh \cdot S^*f$.

Definition. The Ord-monad $\mathbb{S}$ is said to be *simple* if the locally monotone forgetful functor $\mathbb{S}$-Emb $\to \mathsf{C}^2$ has a right adjoint and the induced comonad has underlying functor L and counit Φ.

As shown in [13]:

Proposition. *A lax idempotent monad $\mathbb{S} = (S, \eta, \mu)$ on C is simple if, and only if, for every morphism $f : X \to Y$, there is an adjunction*

$$SLf \dashv \mu_X \cdot Sq_f.$$

Theorem. *If $\mathbb{S}$ is a lax idempotent and simple monad, then $(\mathbb{L}, \mathbb{R})$ is a* LOFS. *Moreover, the left class $\mathcal{L}$ of the weak factorisation system it induces is the class of S-embeddings.*

Proof: (Sketch of the proof; for details see [13].) Simplicity of $\mathbb{S}$ gives the comonad structure for L needed to define the AWFS.

In order to show that $\mathbb{R}$ is a lax idempotent monad, that is, $R\Lambda_f \leq \Lambda_{Rf} = (LRf, 1_Y)$, we denote $R\Lambda_f$ by $((R\Lambda_f)_1, 1_Y)$ and note that, by definition of R, $R^2f \cdot (R\Lambda_f)_1 = Rf$ and $q_{Rf} \cdot (R\Lambda_f)_1 = SLf \cdot q_f$. Then

$$q_{Rf} \cdot (R\Lambda_f)_1 = \mu_{SKf} \cdot \eta_{S^2Kf} \cdot SLf \cdot q_f = SLf \cdot \mu_X \cdot Sq_f \cdot \eta_{Kf} \leq \eta_{Kf},$$

by simplicity of S. Now, by definition of comma-object and by the equalities $R^2f \cdot LRf = Rf$ and $q_f \cdot LRf = \eta_{Kf}$, it follows that $(R\Lambda_f)_1 \leq LRf$ as claimed.

The last assertion follows from the lemma above. ∎

2.6. Submonads of simple monads. Well-behaved submonads of simple monads are simple, as stated below.

Theorem ([13]). *Suppose that $\varphi : \mathbb{S}' \to \mathbb{S}$ is a monad morphism between Ord-monads whose components are pullback-stable $\mathbb{S}$-embeddings, and that $\mathbb{S}$-embeddings are full. If $\mathbb{S}$ is lax idempotent, then $\mathbb{S}'$ is simple whenever $\mathbb{S}$ is so. Moreover, every $\mathbb{S}'$-embedding is an $\mathbb{S}$-embedding.*

(Here by *full morphism* in an Ord-category we mean a morphism $f : X \to Y$ such that, for every $u, v : Z \to X$, $f \cdot u \leq f \cdot v$ implies $u \leq v$.)

3. $(\mathbb{T}, V)$-categories

3.1. The setting. First we describe the setting where we will be working throughout the paper.

A. V is a *commutative and unital quantale*, that is, a complete lattice equipped with a tensor product $\otimes$, with unit $k \neq \bot$ and with right adjoint hom. We denote by V-Rel the bicategory of V*-relations*, having sets as objects, while morphisms $r : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$ are V-relations, i.e. maps $r : X \times Y \to V$; their composition is given by relational composition, that is, for $r : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$ and $s : Y \relbar\joinrel\mapstochar\joinrel\rightarrow Z$,

$$s \cdot r(x,z) = \bigvee_{y \in Y} r(x,y) \otimes s(y,z).$$

Every map $f : X \to Y$ *is* a V-relation $f : X \times Y \to V$ with $f(x,y) = k$ if $f(x) = y$ and $f(x,y) = \bot$ elsewhere. This correspondence defines a bijective on objects and faithful pseudofunctor $\mathsf{Set} \to V$-Rel. V-Rel is a locally ordered and locally complete bicategory, with $r \leq s$ if $r(x,y) \leq s(x,y)$, for $r, s : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$, $x \in X$, $y \in Y$. It has an involution $(\ )^\circ : V$-$\mathsf{Rel} \to V$-Rel assigning to each $r : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$ the V-relation $r^\circ : Y \relbar\joinrel\mapstochar\joinrel\rightarrow X$ defined by $r^\circ(y,x) = r(x,y)$. For each $r : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$ both left and right compositions with r preserve suprema, and therefore we have the following adjunctions

$$\begin{tikzcd} V\text{-}\mathsf{Rel}(Y,Z) \arrow[r, shift left=1.5, "(\ )\cdot r"] \arrow[r, phantom, "\perp"] & V\text{-}\mathsf{Rel}(X,Z) \arrow[l, shift left=1.5, "(\ )\bullet\!\!- r"] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} V\text{-}\mathsf{Rel}(Z,X) \arrow[r, shift left=1.5, "r\cdot(\ )"] \arrow[r, phantom, "\perp"] & V\text{-}\mathsf{Rel}(Z,Y) \arrow[l, shift left=1.5, "r-\!\!\bullet(\ )"] \end{tikzcd}$$

so that, for every $s : Y \relbar\joinrel\mapstochar\joinrel\rightarrow Z$, $s' : X \relbar\joinrel\mapstochar\joinrel\rightarrow Z$, $t : Z \relbar\joinrel\mapstochar\joinrel\rightarrow X$, $t' : Z \relbar\joinrel\mapstochar\joinrel\rightarrow Y$,

$$s \cdot r \leq s' \iff s' \leq s' \bullet\!\!- r \text{ and } r \cdot t \leq t' \iff t \leq r -\!\!\bullet t'.$$

B. $\mathbb{T} = (T, e, m)$ is a non-trivial Set-monad that *satisfies (BC)*; that is, T preserves weak pullbacks and every naturality square of m is a weak pullback. We point out that, in particular, the monad $\mathbb{T}$ is *taut* in the sense of Manes [27] (see [11] for details).

C. $\xi : TV \to V$ is a $\mathbb{T}$-algebra structure on V such that both $\otimes : V \times V \to V$ and $k : 1 \to V$, $* \mapsto k$, are $\mathbb{T}$-algebra homomorphisms, that is, the following diagrams

$$\begin{tikzcd} T1 \arrow[r, "Tk"] \arrow[d, "!"'] & TV \arrow[d, "\xi"] \\ 1 \arrow[r, "k"] & V \end{tikzcd} \qquad \begin{tikzcd} T(V \times V) \arrow[r, "T(\otimes)"] \arrow[d, "<\xi\cdot T\pi_1, \xi\cdot T\pi_2>"'] & TV \arrow[d, "\xi"] \\ V \times V \arrow[r, "\otimes"] & V \end{tikzcd}$$

are commutative, and, for all maps $f : X \to Y$, $\varphi : X \to V$ and $\psi : Y \to V$ with $\psi(y) = \bigvee_{x \in f^{-1}(y)} \varphi(x)$ for every $y \in Y$, the following inequality holds

$$\xi \cdot T\psi(y) \le \bigvee_{\mathfrak{x} \in Tf^{-1}(\mathfrak{y})} \xi \cdot T\varphi(\mathfrak{x}),$$

for every $\mathfrak{y} \in TY$. (For alternative descriptions of the latter condition see [19].)

D. Using ξ we define, for each V-relation $r : X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$, the V-relation $T_\xi r : TX \relbar\joinrel\mapstochar\joinrel\rightarrow TY$ as the composite

$$\begin{tikzcd}
TX \times TY \arrow[r, "T_\xi r"] \arrow[d, "<T\pi_1, T\pi_2>^\circ"', "|" marking] & V \\
T(X \times Y) \arrow[r, "Tr"'] & TV \arrow[u, "\xi"']
\end{tikzcd}$$

that is, for each $\mathfrak{x} \in TX$, $\mathfrak{y} \in TY$,

$$T_\xi r(\mathfrak{x}, \mathfrak{y}) = \bigvee \{\xi(Tr(\mathfrak{w})) \,|\, \mathfrak{w} \in T(X \times Y), T\pi_1(\mathfrak{w}) = \mathfrak{x}, T\pi_2(\mathfrak{w}) = \mathfrak{y}\}.$$

This defines a pseudofunctor $T_\xi : V\text{-}\mathsf{Rel} \to V\text{-}\mathsf{Rel}$ that extends $T : \mathsf{Set} \to \mathsf{Set}$, so that $m : T_\xi T_\xi \to T_\xi$ is a natural transformation while $e : \mathrm{Id}_{V\text{-}\mathsf{Rel}} \to T_\xi$ is an op-lax natural transformation (see [19] for details).

3.2. $(\mathbb{T}, V)$**-categories.** Having fixed these data, a $(\mathbb{T}, V)$-category is a pair (X, a), where X is a set and $a : TX \relbar\joinrel\mapstochar\joinrel\rightarrow X$ is a V-relation such that

$$1_X \le a \cdot e_X \text{ and } a \cdot Ta \le a \cdot m_X.$$

Given $(\mathbb{T}, V)$-categories (X, a), (Y, b), a $(\mathbb{T}, V)$-functor $f : (X, a) \to (Y, b)$ is a map $f : X \to Y$ such that

$$f \cdot a \le b \cdot Tf.$$

We denote the category of $(\mathbb{T}, V)$-categories and $(\mathbb{T}, V)$-functors by $(\mathbb{T}, V)$-Cat. As defined in [14, Section 12], $(\mathbb{T}, V)$-Cat is (pre)ordered-enriched by:

$$f \le g \text{ if } g \le b \cdot e_Y \cdot f,$$

for $f, g : (X, a) \to (Y, b)$. (This structure is in fact inherited from the order-enrichment of V-Rel as explained in 3.5.) Identifying an element x of X with the $(\mathbb{T}, V)$-functor $E = (1, e_1^\circ) \to (X, a)$, $* \mapsto x$, (X, a) becomes (pre)ordered; (X, a) is called *separated*, or *skeletal*, if, for $x, x' \in X$, $x \le x'$ and $x' \le x$

implies $x = x'$. The category of separated $(\mathbb{T}, V)$-categories and $(\mathbb{T}, V)$-functors will be denoted by $(\mathbb{T}, V)\text{-}\mathsf{Cat}_0$.

Examples. Let $\mathbb{T}$ be the identity monad $\mathbb{I}\mathrm{d}$ and $\xi : V \to V$ the identity map.

– When $V = 2$, $(\mathbb{I}\mathrm{d}, 2)\text{-}\mathsf{Cat}$ is the category of (pre)ordered sets and monotone maps.
– Let $V = [0, \infty]_+$ be the complete half-real line $[0, \infty]$ ordered by the greater or equal relation, with $\otimes = +$ and hom the truncated minus, so that $\hom(u, v) = v \ominus u$, which is equal to $v - u$ if $v \geq u$ and 0 otherwise. As shown by Lawvere in [25], $(\mathbb{I}\mathrm{d}, [0, \infty]_+)\text{-}\mathsf{Cat}$ is the category of generalised metric spaces and non-expansive maps.

Let $\mathbb{T}$ be the ultrafilter monad $\mathbb{U}$ and $\xi : TV \to V$ be defined by $\xi(\mathfrak{x}) = \bigvee\{v \in V \,|\, \mathfrak{x} \in T(\uparrow v)\}$.

– When $V = 2$ – as shown by Barr in [2] – $(\mathbb{U}, 2)\text{-}\mathsf{Cat}$ is the category of topological spaces and continuous maps.
– When $V = [0, \infty]_+$ – as shown in [8] – $(\mathbb{U}, [0, \infty]_+)\text{-}\mathsf{Cat}$ is the category of approach spaces and non-expansive maps [26].

3.3. The dual of a $(\mathbb{T}, V)$-category. When $\mathbb{T}$ is the identity monad, $(\mathbb{T}, V)\text{-}\mathsf{Cat}$ is the category $V\text{-}\mathsf{Cat}$ of V-categories and V-functors. In $V\text{-}\mathsf{Cat}$ there is a natural notion of *dual category*, inducing a functor $D : V\text{-}\mathsf{Cat} \to V\text{-}\mathsf{Cat}$, with $D(X, a) = (X, a^\circ)$. To build a dual for a $(\mathbb{T}, V)$-category we first note that the Set-monad $\mathbb{T}$ can be extended to $V\text{-}\mathsf{Cat}$, with $T(X, a) = (TX, Ta)$, and make use of the following adjunction

$$\begin{tikzcd} (V\text{-}\mathsf{Cat})^{\mathbb{T}} \arrow[r, shift left, "N"] \arrow[r, phantom, "\perp"] & (\mathbb{T}, V)\text{-}\mathsf{Cat} \arrow[l, shift left, "M"] \end{tikzcd}$$

where, for a V-category (X, a), a $\mathbb{T}$-algebra structure $\alpha : T(X, a) \to (X, a)$ and a $\mathbb{T}$-homomorphism f, $N((X, a), \alpha) = (X, a \cdot \alpha)$ and $Nf = f$, and, for a $(\mathbb{T}, V)$-category (Y, b) and a $(\mathbb{T}, V)$-functor g, $M(Y, b) = ((TY, Tb \cdot m_Y^\circ), m_Y)$ and $Mg = Tg$. The functor $D : V\text{-}\mathsf{Cat} \to V\text{-}\mathsf{Cat}$ lifts to a functor $D : (V\text{-}\mathsf{Cat})^{\mathbb{T}} \to (V\text{-}\mathsf{Cat})^{\mathbb{T}}$, with $D((X, a), \alpha) = ((X, a^\circ), \alpha)$, and we define the *dual* $(X, a)^{\mathrm{op}}$ *of a* $(\mathbb{T}, V)$*-category* (X, a) as

$$\begin{tikzcd} (V\text{-}\mathsf{Cat})^{\mathbb{T}} \arrow[loop left, "D"] \arrow[r, shift left, "N"] \arrow[r, phantom, "\perp"] & (\mathbb{T}, V)\text{-}\mathsf{Cat} \arrow[l, shift left, "M"] \end{tikzcd}$$

$NDM(X,a) = (TX, m_X \cdot (Ta)^\circ \cdot m_X)$; that is, denoting its structure by a^{op},

$$a^{\mathrm{op}}(\mathfrak{X}, \mathfrak{y}) = \bigvee_{\mathfrak{Y}\,:\,m_X(\mathfrak{Y})=\mathfrak{y}} Ta(\mathfrak{Y}, m_X(\mathfrak{X})),$$

for $\mathfrak{X} \in T^2X$ and $\mathfrak{y} \in TX$ (see [7]).

3.4. V **as a** $(\mathbb{T}, V)$**-category.** As we have in V both a V-categorical structure hom : $V \longrightarrow\!\!\!\!\!\!\!+\;\, V$ and a $\mathbb{T}$-algebra structure $\xi : TV \to V$, which is a V-functor $\xi : (TV, T\hom) \to (V, \hom)$ due to our assumptions, $N((V, \hom), \xi) = (V, \hom_\xi)$ is a $(\mathbb{T}, V)$-category; this structure has a crucial role in our study, as we will see in the next section.

3.5. $(\mathbb{T}, V)$**-bimodules.** Given $(\mathbb{T}, V)$-categories (X, a) and (Y, b), a $(\mathbb{T}, V)$-*bimodule* (or simply a *bimodule*) $\psi : (X, a) \multimap (Y, b)$ is a V-relation $\psi : TX \rightharpoonup Y$ such that $\psi \circ a \leq \psi$ and $b \circ \psi \leq \psi$, where the composition $s \circ r$ of two V-relations $r : TX \rightharpoonup Y$ and $s : TY \rightharpoonup Z$ is given by the *Kleisli convolution* (see [22]), that is

$$s \circ r = s \cdot Tr \cdot m_X^\circ.$$

Under our assumptions bimodules compose, with the $(\mathbb{T}, V)$-categorical structures as identities for this composition. We denote by $(\mathbb{T}, V)$-Mod the category of $(\mathbb{T}, V)$-categories and $(\mathbb{T}, V)$-bimodules. $(\mathbb{T}, V)$-Mod is locally preordered by the preorder inherited from V-Rel.

Every $(\mathbb{T}, V)$-functor $f : (X, a) \to (Y, b)$ induces a pair of bimodules $f_* : (X, a) \multimap (Y, b)$ and $f^* : (Y, b) \multimap (X, a)$, defined by $f_* = b \cdot Tf$ and $f^* = f^\circ \cdot b$; that is, $f_*(\mathfrak{x}, y) = b(Tf(\mathfrak{x}), y)$ and $f^*(\mathfrak{y}, x) = b(\mathfrak{y}, f(x))$, for $\mathfrak{x} \in TX$, $\mathfrak{y} \in TY$, $x \in X$ and $y \in Y$. The Kleisli convolution becomes simpler when composing with these bimodules: for any $\varphi : X \multimap Z$ and $\psi : Z \multimap X$, $f^* \circ \varphi = f^\circ \cdot \varphi$ and $\psi \circ f_* = \psi \cdot Tf$. It is easy to check that $a \leq f^* \circ f_*$ and $f_* \circ f^* \leq b$, that is, $f_* \dashv f^*$. The $(\mathbb{T}, V)$-functor f is said to be *fully faithful* when $f^* \circ f_* = a$, or, equivalently, $a(\mathfrak{x}, x) = b(Tf(\mathfrak{x}), f(x))$, for every $\mathfrak{x} \in TX, x \in X$. The local (pre)order on $(\mathbb{T}, V)$-Cat corresponds to the local (pre)order on $(\mathbb{T}, V)$-Mod: for $(\mathbb{T}, V)$-functors $f, g : (X, a) \to (Y, b)$,

$$f \leq g \iff f^* \leq g^* \iff f_* \geq g_*.$$

4. The presheaf monad and its submonads

4.1. The Yoneda Lemma. The tensor product in V defines a tensor product in $(\mathbb{T},V)$-Cat, with $(X,a)\otimes(Y,b) = (X\times Y, c)$, where $c(\mathfrak{w},(x,y)) = a(T\pi_1(\mathfrak{w}),x)\otimes b(T\pi_2(\mathfrak{w}),y)$, for $\mathfrak{w}\in T(X\times Y)$, $x\in X$, $y\in Y$. Its neutral element is $E=(1,e_1^\circ)$. For each $(\mathbb{T},V)$-category (X,a), the functor $X^{\mathrm{op}}\otimes(\ ) : (\mathbb{T},V)\text{-}\mathsf{Cat}\to(\mathbb{T},V)\text{-}\mathsf{Cat}$ has a right adjoint $(\ )^{X^{\mathrm{op}}} : (\mathbb{T},V)\text{-}\mathsf{Cat}\to(\mathbb{T},V)\text{-}\mathsf{Cat}$.

Proposition ([10])**.** *For $(\mathbb{T},V)$-categories $(X,a),(Y,b)$ and a V-relation $\psi : TX \to\!\!\!\!\!\mapsto Y$, the following conditions are equivalent:*

(i) $\psi : (X,a) \multimap (Y,b)$ *is a bimodule;*
(ii) $\psi : X^{\mathrm{op}}\otimes Y\to V$ *is a $(\mathbb{T},V)$-functor.*

Since $a : (X,a)\multimap(X,a)$ is a bimodule, this result tells us that $a : X^{\mathrm{op}}\otimes X\to V$ is a $(\mathbb{T},V)$-functor, and therefore, from the adjunction $X^{\mathrm{op}}\otimes(\ )\dashv(\ )^{X^{\mathrm{op}}}$, a induces the *Yoneda $(\mathbb{T},V)$-functor*

$$(X,a)\xrightarrow{\ y_X\ } V^{X^{\mathrm{op}}}.$$

The following result provides a Yoneda Lemma for $(\mathbb{T},V)$-categories.

Theorem ([10])**.** *Let (X,a) be a $(\mathbb{T},V)$-category. For all $\psi\in V^{X^{\mathrm{op}}}$ and all $\mathfrak{x}\in TX$,*

$$\widehat{a}(T\,y_X(\mathfrak{x}),\psi)=\psi(\mathfrak{x}),$$

where $\widehat{a}$ denotes the $(\mathbb{T},V)$-categorical structure on $V^{X^{\mathrm{op}}}$. In particular, y_X is fully faithful.

4.2. The presheaf monad. In order to work in an Ord-enriched category, from now on we restrict ourselves to $(\mathbb{T},V)$-Cat_0. We remark that the results of the previous subsection remain valid when we replace $(\mathbb{T},V)$-Cat by $(\mathbb{T},V)$-Cat_0. Denoting $V^{X^{\mathrm{op}}}$ by PX, we point out that, via Theorem 4.1, $PX = \{\varphi : (X,a)\multimap E\,|\,\varphi \text{ bimodule}\}$. Moreover, the Yoneda $(\mathbb{T},V)$-functor y_X turns out to assign to each $x\in X$, that is to each $(\mathbb{T},V)$-functor $x : E\to X$, the bimodule $x^* : X\multimap E$. Each $(\mathbb{T},V)$-functor $f : (X,a)\to(Y,b)$ induces a $(\mathbb{T},V)$-functor $Pf : PX\to PY$, assigning to $\varphi : X\multimap E$ the bimodule $\varphi\circ f^* : Y\multimap E$, that is $Pf = (\ )\circ f^*$. This defines an endofunctor P on $(\mathbb{T},V)$-Cat. From the adjunction $f_*\dashv f^*$, for every $(\mathbb{T},V)$-functor $f : (X,a)\to(Y,b)$ one gets a right adjoint to Pf, $P^*f = (\ )\circ f_* : PY\to PX$. In particular, $P\,y_X : PX\to PPX$ has a right adjoint $m_X : PPX\to PX$, which, together with P and y, defines a lax idempotent monad, the *presheaf* *monad*. Next we show that this monad is simple. In order to do that we use Proposition 2.5.

Theorem. *The presheaf monad* $\mathbb{P}$ *on* $(\mathbb{T},V)$-Cat *is simple.*

Proof: We need to show that, for any $(\mathbb{T},V)$-functor $f:(X,a)\to(Y,b)$, in the diagram below $PLf\dashv m_X\cdot Pq_f$.

$$\begin{tikzcd}
X \arrow[rr, "y_X"] \arrow[dr, "Lf"] \arrow[dd, "f"'] & & PX \arrow[rr, shift left, "P\,y_X"] \arrow[dr, "PLf"] \arrow[dd, "Pf"] & & PPX \arrow[ll, shift left, "m_X"] \\
& Kf \arrow[ur, "q_f"] \arrow[rr, "y_{Kf}"] \arrow[dl, "Rf"] \arrow[dr, phantom, "\geq"] & & PKf \arrow[ur, "Pq_f"'] \arrow[dl, "PRf"] & \\
Y \arrow[rr, "y_Y"'] & & PY & &
\end{tikzcd}$$

First we recall that the comma object $(Kf,\tilde{a})=Pf\downarrow y_Y$ is given by $Kf=\{(\varphi,y)\in PX\times Y\mid Pf(\varphi)\leq y^*\}$, and $\tilde{a}(\mathfrak{w},(\varphi,y))=\widehat{a}(Tq_f(\mathfrak{w}),\varphi)\wedge b(TRf(\mathfrak{w}),y)$, where $\widehat{a}$ is the structure on PX. On one hand, as we observed before, PLf has as right adjoint the $(\mathbb{T},V)$-functor $P^*Lf=(\ )\circ(Lf)_*$. On the other hand, $m_X\cdot Pq_f=P^*\,y_X\cdot Pq_f=(\ )\circ q_f^*\circ(y_X)_*$. Next we will show that $(Lf)_*=q_f^*\circ(y_X)_*:X{\longrightarrow\!\!\!\!\!\!\circ\;}Kf$, which concludes the proof. For each $\mathfrak{x}\in TX$ and $(\varphi,y)\in Kf$,

$$(q_f^*\circ(y_X)_*)(\mathfrak{x},(\varphi,y))=\widehat{a}(T\,y_X(\mathfrak{x}),\varphi)=\varphi(\mathfrak{x}),$$

while

$$(Lf)_*(\mathfrak{x},(\varphi,y))=\tilde{a}(TLf(\mathfrak{x}),(\varphi,y))=\widehat{a}(T\,y_X(\mathfrak{x}),\varphi)\wedge b(Tf(\mathfrak{x}),y).$$

Since $y^*\geq Pf(\varphi)$ and $b(Tf(\mathfrak{x}),y)=\widehat{b}(T\,y_Y(Tf(\mathfrak{x})),y^*)$ by Yoneda Lemma,

$$b(Tf(\mathfrak{x}),y)\geq\widehat{b}(T\,y_Y(Tf(\mathfrak{x})),Pf(\varphi))=\widehat{b}(TPf\cdot T\,y_X(\mathfrak{x}),Pf(\varphi))\geq\widehat{a}(T\,y_X(\mathfrak{x}),\varphi),$$

because Pf is a $(\mathbb{T},V)$-functor, and so

$$(Lf)_*(\mathfrak{x},(\varphi,y))=\widehat{a}(T\,y_X(\mathfrak{x}),\varphi)=\varphi(\mathfrak{x}).$$

■

Proposition. (1) *A* $(\mathbb{T},V)$*-functor is a* P*-embedding if, and only if, it is fully faithful.*

(2) *Fully faithful* $(\mathbb{T},V)$*-functors are pullback stable.*

Proof: (1) If $f:(X,a)\to(Y,b)$ is a $(\mathbb{T},V)$-functor, then Pf has a right adjoint, P^*f. It remains to show that $P^*f\cdot Pf=1_X$ when f is fully faithful; this means $f^*\cdot f_*=a$, and so, for any bimodule $\varphi:X{\longrightarrow\!\!\!\!\!\!\circ\,} E$,

$$P^*f\cdot Pf(\varphi)=\varphi\circ f^*\circ f_*=\varphi\circ a=\varphi.$$

Conversely, if $P^*f\cdot Pf=1_{PX}$, then, for any $x\in X$,

$$a(-,x)=x^*=P^*f\cdot Pf(x^*)=x^*\circ f^*\circ f_*=f^\circ\cdot x^\circ\cdot b\cdot Tf=b(Tf(-),f(x)),$$

that is, f is fully faithful.

(2) As in any topological category, (bijective, fully faithful $(\mathbb{T},V)$-functors) is an orthogonal factorisation system in $(\mathbb{T},V)$-Cat, and therefore fully faithful $(\mathbb{T},V)$-functors are pullback-stable. ∎

4.3. Presheaf submonads. Let Φ be a class of $(\mathbb{T},V)$-bimodules satisfying the conditions:

- (S1) Φ is closed under composition.
- (S2) For every $(\mathbb{T},V)$-functor f, $f^*\in\Phi$.
- (S3) For every $(\mathbb{T},V)$-bimodule $\psi:X{\longrightarrow\!\!\!\!\!\!\circ\,} Y$, $\psi\in\Phi$ provided that $y^*\circ\psi\in\Phi$ for every $y\in Y$.

We call such a class *saturated*. There is a largest saturated class, of all $(\mathbb{T},V)$-bimodules, and a smallest one, $\{f^*\,|\,f$ is a $(\mathbb{T},V)$-functor$\}$. In the last section we will explore other examples.

For each $(\mathbb{T},V)$-category (X,a), we define

$$\Phi X=\{\varphi:X{\longrightarrow\!\!\!\!\!\!\circ\,} E\,|\,\varphi\in\Phi\}\subseteq PX,$$

equipped with the structure $\widehat{a}$ inherited from PX, and, to each $(\mathbb{T},V)$-functor $f:(X,a)\to(Y,b)$ we assign

$$\Phi f:\Phi X\to\Phi Y,\ \text{with}\ \Phi f(\varphi)=\varphi\circ f^*.$$

Since $x^*\in\Phi$ for every $x\in X$, y_X corestricts to ΦX,

$$X\xrightarrow{\;y_X^\Phi\;}\Phi X\ .$$

Moreover, condition (S3) guarantees that $m_X=P^*\,y_X$ (co)restricts to $m_X^\Phi:\Phi\Phi X\to\Phi X$: by the Yoneda Lemma, for all $\varphi\in\Phi X$, $\varphi^*\circ P^*\,y_X=\varphi\in\Phi$. So, (Φ,y^Φ,m^Φ) is a submonad of P.

Theorem. *If Φ is a saturated class of bimodules, then the monad $(\Phi, y^{\Phi}, m^{\Phi})$ is lax idempotent and simple, and so it defines a lax orthogonal factorisation system.*

Proof: Since fully faithful $(\mathbb{T}, V)$-functors are pullback-stable and full, and the inclusion $\Phi X \to PX$ is clearly fully faithful, this result follows directly from Theorem 2.6. ■

5. Examples: The induced lofss

5.1. Examples: the presheaf LOFS**.** From Theorem 2.5 we know that the presheaf monad defines a LOFS $(\mathbb{L}, \mathbb{R})$ in $(\mathbb{T}, V)$-Cat_0, and, consequently, a WFS $(\mathcal{L}, \mathcal{R})$, where $\mathcal{L}$ is the class of fully faithful $(\mathbb{T}, V)$-functors. It is easy to check that they coincide with extremal monomorphisms in $(\mathbb{T}, V)$-Cat_0, that is, *topological embeddings.* Therefore, from Theorem 2.5 we conclude that, for every quantale V and monad $\mathbb{T}$ in the conditions of 3.1, $(\mathbb{T}, V)$-Cat_0 has a WFS $(\mathcal{L}, \mathcal{R})$ where $\mathcal{L}$ is the class of embeddings and $\mathcal{R}$ is the class of morphisms with the right lifting property with respect to embeddings. Since $(\mathbb{L}, \mathbb{R})$ is a LOFS, these morphisms have the KZ-lifting property with respect to embeddings. Such morphisms encompass interesting properties.

In $(\mathbb{I}\mathrm{d}, 2)$-$\mathsf{Cat}_0$, that is in the category of (anti-symmetric) ordered sets and monotone maps, $\mathcal{R}$ is the class of monotone maps characterized by Adámek as *fibre-complete*, *fibrations* and *co-fibrations* (see [1, 30]).

In $(\mathbb{I}\mathrm{d}, [0, \infty]_+)$-$\mathsf{Cat}_0$, that is, in the category of separated generalised metric spaces and contractions, morphisms in $\mathcal{R}$ are the fibrewise version of *hyperconvex metric spaces*, studied for instance by Isbell [23].

In $(\mathbb{U}, 2)$-Cat_0, that is, the category of T0-spaces and continuous maps, morphisms in $\mathcal{R}$ are the so-called *fibrewise continuous lattices*, as studied in [4, 5].

5.2. General description. Now let us fix a saturated class Φ of $(\mathbb{T}, V)$-bimodules as in 4.3. The presheaf submonad Φ induces a LOFS $(\mathbb{L}^{\Phi}, \mathbb{R}^{\Phi})$, and consequently a WFS $(\mathcal{L}^{\Phi}, \mathcal{R}^{\Phi})$ where $\mathcal{L}^{\Phi}$ is the class of Φ-embeddings.

Following [10], we say that a $(\mathbb{T}, V)$-functor f is Φ-dense if $f_* \in \Phi$.

Lemma ([10])**.** *For a $(\mathbb{T}, V)$-functor h, the following conditions are equivalent:*

(i) *h is Φ-dense;*
(ii) *Φh is a left adjoint;*

(iii) Φh *is* Φ*-dense.*

We note that Φh has a right adjoint if and only if the right adjoint P^*h of Ph can be (co)restricted to $\Phi_* h : \Phi Y \to \Phi X$, which is the case precisely when $h_* \in \Phi$.

Proposition. *For a* $(\mathbb{T}, V)$*-functor* $h : (X, a) \to (Y, b)$*, the following conditions are equivalent:*

(i) h *is a* Φ*-embedding;*
(ii) h *is fully faithful and* Φ*-dense.*

Proof: (i)⇒(ii): From Theorem 4.3 we know that a Φ-embedding h is fully faithful, and, by definition, Φh is a left adjoint. (ii)⇒(i): If h is Φ-dense, then Φh has a right adjoint $\Phi^* h$, and so it remains to show that, when $h^* \circ h_* = a$, $\Phi^* h \cdot \Phi h = 1_{PX}$: since $x^* \in \Phi$ for every $x \in X$, the proof follows the arguments used in Proposition 4.2(1). ■

Corollary. *For every* $(\mathbb{T}, V)$*-category* (X, a)*,* y_X^Φ *is a* Φ*-embedding.*

5.3. Examples: the Lawvere LOFS. It has particular relevance the choice of $\Phi = \{\psi \in PX \mid \psi \text{ is right adjoint}\}$. Indeed, as Lawvere showed in [25] (see also [10]), when $\mathbb{T} = \mathbb{I}\mathrm{d}$ and $V = [0, \infty]_+$ the injective objects with respect to Φ-embeddings are the Cauchy-complete metric spaces, that is, a non-expansive map $X \to 1$ belongs to $\mathcal{R}^\Phi$ if and only if X is Cauchy-complete. Therefore, the morphisms in $\mathcal{R}^\Phi$ are good candidates for a fibrewise notion of Cauchy-completeness. This LOFS was studied in [13]. We point out that the non-expansive maps in $\mathcal{R}^\Phi$ do not coincide with Sozubek's L-complete maps [29]. Indeed, Sozubek's define them via an injective property, but his left part – the so called L-equivalences – is a proper subclass of our $\mathcal{L}^\Phi$.

5.4. Further examples. Using the techniques of [15, 16] and [9, 3.7], one can define saturated classes of $(\mathbb{T}, V)$-bimodules Φ_0, Φ_ω and Φ_Ω so that the left parts of the corresponding WFS are $\mathcal{L}^{\Phi_0} = \{\text{dense embeddings}\}$, $\mathcal{L}^{\Phi_\omega} = \{\text{flat embeddings}\}$ and $\mathcal{L}^{\Phi_\Omega} = \{\text{completely flat embeddings}\}$. The simple presheaf submonads they define induce LOFS whose underlying WFS were studied in [4], where $\mathcal{R}^{\Phi_0}$, $\mathcal{R}^{\Phi_\omega}$ and $\mathcal{R}^{\Phi_\Omega}$ give the fibrewise notions of *Scott domain*, *stably compact* and *sober spaces* (cf. also [15, 16, 28]).

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