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SOME REMARKS ON CONNECTORS AND GROUPOIDS IN GOURSAT CATEGORIES

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Abstract: We prove that connectors are stable under quotients in any (regular) Goursat category. As a consequence, the category Conn($\mathbb{C}$) of connectors in $\mathbb{C}$ is a Goursat category whenever $\mathbb{C}$ is. This implies that Goursat categories can be characterised in terms of a simple property of internal groupoids.

Keywords: Goursat categories, 3-permutable varieties, Shifting Lemma, connectors, groupoids.
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Over the last twenty years the property of n-permutability of congruences in a variety of universal algebras has been investigated from a categorical perspective (see [7, 12, 18], for instance, and references therein). When $\mathbb{C}$ is a regular category, the 2-permutability property, usually called the *Mal'tsev* property, is a concept giving rise to a beautiful theory, whose main features are collected in [1]. Many important results still hold when a regular category $\mathbb{C}$ satisfies the strictly weaker property of 3-permutability, namely the *Goursat* property. A nice feature of a (regular) Goursat category $\mathbb{C}$ is that the lattice of equivalence relations on any object in $\mathbb{C}$ is a modular lattice [7], a property that plays a crucial role in commutator theory [9, 16].

The aim of this paper is twofold: first of all we establish some basic properties of Goursat categories in terms of connectors [4], as it was done in [5] for the case of Mal'tsev categories. These results have turned out to be useful to develop a monoidal approach to internal structures [11]. We then give a new characterisation of Goursat categories in terms of properties of (internal) groupoids, on the model of what was done in [10] in the case of Mal'tsev categories.

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In the first section, we recall the main properties of Goursat categories that will be needed throughout the paper. In Section 2 we prove that for any Goursat category $\mathbb{C}$, the category Equiv($\mathbb{C}$) of equivalence relations in $\mathbb{C}$ is also a Goursat category (Proposition 2.2) (see also [2]). We use this result to give some properties of Goursat categories in terms of connectors in Section 3. More precisely, we show that, when $\mathbb{C}$ is a Goursat category, then connectors are stable under quotients in $\mathbb{C}$ (Proposition 3.6), and this implies that the category Conn($\mathbb{C}$) of connectors in $\mathbb{C}$ is again a Goursat category (Theorem 3.8).

We conclude the paper by giving a new characterisation of Goursat categories in terms of properties of groupoids (Theorem 3.9). It turns out that a regular category $\mathbb{C}$ is a Goursat category if and only if the category Grpd($\mathbb{C}$) of groupoids in $\mathbb{C}$ is closed under quotients in the category RG($\mathbb{C}$) of reflexive graphs in $\mathbb{C}$.

1. Preliminaries

In this section we recall some basic definitions and properties of (regular) Goursat categories, needed throughout the article. We shall always assume that the category $\mathbb{C}$ in which we are working is a **regular category**: this means that $\mathbb{C}$ is finitely complete, regular epimorphisms are stable under pullbacks, and kernel pairs have coequalisers. Equivalently, any arrow $f : A \longrightarrow B$ has a unique factorisation $f = i \circ r$ (up to isomorphism), where r is a regular epimorphism and i is a monomorphism and this factorisation is pullback stable.

A **relation** R from X to Y is a subobject $\langle r_1, r_2\rangle : R \rightarrowtail X \times Y$. The opposite relation of R, denoted R^o, is the relation from Y to X given by the subobject $\langle r_2, r_1\rangle : R \rightarrowtail Y \times X$. A relation R from X to X is called a relation on X. We shall identify a morphism $f : X \longrightarrow Y$ with the relation $\langle 1_X, f\rangle : X \rightarrowtail X \times Y$ and write f^o for its opposite relation. Given another relation $\langle s_1, s_2\rangle : S \rightarrowtail Y \times Z$ from Y to Z, one can define the composite relation SR of R and S as the regular image of the arrow $(r_1 \circ p_1, s_2 \circ p_2) : R \times_Y S \longrightarrow X \times Z$, where $(R \times_Y S, p_1, p_2)$ is the pullback of $r_2 : R \longrightarrow Y$ along $s_1 : S \longrightarrow Y$. With the above notations, any relation $\langle r_1, r_2\rangle : R \rightarrowtail X \times Y$ can be seen as the relational composite $r_2 r_1^o$.

The following properties are well known and easy to prove. We collect them in the following lemma:

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Mal'tsev categories and Goursat categories. Similar results hold for Mal'tsev categories with respect to monomorphisms and for Goursat categories with respect to regular epimorphisms.

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Lemma 1.1. *Let $f : X \longrightarrow Y$ be an arrow in a regular category $\mathbb{C}$, and let $i \circ r$ be its (regular epimorphism,monomorphism) factorisation. Then:*

(1) *$f^o f$ is the kernel pair of f, thus $1_X \leqslant f^o f$; moreover, $1_X = f^o f$ if and only if f is a monomorphism;*
(2) *$f f^o$ is (i, i), thus $f f^o \leqslant 1_Y$; moreover, $f f^o = 1_Y$ if and only if f is a regular epimorphism;*
(3) *$f f^o f = f$ and $f^o f f^o = f^o$.*

Definition 1.2. A relation (R, r_1, r_2) on an object X is said to be :

- **reflexive** when there is an arrow $r : X \longrightarrow R$ such that $r_1 \circ r = 1_X = r_2 \circ r$;
- **symmetric** when there is an arrow $\sigma : R \longrightarrow R$ such that $r_2 = r_1 \circ \sigma$ and $r_1 = r_2 \circ \sigma$;
- **transitive** when, by considering the following pullback

$$\begin{array}{ccc} R \times_X R & \xrightarrow{p_2} & R \\ {\scriptstyle p_1}\downarrow & \lrcorner & \downarrow{\scriptstyle r_1} \\ R & \xrightarrow[r_2]{} & X, \end{array}$$

  there is an arrow $t : R \times_X R \longrightarrow R$ such that $r_1 \circ t = r_1 \circ p_1$ and $r_2 \circ t = r_2 \circ p_2$.
- an **equivalence relation** if R is reflexive, symmetric and transitive.

In particular, a kernel pair $\langle f_1, f_2 \rangle : \mathrm{Eq}(f) \rightarrowtail X \times X$ of a morphism $f : X \longrightarrow Y$ is an equivalence relation.

Let $\mathrm{Equiv}(\mathbb{C})$ be the category whose objects are equivalence relations in $\mathbb{C}$ and arrows from $\langle r_1, r_2 \rangle : R \rightarrowtail X \times X$ to $\langle s_1, s_2 \rangle : S \rightarrowtail Y \times Y$ are pairs (f, g) of arrows in $\mathbb{C}$ making the following diagram commute

$$\begin{array}{ccc} R & \xrightarrow{g} & S \\ {\scriptstyle r_1}\downarrow\downarrow{\scriptstyle r_2} & & {\scriptstyle s_1}\downarrow\downarrow{\scriptstyle s_2} \\ X & \xrightarrow[f]{} & Y. \end{array}$$

When $\mathbb{C}$ is a regular category, (R, r_1, r_2) is an equivalence relation on X and $f : X \twoheadrightarrow Y$ a regular epimorphism, we define the **regular image of** R **along** $f : X \longrightarrow Y$ to be the relation $f(R)$ on Y induced by the (regular epimorphism, monomorphism) factorisation $\langle s_1, s_2 \rangle \circ \psi$ of the composite $(f \times$

$f) \circ \langle r_1, r_2 \rangle$:

$$\begin{array}{ccc} R & \overset{\psi}{\dashrightarrow} & f(R) \\ {\scriptstyle \langle r_1, r_2 \rangle}\downarrow & & \downarrow{\scriptstyle \langle s_1, s_2 \rangle} \\ X \times X & \underset{f \times f}{\longrightarrow} & Y \times Y. \end{array}$$

Note that the regular image $f(R)$ can be obtained as the relational composite $f(R) = fRf^o = fr_2r_1^of^o$. When R is an equivalence relation, $f(R)$ is also reflexive and symmetric. In a general regular category $f(R)$ is not necessarily an equivalence relation. This is the case in a *Goursat category* (Theorem 1.4).

Definition 1.3. [7] A regular category $\mathbb{C}$ is called a **Goursat category** when the equivalence relations in $\mathbb{C}$ are 3-permutable, i.e. $RSR = SRS$ for any pair of equivalence relations R and S on the same object.

The following characterisation will be useful in the sequel:

Theorem 1.4. [7] *A regular category $\mathbb{C}$ is a Goursat category if and only if for any regular epimorphism $f : X \twoheadrightarrow Y$ and any equivalence relation R on X, the regular image $f(R) = fRf^o$ of R along f is an equivalence relation.*

There are many important algebraic examples of Goursat categories. Indeed, by a classical theorem in [17], a variety of universal algebras is a Goursat category precisely when its theory has two ternary operations r and s such that the identities $r(x, y, y) = x$, $r(x, x, y) = s(x, y, y)$ and $s(x, x, y) = y$ hold. Accordingly, the categories of groups, abelian groups, modules over some fixed ring, crossed modules, quasi-groups, rings, associative algebras, Heyting algebras and implication algebras are all Goursat categories.

Any regular *Mal'tsev* category is a Goursat category, thus, in particular, so is any semi-abelian category.

Many interesting properties of Goursat categories can be found in the literature (see [7, 14, 15] and references therein). In particular, the following characterisations will be useful for the development of this work:

Theorem 1.5. [14] *Let $\mathbb{C}$ be a regular category. The following conditions are equivalent:*

(i) $\mathbb{C}$ *is a Goursat category;*

- $d \circ i = c$, $c \circ i = d$, $m \circ \langle i, 1_{X_1} \rangle = e \circ c$ and $m \circ \langle 1_{X_1}, i \rangle = e \circ d$.

We write $\mathrm{Grpd}(\mathbb{C})$ for the category of groupoids in $\mathbb{C}$ and $\mathrm{RG}(\mathbb{C})$ for the category of internal reflexive graphs in $\mathbb{C}$.

Theorem 3.9. *Let $\mathbb{C}$ be a regular category. Then the following conditions are equivalent:*

(i) $\mathbb{C}$ *is a Goursat category;*
(ii) $\mathrm{Grpd}(\mathbb{C})$ *is closed in* $\mathrm{RG}(\mathbb{C})$ *under quotients.*

Proof: $(i) \Rightarrow (ii)$ Let

$$\begin{array}{ccc} X_1 & \overset{g}{\twoheadrightarrow} & X_1' \\ {\scriptstyle d}\downarrow\uparrow{\scriptstyle c} & & {\scriptstyle d'}\downarrow\uparrow{\scriptstyle c'} \\ X_0 & \underset{f}{\twoheadrightarrow} & X_0' \end{array}$$

be a regular epimorphism (f, g) in $\mathrm{RG}(\mathbb{C})$ (which means that f and g are regular epimorphisms in $\mathbb{C}$), with

$$X_1 \underset{c}{\overset{d}{\rightleftarrows}} X_0$$

a groupoid in $\mathbb{C}$. Let $\mathrm{Eq}(d)$, $\mathrm{Eq}(c)$, $\mathrm{Eq}(d')$ and $\mathrm{Eq}(c')$ be the kernel pairs of the arrows d, c, d' and c', respectively. Let $\lambda : \mathrm{Eq}(d) \to \mathrm{Eq}(d')$ and $\beta : \mathrm{Eq}(c) \to \mathrm{Eq}(c')$ be the arrows induced by the universal property of kernel pairs $\mathrm{Eq}(d')$ and $\mathrm{Eq}(c')$, respectively. By Theorem 1.5, λ and β are regular epimorphisms, so that $g(\mathrm{Eq}(d)) = \mathrm{Eq}(d')$ and $g(\mathrm{Eq}(c)) = \mathrm{Eq}(c')$. By Proposition 3.5 there is then a connector between $\mathrm{Eq}(d')$ and $\mathrm{Eq}(c')$, thus

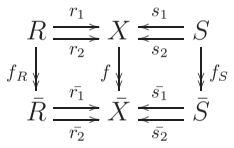
$$X_1' \underset{c'}{\overset{d'}{\rightleftarrows}} X_0'$$

is a groupoid (see Example 3.2 (b)).

$(ii) \Rightarrow (i)$ This implication follows from Theorem 1.4 and the fact that equivalence relations are in particular groupoids (whose domain and codomain morphisms are jointly monomorphic). ∎

Remark 3.10. A result analogous to Theorem 3.9 holds in the Mal'tsev context: a category $\mathbb{C}$ is a Mal'tsev category if and only if $\mathrm{Grpd}(\mathbb{C})$ is closed in $\mathrm{RG}(\mathbb{C})$ under subobjects [3]. Together with the comments made before Proposition 3.5 we observe the existence of a sort of duality between

Let us then consider a regular epimorphism in 2-Eq(ℂ)



(this means that f, f_R and f_S are regular epimorphisms in ℂ) such that its domain (R, S, X) belongs to Conn(ℂ). The equalities $f(R) = \bar{R}$ and $f(S) = \bar{S}$, together with Proposition 3.5, imply that there exists a connector between $\bar{R}$ and $\bar{S}$. ∎

Lemma 3.7. *Let $\mathbb{D}$ be a finitely complete category, and $\mathbb{C}$ a full subcategory of $\mathbb{D}$ closed in $\mathbb{D}$ under finite limits and quotients. Then:*

(1) *$\mathbb{C}$ is regular whenever $\mathbb{D}$ is regular.*
(2) *$\mathbb{D}$ is a Goursat category whenever $\mathbb{C}$ is a Goursat category.*

Proof: The (regular epimorphism, monomorphism) factorisation in $\mathbb{D}$ of an arrow in $\mathbb{C}$ is also its factorisation in $\mathbb{C}$, since $\mathbb{C}$ is closed in $\mathbb{D}$ under quotients. Since finite limits in $\mathbb{C}$ are calculated as in $\mathbb{D}$, it follows that regular epimorphisms are stable under pullbacks. Now the second statement easily follows from the fact that the composition of relations is computed in the same way in $\mathbb{C}$ and in $\mathbb{D}$. ∎

Theorem 3.8. *If $\mathbb{C}$ is a Goursat category then* Conn(ℂ) *is a Goursat category.*

Proof: From Proposition 2.2 one may deduce that 2-Eq(ℂ) is a Goursat category. The result then follows from Proposition 3.6 and Lemma 3.7. ∎

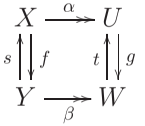
We finally prove that groupoids can be used to characterise Goursat categories. Recall that an **(internal) groupoid** in a category $\mathbb{C}$ with pullbacks is a reflexive graph with a multiplication $m\colon X_1 \times_{X_0} X_1 \to X_1$ and an inverse $i\colon X_1 \to X_1$

$$X_1 \times_{X_0} X_1 \underset{p_2}{\overset{p_1}{\underset{\overset{m}{\longrightarrow}}{\rightrightarrows}}} X_1 \underset{c}{\overset{d}{\underset{\overset{c}{\longleftarrow}}{\rightrightarrows}}} X_0, \qquad i\colon X_1 \to X_1$$

(where $X_1 \times_{X_0} X_1$ is the pullback of d and c) such that:

- $d \circ m = d \circ p_2$, $c \circ m = c \circ p_1$, $m \circ \langle e \circ d, 1_{X_1} \rangle = 1_{X_1} = m \circ \langle 1_{X_1}, e \circ c \rangle$;
- $m \circ (1 \times m) = m \circ (m \times 1)$;

(ii) *any pushout where α and β are regular epimorphisms and f and g are split epimorphisms in ℂ*

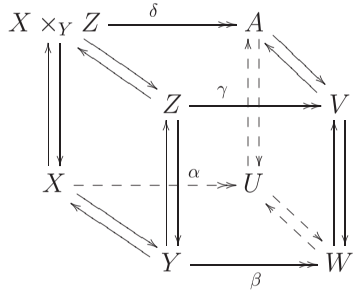


*is a **Goursat pushout**: the morphism $\lambda : \mathrm{Eq}(f) \longrightarrow \mathrm{Eq}(g)$ induced by the universal property of kernel pair* Eq(g) *of g is a regular epimorphism.*

We recall part of Theorem 1.3 in [15]:

Theorem 1.6. [15] *Let $\mathbb{C}$ be a regular category. The following conditions are equivalent:*

(i) *$\mathbb{C}$ is a Goursat category;*
(ii) *for any commutative cube*



where the left square is a pullback of split epimorphisms, the right square is a commutative square of split epimorphisms and the horizontal arrows α, β, γ and δ are regular epimorphisms (commuting also with the splittings), then the right square is a pullback.

2. Equivalence relations in Goursat categories

In this section we prove that Equiv(ℂ) is a Goursat category for any Goursat category ℂ.

The category Equiv(ℂ) is finitely complete whenever ℂ is. Note that kernel pairs in ℂ are computed "levelwise", i.e. the kernel pair of a morphism (f, g)

in Equiv($\mathbb{C}$) is given by the kernel pairs Eq(f) of f and Eq(g) of g in $\mathbb{C}$

$$\begin{array}{ccccc} \mathrm{Eq}(g) & \overset{g_1}{\underset{g_2}{\rightrightarrows}} & R & \xrightarrow{g} & S \\ {\scriptstyle \bar{r}_1}\downarrow\downarrow{\scriptstyle \bar{r}_2} & & {\scriptstyle r_1}\downarrow\downarrow{\scriptstyle r_2} & & {\scriptstyle s_1}\downarrow\downarrow{\scriptstyle s_2} \\ \mathrm{Eq}(f) & \overset{f_1}{\underset{f_2}{\rightrightarrows}} & X & \xrightarrow[f]{} & Y. \end{array} \tag{1}$$

Consequently, a morphism (f, g) is a monomorphism in Equiv($\mathbb{C}$) if and only if f and g are monomorphisms in $\mathbb{C}$. When $\mathbb{C}$ is a Goursat category, a similar property holds with respect to regular epimorphisms:

Lemma 2.1. *Let R and S be two equivalence relations in a Goursat category $\mathbb{C}$ and $(f, g) : R \to S$ a morphism*

$$\begin{array}{ccc} R & \xrightarrow{g} & S \\ {\scriptstyle r_1}\downarrow\downarrow{\scriptstyle r_2} & & {\scriptstyle s_1}\downarrow\downarrow{\scriptstyle s_2} \\ X & \xrightarrow[f]{} & Y \end{array} \tag{2}$$

in Equiv($\mathbb{C}$)*. Then (f, g) is a regular epimorphism in* Equiv($\mathbb{C}$) *if and only if f and g are regular epimorphisms in $\mathbb{C}$.*

Proof: When f and g are regular epimorphisms in $\mathbb{C}$, it is not difficult to check that (f, g) is necessarily the coequaliser of its kernel pair in Equiv($\mathbb{C}$) given in (1) (one uses the fact that $g = coeq(g_1, g_2)$ and $f = coeq(f_1, f_2)$ in $\mathbb{C}$).

Conversely, let (f, g) be a morphism in Equiv($\mathbb{C}$) as in (2) that is a regular epimorphism in Equiv($\mathbb{C}$). Consider the kernel pairs of f and g, the (regular epimorphism, monomorphism) factorisation $f = i \circ q$ of f, and the regular image $(q(R), t_1, t_2)$ of (R, r_1, r_2) along q. We obtain the following

the same object X

$$R \overset{r_1}{\underset{r_2}{\rightrightarrows}} X \overset{s_1}{\underset{s_2}{\leftleftarrows}} S$$

and arrows are triples (f_R, f_S, f) making the following diagram commute:

$$\begin{array}{ccccc} R & \overset{r_1}{\underset{r_2}{\rightrightarrows}} & X & \overset{s_1}{\underset{s_2}{\leftleftarrows}} & S \\ {\scriptstyle f_R}\downarrow & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_S} \\ \bar{R} & \overset{\bar{r}_1}{\underset{\bar{r}_2}{\rightrightarrows}} & \bar{X} & \overset{\bar{s}_1}{\underset{\bar{s}_2}{\leftleftarrows}} & \bar{S}. \end{array} \tag{5}$$

We write Conn($\mathbb{C}$) for the category whose objects (R, S, X, p) are pairs of equivalence relations R and S on an object X with a given connector $p : R \times_X S \to X$; arrows in Conn($\mathbb{C}$) are arrows in 2-Eq($\mathbb{C}$) respecting the connectors. This means that, given a diagram (4) where both (R, S, X) and $(\bar{R}, \bar{S}, \bar{X})$ are in Conn($\mathbb{C}$), with $p : R \times_X S \to X$ and $\bar{p} : \bar{R} \times_Y \bar{S} \to Y$ the corresponding connectors, then the diagram

$$\begin{array}{ccc} R \times_X S & \xrightarrow{\bar{f}} & \bar{R} \times_{\bar{X}} \bar{S} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \bar{p}} \\ X & \xrightarrow[f]{} & \bar{X} \end{array}$$

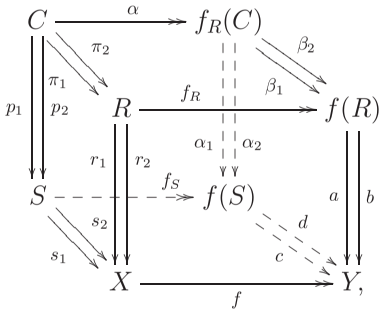
commutes, where $\bar{f}$ is the natural map induced by the universal property of the pullback $\bar{R} \times_{\bar{X}} \bar{S}$.

We say that a subcategory $\mathbb{P}$ is *closed under (regular) quotients* in a category $\mathbb{Q}$ if, for any regular epimorphism $f : A \twoheadrightarrow B$ in $\mathbb{Q}$ such that $A \in \mathbb{P}$, then $B \in \mathbb{P}$.

Proposition 3.6. *If $\mathbb{C}$ is a Goursat category, then* Conn($\mathbb{C}$) *is a full subcategory of 2-*Eq($\mathbb{C}$)*, that is closed in 2-*Eq($\mathbb{C}$) *under quotients.*

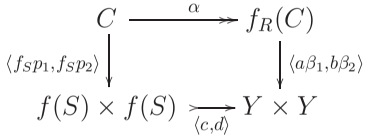
Proof: The fullness of the forgetful functor Conn($\mathbb{C}$) $\to$ 2-Eq($\mathbb{C}$) follows from the results in [6], by taking into account the fact that any Goursat category satisfies the Shifting Property.

We obtain the following diagram



(4)

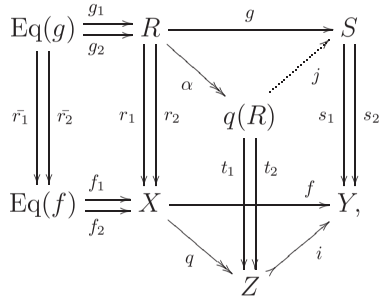
where $(f_R(C), \beta_1, \beta_2)$ is the regular image of the equivalence relation (C,π_1,π_2) along the regular epimorphism f_R. The fact that the diagram



commutes, α is a strong epimorphism and $\langle c, d\rangle$ is a monomorphism, implies the existence of the two arrows $\alpha_1, \alpha_2 : f_R(C) \longrightarrow f(S)$ making diagram (4) commute. The relations $(f_R(C), \beta_1, \beta_2)$, $(f(R), a, b)$ and $(f(S), c, d)$ are all equivalence relations by Theorem 1.4. It is then easy to check that the relation $(f_R(C), \alpha_1, \alpha_2)$ is an equivalence relation on $f(S)$. By assumption all the left squares of (4) are pullbacks, so it follows that all the right squares of (4) are pullbacks as well by Theorem 1.6 (ii). Then $(f_R(C), (\alpha_1, \alpha_2), (\beta_1, \beta_2))$ is a centralizing relation on $f(R)$ and $f(S)$. By Lemma 3.4 there is a connector between $f(R)$ and $f(S)$. ■

We are now going to show that the category whose objects are pairs of equivalence relations equipped with a connector is a Goursat category whenever the base category is a Goursat category. For this, let us first fix some notation: if $\mathbb{C}$ is a finitely complete category, we write 2-Eq($\mathbb{C}$) for the category whose objects (R, S, X) are pairs of equivalence relations R and S on

commutative diagram



(3)

where $(q(R), t_1, t_2) \in \text{Equiv}(\mathbb{C})$ (by Theorem 1.4) and (i, j) is the morphism in Equiv($\mathbb{C}$) such that $(i, j) \circ (q, \alpha) = (f, g)$. Note that j is induced from the fact that $i \times i \circ \langle t_1, t_2\rangle \circ \alpha$ is the (regular epimorphism, monomorphism) factorisation of $\langle s_1, s_2\rangle \circ g$, thus it is a monomorphism. From the fact that (f, g) is the coequaliser of its kernel pair in Equiv($\mathbb{C}$) it easily follows that (i, j) is an isomorphism in Equiv($\mathbb{C}$). This implies that f and g are regular epimorphisms in $\mathbb{C}$. ■

Proposition 2.2. Equiv($\mathbb{C}$) *is a Goursat category whenever* $\mathbb{C}$ *is.*

Proof: The category Equiv($\mathbb{C}$) is finitely complete because $\mathbb{C}$ is so. Lemma 2.1 implies that regular epimorphisms in Equiv($\mathbb{C}$) are stable under pullbacks since regular epimorphisms are stable in $\mathbb{C}$, and regular epimorphisms in Equiv($\mathbb{C}$) are "levelwise" regular epimorphisms. The existence of the (regular epimorphism, monomorphism) factorisation of a morphism (f, g) as in (2) in the category Equiv($\mathbb{C}$) follows from the construction of diagram (3): the (regular epimorphism, monomorphism) factorisation $f = i \circ q$ of f in $\mathbb{C}$ gives rise to the (regular epimorphism, monomorphism) factorisation $g = j \circ \alpha$ of g in $\mathbb{C}$. Thus $(q, \alpha) \circ (i, j)$ is the (regular epimorphism, monomorphism) factorisation of (f, g) in Equiv($\mathbb{C}$). To see that Equiv($\mathbb{C}$) has the Goursat property one uses Theorem 1.4: to check that the regular image of an equivalence relation in the category Equiv($\mathbb{C}$) is again an equivalence in Equiv($\mathbb{C}$) one mainly uses the same property in the category $\mathbb{C}$. ■

3. Connectors and groupoids in Goursat categories

In this section we prove that connectors are stable under quotients in any Goursat category $\mathbb{C}$. We then define the category Conn($\mathbb{C}$) of connectors in

$\mathbb{C}$ whose objects are pairs of equivalence relations equipped with a connector, and prove that $\mathrm{Conn}(\mathbb{C})$ is a Goursat category whenever the base category $\mathbb{C}$ is. We conclude by giving a new characterisation of Goursat categories in terms of proprieties of groupoids.

Definition 3.1. Let (R, r_1, r_2) and (S, s_1, s_2) be two equivalence relations on an object X and $R \times_X S$ the pullback of r_2 along s_1. A **connector** [4] between R and S is an arrow $p : R \times_X S \longrightarrow X$ in $\mathbb{C}$ such that

(1) $xSp(x, y, z)Rz$;
(2) $p(x, x, y) = y$;
(3) $p(x, y, y) = x$;
(4) $p(x, y, p(z, u, v)) = p(p(x, y, z), u, v)$.

Examples 3.2. (a) *If ∇_X is the largest equivalence relation on an object X, then an associative Mal'tsev operation*

$$p : X \times X \times X \longrightarrow X$$

is precisely a connector between ∇_X and ∇_X.

(b) *Given a reflexive graph*

$$X_1 \underset{c}{\overset{d}{\rightleftarrows}} X_0$$

in a finitely complete category $\mathcal{C}$ (i.e. $d \circ e = 1_{X_0} = c \circ e$) then the connectors between $\mathrm{Eq}(c)$ and $\mathrm{Eq}(d)$ are in bijections with the groupoid structures on this reflexive graph [8].

It is well known that Goursat categories satisfy the so-called *Shifting Property* [16, 6]. In this context connectors are unique when they exist (Theorem 2.13 and Proposition 5.1 in [6]): accordingly, for a given pair of equivalence relations on the same object the fact of having a connector becomes a *property*.

Definition 3.3. Let R and S be two equivalence relations on an object X. A **double equivalence relation** on R and S is given by an object $C \in \mathbb{C}$ equipped with two equivalence relations $(\pi_1, \pi_2) : C \rightrightarrows R$ and $(p_1, p_2) : C \rightrightarrows S$ such that the following diagram

$$\begin{array}{ccc} C & \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} & R \\ {\scriptstyle p_1}\downarrow\downarrow{\scriptstyle p_2} & & {\scriptstyle r_1}\downarrow\downarrow{\scriptstyle r_2} \\ S & \underset{s_2}{\overset{s_1}{\rightrightarrows}} & X \end{array}$$

commutes (in the "obvious" way).

A double equivalence relation C on R and S is called a **centralizing relation** [8] when the square

$$\begin{array}{ccc} C & \xrightarrow{\pi_1} & R \\ {\scriptstyle p_1}\downarrow & \lrcorner & \downarrow{\scriptstyle r_1} \\ S & \xrightarrow[s_1]{} & X \end{array}$$

is a pullback. Under this assumption it follows that any of the commutative squares in the definition of a centralizing relation is a pullback.

The following lemma gives the relationship between connectors and centralizing relations.

Lemma 3.4. [5] *If $\mathbb{C}$ is a category with finite limits and R and S are two equivalence relations on the same object X, then the following conditions are equivalent:*

(i) *there exists a connector between R and S;*
(ii) *there exists a centralizing relation on R and S.*

When $\mathbb{C}$ is a Mal'tsev category, R and S are equivalence relations on an object X with a connector and $i : I \rightarrowtail X$ is a monomorphism, then the inverse images $i^{-1}(R)$ and $i^{-1}(S)$ also have a connector [4]. We establish a similar property for Goursat categories, with respect to regular epimorphisms:

Proposition 3.5. *Let $\mathbb{C}$ be a Goursat category, R and S two equivalence relations on an object X, and let $f : X \twoheadrightarrow Y$ be a regular epimorphism. If there exists a connector between R and S, then there exists a connector between the regular images $f(R)$ and $f(S)$.*

Proof: Suppose that there exists a connector between R and S. This implies that there exists a centralizing relation $(C, (\pi_1, \pi_2), (p_1, p_2))$ on R and S. Consider the regular image $(f(R), a, b)$ and $(f(S), c, d)$ of R and S along f.