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DIRECT SEARCH BASED ON PROBABILISTIC FEASIBLE DESCENT FOR BOUND AND LINEARLY CONSTRAINED PROBLEMS

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Abstract: Direct search is a methodology for derivative-free optimization whose iterations are characterized by evaluating the objective function using a set of polling directions. In deterministic direct search applied to smooth objectives, these directions must somehow conform to the geometry of the feasible region and typically consist of positive generators of approximate tangent cones (which then renders the corresponding methods globally convergent in the linearly constrained case). One knows however from the unconstrained case that randomly generating the polling directions leads to better complexity bounds as well as to gains in numerical efficiency, and it becomes then natural to consider random generation also in the presence of constraints.

In this paper, we study a class of direct search based on sufficient decrease for solving smooth linearly constrained problems where the polling directions are randomly generated (in approximate tangent cones). The random polling directions must satisfy probabilistic feasible descent, a concept which reduces to probabilistic descent in the absence of constraints. Such a property is instrumental in establishing almost-sure global convergence and worst-case complexity bounds with overwhelming probability. Numerical results show that the randomization of polling directions compares favorably to the classical deterministic approach. In some cases, one can observe a clear superiority of randomization, as it is suggested by our complexity results.

Keywords: Derivative-free optimization, direct-search methods, positive generators, probabilistic feasible descent, bounds, linear constraints.
AMS Subject Classification (2010): 90C56, 90C30, 90C15.

1. Introduction

In various practical scenarios, derivative-free algorithms are the single way to solve optimization problems for which no derivative information can be computed or approximated. Among the various classes of such schemes, direct search is one of the most popular, due to its simplicity, robustness, and

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easiness of parallelization. Direct-search methods [18, 7] explore the objective function along suitably chosen sets of directions, called polling directions. When there are no constraints on the problem variables and the objective function is smooth, those directions must provide a suitable angle approximation to the unknown negative gradient, and this is guaranteed by assuming that the polling vectors *positively* span the whole space (by means of non-negative linear combinations). In the presence of constraints, such directions must conform to the shape of the feasible region in the vicinity of the current feasible point, and are typically given by the *positive* generators of some approximate tangent cone identified by nearby active constraints. When the feasible region is polyhedral, it is possible to decrease a smooth objective function value along such directions without violating the constraints, provided a sufficiently small step is taken.

Bound constraints are a classical example of such a polyhedral setting. Direct-search methods tailored to these constraints have been introduced by [21] where polling directions can (and must) be chosen among the coordinate directions and their negatives, which naturally conform to the geometry of such a feasible region. A number of other methods have been proposed and analyzed based on these polling directions (see [12, 14, 25] and [17, Chapter 4]).

In the general linearly constrained case, polling directions have to be computed as positive generators of cones tangent to a set of constraints that are nearly active (see [22, 19]). The identification of the nearly active constraints can be tightened to the size of the step taken along the directions [19], and global convergence is guaranteed to a true stationary point as the step size goes to zero. In the presence of constraint linear dependence, the computation of these positive generators is problematic and may require enumerative techniques [1, 20, 30], although the practical performance of the corresponding methods does not seem much affected. Direct-search methods for linearly constrained problems have been successfully combined with global exploration strategies by means of a search step [8, 32, 33].

Several strategies for the specific treatment of linear equality constraints have also been proposed, one way being to remove those constraints by changing of variables [2, 11]. Another possibility is to design the algorithm so that it explores the null space of the linear equality constraints, which is also equivalent to solving the problem in a lower-dimensional subspace [24, 20]. Such techniques may lead to a possibly less separable problem.

All aforementioned strategies involve the *deterministic* generation of the polling directions. Recently, it has been shown in [16] that the random generation of the polling directions outperforms the classical choice of deterministic positive spanning sets for unconstrained optimization (whose cardinality is at least $n+1$, where n is the dimension of the problem). Inspired by the concept of probabilistic trust-region models [3], the authors in [16] introduced the concept of probabilistic descent, which essentially imposes on the random polling directions the existence of a direction that makes an acute angle with the negative gradient with a probability sufficiently large conditioning on the history of the iterations. This approach has been proved globally convergent with probability one. More importantly, a complexity bound of the order $n\epsilon^{-2}$ has been showed (with overwhelming high probability) for the number of function evaluations needed to reach a gradient of norm below a positive threshold ϵ, which contrasts with the $n^2\epsilon^{-2}$ bound of the deterministic case [34] (for which the factor of n^2 cannot be improved [9]). It was also reported a substantial gain in numerical efficiency while generating the polling directions randomly, with the choice of only two directions (supported by the theory) leading to the best observed performance.

Motivated by these results for unconstrained optimization, we introduce in this paper the concept of probabilistic feasible descent for the linearly constrained setting, essentially by considering the projection of the negative gradient on an approximate tangent cone identified by nearby active constraints. We prove for smooth objectives that direct search based on probabilistic feasible descent (and sufficient decrease) enjoys global convergence with probability one and takes (with overwhelming high probability) a number of iterations of the order of ϵ^{-2} to reduce an optimality measure of the problem below $\epsilon > 0$. As a by-product of our work we will also show a similar complexity bound for the deterministic choice of polling directions — left open in the literature until now. We then quantify the number of function evaluations required for the same goal, and we do this for two randomized strategies: one where the directions are a random subset of the positive generators of the approximate tangent cone; and another where one first decomposes the approximate tangent cone into a subspace and a cone orthogonal to the subspace. In the latter case, the subspace component (if nonzero) is handled by generating random directions in it, and the cone component is treated by considering a random subset of its positive generators.

Throughout the paper we particularize our results for the cases where there are only bounds on the variables or there are only linear equality constraints.

We organize our paper as follows. In Section 2, we describe the problem at hand as well as the direct-search framework under consideration. In Section 3 we motivate the concept of feasible descent and show how to derive from this the already known global convergence of [19] — although not providing a new result, this setup is crucial for the rest of the paper. It allows us to establish right away (see Section 4) the complexity of this class of direct-search methods based on sufficient decrease for linearly constrained optimization, establishing bounds for the worst-case effort when measured in terms of number of iterations and function evaluations. In Section 5 we introduce the concept of probabilistic feasible descent and prove almost-sure global convergence for direct-search methods based on it. The corresponding worst-case complexity bounds are established in Section 6. In Section 7 we discuss how to take advantage of subspace information in the random generation of the polling directions. Then we report in Section 8 a numerical comparison between direct search based on probabilistic feasible descent and a built-in MATLAB direct-search solver, which enlightens the potential of random polling directions. Finally, conclusions are drawn in Section 9.

We will use the following notation. Throughout the document, any set of vectors $V = \{v_1, \ldots, v_m\} \subset \mathbb{R}^n$ will be identified with the matrix $[v_1 \cdots v_m] \in \mathbb{R}^{n\times m}$. We will thus allow ourselves to write $v \in V$ even though we may manipulate V as a matrix. Given a closed, convex cone K in $\mathbb{R}^n$, $P_K[x]$ will denote the uniquely defined projection of the vector x onto the cone K (with the convention $P_\emptyset[x] = 0$ for all x). The polar cone of K is the set $\{x : y^\top x \leq 0, \forall y \in K\}$. For every space considered, the norm $\|\cdot\|$ will be the Euclidean one. The notation $\mathcal{O}(A)$ will stand for a scalar times A, with this scalar depending solely on the problem considered or constants from the algorithm. The dependence on the problem dimension n will explicitly appear in A when considered appropriate.

2. Direct search for linearly constrained problems

In this paper, we consider optimization problems given in the following form:

$$\begin{array}{ll} \min\limits_{x\in\mathbb{R}^n} & f(x) \\ \text{s.t.} & Ax = b, \\ & \ell \leq A_{iq}x \leq u, \end{array} \tag{1}$$

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where $f : \mathbb{R}^n \to \mathbb{R}$, $A \in \mathbb{R}^{m\times n}$, $A_{iq} \in \mathbb{R}^{m_{iq}\times n}$, $b \in \mathbb{R}^m$, and $(\ell, u) \in (\mathbb{R}\cup\{-\infty,\infty\})^{m_{iq}}$, with $\ell < u$. We consider that the matrix A can be empty (i.e., m can be zero), in order to encompass unconstrained and bound-constrained problems into this general formulation (when $m_{iq} = n$ and $A_{iq} = I_n$). Whenever $m \geq 1$, we suppose that the matrix A is of full row rank. We define the feasible region as

$$\mathcal{F} = \{x \in \mathbb{R}^n : \; Ax = b, \; \ell \leq A_{iq}x \leq u\}\,.$$

The algorithmic analysis in this paper requires a measure of first-order criticality for problem (1). Given $x \in \mathcal{F}$, we will work with

$$\chi\,(x) \stackrel{\text{def}}{=} \max_{\substack{x+d\in\mathcal{F}\\ \|d\|\leq 1}} d^\top\,[-\nabla f(x)]\,. \tag{2}$$

The criticality measure $\chi\,(\cdot)$ is a non-negative, continuous function that equals zero only at a KKT first-order stationary point of problem (1) (see [35]). It has been successfully used to derive convergence analyzes of direct-search schemes applied to linearly constrained problems [18]. Given an orthonormal basis $W \in \mathbb{R}^{n\times(n-m)}$ for the null space of A, this measure can be reformulated as

$$\chi(x) \;=\; \max_{\substack{x+W\tilde{d}\in\mathcal{F}\\ \|W\tilde{d}\|\leq 1}} \tilde{d}^\top[-W^\top\nabla f(x)] \;=\; \max_{\substack{x+W\tilde{d}\in\mathcal{F}\\ \|\tilde{d}\|\leq 1}} \tilde{d}^\top[-W^\top\nabla f(x)]. \tag{3}$$

Algorithm 2.1 presents the basic direct-search method under analysis in this paper. We suppose that a feasible initial point is provided by the user. At every iteration, using a finite number of polling directions, the algorithm attempts to compute a new feasible iterate that reduces the objective function value by a sufficient amount (measured by the value of a *forcing function* ρ on the step size α_k).

Algorithm 2.1. *Feasible Direct Search based on sufficient decrease.* ***Inputs:*** *$x_0 \in \mathcal{F}$, $\alpha_{\max} \in (0,\infty]$, $\alpha_0 \in (0,\alpha_{\max})$, $\theta \in (0,1)$, $\gamma \in [1,\infty)$, and a forcing function $\rho : (0,\infty) \to (0,\infty)$. For $k = 0, 1, \ldots$ do:* ***Poll Step*** *Choose a finite set D_k of non-zero polling directions. Evaluate f at the polling points $\{x_k + \alpha_k d : d \in D_k\}$ following a chosen order. If a* **feasible** *poll point $x_k + \alpha_k d_k$ is found such that $f(x_k + \alpha_k d_k) < f(x_k) - \rho(\alpha_k)$ then stop polling, set $x_{k+1} = x_k + \alpha_k d_k$, and declare the iteration successful. Otherwise declare the iteration unsuccessful and set $x_{k+1} = x_k$.* ***Step Size Update*** *If the*

iteration is successful, (possibly) increase the step size by setting $\alpha_{k+1} = \min\{\gamma\alpha_k, \alpha_{\max}\}$*; Otherwise, decrease the step size by setting* $\alpha_{k+1} = \theta\alpha_k$.

We consider that such a method runs under the following assumptions.

Assumption 2.1. *For each* $k \geq 0$, D_k *is a finite set of normalized vectors.*

As it was done in [16] for unconstrained optimization, the constants in the derivation of complexity bounds simplify if we assume that all polling directions are normalized. However, all global convergence limits and worst-case complexity orders remain true when the norms of polling directions are only assumed to be above and below certain positive constants.

Assumption 2.2. *The forcing function* ρ *is positive, non-decreasing, and* $\rho(\alpha) = o(\alpha)$ *when* $\alpha \to 0^+$. *There exist constants* $\bar{\theta}$ *and* $\bar{\gamma}$ *satisfying* $0 < \bar{\theta} < 1 \leq \bar{\gamma}$ *such that, for each* $\alpha > 0$, $\rho(\theta\alpha) \leq \bar{\theta}\rho(\alpha)$ *and* $\rho(\gamma\alpha) \leq \bar{\gamma}\rho(\alpha)$.

Typical examples of such functions include monomials of the form $\rho(\alpha) = c\,\alpha^q$, with $c > 0$ and $q > 1$. The case $q = 2$ gives rise to optimal worst-case complexity bounds [34].

The directions used by the algorithm should promote feasible displacements. As the feasible region is polyhedral this can be achieved by selecting polling directions from tangent cones. However, the algorithm is not of active-set type and thus the iterates may get very close to the boundary but never lie at the boundary. In such a case, when no constraint is active, tangent cones promote all directions equally, not reflecting the proximity to the boundary. A possible fix is then to consider tangent cones corresponding to nearby active constraints as in [19, 20].

In this paper we will make use of concepts and properties from [19, 22] where the problem has been stated using a linear inequality formulation. To be able to apply them to problem (1), where the variables are also subject to the equality constraints $Ax = b$, we first consider the feasible region $\mathcal{F}$ reduced to the nullspace of A (by writing $x = \bar{x} + W\tilde{x}$ for a fix $\bar{x}$ such that $A\bar{x} = b$),

$$\tilde{\mathcal{F}} \;=\; \left\{\tilde{x} \in \mathbb{R}^{n-m} : \ell - A_{iq}\bar{x} \leq A_{iq}W\tilde{x} \leq u - A_{iq}\bar{x}\right\}, \tag{4}$$

where, again, W is an orthonormal basis for null(A). Then we define two index sets corresponding to approximate active inequality constraints/bounds,

Then by the fact that U^c is a uniform random subset of V^c and $|U^c| = \lceil p_c|V^c|\rceil$, we have

$$\begin{aligned}\Pr\left(\mathrm{cm}_{T^c}(U^c, -\tilde{G}) = \mathrm{cm}_{T^c}(V^c, -\tilde{G}) \;\middle|\; \sigma \cap \bar{\mathcal{E}}\right) \\ \geq\; \Pr\left(D^* \in U^c \;\middle|\; \sigma \cap \bar{\mathcal{E}}\right) \;=\; \frac{|U^c|}{|V^c|} \;\geq\; p_c.\end{aligned} \tag{40}$$

Let

$$\kappa_c \;=\; \lambda\left(\left\{C = \bar{T} \cap \bar{S}^{\perp} :\; \bar{T} \in \mathbf{T} \text{ and } \bar{S} \;\text{ is a subspace of } \bar{T}\right\}\right),$$

where $\lambda(\cdot)$ is defined as in (8) and $\mathbf{T}$ denotes all possible occurrences for the approximate tangent cone. Then $\kappa_c > 0$ and $\mathrm{cm}_{T^c}(V^c, -\tilde{G}) \geq \kappa_c$. Hence (40) implies

$$\Pr\left(\mathrm{cm}_{T^c}(U^c, -\tilde{G}) \geq \kappa_c \;\middle|\; \sigma \cap \bar{\mathcal{E}}\right) \;\geq\; p_c. \tag{41}$$

Finally, set

$$\kappa \;=\; \frac{1}{\sqrt{2}}\min\left\{\frac{\tau}{\sqrt{\tilde{n}}},\; \kappa_c\right\}, \quad p \;=\; \min\{p_s,\; p_c\}.$$

Then $\kappa \in (0, 1)$, $p \in (p_0, 1)$, and they fulfill (37) and (38) according to (39) and (41). Moreover, κ depends on the geometry of T, and consequently depends on m, n, and m_{iq}, while p depends solely on θ, γ, and p_c. The proof is then completed. ∎

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($\mathcal{E}$ only depends on the past iterations), we have

$$\begin{aligned}
&\Pr\left(\left\{\mathrm{cm}_S(U^s,-\tilde{G})\geq\sqrt{2}\kappa\right\}\cap\mathcal{E}\mid\sigma\right)\\
&=\Pr\left(\mathrm{cm}_S(U^s,-\tilde{G})\geq\sqrt{2}\kappa\mid\sigma\cap\mathcal{E}\right)\Pr(\mathcal{E}|\sigma),\\
&=\Pr\left(\mathrm{cm}_S(U^s,-\tilde{G})\geq\sqrt{2}\kappa\mid\sigma\cap\mathcal{E}\right)\mathbf{1}_{\mathcal{E}},\\
&\Pr\left(\left\{\mathrm{cm}_{T^c}(U^c,-\tilde{G})\geq\sqrt{2}\kappa\right\}\cap\bar{\mathcal{E}}\mid\sigma\right)\\
&=\Pr\left(\mathrm{cm}_{T^c}(U^c,-\tilde{G})\geq\sqrt{2}\kappa\mid\sigma\cap\bar{\mathcal{E}}\right)\Pr(\bar{\mathcal{E}}|\sigma)\\
&=\Pr\left(\mathrm{cm}_{T^c}(U^c,-\tilde{G})\geq\sqrt{2}\kappa\mid\sigma\cap\bar{\mathcal{E}}\right)\mathbf{1}_{\bar{\mathcal{E}}},
\end{aligned}$$

where $\sigma\cap\mathcal{E}$ is the trace σ-algebra [31] of $\mathcal{E}$ in σ, namely $\sigma\cap\mathcal{E}=\{\mathcal{A}\cap\mathcal{E}:\mathcal{A}\in\sigma\}$, and $\sigma\cap\bar{\mathcal{E}}$ is that of $\bar{\mathcal{E}}$. Hence it remains to prove that

$$\Pr\left(\mathrm{cm}_S(U^s,-\tilde{G})\geq\sqrt{2}\kappa\mid\sigma\cap\mathcal{E}\right)\ \geq\ p, \tag{37}$$

$$\Pr\left(\mathrm{cm}_{T^c}(U^c,-\tilde{G})\geq\sqrt{2}\kappa\mid\sigma\cap\bar{\mathcal{E}}\right)\ \geq\ p. \tag{38}$$

Let us examine $\mathrm{cm}_S(U^s,-\tilde{G})$ whenever $\mathcal{E}$ happens (which necessarily means that S is nonzero). Since $\mathcal{E}$ depends only on the past iterations, while U^s is essentially a set of r_s (recall (28)) *i.i.d.* directions from the uniform distribution on the unit sphere in $\mathbb{R}^s$ with $s=\dim(S)\leq\tilde{n}$, one can employ the theory of [16, Appendix B] to justify the existence of $p_s>p_0$ and $\tau>0$ that are independent of $\tilde{n}$ or s (solely depending on θ and γ) and satisfy

$$\Pr\left(\mathrm{cm}_S(U^s,-\tilde{G})\geq\frac{\tau}{\sqrt{\tilde{n}}}\ \middle|\ \sigma\cap\mathcal{E}\right)\ \geq\ p_s. \tag{39}$$

Now consider $\mathrm{cm}_{T^c}(U^c,-\tilde{G})$ under the occurrence of $\bar{\mathcal{E}}$ (in that case, T^c is nonzero and V^c is nonempty). Let D^* be a direction in V^c that achieves

$$\frac{D^{*\top}(-\tilde{G})}{\|D^*\|\|P_{T^c}[\tilde{G}]\|}\ =\ \mathrm{cm}_{T^c}(V^c,-\tilde{G}).$$

namely

$$\begin{aligned}
I_u(x,\alpha)\ &=\ \left\{i:|u_i-[A_{iq}\bar{x}]_i-[A_{iq}W\tilde{x}]_i|\leq\alpha\|W^\top A_{iq}^\top e_i\|\right\}\\
I_\ell(x,\alpha)\ &=\ \left\{i:|\ell_i-[A_{iq}\bar{x}]_i-[A_{iq}W\tilde{x}]_i|\leq\alpha\|W^\top A_{iq}^\top e_i\|\right\},
\end{aligned} \tag{5}$$

where $e_1,\ldots,e_{m_{iq}}$ are the coordinate vectors in $\mathbb{R}^{m_{iq}}$ and α is the step size in the algorithm. The indices in these sets contain the inequality constraints/bounds where the Euclidean distance from x to the corresponding boundary is less than or equal to α. Note that one can assume without loss of generality that $\|W^\top A_{iq}^\top e_i\|\neq 0$, otherwise given that we assume that $\mathcal{F}$ is nonempty, the inequality constraints/bounds $\ell_i\leq[A_{iq}x]_i\leq u_i$ would be redundant.

Now we consider an approximate tangent cone $T(x,\alpha)$ as if these inequality constraints/ bounds were active. This corresponds to considering a normal cone $N(x,\alpha)$ positively generated by the vectors

$$\{W^\top A_{iq}^\top e_i\}_{i\in I_u(x,\alpha)}\cup\{-W^\top A_{iq}^\top e_i\}_{i\in I_\ell(x,\alpha)} \tag{6}$$

and to take $T(x,\alpha)$ as the polar of $N(x,\alpha)$[1].

The feasible directions we are looking for are given by $W\tilde{d}$ with $\tilde{d}\in T(x,\alpha)$, as supported by the lemma below.

Lemma 2.1. *Let $x\in\mathcal{F}$, $\alpha>0$, and $\tilde{d}\in T(x,\alpha)$. If $\|\tilde{d}\|\leq\alpha$, then $x+W\tilde{d}\in\mathcal{F}$.*

Proof: First we observe that $x+W\tilde{d}\in\mathcal{F}$ is equivalent to $\tilde{x}+\tilde{d}\in\tilde{\mathcal{F}}$. Then we apply [19, Proposition 2.2] to the reduced formulation (4). ∎

This results also holds for displacements of norm α in the full space as $\|W\tilde{d}\|=\|\tilde{d}\|$. Hence, by considering steps of the form $\alpha W\tilde{d}$ with $\tilde{d}\in T(x,\alpha)$ and $\|\tilde{d}\|=1$ we are ensured to remain in the feasible region.

We point out that the previous definitions and the resulting analysis can be extended by replacing α in the definition of the nearby active inequality

[1]When developing the convergence theory in the deterministic case, the authors in [22, 19] have considered a formulation only involving linear inequality constraints. In [20] they have suggested to include linear equality constraints by adding to the positive generators of the approximate normal cone the transposed rows of A and their negatives, which in turn implies that the tangent one lies in the null space of A. In this paper we take an equivalent approach by explicitly considering the iterates in null(A). Not only this allows a more direct application of the theory in [22, 19] but it also renders the consideration of the particular cases (only bounds or only linear equalities) simpler.

constraints/bounds by a quantity of the order of α (that goes to zero when α does); see [19]. For matters of simplicity of the presentation we will work with α instead which was also the practical choice suggested in [20].

3. Feasible descent and deterministic global convergence

In the absence of constraints, all that is required for the set of polling directions is the existence of a descent direction, in other words,

$$\mathrm{cm}(D, -\nabla f(x)) \ \geq \ \kappa \quad \text{with} \quad \mathrm{cm}(D, v) \ = \ \max_{d \in D} \frac{d^\top v}{\|d\|\|v\|},$$

for some $\kappa > 0$. The quantity $\mathrm{cm}(D, v)$ is the cosine measure of D given v, introduced in [16] as an extension of the cosine measure of D [18] (and both measures are bounded away from zero for positive spanning sets D). To generalize this concept for problem (1), where the variables are subject to inequality constraints/bounds and equalities, we first consider the feasible region $\mathcal{F}$ in its reduced version (4), so that we can apply in the reduced space the concepts and properties of [19, 22] where the problem has been stated using a linear inequality formulation.

Let $\tilde{\nabla} f(x)$ be the gradient of f reduced to the null space of the matrix A defining the equality constraints, namely $\tilde{\nabla} f(x) = W^\top \nabla f(x)$. Given an approximate tangent cone $T(x, \alpha)$ and a set of directions $\tilde{D} \subset T(x, \alpha)$, we will use

$$\mathrm{cm}_{T(x,\alpha)}(\tilde{D}, -\tilde{\nabla} f(x)) \ \stackrel{\text{def}}{=} \ \max_{\tilde{d} \in \tilde{D}} \frac{\tilde{d}^\top (-\tilde{\nabla} f(x))}{\|\tilde{d}\| \left\| P_{T(x,\alpha)}[-\tilde{\nabla} f(x)] \right\|} \tag{7}$$

as the cosine measure of $\tilde{D}$ given $-\tilde{\nabla} f(x)$. If $P_{T(x,\alpha)}[-\tilde{\nabla} f(x)] = 0$, then we define the quantity in (7) as equal to 1. This cosine measure given a vector is motivated by the analysis given in [19] for the cosine measure (see Condition 1 and Proposition A.1 therein). It is clear that $|\,\mathrm{cm}_{T(x,\alpha)}(\tilde{D}, -\tilde{\nabla} f(x))| \leq 1$, and $\mathrm{cm}_{T(x,\alpha)}(\tilde{D}, -\tilde{\nabla} f(x))$ is close to 1 if $\tilde{D}$ contains a vector that is nearly in the direction of $-\tilde{\nabla} f(x)$.[2]

[2]We note that $\mathrm{cm}(\tilde{D}, -\tilde{\nabla} f(x))$ may well be quite small when $\mathrm{cm}_{T(x,\alpha)}(\tilde{D}, -\tilde{\nabla} f(x))$ is close to 1, as $\left\| P_{T(x,\alpha)}[-\tilde{\nabla} f(x)] \right\|$ can be arbitrarily small compared with $\|\tilde{\nabla} f(x)\|$.

for each κ. Indeed, when $\mathcal{E}$ happens, we have

$$\begin{aligned} \mathrm{cm}_T([U^s \ U^c], -\tilde{G}) \ &\geq \ \mathrm{cm}_T(U^s, -\tilde{G}) \\ = \ \frac{\|P_S[\tilde{G}]\|}{\|P_T[\tilde{G}]\|} \mathrm{cm}_S(U^s, -\tilde{G}) \ &\geq \ \frac{1}{\sqrt{2}} \mathrm{cm}_S(U^s, -\tilde{G}), \end{aligned}$$

and (33) is consequently true; when $\bar{\mathcal{E}}$ happens, then by the orthogonal decomposition

$$\|P_T[\tilde{G}]\|^2 \ = \ \|P_S[\tilde{G}]\|^2 + \|P_{T^c}[\tilde{G}]\|^2,$$

we know that $\|P_{T^c}[\tilde{G}]\|/\|P_T[\tilde{G}]\| \geq 1/\sqrt{2}$, and hence

$$\begin{aligned} \mathrm{cm}_T([U^s \ U^c], -\tilde{G}) \ &\geq \ \mathrm{cm}_{T^c}(U^c, -\tilde{G}) \\ = \ \frac{\|P_{T^c}[\tilde{G}]\|}{\|P_T[\tilde{G}]\|} \mathrm{cm}_{T^c}(U^c, -\tilde{G}) \ &\geq \ \frac{1}{\sqrt{2}} \mathrm{cm}_{T^c}(U^c, -\tilde{G}), \end{aligned}$$

which gives us (34). Since the events on the right-hand sides of (33) and (34) are mutually exclusive, the two inclusions lead us to

$$\begin{aligned} \Pr\left(\mathrm{cm}_T([U^s \ U^c], -\tilde{G}) \geq \kappa \,\middle|\, \sigma\right) \ \geq \ &\Pr\left(\left\{\mathrm{cm}_S(U^s, -\tilde{G}) \geq \sqrt{2}\kappa\right\} \cap \mathcal{E} \,\middle|\, \sigma\right) + \\ &\Pr\left(\left\{\mathrm{cm}_{T^c}(U^c, -\tilde{G}) \geq \sqrt{2}\kappa\right\} \cap \bar{\mathcal{E}} \,\middle|\, \sigma\right). \end{aligned}$$

Then it suffices to show the existence of $\kappa \in (0, 1)$ and $p \in (p_0, 1)$ that fulfill simultaneously

$$\Pr\left(\left\{\mathrm{cm}_S(U^s, -\tilde{G}) \geq \sqrt{2}\kappa\right\} \cap \mathcal{E} \,\middle|\, \sigma\right) \ \geq \ p\, \mathbf{1}_{\mathcal{E}}, \tag{35}$$

$$\Pr\left(\left\{\mathrm{cm}_{T^c}(U^c, -\tilde{G}) \geq \sqrt{2}\kappa\right\} \cap \bar{\mathcal{E}} \,\middle|\, \sigma\right) \ \geq \ p\mathbf{1}_{\bar{\mathcal{E}}}, \tag{36}$$

because $\mathbf{1}_{\mathcal{E}} + \mathbf{1}_{\bar{\mathcal{E}}} = 1$. If $\Pr(\mathcal{E}) = 0$, then (35) holds trivially regardless of κ or p. Similar things can be said about $\bar{\mathcal{E}}$ and (36). Therefore, we assume that both $\mathcal{E}$ and $\bar{\mathcal{E}}$ are of positive probabilities. Then, noticing that $\mathcal{E} \in \sigma$

inverse for $Q^\top$ and N is a matrix whose columns form a basis for the null space of $Q^\top$, then the set

$$Y \;=\; [\,-BR \quad BN \quad -BN\,] \tag{32}$$

positively generates $T(x,\alpha)$.

Note that the number of vectors in Y is then $n_R + 2(n_B - n_R) = 2n_B - n_R$, where n_B is the rank of B and n_R is that of Q (equal to number of columns of R). Since $n_B < \tilde{n}$, we have that $|Y| \leq 2\tilde{n}$.

In the case where $W_i^\top B$ is not full row rank, one could consider all subsets of columns of W_i of largest size that yield full row rank matrices, obtain the corresponding positive generators by Proposition A.1, and then take the union of all these sets [30]. Due to the combinatorial nature of this technique, we adopted a different approach, following the lines of [18, 20] where an algorithm originating from computational geometry called the double description method [13] was applied. We implemented this algorithm to compute a set of *extreme rays* (or positive generators) for a polyhedral cone of the form $\{\tilde{d} \in \mathbb{R}^{\tilde{n}} : B\tilde{d} = 0, C\tilde{d} \geq 0\}$, where we assume that $\mathrm{rank}(B) < \tilde{n}$, and applied it to the approximate tangent cone. Finally, we point out that in the experiments our three variants detected degeneracy (and thus invoked the double description method) on only less than an average of 2% of the iterations.

B. Proof of Proposition 7.1

Proof: To simplify the notation, we omit the index k in the proof. By our convention about $\mathrm{cm}_T(V, -\tilde{G})$ when $P_T[\tilde{G}] = 0$, we only need to consider the situation where $P_T[\tilde{G}]$ is nonzero.

Define the event

$$\mathcal{E} = \left\{ \frac{\|P[\tilde{G}]\|}{\|P_T[\tilde{G}]\|} \geq \frac{1}{\sqrt{2}} \right\}$$

and let $\bar{\mathcal{E}}$ denote its complementary event. We observe that

$$\left\{ \mathrm{cm}_T([U^s \;\; U^c], -\tilde{G}) \geq \kappa \right\} \;\supset\; \left\{ \mathrm{cm}_S(U^s, -\tilde{G}) \geq \sqrt{2}\kappa \right\} \cap \mathcal{E}, \tag{33}$$

$$\left\{ \mathrm{cm}_T([U^s \;\; U^c], -\tilde{G}) \geq \kappa \right\} \;\supset\; \left\{ \mathrm{cm}_{T^c}(U^c, -\tilde{G}) \geq \sqrt{2}\kappa \right\} \cap \bar{\mathcal{E}} \tag{34}$$

Given a finite set $\mathbf{C}$ of cones, where each cone $C \in \mathbf{C}$ is positively generated from a set of vectors $G(C)$, one knows from [22, Proposition 10.3] that

$$\lambda(\mathbf{C}) \;=\; \min_{C \in \mathbf{C}} \left\{ \inf_{\substack{u \in \mathbb{R}^{n-m} \\ P_C[u] \neq 0}} \max_{v \in G(C)} \frac{v^\top u}{\|v\| \|P_C[u]\|} \right\} \;>\; 0. \tag{8}$$

Thus, if $\tilde{D}$ positively generates $T(x,\alpha)$,

$$\mathrm{cm}_{T(x,\alpha)}(\tilde{D}) \;=\; \inf_{\substack{u \in \mathbb{R}^{n-m} \\ P_{T(x,\alpha)}[u] \neq 0}} \max_{\tilde{d} \in \tilde{D}} \frac{\tilde{d}^\top u}{\|\tilde{d}\| \left\| P_{T(x,\alpha)}[u] \right\|} \;\geq\; \kappa_{\min} \;>\; 0,$$

where $\kappa_{\min} = \lambda(\mathbf{T})$ and $\mathbf{T}$ is formed by all possible tangent cones $T(x,\varepsilon)$ (with some associated sets of positive generators), for all possible values of $x \in \mathcal{F}$ and $\varepsilon > 0$. Note that $\mathbf{T}$ is necessarily finite given that the number of constraints also is. This guarantees that the cosine measure (7) of such a $\tilde{D}$ given $-\tilde{\nabla} f(x)$ would be necessarily bounded away from zero.

We can now naturally impose a lower bound on (7) to guarantee the existence of a feasible descent direction in a given $\tilde{D}$. However, we will do it in the full space $\mathbb{R}^n$ by considering directions of the form $d = W\tilde{d}$ for $\tilde{d} \in \tilde{D}$.

Definition 3.1. *Let $x \in \mathcal{F}$ and $\alpha > 0$. Let D be a set of vectors in $\mathbb{R}^n$ of the form $\{d = W\tilde{d}, \tilde{d} \in \tilde{D}\}$ for some $\tilde{D} \subset T(x,\alpha) \subset \mathbb{R}^{n-m}$. Given $\kappa \in (0,1)$, the set D is κ-feasible descent in the approximate tangent cone $WT(x,\alpha)$ if*

$$\mathrm{cm}_{WT(x,\alpha)}(D, -\nabla f(x)) \;\stackrel{\mathrm{def}}{=}\; \max_{d \in D} \frac{d^\top(-\nabla f(x))}{\|d\| \|P_{WT(x,\alpha)}[-\nabla f(x)]\|} \;\geq\; \kappa, \tag{9}$$

where W is an orthonormal basis of the null space of A, and we assume by convention that the above quotient is equal to 1 if $\left\|P_{WT(x,\alpha)}[-\nabla f(x)]\right\| = 0$.

In fact, using both $P_{WT(x,\alpha)}[-\nabla f(x)] = P_{WT(x,\alpha)}[-WW^\top \nabla f(x)] = WP_{T(x,\alpha)}[-W^\top \nabla f(x)]$ and the fact that the Euclidean norm is preserved under multiplication by W, we note that

$$\mathrm{cm}_{WT(x,\alpha)}(D, -\nabla f(x)) \;=\; \mathrm{cm}_{T(x,\alpha)}(\tilde{D}, -W^\top \nabla f(x)), \tag{10}$$

which helps passing from the full to the reduced space. Definition 3.1 characterizes the polling directions of interest to the algorithm. Indeed, if D is a κ-feasible descent set, it contains at least one descent direction at x feasible for a displacement of length α (see Lemma 2.1). Furthermore, the size of κ controls how much away from the projected gradient such a direction is.

In the remaining of the section we will show that the algorithm is globally convergent to a first-order stationary point. There will be no novelty here as the result is the same as in [19], but there is a subtlety as we weaken the assumption in [19] when using polling directions which are feasible descent instead of positive generators. This relaxation will be instrumental when later we derive results based on probabilistic feasible descent, generalizing [16] from unconstrained to linearly constrained optimization.

Assumption 3.1. *The function f is bounded below on the feasible region $\mathcal{F}$ (and let $f_{\text{low}} > -\infty$ be a lower bound).*

It is well known that under the boundedness below of f the step size converges to zero for direct-search methods based on sufficient decrease [18]. This type of reasoning carries naturally from unconstrained to constrained optimization as long as the iterates remain feasible, and it is essentially based on the sufficient decrease condition imposed on successful iterates. We present the result in the stronger form of a convergence of a series, as the series limit will be later needed for complexity purposes. The proof is given in [16] for the unconstrained case but it applies verbatim to the feasible constrained setting.

Lemma 3.1. *Consider a run of Algorithm 2.1 applied to problem* (1) *under Assumptions 2.1, 2.2, and 3.1. Then, $\sum_{k=0}^{\infty} \rho(\alpha_k) < \infty$ (thus $\lim_{k\to\infty} \alpha_k = 0$).*

Lemma 3.1 is central to the analysis (and practical stopping criteria) of direct-search schemes. It is usually combined with a bound of the criticality measure in terms of the step size. For stating such a bound we need the following assumption, standard in the analysis of direct search based on sufficient decrease for smooth problems.

Assumption 3.2. *The function f is continuously differentiable with Lipschitz continuous gradient in an open set containing $\mathcal{F}$ (and let $\nu > 0$ be a Lipschitz constant of ∇f).*

As it will be clearer later, the treatment of the linearly constrained case also requires (see, e.g., [19]):

Assumption 3.3. *The gradient of f is bounded in norm in the feasible set, i.e., there exists $B_g > 0$ such that $\|\nabla f(x)\| \leq B_g$ for all $x \in \mathcal{F}$.*

insisting on always increasing the step size upon a success is not the best approach at least in the unconstrained case.

9. Conclusions

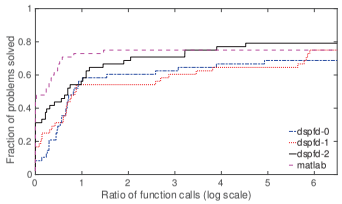
We have shown how to prove global convergence with probability one for direct-search methods applied to linearly constrained optimization when the polling directions are randomly generated. We have also derived a worst-case analysis for the number of iterations and function evaluations. Such worst-case complexity bounds are established with overwhelming probability and are of the order of ϵ^{-2}, where $\epsilon > 0$ is the threshold of criticality. It was instrumental for such probabilistic analyzes to extend the concept of probabilistic descent from unconstrained to linearly constrained optimization. We have also refined such bounds in terms of the problem dimension and number of constraints for two specific subclasses of problems, where it is easier to exhibit an improvement over deterministic direct search. The numerical behavior of our probabilistic strategies was found coherent with those findings, outperforming the one of classical deterministic direct searches.

A natural extension of the presented work is the treatment of general, nonlinear constraints. One possibility is to apply the augmented Lagrangian method, where a sequence of subproblems containing all the possible original linear constraints is solved by direct search, as it was done in [23] for the deterministic case. Although it is unclear at the moment whether the extension of the probabilistic analysis would be straightforward, it represents an interesting perspective of the present work. Another avenue focuses on the linearization of the constraints, see [26], and adapting the random generation techniques of the polling directions to this setting would also represent a continuation of our research.

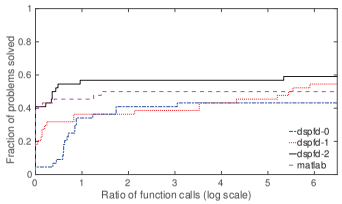
A. Deterministic computation of positive cone generators

Provided a certain linear independence condition is satisfied, the positive generators of the approximate tangent cone can be computed as follows (see [20, Proposition 5.2]).

Proposition A.1. *Let $x \in \mathcal{F}$, $\alpha > 0$, and assume that the set of generators of $N(x, \alpha)$ has the form $[\,W_e \;\; -W_e \;\; W_i\,]$. Let B be a basis for the null space of $W_e^\top$, and suppose that $W_i^\top B = Q^\top$ has full row rank. If R is a right*

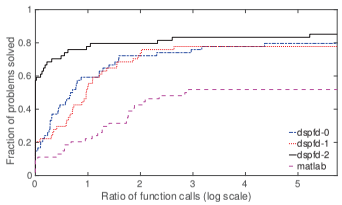


(a) $\varepsilon = 10^{-3}$.

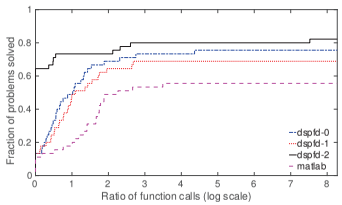


(b) $\varepsilon = 10^{-6}$.

FIGURE 4. Performance of three variants of Algorithm 2.1 versus MATLAB `patternsearch` on problems with at least one linear inequality constraint.



(a) $\varepsilon = 10^{-3}$.



(b) $\varepsilon = 10^{-6}$.

FIGURE 5. Performance of three variants of Algorithm 2.1 versus MATLAB `patternsearch` on problems with at least one linear equality constraint.

the potential benefit of randomly generating in subspaces. Although not plotted here, this conclusion is even more visible for the 13 problems with only linear equality constraints where **dspfd-2** is by far the best method, which does not come as a surprise given what was reported for the unconstrained case [16].

We have also run variant **dspfd-0** with $\gamma_2 = 1$ (i.e., no step-size increase at successful iterations), and although this has led to an improvement of efficiency for bound-constrained problems, the relative position of the profiles did not seem much affected in the general linearly constrained setting. We recall that **dspfd-0** configures a case of deterministic direct search from the viewpoint of the choice of the polling directions, and that it is known that

The next lemma shows that the criticality measure is of the order of the step size for unsuccessful iterations under our assumption of feasible descent. It is a straightforward extension of what can be proved using positive generators [19] but later fundamental in the probabilistic setting.

Lemma 3.2. *Consider a run of Algorithm 2.1 applied to problem* (1) *under Assumptions 2.1 and 3.2. Let D_k be κ-feasible descent in the approximate tangent cone $WT(x_k, \alpha_k)$. Then, denoting $T_k = T(x_k, \alpha_k)$ and $g_k = \nabla f(x_k)$, if the k-th iteration is unsuccessful,*

$$\left\| P_{T_k}[-W^\top g_k] \right\| \leq \frac{1}{\kappa} \left(\frac{\nu}{2} \alpha_k + \frac{\rho(\alpha_k)}{\alpha_k} \right). \tag{11}$$

Proof: The result clearly holds whenever the left-hand side in (11) is equal to zero, therefore we will assume for the rest of the proof that $P_{T_k}[-W^\top g_k] \neq 0$ (T_k is thus non-empty). From (10) we know that $\mathrm{cm}_{T_k}(\tilde{D}_k, -W^\top g_k) \geq \kappa$ and thus there exists a $\tilde{d}_k \in \tilde{D}_k$ such that

$$\frac{\tilde{d}_k^\top [-W^\top g_k]}{\|\tilde{d}_k\| \, \|P_{T_k}[-W^\top g_k]\|} \geq \kappa.$$

On the other hand, from Lemma 2.1 and Assumption 2.1, we also have $x_k + \alpha_k W \tilde{d}_k \in \mathcal{F}$. Hence, using the fact that k is the index of an unsuccessful iteration, followed by a Taylor expansion,

$$\begin{aligned} -\rho(\alpha_k) \leq f(x_k + \alpha_k W \tilde{d}_k) - f(x_k) &\leq \alpha_k \tilde{d}_k^\top W^\top g_k + \frac{\nu}{2} \alpha_k^2 \\ &\leq -\kappa \alpha_k \|\tilde{d}_k\| \|P_{T_k}[-W^\top g_k]\| + \frac{\nu}{2} \alpha_k^2, \end{aligned}$$

which leads to (11). ∎

In order to state a result involving the criticality measure $\chi(x_k)$, one needs to also bound the projection of the reduced gradient onto the polar of $T(x_k, \alpha_k)$. As in [19], one uses the following uniform bound (derived from polyhedral geometry) on the normal component of a feasible vector.

Lemma 3.3. [19, Proposition B.1] *Let $x \in \mathcal{F}$ and $\alpha > 0$. Then, for any vector $\tilde{d}$ such that $x + W\tilde{d} \in \mathcal{F}$, one has*

$$\left\| P_{N(x,\alpha)}[\tilde{d}] \right\| \leq \frac{\alpha}{\eta_{\min}}, \tag{12}$$

where $\eta_{\min} = \lambda(\mathbf{N})$ *and* $\mathbf{N}$ *is formed by all possible approximate normal cones* $N(x,\varepsilon)$ *(generated positively by the vectors in (6)) for all possible values of* $x \in \mathcal{F}$ *and* $\varepsilon > 0$.

We remark that the definition of $\eta_{\min}$ is independent of x and α.

Lemma 3.4. *Consider a run of Algorithm 2.1 applied to problem* (1) *under Assumptions 2.1, 3.2, and 3.3. Let* D_k *be* κ*-feasible descent in the approximate tangent cone* $WT(x_k, \alpha_k)$*. Then, if the* k*-th iteration is unsuccessful,*

$$\chi(x_k) \;\leq\; \left[\frac{\nu}{2\kappa} + \frac{B_g}{\eta_{\min}}\right]\alpha_k + \frac{\rho(\alpha_k)}{\kappa\alpha_k}. \tag{13}$$

Proof: We make use of the classical Moreau decomposition [29], which states that any vector $v \in \mathbb{R}^{n-m}$ can be decomposed as $v = P_{T_k}[v] + P_{N_k}[v]$ with $N_k = N(x_k, \alpha_k)$ and $P_{T_k}[v]^\top P_{N_k}[v] = 0$, and write

$$\begin{aligned}
\chi(x_k) &= \max_{\substack{x_k + W\tilde{d} \in \mathcal{F} \\ \|\tilde{d}\| \leq 1}} \left(\tilde{d}^\top P_{T_k}[-W^\top g_k] + \left(P_{T_k}[\tilde{d}] + P_{N_k}[\tilde{d}]\right)^\top P_{N_k}[-W^\top g_k]\right) \\
&\leq \max_{\substack{x_k + W\tilde{d} \in \mathcal{F} \\ \|\tilde{d}\| \leq 1}} \left(\tilde{d}^\top P_{T_k}[-W^\top g_k] + P_{N_k}[\tilde{d}]^\top P_{N_k}[-W^\top g_k]\right) \\
&\leq \max_{\substack{x_k + W\tilde{d} \in \mathcal{F} \\ \|\tilde{d}\| \leq 1}} \left(\|\tilde{d}\|\|P_{T_k}[-W^\top g_k]\| + \|P_{N_k}[\tilde{d}]\|\|P_{N_k}[-W^\top g_k]\|\right).
\end{aligned}$$

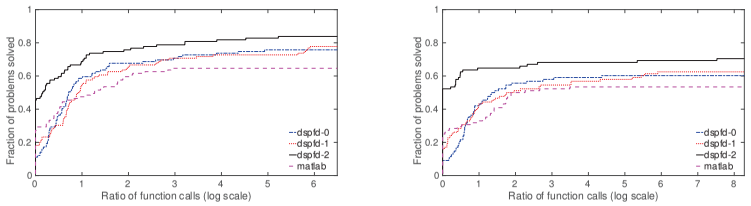
We bound the first term in the maximum using $\|\tilde{d}\| \leq 1$ and Lemma 3.2. For the second term, we apply Lemma 3.3 together with Assumption 3.3, leading to

$$\|P_{N_k}[\tilde{d}]\|\|P_{N_k}[-W^\top g_k]\| \;\leq\; \frac{\alpha_k}{\eta_{\min}} B_g,$$

yielding the desired conclusion. ∎

The use of feasible descent enables us then to establish a global convergence result. Note that one can easily show from Lemma 3.1 that there must exist an infinite sequence of unsuccessful iterations with step size converging to zero, to which one then applies Lemma 3.4 and concludes the following result.

Theorem 3.1. *Consider a run of Algorithm 2.1 applied to problem* (1) *under Assumptions 2.1, 2.2, 3.1, 3.2, and 3.3. Suppose that* D_k *is* κ*-feasible descent*



(a) $\varepsilon = 10^{-3}$. (b) $\varepsilon = 10^{-6}$.

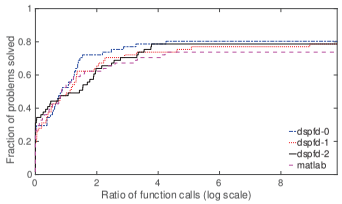
FIGURE 3. Performance of three variants of Algorithm 2.1 versus MATLAB `patternsearch` on problems subject to general linear constraints.

both take a high number of directions per iteration. On the contrary, **dspfd-1** and **dspfd-2**, having a lower iteration cost thanks to randomness, present better profiles, and clearly outperform `patternsearch`. These results concur with those of Section 7, as they show the superiority of **dspfd-2** when the size of the problem is significantly higher than the number of nearby bounds.

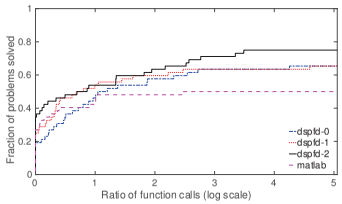
8.2. Linearly constrained problems. Our second experiment considers a benchmark of 106 `CUTEst` problems for which at least one linear constraint (other than a bound) is present. The dimensions vary from 2 to 96, while the number of linear inequalities (when present) lies between 1 and 2000 (with only 14 problems with more than 100 of those constraints).

Figure 3 presents the results of our approaches and of the MATLAB function on those problems. Variant **dspfd-2** significantly outperforms the other variant, in both efficiency and robustness. The other **dspfd** variants seem more expensive in that they rely on a possibly larger set of tangent cone generators; yet, they manage to compete with `patternsearch` in terms of robustness.

We further present the results for two sub-classes of these problems. Figure 4 restricts the profiles to the 48 `CUTEst` problems for which at least one linear inequality constraint was enforced on the variables. The MATLAB routine performs quite well on these problems; still, **dspfd-2** is competitive and more robust. Figure 5 focuses on the 61 problems for which at least one equality constraint is present (and note that 3 problems have both linear equalities and inequalities). In this context, the **dspfd-2** profile highlights

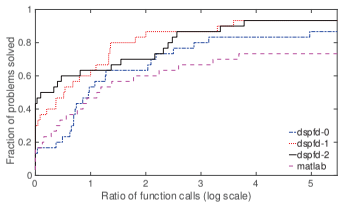


(a) $\varepsilon = 10^{-3}$.

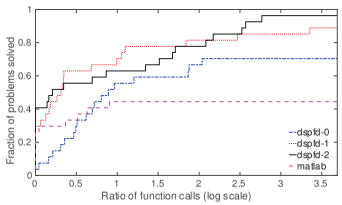


(b) $\varepsilon = 10^{-6}$.

FIGURE 1. Performance of three variants of Algorithm 2.1 versus MATLAB `patternsearch` on bound-constrained problems.



(a) $\varepsilon = 10^{-3}$.



(b) $\varepsilon = 10^{-6}$.

FIGURE 2. Performance of three variants of Algorithm 2.1 versus MATLAB `patternsearch` on larger bound-constrained problems.

variables, with dimensions varying between 2 and 20. Figure 1 presents the results obtained while comparing our three variants with the built-in `patternsearch` function. One observes that the **dspfd-2** variant has the highest percentage of problems on which it is the most efficient (i.e., the highest curve leaving the y-axis). In terms of robustness (large value of the ratio of function calls), the **dspfd** methods outperform `patternsearch`, with **dspfd-0** and **dspfd-2** emerging as the best variants.

To further study the impact of the random generation, we selected 31 problems and increased their dimensions so that all problems had at least 10 variables, with fifteen of them having between 20 and 52 variables. Figure 2 presents the corresponding profiles. One sees that the performance of **dspfd-0** and of the MATLAB function has significantly deteriorated, as they

in the approximate tangent cone $WT(x_k, \alpha_k)$ *for all* k. *Then,*

$$\liminf_{k\to\infty} \chi(x_k) \;=\; 0. \tag{14}$$

4. Complexity in the deterministic case

In this section, our goal is to provide an upper bound on the number of iterations and function evaluations sufficient to achieve

$$\min_{0\le l\le k} \chi(x_l) \;\le\; \epsilon, \tag{15}$$

for a given threshold $\epsilon > 0$. Such complexity bounds have already been obtained for derivative-based methods addressing linearly constrained problems. In particular, it was shown that adaptive cubic regularization methods [5] take $\mathcal{O}(\epsilon^{-1.5})$ iterations to satisfy (15) in the presence of second-order derivatives. Higher-order regularization algorithms require $\mathcal{O}(\epsilon^{-(q+1)/q})$ iterations provided derivatives up to the q-th order are used [4]. A short-step gradient algorithm was also proposed in [6] to address general equality-constrained problems requiring $\mathcal{O}(\epsilon^{-2})$ iterations to reduce the corresponding criticality measure below ϵ.

A worst-case complexity bound on the number of iterations taken by direct search based on sufficient decrease for linearly constrained problems can be derived based on the simple evidence that the methods share the iteration mechanism of the unconstrained case [34]. In the remaining of this section, we will assume that $\rho(\alpha) = c\,\alpha^2/2$, as $q = 2$ is the power q in $\rho(\alpha) = \text{constant}\times\alpha^q$ that leads to the least negative power of ϵ in the complexity bounds for the unconstrained case [34].

In fact, since feasibility is maintained, the sufficient decrease condition for accepting new iterates is precisely the same as for unconstrained optimization. Moreover, Lemma 3.4 gives us a bound on the criticality measure for unsuccessful iterations that only differs (from the unconstrained case) in the multiplicative constants. So, we can state the complexity for the linearly constrained case in Theorem 4.1, and refer the reader to [34] for a proof which is verbatim the same with $\|g_k\|$ replaced by $\chi(x_k)$ up to the multiplicative constants.

Theorem 4.1. *Consider a run of Algorithm 2.1 applied to problem* (1) *under the assumptions of Theorem 3.1. Suppose that* D_k *is* κ*-feasible descent in the approximate tangent cone* $WT(x_k, \alpha_k)$ *for all* k. *Suppose further that*

$\rho(\alpha) = c\,\alpha^2/2$. *Then, the first index* k_ϵ *satisfying* (15) *is such that*

$$k_\epsilon \;\leq\; \left\lceil E_1 \epsilon^{-2} + E_2 \right\rceil,$$

where

$$E_1 \;=\; (1 - \log_\theta(\gamma)) \, \frac{f(x_{k_0}) - f_{\text{low}}}{0.5\theta^2 L_1^2} - \log_\theta(\exp(1)),$$

$$E_2 \;=\; \log_\theta \left(\frac{\theta L_1 \exp(1)}{\alpha_{k_0}} \right) + \frac{f(x_0) - f_{\text{low}}}{0.5\alpha_0^2},$$

$$L_1 \;=\; \min(1, L_2^{-1}), \quad L_2 \;=\; \mathcal{C} \;=\; \frac{\nu + c}{2\kappa} + \frac{B_g}{\eta_{\min}}.$$

and k_0 *is the index of the first unsuccessful iteration (assumed* $\leq k_\epsilon$*). The constants* $\mathcal{C}, \kappa, \eta_{\min}$ *depend on* n, m, m_{iq}*:* $\mathcal{C} = \mathcal{C}_{n,m,m_{iq}}$, $\kappa = \kappa_{n,m,m_{iq}}$, $\eta_{\min} = \eta_{n,m,m_{iq}}$.

Under the assumptions of Theorem 4.1, the number of iterations k_ϵ and the number of function evaluations k_ϵ^f sufficient to meet (15) satisfy

$$k_\epsilon \;\leq\; \mathcal{O}\left(\mathcal{C}_{n,m,m_{iq}}^2 \epsilon^{-2} \right), \quad k_\epsilon^f \;\leq\; \mathcal{O}\left(r_{n,m,m_{iq}} \mathcal{C}_{n,m,m_{iq}}^2 \epsilon^{-2} \right), \tag{16}$$

where $r_{n,m,m_{iq}}$ is a uniform upper bound on $|D_k|$. The dependence of $\kappa_{n,m,m_{iq}}$ and $\eta_{n,m,m_{iq}}$ (and consequently of $\mathcal{C}_{n,m,m_{iq}}$ which is $\mathcal{O}(\max\{\kappa_{n,m,m_{iq}}^{-2}, \eta_{n,m,m_{iq}}^{-2}\})$) in terms of the numbers of variables n, equality constraints m, and inequality constraints/bounds m_{iq} is not straightforward. Indeed, those quantities depend on the polyhedral structure of the feasible region, and can significantly differ from one problem to another. Regarding $r_{n,m,m_{iq}}$, in the presence of a non-degenerate condition such as the one of Proposition A.1, one can say that $r_{n,m,m_{iq}} \leq 2n$ (see the sentence after this proposition in Appendix A). In the general possibly degenerate case, the combinatorial aspect of the faces of the polyhedral may lead to an $r_{n,m,m_{iq}}$ that depends exponentially on n (see [30]).

Only bounds. When only bounds are enforced on the variables ($m = 0$, $m_{iq} = n$, $A_{iq} = W = I_n$), the numbers κ, η, r depend only on n, and the set $D_\oplus = [I_n \;\; -I_n]$ formed by the coordinate directions and their negatives is the preferred choice for the polling directions. In fact, given $x \in \mathcal{F}$ and $\alpha > 0$, the cone $T(x, \alpha)$ defined in Section 2 is always generated by a set $G \subset D_\oplus$, while its polar $N(x, \alpha)$ will be generated by $D_\oplus \setminus G$. The set of all possible occurrences for $T(x, \alpha)$ and $N(x, \alpha)$ thus coincide, and it can be shown (see [21] and [18, Proposition 8.1]) that $\kappa_n \geq \kappa_{\min} = 1/\sqrt{n}$ as well

the largest subspace when there are either only bounds or only linear constraints, and benefits in the general case from the construction described in Proposition A.1. To determine the approximate active linear inequalities or bounds in (5), we used $\min\{10^{-3}, \alpha_k\}$ instead of α_k — as we pointed out in Section 3, our analysis can be trivially extended to this setting. The forcing function was $\rho(\alpha) = 10^{-4}\alpha^2$.

In our experiments, we also used the built-in MATLAB `patternsearch` function [27] for comparison. This algorithm is a direct-search-type method that accepts a point as the new iterate if it satisfies a *simple decrease* condition, i.e., provided the function value is reduced. The options of this function can be set so that it uses a so-called Generating Set Search strategy inspired from [18]: columns of $D_\oplus$ are used for bound-constrained problems, while for linearly constrained problems the algorithm attempts to compute positive generators of an approximate tangent cone based on the technique of Proposition A.1 (with no provision for degeneracy).

For all methods including the MATLAB one, we set $\alpha_0 = 1$, $\gamma = 2$, and $\theta = 0.5$. We adopted the `patternsearch` default settings by allowing infeasible points to be evaluated as long as the constraint violation does not exceed 10^{-3} (times the norm of the corresponding row of A_{iq}). All methods rely upon the built-in MATLAB `null` function to compute the orthogonal basis W when needed. We ran all algorithms on a benchmark of problems from the CUTEst collection [15], stopping whenever a budget of $2000n$ function evaluations was consumed or the step size fell below $10^{-6}\alpha_0$ (for our methods based on random directions, ten runs were performed and the mean of the results was used). For each problem, this provided a best obtained value f_{best}. Given a tolerance $\varepsilon \in (0, 1)$, we then looked at the number of function evaluations needed by each method to reach an iterate x_k such that

$$f(x_k) - f_{best} \;<\; \varepsilon \, (f(x_0) - f_{best}), \tag{31}$$

where the run was considered as a failure if the method stopped before (31) was satisfied. Performance profiles [10, 28] were built to compare the algorithms, using the number of function evaluations as a performance indicator (normalized for each problem in order to remove the dependence on the dimension).

8.1. Bound-constrained problems. We first present results on 63 problems from the `CUTEst` collection that only enforce bound constraints on the

Appendix B) and η by $1/\sqrt{n}$ so that (25) becomes

$$\Pr\left(K_\epsilon^f \leq \left(\left\lfloor \log_2\left(1-\frac{\ln\theta}{\ln\gamma}\right)\right\rfloor + 1 + n_b\right)\left\lceil \frac{\bar{\mathcal{C}} n\epsilon^{-2}}{p-p_0}\right\rceil\right) \geq 1-\exp\left[-\frac{\bar{\mathcal{C}}(p-p_0)\epsilon^{-2}}{8p}\right] \tag{30}$$

for some positive constant $\bar{\mathcal{C}}$ independent of n_b and n. The resulting bound is $\mathcal{O}\left(n_b n\epsilon^{-2}\right)$. If n_b is substantially smaller than n, then this bound represents an improvement over those obtained in Sections 4 and 6, reflecting the fact that a small number of bounds on the variables brings the problem closer to an unconstrained one.

Improving the complexity when there are only linear equalities. In this setting, our subspace generation technique is essentially the same as the one for unconstrained optimization [16]. We can see that (25) renders an improvement from the bound $\mathcal{O}((n-m)^2\epsilon^{-2})$ of Section 6 to $\mathcal{O}((n-m)\epsilon^{-2})$ (also with overwhelming probability), which is coherent with the bound for the unconstrained case obtained in [16].

8. Numerical results

This section illustrates the practical impact of our probabilistic strategies. We implemented Algorithm 2.1 in MATLAB with the following choices of polling directions. The first choice corresponds to randomly selecting a subset of the positive generators of the approximate tangent cone (**dspfd-1**, see Section 6). In the second one we first try to identify a subspace of the cone, ignore the corresponding positive generators, and then randomly generate directions in that subspace (**dspfd-2**, see Section 7). As a term of comparison, we also tested a version based on the complete set of positive generators, but randomly ordering them at each iteration (**dspfd-0**) — such a variant can be analyzed in the classic deterministic way despite the random order. The three variants require the computation of a set of positive generators for the approximate tangent cone, a process we describe in Appendix A. For variant **dspfd-2**, subspaces are detected by identifying opposite vectors in the set of positive generators: this forms our set V_k^s (following the notation of Section 7, and we obtain V_k^c by orthogonalizing the remaining positive generators with respect to those in V_k^s. Such a technique always identifies

as $\eta_n = \eta_{\min} = 1/\sqrt{n}$. In particular, $D_\oplus$ is $(1/\sqrt{n})$-feasible descent in the approximate tangent cone $WT(x,\alpha)$ for all possible values of x and α. Since $r_n \leq 2n$, one concludes from (16) that $\mathcal{O}(n^2\epsilon^{-2})$ evaluations are taken in the worst case, matching the known bound for unconstrained optimization.

Only linear equalities. When there are no inequality constraints/bounds on the variables and only equalities ($m > 0$), the approximate normal cones $N(x,\alpha)$ are empty and $T(x,\alpha) = \mathbb{R}^{n-m}$, for any feasible x and any $\alpha > 0$. Thus, $\kappa_{n,m} \geq 1/\sqrt{n-m}$ and by convention $\eta_{n,m} = \infty$ (or $B_g/\eta_{n,m}$ could be replaced by zero). Since $r_{n,m} \leq 2(n-m)$, one concludes from (16) that $\mathcal{O}((n-m)^2\epsilon^{-2})$ evaluations are taken in the worst case. A similar bound [34] would be obtained by first rewriting the problem in the reduced space and then considering the application of deterministic direct search (based on sufficient decrease) on the unconstrained reduced problem — and we remark that the factor of $(n-m)^2$ cannot be improved [9] using positive spanning vectors.

5. Probabilistic feasible descent and almost-sure global convergence

We now consider that the polling directions are independently randomly generated from some distribution in $\mathbb{R}^n$. Algorithm 2.1 will then represent a realization of the corresponding generated stochastic process. We will use $X_k, G_k, \mathfrak{D}_k, A_k$ to represent the random variables corresponding to the k-th iterate, gradient, set of polling directions, and step size, whose realizations are respectively denoted by $x_k, g_k = \nabla f(x_k), D_k, \alpha_k$ (as in the previous sections of the paper).

The first step towards the analysis of the corresponding randomized algorithm is to pose the feasible descent property in a probabilistic form. This is done below in Definition 5.1, generalizing the definition of probabilistic descent given in [16] for the unconstrained case.

Definition 5.1. *Given $\kappa, p \in (0,1)$, the sequence $\{\mathfrak{D}_k\}$ in Algorithm 2.1 is said to be* p-probabilis- tically κ-feasible descent*, or* (p,κ)-feasible descent *in short, if*

$$\Pr\left(\mathrm{cm}_{WT(x_0,\alpha_0)}(\mathfrak{D}_0, -G_0) \geq \kappa\right) \geq p, \tag{17}$$

and, for each $k \geq 1$,

$$\Pr\left(\mathrm{cm}_{WT(X_k,A_k)}(\mathfrak{D}_k, -G_k) \geq \kappa \middle| \mathfrak{D}_0, \ldots, \mathfrak{D}_{k-1}\right) \geq p. \tag{18}$$

Observe that at iteration $k \geq 1$, the probability (18) involves conditioning on the σ-algebra generated by $\mathfrak{D}_0, \ldots, \mathfrak{D}_{k-1}$, expressing that the feasible descent property is to be ensured with probability at least p regardless of the outcome of iterations 0 to $k-1$.

As in the deterministic case of Section 3, the results from Lemmas 3.1 and 3.4 will form the necessary tools to now establish global convergence with probability one. Lemma 3.1 states that for every realization of Algorithm 2.1, thus for every realization of the random sequence $\{\mathfrak{D}_k\}$, the step size sequence converges to zero. Lemma 3.4 is first rewritten in its reciprocal form.

Lemma 5.1. *Consider Algorithm 2.1 applied to problem* (1)*, and under Assumptions 2.1, 3.2, and 3.3. The k-th iteration in Algorithm 2.1 is successful if*

$$\mathrm{cm}_{WT(x_k,\alpha_k)}(D_k, -g_k) \;\geq\; \kappa \quad \textit{and} \quad \alpha_k \;<\; \varphi(\kappa\chi(x_k)), \tag{19}$$

where

$$\varphi(t) \;\stackrel{\text{def}}{=}\; \inf\left\{\alpha : \alpha > 0, \frac{\rho(\alpha)}{\alpha} + \left[\frac{\nu}{2} + \frac{\kappa B_g}{\eta}\right]\alpha \geq t\right\}. \tag{20}$$

Note that this is exactly [16, Lemma 2.1] but with $\mathrm{cm}_{WT(x_k,\alpha_k)}(D_k, -g_k)$ and $\chi(x_k)$ in the respective places of $\mathrm{cm}(D_k, -g_k)$ and $\|g_k\|$.

For any $k \geq 0$, let Y_k be the indicator function of the event

$$\{\text{the } k\text{th iteration is successful}\}$$

and Z_k be the indicator function of the event

$$\left\{\mathrm{cm}_{WT(X_k,A_k)}(\mathfrak{D}_k, -G_k) \geq \kappa\right\},$$

with y_k, z_k denoting their respective realizations. One can see from Theorem 4.1 that, if the feasible descent property is satisfied at each iteration, that is $z_k = 1$ for each k, then Algorithm 2.1 is guaranteed to converge in the sense that $\liminf_{k\to\infty} \chi(x_k) = 0$. Conversely, if $\liminf_{k\to\infty} \chi(x_k) > 0$, then it is reasonable to infer that $z_k = 1$ did not happen sufficiently often during the iterations. This is the intuition behind the following lemma which does not assume any a priori property on the sequence $\{\mathfrak{D}_k\}$.

Lemma 5.2. *Under the assumptions of Lemmas 3.1 and 5.1, it holds for the stochastic processes $\{\chi(X_k)\}$ and $\{Z_k\}$ that*

$$\left\{\liminf_{k\to\infty} \chi(X_k) > 0\right\} \;\subset\; \left\{\sum_{k=0}^{\infty} [Z_k \ln\gamma + (1 - Z_k)\ln\theta] = -\infty\right\}. \tag{21}$$

Let U_k^s be a set of r_s vectors generated on the unit sphere of S_k, where

$$r_s \;=\; \left\lfloor \log_2\left(1 - \frac{\ln\theta}{\ln\gamma}\right)\right\rfloor + 1. \tag{28}$$

and U_k^c be a set of $\lceil p_c|V_k^c|\rceil$ vectors chosen uniformly at random within V_k^c such that

$$p_0 \;<\; p_c \;<\; 1. \tag{29}$$

Then, there exist $\kappa \in (0,1)$ and $p \in (p_0, 1)$, with p_0 given in (22), such that

$$\Pr\left(\mathrm{cm}_{T_k}([U_k^s\ U_k^c], -\tilde{G}_k) \geq \kappa \middle| \sigma_{k-1}\right) \;\geq\; p,$$

where σ_{k-1} represents the σ-algebra generated by $\mathfrak{D}_0, \ldots, \mathfrak{D}_{k-1}$.

The constant κ depend on n, m, m_{iq}: $\kappa = \kappa_{n,m,m_{iq}}$, while p depends solely on θ, γ, p_c.

In the procedure presented in Proposition 7.1, we conventionally set $U_k^s = \emptyset$ when S_k is zero, and $U_k^c = \emptyset$ when T_k^c is zero. As a corollary of Proposition 7.1, the sequence of polling directions $\{\mathfrak{D}_k = W[U_k^s\ U_k^c]\}$ corresponding to the subspace exploration is (p,κ)-feasible descent. By Theorem 5.1, such a technique guarantees almost-sure convergence and it also falls into the assumptions of Theorem 6.1, thereby admitting a general complexity bound in terms of iterations and evaluations. Contrary to Section 6, we can now show improvement in specific instances of linearly constrained problems.

Improving the complexity when there are only bounds. Suppose that the cones T_k always include an $(n - n_b)$-dimensional subspace with $n_b \geq 1$ (this is the case if only $n_b < n$ problem variables are actually subject to bounds). Then, using[3] $|V_k^c| \leq n_b$, the number of polling directions to be generated per iteration varies between r_s and $r_s + p_c n_b$, where r_s given in (28) does not depend on n nor on n_b, and p_c is a constant in $(p_0, 1)$. This implies that we can replace r by $r_s + n_b$, κ by $t/\sqrt{n}$ with $t \in (0, 1]$ independent of n (see

[3] V_k^c corresponds to the coordinate vectors in T_k for which their negatives do not belong to V_k, and hence the corresponding variables must be subject to bound constraints. Since there only are n_b bound constraints, one has $0 \leq |V_k^c| \leq n_b$.

Only linear equalities. As in the bound-constrained case, one can also here randomly select the polling directions from $[W \ -W]$ (of cardinal $2(n-m)$), leading to a complexity bound of $\mathcal{O}((n-m)^2\epsilon^{-2})$ with overwhelming probability. However, if instead we uniformly randomly generate on the unit sphere of $\mathbb{R}^{n-m}$ (and post multiply by W), the complexity bound reduces to $\mathcal{O}((n-m)\epsilon^{-2})$, a fact that translates directly from unconstrained optimization [16] (see also the argument given in Section 7 as this corresponds to explore the null space of A).

7. Randomly generating directions while exploring subspaces

A random generation procedure for the polling directions in the approximate tangent cone should explore possible subspace information contained in the cone. In fact, if a subspace is contained in the cone, one can generate the polling directions as in the unconstrained case (with as few as two directions within the subspace), leading to significant gains in numerical efficiency as we will see in Section 8.

In this section we will see that such a procedure still yields a condition sufficient for global convergence and worst-case complexity. For this purpose, consider the k-th iteration of Algorithm 2.1. To simplify the presentation, we will set $\tilde{n} = n - m$ for the dimension of the reduced space, $\tilde{g}_k = W^\top g_k$ for the reduced gradient, and T_k for the approximate tangent cone. Let S_k be a linear subspace included in T_k and let $T_k^c = T_k \cap S_k^\perp$ be the portion of T_k orthogonal to S_k. Note that any set of positive generators for T_k can be transformed into a set $V_k = [V_k^s \ V_k^c]$, with V_k,V_k^s,V_k^c being sets of positive generators for T_k,S_k,T_k^c, respectively. Proposition 7.1 describes a procedure to compute a probabilistically feasible descent set of directions by generating random directions on the unit sphere of S_k and by randomly selecting positive generators from V_k^c (in the latter case as it was already mentioned in Section 6 for the bound-constrained case). Its proof is left to Appendix B. Details on how to compute S_k and V_k^c are given in the next section.

Proposition 7.1. *Consider an iteration of Algorithm 2.1 and let T_k be the associated approximate tangent cone. Suppose that T_k contains a linear subspace S_k and let $T_k^c = T_k \cap S_k^\perp$. Let V_k^c be a set of positive generators for T_k^c.*

Proof: The proof is identical to that of [16, Lemma 3.2], using $\chi(x_k)$ instead of $\|g_k\|$. ∎

Our goal is then to refute with probability one the occurrence of the event on the right-hand side of (21). To this end, we ask the algorithm to always increase the step size in successful iterations ($\gamma > 1$), and we suppose that the sequence $\{\mathfrak{D}_k\}$ is (p_0, κ)-feasible descent, with

$$p_0 \ = \ \frac{\ln \theta}{\ln(\gamma^{-1}\theta)}. \tag{22}$$

Under this condition, one can proceed as in [3, Theorem 4.1] to show that the random process

$$\sum_{l=0}^{k} [Z_l \ln \gamma + (1 - Z_l) \ln \theta]$$

is a submartingale with bounded increments. Such sequences have a zero probability of diverging to $-\infty$ ([3, Theorem 4.2]), thus the right-hand event in (21) also happens with probability zero. This finally leads to the following almost-sure global convergence result.

Theorem 5.1. *Consider Algorithm 2.1 applied to problem* (1)*, with $\gamma > 1$, and under Assumptions 2.1, 2.2, 3.1, 3.2, and 3.3. If $\{\mathfrak{D}_k\}$ is (p_0, κ)-feasible descent, then*

$$\Pr\left(\liminf_{k \to \infty} \chi\left(X_k\right) = 0 \right) \ = \ 1. \tag{23}$$

The minimum probability p_0 is essential for applying the martingale arguments that ensure convergence. Note that it depends solely on the constants θ and γ in a way directly connected to the updating rules of the step size.

6. Complexity in the probabilistic case

The derivation of an upper bound for the effort taken by the probabilistic variant to reduce the criticality measure below a tolerance also follows closely its counterpart for unconstrained optimization [16]. As in the deterministic case, we focus on the most favorable forcing function $\rho(\alpha) = c\,\alpha^2/2$, which renders the function $\varphi(t)$ defined in (20) linear in t,

$$\varphi(t) \ = \ \left[\frac{\nu + c}{2} + \frac{B_g \kappa}{\eta} \right]^{-1} t.$$

A complexity bound for probabilistic direct search is given in Theorem 6.1 below for the linearly constrained case, generalizing the one proved in [16, Corollary 4.2] for unconstrained optimization. The proof is exactly the same but with $\mathrm{cm}_{WT(X_k,A_k)}(D_k,-G_k)$ and $\chi(X_k)$ in the respective places of $\mathrm{cm}(D_k,-G_k)$ and $\|G_k\|$. Let us quickly recall the road map of the proof. First, it is derived a bound on the probability $\Pr\left(\min_{0\leq l\leq k}\chi(X_l)\leq\epsilon\right)$ in terms of a probability involving $\sum_{l=0}^{k-1}Z_l$ [16, Lemma 4.3]. Assuming that the set of polling directions is (p,κ)-feasible descent with $p>p_0$, a Chernoff-type bound is then obtained for the tail distribution of such a sum, yielding a lower bound on the probability $\Pr\left(\min_{0\leq l\leq k}\chi(X_l)\leq\epsilon\right)$ [16, Theorem 4.1]. By expressing the event $\min_{0\leq l\leq k}\chi(X_l)\leq\epsilon$ as $K_\epsilon\leq k$, where K_ϵ is the random variable counting the number of iterations performed until the satisfaction of the approximate optimality criterion (15), one reaches an upper bound on K_ϵ of the order of ϵ^{-2}, holding with overwhelming probability (more precisely, with probability at least $1-\exp(\mathcal{O}(\epsilon^{-2}))$; see [16, Theorem 4.2 and Corollary 4.2]). We point out that in the linearly constrained setting the probability p of feasible descent may depend on n, m, and m_{iq}, and hence we will write $p=p_{n,m,m_{iq}}$.

Theorem 6.1. *Consider Algorithm 2.1 applied to problem* (1)*, with $\gamma>1$, and under the assumptions of Theorem 5.1. Suppose that $\{\mathfrak{D}_k\}$ is (p,κ)-feasible descent with $p>p_0$. Suppose also that $\rho(\alpha)=c\,\alpha^2/2$ and that $\epsilon>0$ satisfies*

$$\epsilon\ \leq\ \frac{\mathcal{C}\alpha_0}{2\gamma}.$$

Then, the first index K_ϵ for which (15) *holds satisfies*

$$\Pr\left(K_\epsilon\leq\left\lceil\frac{\beta\mathcal{C}^2}{c(p-p_0)}\epsilon^{-2}\right\rceil\right)\ \geq\ 1-\exp\left[-\frac{\beta(p-p_0)\mathcal{C}^2}{8cp}\epsilon^{-2}\right],\qquad(24)$$

where $\beta=\frac{2\gamma^2}{c(1-\theta)^2}\left[\frac{c}{2}\gamma^{-2}\alpha_0^2+f_0-f_{\mathrm{low}}\right]$ is an upper bound on $\sum_{k=0}^{\infty}\alpha_k^2$ (see [16, Lemma 4.1]*). The constants $\mathcal{C},\kappa,p$ depend on n,m,m_{iq}: $\mathcal{C}=\mathcal{C}_{n,m,m_{iq}}$, $\kappa=\kappa_{n,m,m_{iq}}$, $p=p_{n,m,m_{iq}}$.*

Let K_ϵ^f be the random variable counting the number of function evaluations performed until satisfaction of the approximate optimality criterion (15).

From Theorem 6.1, we have:

$$\Pr\left(K_\epsilon^f\leq r_{n,m,m_{iq}}\left\lceil\frac{\beta\mathcal{C}_{n,m,m_{iq}}^2}{c(p_{n,m,m_{iq}}-p_0)}\epsilon^{-2}\right\rceil\right)$$
$$\geq\ 1-\exp\left[-\frac{\beta(p_{n,m,m_{iq}}-p_0)\mathcal{C}_{n,m,m_{iq}}^2}{8c\,p_{n,m,m_{iq}}}\epsilon^{-2}\right].\qquad(25)$$

As $\mathcal{C}_{n,m,m_{iq}}^2=\mathcal{O}(\max\{\kappa_{n,m,m_{iq}}^{-2},\eta_{n,m,m_{iq}}^{-2}\})$, and emphasizing the dependence on the numbers of variables n, equality constraints m, and linear inequality/bounds m_{iq}, one can finally assure with overwhelming probability that

$$K_\epsilon^f\leq\mathcal{O}\left(\frac{r_{n,m,m_{iq}}\max\{\kappa_{n,m,m_{iq}}^{-2},\eta_{n,m,m_{iq}}^{-2}\}}{p_{n,m,m_{iq}}-p_0}\epsilon^{-2}\right).\qquad(26)$$

Only bounds. Using the simpler setting where there are only bounds on the variables ($m=0$, $m_{iq}=n$, $A_{iq}=W=I_n$), we will show now that probabilistic direct search does not yield a better complexity bound when a certain subset of positive generators of $T(x_k,\alpha_k)$ is randomly selected for polling at each iteration. Without loss of generality, we reason around the worst case $T(x_k,\alpha_k)=\mathbb{R}^n$. Suppose that instead of selecting the entire set $D_\oplus$, we restrict ourselves to a subset of uniformly chosen $\lceil 2np\rceil$ elements, with p being a constant in $(p_0,1)$. Then the corresponding sequence of polling directions is $(p,1/\sqrt{n})$-feasible descent (this follows an argument included in the proof of Proposition 7.1; see (40) in Appendix B). The complexity result (25) can then be refined by setting $r_n=\lceil 2np\rceil$, $\kappa_n\geq\kappa_{\min}=1/\sqrt{n}$, and $\eta_n=\kappa_{\min}=1/\sqrt{n}$, leading to

$$\Pr\left(K_\epsilon^f\leq\lceil 2np\rceil\left\lceil\frac{\bar{\mathcal{C}}n\epsilon^{-2}}{p-p_0}\right\rceil\right)\ \geq\ 1-\exp\left[-\frac{\bar{\mathcal{C}}(p-p_0)\epsilon^{-2}}{8p}\right],\qquad(27)$$

for some positive constant $\bar{\mathcal{C}}$ independent of n. This bound on K_ϵ^f is thus $\mathcal{O}(n^2\epsilon^{-2})$ (with overwhelming probability), showing that such a random strategy does not lead to an improvement over the deterministic setting. To achieve such an improvement we will need to uniformly generate directions on the unit sphere of the subspaces contained in $T(x_k,\alpha_k)$ as it will be shown in Section 7.