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FREELY GENERATED n-CATEGORIES, COINSERTERS AND PRESENTATIONS OF LOW DIMENSIONAL CATEGORIES

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ABSTRACT: Composing with the inclusion $\mathsf{Set} \to \mathsf{Cat}$, a graph G internal to Set becomes a graph of discrete categories, the coinserter of which is the category freely generated by G. Introducing a suitable definition of n-computad, we show that a similar approach gives the n-category freely generated by an n-computad. Suitable n-categories with relations on n-cells are presented by these $(n+1)$-computads, which allows us to prove results on presentations of thin groupoids and thin categories. So motivated, we introduce a notion of deficiency of (a presentation of) a groupoid via computads and prove that every small connected thin groupoid has deficiency 0. We compare the resulting notions of deficiency and presentation with those induced by monads. In particular, we find our notion of group deficiency to coincide with the classical one. Finally, we study presentations of 2-categories via 3-computads, focusing on locally thin groupoidal 2-categories. Under suitable hypotheses, we give efficient presentations of some locally thin and groupoidal 2-categories. A fundamental tool is a 2-dimensional analogue of the association of a "topological graph" to every graph internal to Set. Concretely, we construct a left adjoint $\mathcal{F}_{\mathsf{Top}_2} : 2\text{-}\mathsf{cmp} \to \mathsf{Top}$ associating a 2-dimensional CW-complex to each small 2-computad. Given a 2-computad $\mathfrak{g}$, the groupoid it presents is equivalent to the fundamental groupoid of $\mathcal{F}_{\mathsf{Top}_2}(\mathfrak{g})$. Finally, we sketch the 3-dimensional version $\mathcal{F}_{\mathsf{Top}_3}$.

KEYWORDS: 2-categories, higher categories, computads, weighted (co)limits, coinserter, icons, homotopy groups, fundamental groupoid, Euler characteristic, graphs, rewriting systems, 2-monads, CW-complexes, deficiency of groups, deficiency of algebras, deficiency of groupoids.
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Introduction

The category of small categories cat is monadic over the category of small graphs grph. The left adjoint $\mathcal{F}_1 : \mathsf{grph} \to \mathsf{cat}$ is defined as follows: $\mathcal{F}_1(G)$ has the same objects of G and the morphisms between two objects are lists of

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composable arrows in G between them. The composition of such morphisms is defined by juxtaposition of composable lists and the identities are the empty lists.

Recall that a small graph is a functor $G : \mathfrak{G}^{\mathrm{op}} \to \mathsf{Set}$ in which $\mathfrak{G}$ is the category with two objects and two parallel morphisms between them. If we compose G with the inclusion $\mathsf{Set} \to \mathsf{cat}$, we get a diagram $G' : \mathfrak{G}^{\mathrm{op}} \to \mathsf{cat}$. The benefit of this perspective is that the category freely generated by G is the *coinserter* of G', which is a type of (weighted) 2-colimit introduced in [17].

In the higher dimensional context, we have as primary structures the so called *n-computads*, firstly introduced for dimension 2 in [31]. There are further developments of the theory of computads [29, 5, 19, 27, 13], including generalizations such as in [1] and the proof of the monadicity of the category of the strict ω-categories over the category of ω-computads in [20].

In this paper, we give a concise definition of the classical (strict) n-computad such that the (strict) n-category freely generated by a computad is the coinserter of this computad. More precisely, we define an n-computad as a graph of $(n-1)$-categories satisfying some properties (given in Remark 8.12) and, then, we demonstrate that the n-category freely generated by it is just the coinserter of this graph composed with the inclusion $(n-1)$-$\mathsf{Cat} \to n\mathsf{Cat}$, getting in this way the free n-category functor whose induced monad is denoted by $\overline{\mathcal{F}_n}$. More generally, we show that this approach works for an n-dimensional analogue of the notion of *derivation scheme* introduced in [34].

Since we are talking about coinserters, we of course consider a 2-category of n-categories. Instead of n-natural transformations, we have to consider the n-dimensional analogues of icons introduced in [23]. We get, then, 2-categories $n\mathsf{Cat}$ of n-categories, n-functors and n-icons. In dimension 2, this allows to get the bicategories freely generated by a computad as a coinserter of the 2-category Bicat of bicategories, pseudofunctors and icons as well.

Every monad $\mathcal{T}$ on a category $\mathfrak{X}$ induces a notion of presentation of algebras given in Definition 3.1, which we refer as $\mathcal{T}$-presentation. If, furthermore, $\mathfrak{X}$ has a strong notion of measure μ of objects, we also get a (possibly naive) notion of deficiency (of a presentation) of a $\mathcal{T}$-algebra induced by μ (given in Remark 6.16). In the case of algebras over Set (together with *cardinality of sets*) given in 6.1, we get the classical notions of deficiency of a (presentation of a) finitely presented group, deficiency of a (presentation of a) finitely presented monoid and dimension of a finitely presented vector space.

Higher computads also give notions of presentations of higher categories. More precisely, using the description of n-computads of this paper, the coequalizer of an n-computad $\mathfrak{g} : \mathfrak{G}^{\mathrm{op}} \to (n-1)\text{-}\mathsf{Cat}$, denoted by $\mathcal{P}_{(n-1)}(\mathfrak{g})$, is what we call the $(n-1)$-category presented by this n-computad in which the n-cells of the computad correspond to "relations of the presentation". In this context, an n-computad gives a presentation of an $(n-1)$-category with only equations between $(n-1)$-cells.

We show that every presentation of $(n-1)$-categories via n-computads are indeed particular cases of $\overline{\mathcal{F}_n}$-presentations. Moreover, on one hand, the notion of $\overline{\mathcal{F}_1}$-presentation of a monoid does not coincide with the (classical) notion of $\overline{\mathcal{F}_0}$-presentation, since there are $\overline{\mathcal{F}_1}$-presentations that are not $\overline{\mathcal{F}_0}$-presentations of a monoid. On the other hand, the notion of presentation of a monoid via computads does coincide with the classical one.

We present, then, the topological aspects of this theory. In order to do so, we construct two particular adjunctions. Firstly, we give the construction in Remark 5.1 of the left adjoint functor $\mathcal{F}_{\mathsf{Top}_1} : \mathsf{Grph} \to \mathsf{Top}$ which gives the "topological graph" associated to each graph $\mathfrak{G}^{\mathrm{op}} \to \mathsf{Set}$ via a topological enriched version of the coinserter. Secondly, we show how the usual concatenation of continuous paths in a topological space gives rise to a monad functor/morphism $\overline{\mathcal{F}_1} \longrightarrow \overline{\mathcal{F}_{\mathsf{Top}_1}}$ between the free category monad and the monad induced by the left adjoint $\mathcal{F}_{\mathsf{Top}_1}$. Finally, using this monad morphism, we construct a left adjoint functor $\mathcal{F}_{\mathsf{Top}_2} : 2\text{-}\mathsf{cmp} \to \mathsf{Top}$.

The adjunction $\mathcal{F}_{\mathsf{Top}_2} \dashv \mathcal{C}_{\mathsf{Top}_2}$ gives a way of describing the fundamental groupoid of a topological space: $\mathcal{C}_{\mathsf{Top}_2}(X)$ presents the fundamental groupoid of X. More precisely, it is clear that $\mathcal{P}_1\mathcal{C}_{\mathsf{Top}_2}(X) \cong \Pi(X)$ which we adopt as the definition of the fundamental groupoid of a topological space X in Section 5.

Denoting by $\mathcal{L}_1 : \mathsf{cat} \to \mathsf{gr}$ the functor left adjoint to the inclusion of the category of small groupoids into the category of small categories, we show that the fundamental groupoid of a graph is equivalent to the groupoid freely generated by this graph, proving that there is a natural transformation which is objectwise an equivalence between $\mathcal{L}_1\mathcal{F}_1$ and $\Pi\mathcal{F}_{\mathsf{Top}_1} \cong \mathcal{P}_1\mathcal{C}_{\mathsf{Top}_2}\mathcal{F}_{\mathsf{Top}_1}$. We also show that, given a small 2-computad $\mathfrak{g}$, there is an equivalence $\mathcal{P}_1\overline{\mathcal{F}_{\mathsf{Top}_2}}(\mathfrak{g}) \simeq \mathcal{L}_1\mathcal{P}_1(\mathfrak{g})$, which means that there is an equivalence between the fundamental groupoid of the CW-complex/topological space associated to a small 2-computad $\mathfrak{g}$ and the groupoid presented by $\mathfrak{g}$.

In the context of presentation of groups via computads, the left adjoint functor $\mathcal{F}_{\mathsf{Top}_2}$ formalizes the usual association of each classical ($\overline{\mathcal{L}_0\mathcal{F}_0}$-)presentation of a group G with a 2-dimensional CW-complex X such that $\pi_1(X) \cong G$.

We study freely generated categories and presentations of categories via computads, focusing on the study of thin groupoids and thin categories. By elementary results on Euler characteristic of CW-complexes, the results on $\mathcal{F}_{\mathsf{Top}_2}$ described above imply in Theorem 6.7 which, together with Theorem 6.13, motivate the definition of deficiency of a groupoid (w.r.t. presentations via groupoidal computads). We compare this notion of deficiency with the previously presented ones: for instance, in Remark 6.16, we compare with the notion of deficiency induced by the free groupoid monad $\overline{\mathcal{L}_1\mathcal{F}_1}$ together with the "measure" Euler characteristic, while in Propostion 6.11 we show that the classical concept of deficiency of groups coincides with the concept of deficiency of the suspension of a group w.r.t. presentations via computads.

By Theorem 6.13 and Theorem 6.7, the deficiency of thin "finitely generated" groupoids are 0, what generalizes the elementary fact that the trivial group has deficiency 0. Moreover, this implies that Theorem 6.13 gives efficient presentations, meaning that it has the least number of 2-cells (equations) of the finitely presented thin groupoids.

We lift some of these results to presentations of thin categories and give some further aspects of presentations of thin categories as well. Finally, supported by these results and the characterization of thin categories that are free $\overline{\mathcal{F}_1}$-algebras, we give comments towards the deficiency of a thin finitely presented category, considering a naive generalization of the concept of deficiency of groupoid introduced previously.

The final topic of this paper is the study of presentations of locally thin 2-categories via 3-computads. Similarly to the 1-dimensional case, we firstly describe aspects of freely generated 2-categories, including straightforward sufficient conditions to conclude that a given (locally thin) 2-category is not a free $\overline{\mathcal{F}_2}$-algebra. We conclude that there are interesting locally thin 2-categories that are not free, what gives a motivation to study presentations of locally thin 2-categories. In order to study such locally thin 2-categories, we study presentations of some special locally thin $(2,0)$-categories which are, herein, 2-categories with only invertible cells. With suitable conditions, we can lift such presentations to presentations of locally thin and groupoidal 2-categories.

We also give a sketch of the construction of a left adjoint functor $\mathcal{F}_{\mathsf{Top}_3}$: $3\text{-}\mathsf{cmp} \to \mathsf{Top}$ which allows us to give a result (Corollary 10.16) towards a 3-dimensional version of Theorem 6.7. This result shows that the presentations of $(2,0)$-categories given previously have the least number of 3-cells (equations): they are efficient presentations.

In [24, 25, 26], we introduce 2-dimensional versions of bicategorical replacements of the category $\dot{\Delta}'_3$ of the ordinals $\mathsf{0}$, $\mathsf{1}$, $\mathsf{3}$ and order-preserving functions between them without nontrivial morphisms $\mathsf{3} \to \mathsf{n}$. We apply our theory to give an efficient presentation of the bicategorical replacement of the category $\dot{\Delta}_2$ and study the presentation of the locally groupoidal 2-category $\dot{\Delta}_{\mathrm{Str}}$ introduced in [25].

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1. Preliminaries

The most important hypothesis is that $\mathsf{Cat}, \mathsf{CAT}$ are cartesian closed categories of categories such that Cat is an internal category of CAT. So, herein, a *category* $\mathfrak{X}$ means an object of CAT. Moreover, if X, Y are objects of Cat, we denote by $\mathsf{Cat}[X, Y]$ its internal hom and by $\mathsf{Cat}(X, Y)$ the *discrete category* of functors between X and Y. We also assume that the category of sets Set is an object of Cat. The category of *small categories* is $\mathsf{cat} := \mathsf{int}(\mathsf{Set})$, that is to say, cat is the category of internal categories of the category of sets.

If V is a symmetric monoidal closed category, we denote by $V\text{-}\mathsf{Cat}$ the category of V-enriched categories. We refer the reader to [16, 9, 3] for *enriched categories* and *weighted limits*. It is important to ratify that herein the collection of objects of a V-category X of $V\text{-}\mathsf{Cat}$ is a discrete category in Cat, while $V\text{-}\mathsf{cat}$ denotes the category of small V-categories.

Inductively, we define the category $n\text{-}\mathsf{Cat}$ by $(n+1)\text{-}\mathsf{Cat} := (n\text{-}\mathsf{Cat})\text{-}\mathsf{Cat}$ and $1\text{-}\mathsf{Cat} := \mathsf{Cat}$. Therefore there are full inclusions $(n+1)\text{-}\mathsf{Cat} \to \mathsf{int}(n\text{-}\mathsf{Cat})$ and $n\text{-}\mathsf{Cat}$ is cartesian closed, being an $(n+1)$-category which is not an object of $(n+1)\text{-}\mathsf{Cat}$. In particular, Cat is a 2-category which is not an object of $2\text{-}\mathsf{Cat}$.

We deal mainly with weighted limits in the Cat-enriched context, the so called 2-categorical limits. The basic references are [31, 17]. Let $\mathcal{W} : \mathfrak{S} \to \mathsf{Cat}$, $\mathcal{W}' : \mathfrak{S}^{\mathrm{op}} \to \mathsf{Cat}$ and $\mathcal{D} : \mathfrak{S} \to \mathfrak{A}$ be 2-functors with a small domain. If it exists, we denote the *weighted limit* of $\mathcal{D}$ with weight $\mathcal{W}$ by $\{\mathcal{W}, \mathcal{D}\}$. Dually, we denote by $\mathcal{W}' * \mathcal{D}$ the *weighted colimit* provided that it exists. Recall that, by definition, there is a 2-natural isomorphism (in X)

$$\mathfrak{A}(\mathcal{W}' * \mathcal{D}, X) \cong [\mathfrak{S}^{\mathrm{op}}, \mathsf{Cat}]\,(\mathcal{W}', \mathfrak{A}(\mathcal{D}-, X)) \cong \{\mathcal{W}', \mathfrak{A}(\mathcal{D}-, X)\}$$

in which $[\mathfrak{S}^{\mathrm{op}}, \mathsf{Cat}]$ denotes the 2-category of 2-functors $\mathfrak{S}^{\mathrm{op}} \to \mathsf{Cat}$, 2-natural transformations and modifications.

In the last section, we apply 2-monad theory to construct 2-categories $n\mathsf{Cat}$ for each natural number n. We refer the reader to [4] for the basics of 2-monad theory. The category $n\mathsf{Cat}$ is one of the possible higher dimensional analogues of the 2-category of 2-categories, 2-functors and icons introduced in [23].

The category $\dot{\Delta}$ is the *category of finite ordinals*, denoted by $\mathsf{0}, \mathsf{1}, \mathsf{2}, \ldots, \mathsf{n}, \ldots$, and order-preserving functions between them. We denote by Δ the full *subcategory of nonempty ordinals*. There are full inclusions $\Delta \to \dot{\Delta} \to \mathsf{cat} \to \mathsf{Cat}$. Often, we use n also to denote its image by these inclusions. Thereby the category n is the category

$$\mathsf{0} \to \mathsf{1} \to \cdots \to \mathsf{n-1}.$$

For each n of Δ, the n-*truncated category of* $\dot{\Delta}$, denoted by $\dot{\Delta}_{\mathsf{n}}$, is the full subcategory of $\dot{\Delta}$ with only $\mathsf{0}, \mathsf{1}, \ldots, \mathsf{n}$ as objects. The truncated category Δ_{n} is analogously defined. For instance, the category Δ_{2} is generated by the faces d^0, d^1 and by the degeneracy s^0 as follows:

$$\begin{tikzcd} \mathsf{1} \arrow[r, shift left=2, "d^1"] \arrow[r, shift right=2, "d^0"'] & \mathsf{2} \arrow[l, "s^0" description] \end{tikzcd}$$

in which, after composing with the inclusions $\Delta_{\mathsf{2}} \to \Delta \to \mathsf{Cat}$, d^0 and d^1 are respectively the inclusions of the codomain and the domain of morphism $\mathsf{0} \to \mathsf{1}$ of $\mathsf{2}$.

Moreover, the *category* $\mathfrak{G}$ is, herein, the subcategory of Δ_{2} without the degeneracy $\mathsf{2} \to \mathsf{1}$ and with all the faces $\mathsf{1} \to \mathsf{2}$ of Δ_{2} as it is shown below. Again, considering $\mathfrak{G}$ as a subcategory of Cat, d^1 is the inclusion of the domain and d^0 is the inclusion of the codomain.

$$\begin{tikzcd} \mathsf{1} \arrow[r, shift left, "d^1"] \arrow[r, shift right, "d^0"'] & \mathsf{2} \end{tikzcd}$$

We denote by $\mathcal{I} : \mathfrak{G} \to \mathsf{Cat}$ the inclusion given by the composition of the inclusions $\mathfrak{G} \to \Delta_{\mathsf{2}} \to \mathsf{Cat}$. The 2-functor $\mathcal{I}$ defines the weight of the limits called *inserters*, while $\mathfrak{I}$-weighted colimits are called *coinserters*. Also, we have the weight $\mathfrak{U}_1\mathcal{L}_1\mathfrak{I} : \mathfrak{G} \to \mathsf{Cat}$ which gives the notions of *isoinserter* and *isocoinserter*, defined as follows:

$$1 \rightrightarrows \nabla 2$$

in which $\nabla 2$ is the category with two objects and one isomorphism between them and $\mathfrak{U}_1\mathcal{L}_1\mathfrak{I}(d^0), \mathfrak{U}_1\mathcal{L}_1\mathfrak{I}(d^1)$ are the inclusions of the two different objects.

Let 2_2 be the 2-category below with two parallel nontrivial 1-cells and only one nontrivial 2-cell between them. We define the weight J_{2_2} by

$$\mathrm{J}_{2_2} : 2_2 \to \mathsf{Cat}$$
$$\begin{tikzcd} * \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & * \end{tikzcd} \longmapsto \begin{tikzcd} 1 \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \nabla 2, \end{tikzcd}$$

in which the image of the 2-cell is the only possible natural isomorphism between the inclusion of the domain and the inclusion of the codomain. The J_{2_2}-weighted colimits are called *coinverters*.

Finally, let $\mathfrak{G}_2$ be the 2-category with two parallel nontrivial 1-cells and only two parallel nontrivial 2-cells between them. We define the weight $\mathrm{J}_{\mathfrak{G}_2}$ by

$$\mathrm{J}_{\mathfrak{G}_2} : \mathfrak{G}_2 \to \mathsf{Cat}$$
$$\begin{tikzcd} * \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\,\Downarrow"] & * \end{tikzcd} \longmapsto \begin{tikzcd} 1 \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & 2, \end{tikzcd}$$

in which the images of the 2-cells are the only possible natural transformation between the inclusion of the domain and the inclusion of the codomain. The $\mathrm{J}_{\mathfrak{G}_2}$-weighted colimits are called *coequifiers*.

1.1. Thin Categories and Groupoids. A category X is a groupoid if every morphism of X is invertible. The 2-category of groupoids of Cat is denoted by Gr. The inclusion $\mathfrak{U}_1 : \mathsf{Gr} \to \mathsf{Cat}$ has a left 2-adjoint $\mathcal{L}_1$. Also, the category of *locally groupoidal 2-categories* is, by definition, Gr-Cat and the previous adjunction induces a left adjoint $\mathcal{L}_2$ to the inclusion $\mathfrak{U}_2 : \mathsf{Gr}\text{-}\mathsf{Cat} \to 2\text{-}\mathsf{Cat}$.

Definition 1.1. [Connected Category] A category X of Cat is *connected* if every object of $\mathcal{L}_1(X)$ is weakly terminal. In particular, a groupoid Y is connected if and only if every object of Y is weakly terminal.

A category X of Cat is *thin* if between any two objects of X there is at most one morphism. Again, we can consider *locally thin 2-categories*, which are categories enriched over the category of thin categories of Cat. We denote by Prd the category of thin categories. The inclusion $M_1 : \mathsf{Prd} \to \mathsf{Cat}$ has a left 2-adjoint $\mathcal{M}_1$. Again, it induces a left adjoint $\mathcal{M}_2$ to the inclusion $\mathsf{Prd\text{-}Cat} \to \mathsf{2\text{-}Cat}$.

Remark 1.2. The 2-functors $\mathcal{U}_1 : \mathsf{Gr} \to \mathsf{Cat}, M_1 : \mathsf{Prd} \to \mathsf{Cat}$ are 2-monadic and the 2-monads induced by them are idempotent, since $\mathcal{U}_1, M_1$ are fully faithful. Therefore $\mathcal{U}_1, M_1$ create 2-limits.

The functor $\mathcal{U}_1$ is left adjoint: hence, as $\mathcal{U}_1$ is monadic, $\mathcal{U}_1$ creates coequalizers and coproducts. But it does not preserve tensor with 2. Finally, Prd is isomorphic to the 2-category of categories enriched over 2 and, hence, it is 2-cocomplete.

Proposition 1.3. *Let X be an object of Gr or Cat. We have that X is a thin category if and only if X is (isomorphic to) the coequifier of*

$$\begin{tikzcd} (\mathfrak{G}^{\mathrm{op}} \pitchfork X)_0 \arrow[r, shift left] \arrow[r, shift right] \arrow[r, phantom, "\alpha\Downarrow\ \beta\Downarrow"] & X \end{tikzcd}$$

in which $\mathsf{Cat}(\mathfrak{G}^{\mathrm{op}}, X) \cong (\mathfrak{G}^{\mathrm{op}} \pitchfork X)_0$ is the discrete category of internal graphs of X, $\alpha_G = G(d^0)$ and $\beta_G = G(d^1)$.

Theorem 1.4. *There are categories X, Y in Cat such that $\mathcal{L}_1(X)$ and Y are thin, but X and $\mathcal{L}_1(Y)$ are not thin. In particular, $\mathcal{L}_1$ is not faithful.*

Proof: For instance, we define Y to be the category generated by the graph

$$\begin{tikzcd} * \arrow[r] \arrow[rrr, bend left] & * & * \arrow[l] \arrow[r] & * \end{tikzcd} \qquad (\textit{example of weak tree})$$

in which there is no nontrivial composition and X can be defined as

$$\begin{tikzcd} * \arrow[r, "h"] & * \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & * \end{tikzcd}$$

satisfying the equation $fh = gh$. ■

A category X satisfies the *cancellation law* if every morphism of X is a monomorphism and an epimorphism.

Theorem 1.5. *If X satisfies the cancellation law and $\mathcal{L}_1(X)$ is a thin groupoid, then X is a thin category.*

Proof: The components of the unit on the categories that satisfy the cancellation law of the adjunction $\mathcal{L}_1 \dashv \mathcal{U}_1$ are faithful. Thereby, if X satifies the cancellation law and $\mathcal{L}_1(X)$ is thin, X is thin. ■

Theorem 1.6. *Let X be an object of Prd or Cat. X is a groupoid if X is the coinverter of*

$$\begin{tikzcd} (2 \pitchfork X)_0 \arrow[r, shift left] \arrow[r, shift right] \arrow[r, phantom, "\alpha\Downarrow"] & X \end{tikzcd}$$

in which $(2\pitchfork X)_0$ *is the discrete category of morphisms in* X *and* $\alpha_f = f$*.*

Remark 1.7. As a consequence, since $\mathcal{M}_1$ preserves 2-colimits and, for each Y in Prd, the induced functor $\mathsf{Prd}((2\pitchfork\mathcal{M}_1(X))_0, Y) \to \mathsf{Prd}(\mathcal{M}_1(2\pitchfork X)_0, Y)$ is fully faithful, $\mathcal{M}_1$ preserves groupoids.

2. Graphs

We start studying aspects of graphs and freely generated categories. An *internal graph* of a category $\mathfrak{X}$ is a functor $G : \mathfrak{G}^{\mathrm{op}} \to \mathfrak{X}$, while the *category of graphs internal to* $\mathfrak{X}$, denoted by $\mathsf{Grph}(\mathfrak{X})$, is the category of functors and natural transformations $\mathsf{CAT}[\mathfrak{G}^{\mathrm{op}}, \mathfrak{X}]$.

Herein, a *graph* is an internal graph of discrete categories in Cat. That is to say, a graph is a functor $G : \mathfrak{G}^{\mathrm{op}} \to \mathsf{Cat}$ that factors through the inclusion of the discrete categories $\mathsf{SET} \to \mathsf{Cat}$. This defines the category of graphs $\mathsf{Grph} := \mathsf{Grph}(\mathsf{SET})$. Although the basic theory works for larger graphs and computads in the setting of Section 1, the combinatorial part is of course just suited for small graphs and computads. We define the *category of small graphs* by $\mathsf{grph} := \mathsf{Cat}[\mathfrak{G}^{\mathrm{op}}, \mathsf{Set}]$, while the category of *finite/countable graphs* is the full subcategory of small graphs G such that $G(1)$ is finite/countable.

If $G : \mathfrak{G}^{\mathrm{op}} \to \mathsf{Cat}$ is a graph, $G(1)$ is the *discrete category/collection of objects of* G, while $G(2)$ is *the discrete category/collection of arrows (or edges) of* G. An arrow a of G is denoted by $a : x \to z$, if $G(d^0)(a) = z$ and $G(d^1)(a) = x$. As usual, in this case, z is called the *codomain* and x is called the *domain of the edge* a.

We also consider *the category of reflexive graphs* $\mathsf{RGrph} := \mathsf{Cat}\left[\Delta_2^{\mathrm{op}}, \mathsf{SET}\right]$ and *the category of small reflexive graphs* $\mathsf{Rgrph} := \mathsf{Cat}\left[\Delta_2^{\mathrm{op}}, \mathsf{Set}\right]$. If G is a reflexive graph, the collection in the image of $G(s^0)$ is called the collection/discrete category of *trivial arrows/identity arrows/identities* of G.

The inclusion $\mathfrak{G}^{\mathrm{op}} \to \Delta_2^{\mathrm{op}}$ induces a forgetful functor $\mathcal{R} : \mathsf{RGrph} \to \mathsf{Grph}$ and the left Kan extensions along this inclusion provide a left adjoint to this forgetful functor, denoted by $\mathcal{E}$.

Lemma 2.1. *The forgetful functor* $\mathcal{R} : \mathsf{RGrph} \to \mathsf{Grph}$ *has a left adjoint* $\mathcal{E}$*.*

Remark 2.2. The terminal object of RGrph is denoted by $\bullet$. It has only one object and its trivial arrow. It should be noted that RGrph is not equivalent to Grph, since $\bullet$ is also weakly initial in RGrph while the terminal graph $\mathcal{R}(\bullet) \cong \bigcirc$ is not.

The inclusion $\mathsf{SET} \to \mathsf{Cat}$ has a right adjoint $(-)_0 : \mathsf{Cat} \to \mathsf{SET}$, the forgetful functor. The comonad induced by this adjunction is also denoted by $(-)_0$. On one hand, we define $\mathcal{C}_1 : \mathsf{Cat} \to \mathsf{Grph}$ by $\mathcal{C}_1(X) := \mathsf{Cat}(\mathcal{I}-, X) = (\mathsf{Cat}\,[\mathcal{I}-, X])_0$. On the other hand, if $G : \mathfrak{G}^{\mathrm{op}} \to \mathsf{Cat}$ is any 2-functor, we have:

$$\mathsf{Cat}[\mathcal{I} * G, X] \cong [\mathfrak{G}^{\mathrm{op}}, \mathsf{Cat}]\,(G, \mathsf{Cat}[\mathcal{I}-, X]),$$

since $\mathcal{I} * G \cong G * \mathcal{I}$. This induces an adjunction between the category of categories and the category of internal graphs of Cat. If $G(\mathsf{2})$ is a set, this shows how the coinserter encompasses the notion of freely adding morphisms to a category $G(\mathsf{1})$. In particular, if G is a graph, this induces a (natural) bijection between natural transformations $G \longrightarrow \mathsf{Cat}(\mathcal{I}-, X)$ and functors $\mathcal{I} * G \longrightarrow X$. Therefore:

Lemma 2.3. $\mathcal{F}_1 : \mathsf{Grph} \to \mathsf{Cat}, \mathcal{F}_1(G) = \mathcal{I} * G$ *gives the left adjoint to* $\mathcal{C}_1$.

Informally, we get the result above once we realize that if X is a category in Cat then a functor $f : \mathcal{F}_1(G) \to X$ needs to correspond to a pair (f_0, α^f), in which $f_0 : G(\mathsf{1}) \to (X)_0$ is a morphism of SET and $\alpha^f : f_0 G(d^1) \longrightarrow f_0 G(d^0)$ is a natural transformation. This is precisely an object of the inserter of $\mathsf{Cat}(G-, X)$.

Remark 2.4. [Categories freely generated by reflexive graphs] We can also consider the inclusion $\mathcal{I}^{\mathcal{R}} : \Delta_2 \to \mathsf{Cat}$ and this inclusion induces the functor $\mathcal{C}_1^{\mathcal{R}} : \mathsf{Cat} \to \mathsf{RGrph}$, $\mathcal{C}_1^{\mathcal{R}}(X) = \mathsf{Cat}(\mathcal{I}^{\mathcal{R}}-, X)$. Analogously, this functor has a left adjoint defined by $\mathcal{F}_1^{\mathcal{R}}(G) = \mathcal{I}^{\mathcal{R}} * G$. It is easy to verify that there is a natural isomorphism $\mathcal{F}_1 \cong \mathcal{F}_1^{\mathcal{R}}\mathcal{E}$.

If G is a reflexive graph and x is an object of G, we say that $G(s^0)(x)$ is the trivial arrow/identity arrow of x. In particular, the image of $G(s^0)$ is called the discrete category/collection of the trivial arrows of G.

Remark 2.5. Since $(\mathsf{1} \coprod \mathsf{1}) \cong (\mathsf{2})_0$ in Cat and Cat is lextensive, recall that $X \times (\mathsf{2})_0 \cong (X \times \mathsf{1}) \coprod (X \times \mathsf{1})$ for any object X of Cat.

If G is an object of Grph, we can construct $\mathcal{F}_1(G)$ via the pushout of the morphism $G(\mathsf{2}) \times (\mathsf{2})_0 \to G(\mathsf{1})$ induced by $(G(d^0), G(d^1))$ along the functor $G(\mathsf{2}) \times (\mathsf{2})_0 \to G(\mathsf{2}) \times \mathsf{2}$ given by the product of the identity with the inclusion $(\mathsf{2})_0 \to \mathsf{2}$ induced by the counit of the comonad $(-)_0 : \mathsf{Cat} \to \mathsf{Cat}$.

Remark 2.6. The functor $\mathcal{C}_1$ is monadic since it is right adjoint, reflects isomorphisms, preserves coequalizers and Cat is cocomplete. Hence, each component of the counit of $\mathcal{F}_1 \dashv \mathcal{C}_1$ gives a functor $\mathrm{comp}_X : \mathcal{F}_1\mathcal{C}_1(X) \to X$ which is a regular epimorphism.

The forgetful functor $\mathcal{C}_1\mathcal{U}_1 : \mathsf{Gr} \to \mathsf{Grph}$ has an obvious left adjoint given by $\mathcal{L}_1\mathcal{F}_1 : \mathsf{Grph} \to \mathsf{Gr}$. If $G : \mathfrak{G}^{\mathrm{op}} \to \mathsf{Cat}$ is a graph, $\mathcal{L}_1\mathcal{F}_1(G)$ is called the *groupoid freely generated by G.*

We denote respectively by $\overline{\mathcal{F}_1}$ and $\overline{\mathcal{L}_1\mathcal{F}_1}$ the monads induced by the adjunctions $\mathcal{F}_1 \dashv \mathcal{C}_1$ and $\mathcal{L}_1\mathcal{F}_1 \dashv \mathcal{C}_1\mathcal{U}_1$.The free $\overline{\mathcal{F}_1}$-algebras are called *free categories*, while we call *free groupoids* the free algebras of the monad $\overline{\mathcal{L}_1\mathcal{F}_1}$.

Lemma 2.7. $\mathcal{L}_1\mathcal{F}_1(G) \cong \mathfrak{I} \ast (\mathcal{L}_1 G) \cong (\mathcal{U}_1\mathcal{L}_1\mathfrak{I}) \ast (\mathcal{L}_1 G)$ *gives the left adjoint to* $\mathcal{C}_1$.

Observe that $\mathcal{L}_1 G : \mathfrak{G}^{\mathrm{op}} \to \mathsf{Gr}$ is nothing but G itself as an internal graph of discrete groupoids since $\mathcal{L}_1$ takes discrete categories to discrete groupoids. Also, $\mathcal{U}_1\mathcal{L}_1(\mathcal{F}_1(G)) \cong (\mathcal{U}_1\mathcal{L}_1\mathfrak{I}) \ast G$ in Cat. That is to say, the groupoid freely generated by G is its isocoinseter in Cat.

Remark 2.8. [Characterization of Free Categories [34]] The category Grph has terminal object $\bigcirc$, namely the graph with only one object and only one arrow. If we denote by $\Sigma(\mathbb{N})$ the resulting category from the suspension of the monoid of non-negative integers $\mathbb{N}$, we have that $\mathcal{F}_1(\bigcirc) \cong \Sigma(\mathbb{N})$. Therefore, every graph G comes with a functor

$$\ell^G : \mathcal{F}_1(G) \to \Sigma(\mathbb{N})$$

which is by definition the morphism $\mathcal{F}_1(G \to \bigcirc)$. The functor ℓ^G is called *length functor*. It satisfies a property called *unique lifting of factorizations*, usually refereed as *ulf*. In this case, this means in particular that, if $\ell^G(f) = m$, then there are unique morphisms $f_m, \ldots, f_1, f_0$ such that

– $f_m \cdots f_1 f_0 = f$;
– $\ell^G(f_t) = 1\ \forall t \in \{1, \ldots, m\}$ and f_0 is the identity.

This property characterizes free categories. More precisely, $X \cong \mathcal{F}_1(G)$ for some graph G if and only if there is a functor $\ell_X : X \to \Sigma(\mathbb{N})$ satisfying the unique lifting of factorizations property.

A morphism f has length m if $\ell^G(f) = m$. It is easy to see that the morphisms of $\mathcal{F}_1(G)$ with length 1 correspond to the edges of G. Roughly, the unique lifting property of ℓ^G says that every morphism $f : x \to z$ is a composition $f = a_1 \ldots a_m$ of arrows with length 1 which corresponds to a list of arrows in G satisfying $G(d^1)(a_t) = G(d^0)(a_{t+1})$ for all $t \in \{1, \ldots, m-1\}$, while the identities of $\mathcal{F}_1(G)$ correspond to empty lists. Following this viewpoint, the composition is given by juxtaposition of these lists. A morphism of $f : x \to z$ of $\mathcal{F}_1(G)$ is often called a *path (of length $\ell^G(f)$) between x an z in the graph G.*

It is clear that the length functors reflect isomorphisms. More precisely, if ℓ^G is a length functor, then $\ell^G(f) = 0$ implies that $f = \mathrm{id}$.

As a particular consequence of the characterization given in Remark 2.8, we get that:

Theorem 2.9. *For any graph G, $\mathcal{F}_1(G)$ satisfies the cancellation law.*

Remark 2.10. Let X be a category. By the natural isomorphism of Remark 2.4, we have that $X \cong \mathcal{F}_1(G)$ for some graph G if and only if $X \cong \mathcal{F}_1^{\mathcal{R}}\mathcal{E}(G)$. Also, $X \cong \mathcal{F}_1^{\mathcal{R}}(G)$ for some reflexive graph G if and only if $X \cong \mathcal{F}_1(G^{\mathcal{E}})$, in which $G^{\mathcal{E}} : \mathfrak{G}^{\mathrm{op}} \to \mathsf{Set}$ has the same objects of G and the nontrivial arrows of G. More precisely, $G^{\mathcal{E}}(\mathsf{2}) = G(\mathsf{2}) - G(s^0)(G(\mathsf{1}))$, $G^{\mathcal{E}}(\mathsf{1}) = G(\mathsf{1})$. Therefore the characterization of categories freely generated by reflexive graphs is equivalent to the characterization given in Remark 2.8.

It should be noted that $(-)^{\mathcal{E}}$ is a functor between the subcategories of monomorphisms of RGrph and Grph.

Remark 2.11. [Characterization of Free Groupoids] A natural extension of the Remark 2.8 gives a characterization of free groupoids. More precisely, for each graph G, there is functor

$$\mathcal{L}_1(\ell^G) = \mathcal{L}_1\mathcal{F}_1(G \to \bigcirc) : \mathcal{L}_1\mathcal{F}_1(G) \to \Sigma(\mathbb{Z})$$

in which $\Sigma(\mathbb{Z})$ is the suspension of the group of integers. This functor has the ulf property. In this case, this means that, if $\mathcal{L}_1(\ell^G)(f) = m$, then there are unique morphisms $f_n, \ldots, f_1, f_0$ such that

- $f_n \cdots f_1 f_0 = f$;
- $\mathcal{L}_1(\ell^G)(f_t) \in \{-1, 1\}$, for all $t \in \{1, \ldots, n\}$ and f_0 is identity;
- $\sum \mathcal{L}_1(\ell^G)(f_t) = m$.

This property characterizes free groupoids. That is to say, $X \cong \mathcal{L}_1\mathcal{F}_1(G)$ for some graph G if and only if X is a groupoid and there is a functor $\ell_X : X \to \Sigma(\mathbb{Z})$ satisfying the unique lifting of factorizations property.

It is easy to see that the morphisms of $\mathcal{L}_1\mathcal{F}_1(G)$ with length 1 correspond to the arrows of G, while the morphisms with length -1 correspond to formal inversions of arrows of G.

Definition 2.12. A graph G is called:

– *connected* if $\mathcal{F}_1(G)$ is connected;
– a *weak forest* if $\mathcal{F}_1(G)$ is thin;
– a *forest* if $\mathcal{L}_1\mathcal{F}_1(G)$ is thin;
– a *tree/weak tree* if G is a connected forest/weak forest.

Theorem 2.13. *If G is a forest, then it is a weak forest as well.*

Proof: By Theorem 1.5 and Theorem 2.9, if $\mathcal{L}_1\mathcal{F}_1(G)$ is thin, then $\mathcal{F}_1(G)$ is thin as well. ■

The converse of Theorem 2.13 is not true. For instance, a counterexample is given in Remark 2.16.

Remark 2.14. [Maximal Tree] By Zorn's Lemma, every small connected graph G has maximal trees and maximal weak trees. This means that, given a small connected graph G, the preordered set of trees and the preordered set of weak trees of G have maximal objects. Of course, these results do not depend on Zorn's Lemma if G is countable.

Lemma 2.15. *G_{mtree} is a maximal tree of a connected graph G if and only if the following properties are satisfied:*

– *G_{mtree} is a subgraph of G;*
– *G_{mtree} is a tree;*
– *G_{mtree} has every object of G.*

Remark 2.16. By the last result, a tree in a small connected graph G is maximal if and only if it has all the objects of G. Such a characterization does not hold for maximal weak trees. For instance, the graph $\mathtt{T}$ given by the *example of weak tree* is a weak tree which is not a tree. Hence, the maximal tree of this graph is an example of a weak tree that has all the objects of the graph $\mathtt{T}$ without being a maximal weak tree. However, one of the directions holds. Namely, every maximal weak tree of a small connected graph G has every object of G.

Remark 2.17. All definitions and results related to trees and forests have analogues for reflexive graphs. In fact, for instance, a reflexive graph G is a *reflexive tree* if $\mathcal{F}_1^{\mathcal{R}}(G)$ is a connected thin category. Then, we get that G is a *reflexive tree* if and only if the graph $G^{\mathcal{E}}$ (defined in Remark 2.10) is a tree.

In particular, G_{mtree} is a maximal reflexive tree of a connected reflexive graph G if and only if $G^{\mathcal{E}}_{\mathrm{mtree}}$ is a maximal tree of the graph $G^{\mathcal{E}}$.

Definition 2.18. [Fair Graph] An object G of Grph is a *fair graph* if it has a maximal weak tree which is a tree.

Remark 2.19. From Zorn's Lemma, we also get that every small graph G has a maximal fair subgraph which contains a maximal tree of G. Again, we can avoid Zorn's Lemma if we restrict our attention to countable graphs.

There are thin categories which are not free $\overline{\mathcal{F}_1}$-algebras. For instance, as a particular case of Lemma 2.20, the category $\nabla 2$ is thin and is not a free category. Furthermore, by Theorem 2.22, $\mathbb{R}$ and $\mathbb{Q}$ are examples of small thin categories without nontrivial isomorphisms that are not free categories.

Lemma 2.20. *If X is a category and it has a nontrivial isomorphism, then X is not a free category.*

Proof: There is only one isomorphism in $\Sigma(\mathbb{N})$, namely the identity 0. If f is an isomorphism of $\mathcal{F}_1(G)$, then $\ell^G(f) = 0$. Since ℓ^G reflects identities, we conclude that f is an identity. ■

We can also consider the *thin category freely generated by a graph G*, since $\mathcal{M}_1\mathcal{F}_1 \dashv \mathcal{C}_1 M_1$. It is clear that $\mathcal{C}_1 M_1$ is fully faithful and, hence, it induces an idempotent monad $\overline{\mathcal{M}_1\mathcal{F}_1}$. In particular, every $\overline{\mathcal{M}_1\mathcal{F}_1}$-algebra is a free $\overline{\mathcal{M}_1\mathcal{F}_1}$-algebra. That is to say, every thin category is a thin category freely generated by a graph.

Proposition 2.21. *If $\mathcal{F}_1(G)$ is a totally ordered set then, for each object x of $\mathcal{F}_1(G)$ and each length m, there is at most one morphism of length m with x as domain in $\mathcal{F}_1(G)$. Moreover, if x is not the terminal object, then there is a unique morphism of length 1 with x as domain in $\mathcal{F}_1(G)$.*

Proof: In fact, suppose there are morphisms $b : x \to z', a : x \to z$ of length m. Since $\mathcal{F}_1(G)$ is totally ordered, we can assume without loosing generality that there is a morphism $c : z \leqslant z'$ of some length n.

As $\mathcal{F}_1(G)$ is thin, $ca = b$. In particular, $n + m = \ell^G(ca) = \ell^G(b) = m$. Hence $n = 0$. This means that c is the empty path (identity) and $z = z'$. Again, since $\mathcal{F}_1(G)$ is thin, $a = b$.

It remains to prove the existence of a morphism of length 1 with x as domain whenever x is not the top element. In this case, there is a morphism $x \to z''$ of length $m > 0$ By Remark 2.8, we conclude that there is a unique list $x < z_1 < \ldots < z_{m-1} < z''$ such that $z_t < z_{t+1}$ corresponds to a morphism of length 1. In particular, $x < z_1$ has length 1. ■

Theorem 2.22. *If $\mathcal{F}_1(G)$ is a totally ordered set, then it is isomorphic to one of the following ordered sets:*

- *The finite ordinals* $\mathsf{0}, \mathsf{1}, \ldots, \mathsf{n}, \ldots$*;*
- *The totally ordered sets* $\mathbb{N}, \mathbb{N}^{\mathrm{op}}$ *and* $\mathbb{Z}$*.*

Proof: – If $\mathcal{F}_1(G)$ has bottom $\perp$ and top $\top$ elements:
If $\mathsf{1} \not\cong \mathcal{F}_1(G) \not\cong \mathsf{2}$, $\perp \to \top$ has a length, say $m - 1 \geqslant 2$. This means that $\mathcal{F}_1(G) \cong \{\perp < 1 < \ldots < m-2 < \top\} \cong \mathsf{m}$.

– If $\mathcal{F}_1(G)$ has a bottom element $\perp$ but it does not have a top element: We can define $s : \mathbb{N} \to \mathcal{F}_1(G)$ in which $s(0) := \perp$ and $s(n+1)$ is the codomain of the unique morphism of length 1 with $s(n)$ as domain. Of course, s is order preserving.

It is easy to see by induction that $\perp < s(n)$ has length n. Hence it is obvious that s is injective. Also, given an object x of $\mathcal{F}_1(G)$, there is m' such that $\perp \to x$ has length m'. By Proposition 2.21, it follows that $s(m') = x$. This proves that s is actually a bijection.

– If $\mathcal{F}_1(G)$ has a top element $\top$ but it does not have a bottom element: By duality, we get that $\mathbb{N}^{\mathrm{op}} \cong \mathcal{F}_1(G)$.

– If $\mathcal{F}_1(G)$ does not have top nor bottom elements:
If $\mathcal{F}_1(G) \not\cong \mathsf{0}$, given an object y of $\mathcal{F}_1(G)$, take the subcategories $\{x \in \mathcal{F}_1(G) : x \leqslant y\}$ and $\{x \in \mathcal{F}_1(G) : y \leqslant x\}$. By what we proved, these subcategories are isomorphic respectively to $\mathbb{N}^{\mathrm{op}}$ and $\mathbb{N}$. By the uniqueness of pushouts, we get $\mathcal{F}_1(G) \cong \mathbb{Z}$. ■

Corollary 2.23. *If $\mathcal{F}_1(G)$ is a small thin category, then it is isomorphic to a colimit of ordinals* $\mathsf{0}, \mathsf{1}, \ldots, \mathsf{n}, \ldots$ *or/and* $\mathbb{N}, \mathbb{N}^{\mathrm{op}}, \mathbb{Z}$*.*

Remark 2.24. There are non-free categories which are subcategories of free categories. But subgroupoids of freely generated small groupoids are freely generated. In fact, this follows from:

Theorem 2.25. *A small groupoid is free if and only if its skeleton is free. In particular, freeness is a property preserved by equivalences of groupoids. As a consequence, subgroupoids of free groupoids are free.*

Proof: Since $\mathcal{L}_1\mathcal{F}_1$ creates coproducts and every groupoid is a coproduct of connected groupoids, it is enough to prove the statement for connected groupoids.

If a connected groupoid is free, this means that it is isomorphic to $\mathcal{L}_1\mathcal{F}_1(G)$ for a connected graph G. It is easy to see that the skeleton of $\mathcal{L}_1\mathcal{F}_1(G)$ is isomorphic to $\mathcal{L}_1\mathcal{F}_1(G)/\mathcal{L}_1\mathcal{F}_1(G_{\mathrm{mtree}})$ for any maximal tree G_{mtree}. Therefore $\mathcal{L}_1\mathcal{F}_1(G/G_{\mathrm{mtree}})$ is isomorphic to the skeleton and, hence, the skeleton is free.

Reciprocally, if the skeleton of a connected groupoid X is free, it follows that for any object y of X, the full subgroupoid with only x as object, often denoted by $\pi(X,y)$, is free. We take $\pi(X,y) \cong \mathcal{F}_1(H)$ and define the graph $G : \mathfrak{G}^{\mathrm{op}} \to \mathsf{cat}$ by:

– $G(2) := H(2) \coprod (\mathsf{cat}(1,X) - \{y\})$;

– $G(1) := \mathsf{cat}(1,X)$;

– $G(d^1)$ is constant equal to y;

– $G(d^0)(a) := y$ if $a \in H(2)$;

– $G(d^0)(z) := z$ if $z \in \mathsf{cat}(1,X) - \{y\}$.

Of course, $\mathcal{L}_1\mathcal{F}_1(G) \cong X$. The consequence follows from Nielsen-Schreier theorem for groups, since every small groupoid is equivalent to a coproduct of groups. ∎

3. Presentations

If $\mathcal{T} = (\mathcal{T}, m, \eta)$ is a monad on a category $\mathfrak{X}$, we denote respectively by $\mathfrak{X}^{\mathcal{T}}$ and $\mathfrak{X}_{\mathcal{T}}$ the category of Eilenberg-Moore $\mathcal{T}$-algebras and the Kleisli category. Every such monad comes with a notion of presentation of a $\mathcal{T}$-algebra. More precisely, a diagram in $\mathfrak{X}$

$$G_2 \rightrightarrows \mathcal{T}(G_1) \qquad (\mathcal{T}\text{-presentation diagram})$$

can be seen as a graph in $\mathfrak{X}_{\mathcal{T}}$ and, hence, it can be seen as a graph $\mathfrak{G}^{\mathrm{op}} \to \mathfrak{X}^{\mathcal{T}}$ of free $\mathcal{T}$-algebras in $\mathfrak{X}^{\mathcal{T}}$. We say that the graph above is a *presentation* of the $\mathcal{T}$-algebra $(G', \mathcal{T}(G') \to G')$ if this algebra is (isomorphic to) the coequalizer of the corresponding diagram $\mathfrak{G}^{\mathrm{op}} \to \mathfrak{X}^{\mathcal{T}}$ of free $\mathcal{T}$-algebras in $\mathfrak{X}^{\mathcal{T}}$. Every $\mathcal{T}$-algebra admits a presentation, since every $\mathcal{T}$-algebra is a coequalizer of free $\mathcal{T}$-algebras. If $\mathfrak{X}^{\mathcal{T}}$ has all coequalizers of free algebras, denoting by $\mathsf{Grph}(\mathfrak{X}_{\mathcal{T}}) = \mathsf{Cat}[\mathfrak{G}^{\mathrm{op}}, \mathfrak{X}_{\mathcal{T}}]$ the category of graphs internal to the Kleisli category, there is a functor $\mathsf{Grph}(\mathfrak{X}_{\mathcal{T}}) \to \mathfrak{X}^{\mathcal{T}}$ which takes each graph to the category presented by it.

Definition 3.1. [$\mathcal{T}$-presentation] Let $\mathcal{T} = (\mathcal{T}, m, \eta)$ be a monad on a category $\mathfrak{X}$. Consider the comma category $(\mathrm{Id}_{\mathfrak{X}}/\mathcal{T})$. We have a functor $K_{\mathcal{T}} : (\mathrm{Id}_{\mathfrak{X}}/\mathcal{T}) \to \mathsf{Grph}(\mathfrak{X}^{\mathcal{T}})$ given by the composition of the comparisons $(\mathrm{Id}_{\mathfrak{X}}/\mathcal{T}) \to \mathsf{Grph}(\mathfrak{X}_{\mathcal{T}}) \to \mathsf{Grph}(\mathfrak{X}^{\mathcal{T}})$.

Consider also the full subcategory $\mathsf{Grph}'(\mathfrak{X}^{\mathcal{T}})$ of $\mathsf{Grph}(\mathfrak{X}^{\mathcal{T}})$ whose objects are graphs G such that the coequalizer of G exists in $\mathfrak{X}^{\mathcal{T}}$. The *category of* $\mathcal{T}$*-presentations*, denoted by $\mathcal{P}\mathrm{re}(\mathcal{T})$, is the pullback of $K_{\mathcal{T}}$ along the inclusion $\mathsf{Grph}'(\mathfrak{X}^{\mathcal{T}}) \to \mathsf{Grph}(\mathfrak{X}^{\mathcal{T}})$.

We get then a natural functor $K'_{\mathcal{T}} : \mathcal{P}\mathrm{re}(\mathcal{T}) \to \mathsf{Grph}'(\mathfrak{X}^{\mathcal{T}})$. The *functor presentation*, denoted by $\mathcal{P}_{\mathcal{T}} : \mathcal{P}\mathrm{re}(\mathcal{T}) \to \mathfrak{X}^{\mathcal{T}}$, is the composition of the coequalizer $\mathsf{Grph}'(\mathfrak{X}^{\mathcal{T}}) \to \mathfrak{X}^{\mathcal{T}}$ with $K'_{\mathcal{T}}$.

Lemma 3.2. *$\mathcal{P}_{\mathcal{T}}$ is essentially surjective. This means that every $\mathcal{T}$-algebra has at least one presentation.*

Remark 3.3. We ratify that if $\mathcal{T}$ is a monad such that $\mathfrak{X}^{\mathcal{T}}$ has coequalizers of free algebras, then the definition of $\mathcal{P}\mathrm{re}(\mathcal{T})$ is easier. More precisely, $\mathcal{P}\mathrm{re}(\mathcal{T}) := (\mathrm{Id}_{\mathfrak{X}}/\mathcal{T})$.

We denote by $\overline{\mathcal{L}_0\mathcal{F}_0}$ the *free group monad* on Set whose category of algebras is the category of groups Group. A $\overline{\mathcal{L}_0\mathcal{F}_0}$*-presentation of a group* is a pair $\langle S, R\rangle$ in which S is a set and $R : \mathfrak{G}^{\mathrm{op}} \to \mathsf{Set}$ is a small graph such that $R(\mathsf{1}) = \overline{\mathcal{L}_0\mathcal{F}_0}(S)$. This induces a graph $\overline{R} : \mathfrak{G}^{\mathrm{op}} \to \mathsf{Group}$ of free groups. The coequalizer of this graph is precisely the *group presented by* $\langle S, R\rangle$. Analogously, we get the notion of $\overline{\mathcal{F}_0}$*-presentation of monoids* induced by the *free monoid monad* $\overline{\mathcal{F}_0}$ on Set.

Remark 3.4. Recall, for instance, the basics of presentations of groups [15]. The classical definition of a presentation of a group is not usually given explicitly by a graph as it is described above. Instead, the usual definition of a presentation of a group is given by a pair $\langle S, R\rangle$ in which S is a set and R is a "set of relations or equations". However, this is of course the same as an $\overline{\mathcal{L}_0\mathcal{F}_0}$-presentation. That is to say, it is a graph

$$R \rightrightarrows \overline{\mathcal{L}_0\mathcal{F}_0}(S)$$

in Set such that the first arrow gives one side of the equations and the second arrow gives the other side of the equations. For instance, in computing the fundamental group of a torus via the Van Kampen Theorem and the quotient of the square [18], one usually gets it via the presentation $\langle \{a, b\}, ab = ba\rangle$. This is the same as the graph

$$* \rightrightarrows \overline{\mathcal{L}_0\mathcal{F}_0}(\{a, b\})$$

in which the image of $*$ by the first arrow is the word ab and the image by the second arrow is ba. Of course, this is the presentation of $\mathbb{Z}\times\mathbb{Z}$, as $\mathbb{Z}\times\mathbb{Z}$ is the coequalizer of the corresponding diagram of free groups in the category of groups.

The free category monad $\overline{\mathcal{F}_1}$ on Grph induces a notion of presentation of categories. More precisely, an $\overline{\mathcal{F}_1}$-presentation of a category X is a graph $\mathfrak{g} : \mathfrak{G} \to \mathsf{Grph}_{\overline{\mathcal{F}_1}}$ such that, after composing $\mathfrak{g}$ with $\mathsf{Grph}_{\overline{\mathcal{F}_1}} \to \mathsf{Grph}^{\overline{\mathcal{F}_1}} \simeq \mathsf{Cat}$, its coequalizer in Cat is isomorphic to X. Analogously, the free groupoid monad $\overline{\mathcal{L}_1\mathcal{F}_1}$ gives rise to the notion of $\overline{\mathcal{L}_1\mathcal{F}_1}$-presentation of groupoids.

Remark 3.5. [Suspension] The forgetful functor $\mathfrak{u}_1 : \mathsf{Grph} \to \mathsf{SET}$ has left and right adjoints. The left adjoint $\mathfrak{i}_1 : \mathsf{SET} \to \mathsf{Grph}$ is defined by $\mathfrak{i}_1(X)(\mathsf{2}) = \emptyset$ and $\mathfrak{i}_1(X)(\mathsf{1}) = X$. The right adjoint $\sigma_1 = \Sigma' : \mathsf{SET} \to \mathsf{Grph}$ is defined by $\Sigma'(X)(\mathsf{2}) = X$ and $\Sigma'(X)(\mathsf{1}) = *$ is the terminal set.

Indeed, σ_1 is part of monad (mono)morphisms $\overline{\mathcal{F}_0} \to \overline{\mathcal{F}_1}$ and $\overline{\mathcal{L}_0\mathcal{F}_0} \to \overline{\mathcal{L}_1\mathcal{F}_1}$. We conclude that presentation of monoids are particular cases of presentations of categories and presentations of groups are particular cases of presentations of groupoids. More precisely, there are inclusions

$$\begin{tikzcd}
\mathcal{P}\mathrm{re}(\overline{\mathcal{L}_0\mathcal{F}_0}) \arrow[r] \arrow[d] & \mathcal{P}\mathrm{re}(\overline{\mathcal{F}_0}) \arrow[d] \\
\mathcal{P}\mathrm{re}(\overline{\mathcal{L}_1\mathcal{F}_1}) \arrow[r] & \mathcal{P}\mathrm{re}(\overline{\mathcal{F}_1})
\end{tikzcd}$$

but it is important to note that they are not essentially surjective.

Roughly, $\overline{\mathcal{F}_1}$-presentations and $\overline{\mathcal{L}_1\mathcal{F}_1}$-presentations can be seen as freely generated graphs with equations between 1-cells and equations between 0-cells. More precisely, we have:

Definition 3.6. If $\mathfrak{g} : \mathfrak{G}^{\mathrm{op}} \to \mathsf{Grph}_{\overline{\mathcal{F}_1}}$ is a presentation of a category, we denote by $\mathfrak{g}(d^0)_1$ the component of the graph morphism $\mathfrak{g}(d^0)$ in $\mathsf{1}$. If $\mathfrak{g}(d^0)_1 = \mathfrak{g}(d^1)_1$ and they are inclusions, $\mathfrak{g} : \mathfrak{G}^{\mathrm{op}} \to \mathsf{Grph}_{\overline{\mathcal{F}_1}}$ is called an 1-cell presentation.

Theorem 3.7. *If* $\mathfrak{g} : \mathfrak{G}^{\mathrm{op}} \to \mathsf{Grph}_{\overline{\mathcal{F}_1}}$

$$\begin{tikzcd}
\mathfrak{g}(\mathsf{2}) \arrow[r, shift left, "\mathfrak{g}(d^0)"] \arrow[r, shift right, "\mathfrak{g}(d^1)"'] & \overline{\mathcal{F}_1}(\mathfrak{g}_1)
\end{tikzcd}$$

is a presentation of a category X, then there is an induced 1-cell presentation $\underline{\mathfrak{g}}$ of X

$$\underline{\mathfrak{g}}(2) \overset{\underline{\mathfrak{g}}(d^0)}{\underset{\underline{\mathfrak{g}}(d^1)}{\rightrightarrows}} \overline{\mathcal{F}_1}(\underline{\mathfrak{g}}_1)$$

in which $\underline{\mathfrak{g}}(2)(1)$ is the coequalizer of the graph of objects induced by $\underline{\mathfrak{g}}$.

Remark 3.8. We denote by $\widehat{2}$ the graph such that $\mathcal{F}_1(\widehat{2}) = 2$. It is clear that $\mathcal{I}$ can be lifted through $\mathcal{C}_1$. That is to say, there is a functor $\widehat{\mathcal{I}} : \mathfrak{G} \to \mathsf{Grph}$ such that $\mathcal{C}_1\widehat{\mathcal{I}} = \mathcal{I}$. Then $\mathcal{F}_1\widehat{\mathcal{I}}$ composed with the isomorphism $\mathfrak{G}^{\mathrm{op}} \cong \mathfrak{G}$ gives a graph of free $\overline{\mathcal{F}_1}$-algebras. Therefore it gives an $\overline{\mathcal{F}_1}$-presentation

$$\bullet \rightrightarrows \overline{\mathcal{F}_1}(\widehat{2})$$

of the category (suspension of the monoid) $\Sigma(\mathbb{N})$. Actually, the corresponding 1-cell presentation is just

$$\varnothing \rightrightarrows \overline{\mathcal{F}_1}(\bigcirc) \cong \Sigma(\mathbb{N}).$$

Remark 3.9. Of course, we also have the notion of $\overline{\mathcal{F}_1^{\mathcal{R}}}$-presentations of categories. Although the category of $\overline{\mathcal{F}_1}$-presentations is not isomorphic to the category of $\overline{\mathcal{F}_1^{\mathcal{R}}}$-presentations, we have an obvious inclusion between these categories which is essentially surjective.

4. Definition of Computads

In Section 8, we give the definition of the n-category freely generated by an n-computad by induction. The starting point of the induction is the definition of a category freely generated by a graph. Thereby graphs are called 1-computads and we define respectively the *category of 1-computads* and the category of *small 1-computads* by $1\text{-}\mathsf{Cmp} := \mathsf{Grph}$ and $1\text{-}\mathsf{cmp} := \mathsf{grph}$.

In the present section, we give a concise definition of 2-computads and of the category $2\text{-}\mathsf{Cmp}$. This concise definition is precisely what allows us to get its freely generated 2-category via a coinserter. We also introduce the notion of a category presented by a computad, which is going to be our canonical notion of presentation of categories.

Definition 4.1. [Derivation Schemes and Computads] Consider the functor $(- \times \mathfrak{G}) : \mathsf{SET} \to \mathsf{Cat}, Y \mapsto Y \times \mathfrak{G}$ and the functor $\mathcal{F}_1 : \mathsf{Grph} \to \mathsf{Cat}$. The *category of derivation schemes* is the comma category $\mathsf{Der} := (- \times \mathfrak{G}/\mathrm{Id}_{\mathsf{Cat}})$. The *category of 2-computads* is the comma category $2\text{-}\mathsf{Cmp} := (- \times \mathfrak{G}/\mathcal{F}_1)$.

Considering the restrictions $(- \times \mathfrak{G}) : \mathsf{Set} \to \mathsf{cat}$ and $\mathcal{F}_1 : \mathsf{grph} \to \mathsf{cat}$, we define the *category of small 2-computads* as $2\text{-}\mathsf{cmp} := (- \times \mathfrak{G}/\mathcal{F}_1)$. We also define the *category of small computads over reflexive graphs* (or just *category of reflexive computads*) as $\mathsf{Rcmp} := (- \times \mathfrak{G}/\mathcal{F}_1^{\mathcal{R}})$. There is an obvious left adjoint inclusion $\mathsf{cmp} \to \mathsf{Rcmp}$ induced by $\mathcal{E}$. We denote the induced adjunction by $\mathcal{E}_{\mathsf{cmp}} \dashv \mathcal{R}_{\mathsf{cmp}}$.

Derivations schemes were first defined in [10, 34]. Respecting the original terminology of [31], the word *computad* without any index means 2-computad. Also, we set the notation: $\mathsf{Cmp} := 2\text{-}\mathsf{Cmp}$ and $\mathsf{cmp} := 2\text{-}\mathsf{cmp}$.

The pushout of the inclusion $(2)_0 \to 2$ of Remark 2.5 along itself is (isomorphic to) $\mathfrak{G}$. Hence, by definition, a *derivation scheme* is pair $(\mathfrak{d}, \mathfrak{d}_2)$ in which $\mathfrak{d}_2$ is a discrete category and $\mathfrak{d} : \mathfrak{G}^{\mathrm{op}} \to \mathsf{Cat}$ is an internal graph

$$\mathfrak{d}_2 \times \mathsf{2} \rightrightarrows \mathfrak{d}(1) \qquad (\mathfrak{d}\text{-diagram})$$

such that, for every α of $\mathfrak{d}_2$:

$$\mathfrak{d}(d^0)(\alpha, 0) = \mathfrak{d}(d^1)(\alpha, 0) \qquad \mathfrak{d}(d^0)(\alpha, 1) = \mathfrak{d}(d^1)(\alpha, 1).$$

In this direction, by definition, a *computad* is a triple $(\mathfrak{g}, \mathfrak{g}_2, G)$ in which $(\mathfrak{g}, \mathfrak{g}_2)$ is a derivation scheme and $G : \mathfrak{G}^{\mathrm{op}} \to \mathsf{Cat}$ is a graph such that $\mathfrak{g}(1) = \mathcal{F}_1(G)$. We usually adopt this viewpoint.

Definition 4.2. [Groupoidal Computad] Consider the functor $(- \times \mathcal{L}_1(\mathfrak{G})) : \mathsf{SET} \to \mathsf{Gr}, X \mapsto X \times \mathcal{L}_1(\mathfrak{G})$ and the functor $\mathcal{L}_1\mathcal{F}_1 : \mathsf{Grph} \to \mathsf{Gr}$. The category of *groupoidal computads* is the comma category $\mathsf{Cmp}_{\mathsf{Gr}} := (- \times \mathcal{L}_1(\mathfrak{G})/\mathcal{L}_1\mathcal{F}_1)$. Analogously, the *category of groupoidal computads over reflexive graphs* is defined by $\mathsf{Rcmp}_{\mathsf{gr}} := (- \times \mathcal{L}_1(\mathfrak{G})/\mathcal{L}_1\mathcal{F}_1^{\mathcal{R}})$.

We denote by $\widehat{\mathfrak{G}}$ the graph below with two objects and two arrows between them. It is clear that $\mathcal{F}_1(\widehat{\mathfrak{G}}) \cong \mathfrak{G}$. Hence, there is a natural morphism $\widehat{\mathfrak{G}} \to \mathcal{C}_1(\mathfrak{G})$ induced by the unit of $\mathcal{F}_1 \dashv \mathcal{C}_1$. Moreover, it is important to observe that $\widehat{\mathfrak{G}}$ is not isomorphic to $\mathcal{C}_1(\mathfrak{G})$.

$$* \rightrightarrows *$$

Theorem 4.3. *Consider the functor* $(\mathfrak{i}_1(-) \times \widehat{\mathfrak{G}}) : \mathsf{Set} \to \mathsf{grph}, X \mapsto \mathfrak{i}_1(X) \times \widehat{\mathfrak{G}}$. *There are isomorphisms of categories* $\mathsf{Cmp} \cong (\mathfrak{i}_1(-) \times \widehat{\mathfrak{G}}/\overline{\mathcal{F}_1})$ *and* $\mathsf{Cmp}_{\mathsf{Gr}} \cong (\mathfrak{i}_1(-) \times \widehat{\mathfrak{G}}/\overline{\mathcal{L}_1\mathcal{F}_1})$.

Moreover, considering suitable restrictions of $(\mathfrak{i}_1(-) \times \widehat{\mathfrak{G}})$ *and* $\overline{\mathcal{F}_1}$ *(to* Set *and* grph *respectively), we have that* $\mathsf{cmp} \cong (\mathfrak{i}_1(-) \times \widehat{\mathfrak{G}}/\overline{\mathcal{F}_1})$*. Analogously,* $\mathsf{cmp}_{\mathsf{Gr}} \cong (\mathfrak{i}_1(-) \times \widehat{\mathfrak{G}}/\overline{\mathcal{L}_1\mathcal{F}_1})$*.*

Definition 4.4. [Presentation of a category via a computad] We say that a computad $(\mathfrak{g}, \mathfrak{g}_2, G)$ *presents a category* X if the coequalizer of $\mathfrak{g} : \mathfrak{G}^{\mathrm{op}} \to \mathsf{Cat}$ is isomorphic to X. We have, then, a functor $\mathcal{P}_1 : \mathsf{Cmp} \to \mathsf{Cat}$ which gives the category presented by each computad. Of course, there is also a presentation functor $\mathcal{P}_1^{\mathfrak{R}} : \mathsf{Rcmp} \to \mathsf{cat}$.

Analogously, we say that a groupoidal computad $(\mathfrak{g}, \mathfrak{g}_2, G)$ *presents a groupoid* X if the coequalizer of $\mathfrak{g} : \mathfrak{G}^{\mathrm{op}} \to \mathsf{Gr}$ is isomorphic to X. Again, we have presentation functors $\mathcal{P}_{(1,0)} : \mathsf{Cmp}_{\mathsf{Gr}} \to \mathsf{Gr}$ and $\mathcal{P}^{\mathfrak{R}}_{(1,0)} : \mathsf{Rcmp}_{\mathsf{gr}} \to \mathsf{gr}$.

Theorem 4.5. *Every presentation via computads is an* $\overline{\mathcal{F}_1}$*-presentation. That is to say, there is a natural inclusion* $\mathsf{Cmp} \to \mathcal{P}\mathrm{re}(\overline{\mathcal{F}_1})$*. Analogously, every groupoidal computad is an* $\overline{\mathcal{L}_1\mathcal{F}_1}$*-presentation.*

Proof: By Theorem 4.3, $\mathsf{Cmp} \cong (\mathfrak{i}_1(-) \times \widehat{\mathfrak{G}}/\overline{\mathcal{F}_1})$. So, it is enough to consider the natural inclusion between comma categories

$$(\mathfrak{i}_1(-) \times \widehat{\mathfrak{G}}/\overline{\mathcal{F}_1}) \to (\mathrm{Id}_{\mathsf{Grph}}/\overline{\mathcal{F}_1}).$$

■

Every category admits a presentation via a computad and, analogously, every groupoid admits a presentation via a groupoidal computad. These results follow from Theorem 3.7 and:

Theorem 4.6. *There is a functor* $\mathsf{Cmp} \to \mathcal{P}\mathrm{re}(\overline{\mathcal{F}_1}), \mathfrak{g} \mapsto \underline{\mathcal{C}_1\mathfrak{g}}$ *which is essentially surjective in the subcategory of* 1*-cell presentations* $\mathfrak{g} : \mathfrak{G}^{\mathrm{op}} \to \mathsf{Grph}$ *of categories such that the graph* $\mathfrak{g}(2)$ *has no isolated objects (that is to say, every object is the domain or codomain of some arrow). Moreover, there is a natural isomorphism*

$$\begin{tikzcd}
\mathsf{Cmp} \arrow[rr] \arrow[dr, "\mathcal{P}_1"'] & \arrow[d, phantom, "\cong"] & \mathcal{P}\mathrm{re}(\overline{\mathcal{F}_1}) \arrow[dl, "\mathcal{P}_{\overline{\mathcal{F}_1}}"] \\
& \mathsf{Cat} &
\end{tikzcd}$$

Remark 4.7. The truncated category $\dot{\Delta}_2$ is usually presented by the computad $(\mathfrak{g}^{\dot{\Delta}_2}, \mathfrak{g}_2^{\dot{\Delta}_2}, G_{\dot{\Delta}_2})$ defined as follows:

$$
\begin{array}{rlrl}
G_{\dot{\Delta}_2}(1) & := \{0,1,2\} & G_{\dot{\Delta}_2}(2) & := \left\{d, s^0, d^0, d^1\right\} \\
G_{\dot{\Delta}_2}(d^1)(d^i) & := 1, \forall i & G_{\dot{\Delta}_2}(d^1)(s^0) & := 2 \\
G_{\dot{\Delta}_2}(d^0)(d^i) & := 2, \forall i & G_{\dot{\Delta}_2}(d^0)(s^0) & := 1 \\
G_{\dot{\Delta}_2}(d^1)(d) & := 0, \forall i & G_{\dot{\Delta}_2}(d^0)(d) & := 1, \forall i
\end{array}
$$

$$\mathfrak{g}_2^{\dot{\Delta}} := \{n_0, n_1, \vartheta\}$$

$$
\begin{array}{rlrl}
\mathfrak{g}^{\dot{\Delta}_2}(d^1)(n_0, 0 \to 1) & := s^0 \cdot d^0 & \mathfrak{g}^{\dot{\Delta}_2}(d^0)(n_1, 0 \to 1) & := s^0 \cdot d^1 \\
\mathfrak{g}^{\dot{\Delta}_2}(d^0)(n_0, 0 \to 1) & := \mathrm{id}_1 & \mathfrak{g}^{\dot{\Delta}_2}(d^1)(n_1, 0 \to 1) & := \mathrm{id}_1 \\
\mathfrak{g}^{\dot{\Delta}_2}(d^0)(\vartheta, 0 \to 1) & := d^0 \cdot d & \mathfrak{g}^{\dot{\Delta}_2}(d^1)(\vartheta, 0 \to 1) & := d^1 \cdot d.
\end{array}
$$

This computad can also be described by the graph

$$
\begin{tikzcd}
0 \arrow[r, "d"] & 1 \arrow[r, shift left=2, "d^1"] \arrow[r, shift right=2, "d^0"'] & 2 \arrow[l, "s^0" description]
\end{tikzcd}
$$

with the following 2-cells:

$$n_0 : s^0 \cdot d^0 \Rightarrow \mathrm{id}_1, \qquad n_1 : \mathrm{id}_1 \Rightarrow s^0 \cdot d^1, \qquad \vartheta : d^1 \cdot d \Rightarrow d^0 \cdot d.$$

Lemma 4.8. *The category $\dot{\Delta}_2$ is the coequalizer of the computad $\mathfrak{g}^{\dot{\Delta}_2}$.*

Remark 4.9. The usual presentation of the category $\dot{\Delta}$ via faces and degeneracies is given by the computad $(\mathfrak{g}^{\dot{\Delta}}, \mathfrak{g}_2^{\dot{\Delta}}, G_{\dot{\Delta}})$ which is defined by

$$\mathfrak{g}_2^{\dot{\Delta}} \times 2 \rightrightarrows \mathcal{F}_1(G_{\dot{\Delta}})$$

in which $G_{\dot{\Delta}}(1) := (\mathbb{N})_0$ is the discrete category of the non-negative integers and

$$G_{\dot{\Delta}}(2) := \left\{(d^i, m) : (i,m) \in \mathbb{N}^2, i \leqslant m\right\} \cup \left\{(s^k, m) : (k,m) \in \mathbb{N}^2, k \leqslant m-1 \geqslant 0\right\}$$

$$
\begin{array}{rlrl}
G_{\dot{\Delta}}(d^1)(d^i, m) & := m & G_{\dot{\Delta}}(d^1)(s^k, m) & := m+1 \\
G_{\dot{\Delta}}(d^0)(d^i, m) & := m+1 & G_{\dot{\Delta}}(d^0)(s^k, m) & := m
\end{array}
$$

$$
\begin{aligned}
\mathfrak{g}_2^{\dot{\Delta}} \;:=\; & \left\{(d^k, d^i, m) : (i,k,m) \in \mathbb{N}^3, m \geqslant i < k\right\} \\
& \cup \left\{(s^k, s^i, m) : (i,k,m) \in \mathbb{N}^3, 0 \leqslant m-1 \geqslant k \geqslant i\right\} \\
& \cup \left\{(s^k, d^i, m) : (i,k,m) \in \mathbb{N}^3, k \leqslant m-1 \geqslant 0\right\}
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{g}^{\dot{\Delta}}(d^1)((d^k, d^i, m), \mathsf{0} \to \mathsf{1}) &:= (d^k, m+1) \cdot (d^i, m) \\
\mathfrak{g}^{\dot{\Delta}}(d^1)((s^k, s^i, m), \mathsf{0} \to \mathsf{1}) &:= (s^k, m) \cdot (s^i, m+1) \\
\mathfrak{g}^{\dot{\Delta}}(d^1)((s^k, d^i, m), \mathsf{0} \to \mathsf{1}) &:= (s^k, m+1) \cdot (d^i, m) \\
\mathfrak{g}^{\dot{\Delta}}(d^0)((d^k, d^i, m), \mathsf{0} \to \mathsf{1}) &:= (d^i, m+1) \cdot (d^{k-1}, m) \\
\mathfrak{g}^{\dot{\Delta}}(d^0)((s^k, s^i, m), \mathsf{0} \to \mathsf{1}) &:= (s^i, m) \cdot (s^{k+1}, m+1) \\
\mathfrak{g}^{\dot{\Delta}}(d^0)((s^k, d^i, m), \mathsf{0} \to \mathsf{1}) &:= (d^i, m-1) \cdot (s^{k-1}, m), \text{ if } k > i \\
\mathfrak{g}^{\dot{\Delta}}(d^0)((s^k, d^i, m), \mathsf{0} \to \mathsf{1}) &:= \mathrm{id}_{m}, \text{ if } i = k \text{ or } i = k+1 \\
\mathfrak{g}^{\dot{\Delta}}(d^0)((s^k, d^i, m), \mathsf{0} \to \mathsf{1}) &:= (d^{i-1}, m-1) \cdot (s^k, m-1), \text{ if } i > k+1.
\end{aligned}
$$

Lemma 4.10. *The category $\dot{\Delta}$ is the coequalizer of the computad $\mathfrak{g}^{\dot{\Delta}}$.*

Every computad induces a presentation of groupoids via a groupoidal computad, since we have an obvious functor $\mathsf{Cmp} \to \mathsf{Cmp}_{\mathsf{Gr}}$ induced by $\mathcal{L}_1$. More precisely, the functor $\mathcal{L}_1^{\mathsf{Cmp}} : \mathsf{Cmp} \to \mathsf{Cmp}_{\mathsf{Gr}}$ is defined by $\mathfrak{g} \mapsto \mathcal{L}_1\mathfrak{g}$. Observe that the groupoidal computad $\mathcal{L}_1\mathfrak{g}$ gives a presentation of the coequalizer of $\mathcal{L}_1\mathfrak{g}$ in Gr which is (isomorphic to) $\mathcal{L}_1\mathcal{P}_1(\mathfrak{g})$. In this case, we say that the computad $\mathfrak{g}$ presents the groupoid $\mathcal{L}_1\mathcal{P}_1(\mathfrak{g})$.

Proposition 4.11. *There is a natural isomorphism $\mathcal{P}_{(1,0)}\mathcal{L}_1^{\mathsf{Cmp}} \cong \mathcal{L}_1\mathcal{P}_1$.*

Remark 4.12. If $\mathcal{P}_1(\mathfrak{g})$ is a groupoid, there is no confusion between the groupoid presented by $\mathfrak{g}$ and the category presented by $\mathfrak{g}$, since, in this case, they are actually isomorphic. More precisely, in this case, $\mathcal{L}_1\mathcal{P}_1(\mathfrak{g}) \cong \mathcal{P}_1(\mathfrak{g})$.

Theorem 4.13. *If the groupoid presented by a computad $(\mathfrak{g}, \mathfrak{g}_2, G)$ is thin, then $(\mathfrak{g}, \mathfrak{g}_2, G)$ presents a thin category as well provided that $\mathcal{P}_1(\mathfrak{g}, \mathfrak{g}_2, G)$ satisfies the cancellation law.*

Proof: By Theorem 1.5, if $\mathcal{L}_1\mathcal{P}_1(\mathfrak{g})$ is thin, then $\mathcal{P}_1(\mathfrak{g})$ is thin. ■

Definition 4.14. [2-cells of computads] Let $(\mathfrak{g}, \mathfrak{g}_2, G)$ be a computad. The discrete category $\mathfrak{g}_2$ is called *the discrete category of the 2-cells of the computad* $\mathfrak{g}$. Moreover, we say that α is a 2-cell between f and g, denoted by

$\alpha : f \Rightarrow g$, if $\mathfrak{g}(d^1)(\alpha, \mathsf{0} \to \mathsf{1}) = f$ and $\mathfrak{g}(d^0)(\alpha, \mathsf{0} \to \mathsf{1}) = g$. In this case, the *domain* of α is f while the *codomain* is g.

Sometimes, we need to be even more explicit and denote the 2-cell α by $\alpha : f \Rightarrow g : x \to y$ whenever $\mathfrak{g}(d^1)(\alpha, \mathsf{0} \to \mathsf{1}) = f$, $\mathfrak{g}(d^0)(\alpha, \mathsf{0} \to \mathsf{1}) = g$, $\mathfrak{g}(d^0)(\alpha, \mathsf{0}) = x$ and $\mathfrak{g}(d^0)(\alpha, \mathsf{1}) = y$.

In the context of presentation of categories, the 2-cells of a computad $(\mathfrak{g}, \mathfrak{g}_2, G)$ correspond to the equations of the presentation induced by this computad. If $\mathfrak{g}$ has more than one 2-cell between two arrows of $\mathfrak{g}(\mathsf{1})$, then it is a redundant presentation of the coequalizer of $\mathfrak{g}$. Yet, we also have interesting examples of redundant presentations. For instance, in the next section, we give the definition of the fundamental groupoid via a redundant presentation.

Remark 4.15. [Sigma] There is an obvious forgetful functor $\mathfrak{u}_2 : \mathsf{cmp} \to \mathsf{grph}$. This forgetful functor has left and right adjoints. The left adjoint $\mathfrak{i}_2 : \mathsf{grph} \to \mathsf{cmp}$ is defined by $\mathfrak{i}_2(G) = (G^{\mathfrak{i}_2}, \emptyset, G)$. Sometimes, we denote $G^{\mathfrak{i}_2}$ by $\mathfrak{i}_2(G)$ and, of course, it is defined as follows:

$$\mathfrak{i}_2(G) : \emptyset \rightrightarrows \mathcal{F}_1(G).$$

The right adjoint $\sigma_2 : \mathsf{grph} \to \mathsf{cmp}$ is defined by $\sigma_2(G) = (G^{\sigma_2}, G_2^{\sigma_2}, G)$ in which $\sigma_2(G)(\mathsf{2}) = G_2^{\sigma_2} \times \mathsf{2}$ and the set of 2-cells $G_2^{\sigma_2}$ is the pullback of $(\mathcal{F}_1(G)(d^1), \mathcal{F}_1(G)(d^0)) : \mathcal{F}_1(G)(\mathsf{2}) \to \mathcal{F}_1(G)(\mathsf{1}) \times \mathcal{F}_1(G)(\mathsf{1})$ along itself. Finally, the images of $G^{\sigma_2}(G)(d^1), G^{\sigma_2}(d^0)$ are induced by the obvious projections. Sometimes we write $\sigma_2(G) = (\sigma_2(G), \sigma_2(G)_2, G)$ as follows

$$\sigma_2(G) : G_2^{\sigma_2} \times \mathsf{2} \rightrightarrows \mathcal{F}_1(G).$$

Remark 4.16. [Sigma$^{\mathsf{Gr}}$] Of course, we also have a forgetful functor $\mathfrak{u}_2^{\mathsf{Gr}}$: $\mathsf{cmp}_{\mathsf{Gr}} \to \mathsf{grph}$. The left adjoint of this functor is defined by $\mathfrak{i}_2^{\mathsf{Gr}} := \mathcal{L}_1^{\mathsf{Cmp}}\mathfrak{i}_2$, while the right adjoint is defined by $\sigma_2^{\mathsf{Gr}} := \mathcal{L}_1^{\mathsf{Cmp}}\sigma_2$.

Proposition 4.17. *There is a natural isomorphism* $\mathcal{P}_1\mathfrak{i}_2 \cong \mathcal{F}_1$.

Definition 4.18. [Connected Computad] A computad $(\mathfrak{g}, \mathfrak{g}_2, G)$ is *connected* if $\mathfrak{u}_2(\mathfrak{g}, \mathfrak{g}_2, G) = G$ is connected.

Remark 4.19. Let X be a group. We consider the full subcategory $\mathcal{P}\mathrm{re}(\overline{\mathcal{L}_0\mathcal{F}_0}, X)$ of $\mathcal{P}\mathrm{re}(\overline{\mathcal{L}_0\mathcal{F}_0})$ consisting of the presentations of X. This subcategory is isomorphic to the full subcategory of $\mathsf{cmp}^{\mathsf{Gr}}$ consisting of the groupoidal computads which presents $\Sigma(X)$. This fact shows that presentations of groupoids by groupoidal computads generalizes the notion of $\overline{\mathcal{L}_0\mathcal{F}_0}$-presentations of groups. Moreover, unlike the case of $\overline{\mathcal{L}_1\mathcal{F}_1}$-presentations, the notion of presentations of (suspensions of) groups by groupoidal computads is precisely the same of $\overline{\mathcal{L}_0\mathcal{F}_0}$-presentations.

Analogously, given a monoid Y the category of $\overline{\mathcal{F}_0}$-presentations $\mathcal{P}\mathrm{re}(\overline{\mathcal{F}_0}, Y)$ is isomorphic to the category of computads which presents $\Sigma(Y)$.

5. Topology and Computads

We introduce topological aspects of our theory. We refer the reader to [28] for basic notions and results of algebraic topology, including the Van Kampen theorem for fundamental groupoids.

We start with the relation between the fundamental groupoids and groupoids freely generated by small graphs. By the classical Van Kampen theorem, the fundamental group of a (topological) graph with only one object is the group freely generated by the set of edges/arrows. We show that it also holds for fundamental groupoids: roughly, the groupoid freely generated by a small graph G is equivalent to its fundamental groupoid. Although this is a straightforward result, this motivates the relation between topology and small computads: that is to say, the association of each small computad with a CW-complex presented in 5.1.

We always consider small computads, small graphs and small categories throughout this section. Moreover, we use the appropriate restrictions of the functors $\mathcal{F}_1, \mathcal{L}_1, \mathcal{U}_1, \mathcal{C}_1$. Finally, Top denotes any suitable cartesian closed category of topological spaces: for instance, compactly generated spaces. Then we can consider weighted colimits in Top w.r.t. the Top-enrichment.

Remark 5.1. [Topological Graph] There is an obvious left adjoint inclusion $D_2 : \mathsf{cat} \to \mathsf{Top\text{-}Cat}$ induced by the fully faithful (discrete topology) functor $D : \mathsf{Set} \to \mathsf{Top}$ left adjoint to the forgetful functor $\mathsf{Top} \to \mathsf{Set}$. We denote by $\mathfrak{G}$ and $\mathfrak{G}^{\mathrm{op}}$ the images $D_2(\mathfrak{G})$ and $D_2(\mathfrak{G}^{\mathrm{op}})$ respectively, whenever there is no confusion. If $I = [0, 1]$ is the unit interval with the usual topology and $*$ is the terminal topological space, then the Top-weight $\mathfrak{I}_{\mathsf{Top}_1} : \mathfrak{G} \to \mathsf{Top}$ defined by

$$* \underset{0}{\overset{1}{\rightrightarrows}} I$$

gives the definition of Top-*isoinserters* and Top-*isocoinserters.*

If $G : \mathfrak{G}^{\mathrm{op}} \to \mathsf{Set}$ is a small graph, $DG : \mathfrak{G}^{\mathrm{op}} \to \mathsf{Top}$ is actually compatible with the Top-enrichment. More precisely, since D_2 is left adjoint, there is a Top-functor $D_2(\mathfrak{G}^{\mathrm{op}}) \to \mathsf{Top}$ which is the mate of $DG : \mathfrak{G}^{\mathrm{op}} \to \mathsf{Top}$. Again, by abuse of notation, the mate $D_2(\mathfrak{G}^{\mathrm{op}}) \to \mathsf{Top}$ is also denoted by $DG : \mathfrak{G}^{\mathrm{op}} \to \mathsf{Top}$.

Any small graph $G : \mathfrak{G}^{\mathrm{op}} \to \mathsf{Set}$ has an associated topological (undirected) graph given by the Top-isocoinserter of the Top-functor DG. This gives a functor $\mathcal{F}_{\mathsf{Top}_1} : \mathsf{grph} \to \mathsf{Top}$ which is left adjoint to the functor $\mathcal{C}_{\mathsf{Top}_1} : \mathsf{Top} \to \mathsf{grph}$, $E \mapsto \mathsf{Top}(\mathcal{I}_{\mathsf{Top}_1}-, E)$. We denote the monad induced by this adjunction by $\overline{\mathcal{F}_{\mathsf{Top}_1}}$.

A *path in a topological space* E is an edge of $\mathcal{C}_{\mathsf{Top}_1}(E)$, that is to say, a path in E is a continuous map $a : I \to E$.

Lemma 5.2. *A small graph G is connected if and only if $\mathcal{F}_{\mathsf{Top}_1}(G)$ is a path connected topological space.*

Remark 5.3. We also have an adjunction $\mathcal{F}^{\mathcal{R}}_{\mathsf{Top}_1} \dashv \mathcal{C}^{\mathcal{R}}_{\mathsf{Top}_1}$ in which $\mathcal{C}^{\mathcal{R}}_{\mathsf{Top}_1} = \mathcal{C}_{\mathsf{Top}_1}\mathcal{R}$. This adjunction is induced by a weight analogue of $\mathcal{I}_{\mathsf{Top}_1}$. Namely, if we denote by Δ_2 the image of itself by $\mathsf{cat} \to \mathsf{Top}\text{-}\mathsf{Cat}$, the Top-functor $\mathcal{I}^{\mathcal{R}}_{\mathsf{Top}_1} : \Delta_2 \to \mathsf{Top}$ defined by

$$\begin{tikzcd} * \arrow[r, shift left=2, "1"] \arrow[r, shift right=2, "0"'] & I \arrow[l] \end{tikzcd}$$

in which $\mathcal{I}^{\mathcal{R}}_{\mathsf{Top}_1}$ composed with the inclusion $\mathfrak{G} \to \Delta_2$ is equal to $\mathcal{I}_{\mathsf{Top}_1}$. This weight gives rise to the notion of *reflexive* Top-*isoinserters* and *reflexive* Top-*isocoinserters*. Finally, $\mathcal{F}^{\mathcal{R}}_{\mathsf{Top}_1}(G) = \mathcal{I}^{\mathcal{R}}_{\mathsf{Top}_1} \circledast DG$ and $\mathcal{C}_{\mathsf{Top}_1} : \mathsf{Top} \to \mathsf{Rgrph}$, $E \mapsto \mathsf{Top}(\mathcal{I}^{\mathcal{R}}_{\mathsf{Top}_1}-, E)$.

Given an arrow f of $\overline{\mathcal{F}_1}\mathcal{C}_{\mathsf{Top}_1}(E)$, we have that there is a unique finite list of arrows $a_0^f, \ldots, a_{m-1}^f$ of $\mathcal{C}_{\mathsf{Top}_1}(E)$ such that $f = a_{m-1}^f \cdots a_0^f$ by the ulf property of the length functor. Since, by definition, $a_0^f, \ldots, a_{m-1}^f$ are continuous maps $I \to E$, we can define a continuous map $\lceil \underline{f} \rceil_E : I \to E$ by $\lceil \underline{f} \rceil_E(t) = a_n^f(mt - n)$ whenever $t \in [n/m, (n+1)/m]$. This gives a morphism of graphs

$$\lceil \_ \rceil_E : \mathcal{C}_1\mathcal{F}_1\mathcal{C}_{\mathsf{Top}_1}(E) \to \mathcal{C}_{\mathsf{Top}_1}(E)$$

which is identity on objects and takes each arrow $f = a_{m-1}^f \cdots a_0^f$ of length m to the arrow $\lceil \underline{f} \rceil_E$ of $\mathcal{C}_{\mathsf{Top}_1}(E)$. These graph morphisms define a natural transformation

$$\lceil \_ \rfloor : \overline{\mathcal{F}_1}\mathcal{C}_{\mathsf{Top}_1} \longrightarrow \mathcal{C}_{\mathsf{Top}_1}.$$

Remark 5.4. It is very important to observe that, if f is an arrow of $\mathcal{C}_1\mathcal{F}_1\mathcal{C}_{\mathsf{Top}_1}(E)$ of length $m > 1$, then $\lceil \underline{f} \rfloor_E : x \to z$ is not the same as the morphism $f : x \to z$ itself. The former is an edge of $\mathcal{C}_{\mathsf{Top}_1}(E)$, which means that, as morphism of $\overline{\mathcal{F}_1}\mathcal{C}_{\mathsf{Top}_1}(E)$, its length is 1.

Remark 5.5. We have also a natural transformation $\lceil \_ \rfloor^{\mathsf{Gr}} : \overline{\mathcal{L}_1\mathcal{F}_1}\mathcal{C}_{\mathsf{Top}_1} \longrightarrow \mathcal{C}_{\mathsf{Top}_1}$. Observe that, by the ulf property of the length functor and by the definition of $\mathcal{C}_{\mathsf{Top}_1}$, if f is an arrow of $\overline{\mathcal{L}_1\mathcal{F}_1}\mathcal{C}_{\mathsf{Top}_1}(E)$ of lenght k, then $f = a^f_{m-1}\cdots a^f_0$ for a unique list $(a^f_{m-1},\ldots,a^f_0)$ of paths or formal inverses of paths in E and we can define $\lceil \underline{f} \rfloor^{\mathsf{Gr}}_E : I \to E$ by:

$$\lceil \underline{f} \rfloor^{\mathsf{Gr}}_E(t) = \begin{cases} a^f_n(mt-n), & \text{if } t \in [n/m,(n+1)/m] \text{ and } a^f_n \text{ is a path in } E,\\ \\ b^f_n(-mt+n+1), & \text{if } t \in [n/m,(n+1)/m] \text{ and } a^f_n \\ & \quad \text{is a formal inverse of an arrow } b^f_n \text{ of } \mathcal{C}_{\mathsf{Top}_1}(E). \end{cases}$$

On one hand, this defines morphisms of graphs $\overline{\mathcal{L}_1\mathcal{F}_1}\mathcal{C}_{\mathsf{Top}_1}(E) \longrightarrow \mathcal{C}_{\mathsf{Top}_1}(E)$ for each topological space E. On the other hand, these morphisms define the natural transformation $\lceil \_ \rfloor^{\mathsf{Gr}} : \overline{\mathcal{L}_1\mathcal{F}_1}\mathcal{C}_{\mathsf{Top}_1} \longrightarrow \mathcal{C}_{\mathsf{Top}_1}$.

Theorem 5.6. *The mate of* $\lceil \_ \rfloor : \overline{\mathcal{F}_1}\mathcal{C}_{\mathsf{Top}_1} \longrightarrow \mathcal{C}_{\mathsf{Top}_1}$ *under the adjunction* $\mathcal{F}_{\mathsf{Top}_1} \dashv \mathcal{C}_{\mathsf{Top}_1}$ *and the identity adjunction is a natural transformation*

$$\lceil\ \rceil : \overline{\mathcal{F}_1} \longrightarrow \overline{\mathcal{F}_{\mathsf{Top}_1}}$$

which is a part of a monad functor/morphism $(\mathrm{Id}_{\mathsf{Grph}}, \lceil\ \rceil) : \overline{\mathcal{F}_{\mathsf{Top}_1}} \to \overline{\mathcal{F}_1}$. *Analogously, the mate* $\lceil \_ \rfloor^{\mathsf{Gr}}$ *under the same adjunctions is a natural transformation* $\lceil\ \rceil^{\mathsf{Gr}}$ *which is part of a monad functor* $(\mathrm{Id}_{\mathsf{Grph}}, \lceil\ \rceil^{\mathsf{Gr}}) : \overline{\mathcal{F}_{\mathsf{Top}_1}} \to \overline{\mathcal{L}_1\mathcal{F}_1}$.

Remark 5.7. It is also important to consider the mate $\lceil\cdot\rceil : \mathcal{F}_{\mathsf{Top}_1}\overline{\mathcal{F}_1} \longrightarrow \mathcal{F}_{\mathsf{Top}_1}$ of the natural transformation $\lceil \_ \rfloor : \overline{\mathcal{F}_1}\mathcal{C}_{\mathsf{Top}_1} \longrightarrow \mathcal{C}_{\mathsf{Top}_1}$ under the adjunction $\mathcal{F}_{\mathsf{Top}_1} \dashv \mathcal{C}_{\mathsf{Top}_1}$ and itself. Again, we can consider the case of groupoids: the mate of $\lceil \_ \rfloor^{\mathsf{Gr}}$ under $\mathcal{F}_{\mathsf{Top}_1} \dashv \mathcal{C}_{\mathsf{Top}_1}$ and itself is denoted by $\lceil\cdot\rceil^{\mathsf{Gr}} : \mathcal{F}_{\mathsf{Top}_1}\overline{\mathcal{L}_1\mathcal{F}_1} \longrightarrow \mathcal{F}_{\mathsf{Top}_1}$.

Let S^1 be the circle (complex numbers with norm 1) and B^2 the closed ball (complex numbers whose norm is smaller than or equal to 1). We denote the usual inclusion by $\mathrm{h} : S^1 \to B^2$. We consider also the embeddings:

$$\mathrm{h}_0 : I \to B^2,\ t \mapsto e^{\pi i t} \qquad\qquad \mathrm{h}_1 : I \to B^2,\ t \mapsto e^{\pi i(-t)}.$$

Recall that, if E is a topological space and $a, b : I \to E$ are continuous maps, a *homotopy of paths* $H : a \simeq b$ is a continuous map $H : B^2 \to E$ such that $H\mathrm{h}_0 = a$ and $H\mathrm{h}_1 = b$. If there is such a homotopy, we say that a and b are *homotopic*.

There is a functor $\mathcal{C}_{\mathsf{Top}_2} : \mathsf{Top} \to \mathsf{cmp}$ given by $\mathcal{C}_{\mathsf{Top}_2}(E) = (\mathfrak{g}^E, \mathfrak{g}_2^E, G^E)$ in which $G^E := \mathcal{C}_{\mathsf{Top}_1}(E)$ and

$$\mathfrak{g}_2^E \;:=\; \Big\{(f, g, H : \lceil \underline{f} \rceil_E \simeq \lceil \underline{g} \rceil_E) : \\ H \text{ is a homotopy of paths and } f, g \in \overline{\mathcal{F}_1}\mathcal{C}_{\mathsf{Top}_1}(E)(2)\Big\}.$$

Also, $\mathfrak{g}^E(d^1)(f, g, H : \lceil \underline{f} \rceil_E \simeq \lceil \underline{g} \rceil_E, \mathsf{0} \to \mathsf{1}) := f$ and $\mathfrak{g}^E(d^0)(f, g, H : \lceil \underline{f} \rceil_E \simeq \lceil \underline{g} \rceil_E, \mathsf{0} \to \mathsf{1}) := g$. By an elementary result of algebraic topology, the image of $\mathcal{P}_1\mathcal{C}_{\mathsf{Top}_2} : \mathsf{Top} \to \mathsf{Cat}$ is inside *the category of small groupoids* gr. More precisely, there is a functor $\Pi : \mathsf{Top} \to \mathsf{gr}$ such that $\mathcal{U}_1\Pi \cong \mathcal{P}\mathcal{C}_{\mathsf{Top}_2}$. If E is a topological space, $\Pi(E)$ is called the *fundamental groupoid of E*. Given a point $e \in E$, recall that the fundamental group $\pi_1(E, e)$ is by definition the full subcategory of $\Pi(E)$ with only e as object.

Remark 5.8. The Van Kampen theorem [8] for groupoids (see, for instance, [8, 6]) gives the fundamental groupoid $\Pi(S^1)$ by the pushout of the inclusion $\{0, 1\} \to \Pi(I)$ along itself. This is equivalent to the pushout of the inclusion $(\mathsf{2})_0 \to \mathsf{2}$ of Remark 2.5 along $(\mathsf{2})_0 \to \mathsf{1}$, which is given by the $\overline{\mathcal{L}_1\mathcal{F}_1}$-presentation

$$\bullet \rightrightarrows \overline{\mathcal{L}_1\mathcal{F}_1}(\hat{\mathsf{2}})$$

induced by the $\overline{\mathcal{F}_1}$-presentation of Example 3.8. We conclude that this is isomorphic to $\mathcal{L}_1(\Sigma(\mathbb{N})) \cong \Sigma(\mathbb{Z})$.

Proposition 5.9. *There is a natural isomorphism* $\mathcal{C}_{\mathsf{Top}_1} \cong \mathfrak{u}_2\mathcal{C}_{\mathsf{Top}_2}$.

Remark 5.10. The groupoid freely generated by a given small graph is equivalent to the fundamental groupoid of the respective topological graph. To see that, since $\mathcal{F}_1^{\mathcal{R}}\mathcal{E} \cong \mathcal{F}_1$ and $\mathcal{F}_{\mathsf{Top}_1}^{\mathcal{R}}\mathcal{E} \cong \mathcal{F}_{\mathsf{Top}_1}$, it is enough to prove that, for each small reflexive graph G,

$$\mathcal{L}_1\mathcal{F}_1^{\mathcal{R}}(G) \simeq \Pi\mathcal{F}_{\mathsf{Top}_1}^{\mathcal{R}}(G).$$

On one hand, if G is a reflexive tree, then both $\mathcal{L}_1\mathcal{F}_1^{\mathcal{R}}(G), \Pi\mathcal{F}_{\mathsf{Top}_1}^{\mathcal{R}}(G)$ are thin (and connected): therefore, they are equivalent. On the other hand, if G is a reflexive graph with only one object, then $\mathcal{L}_1\mathcal{F}_1^{\mathcal{R}}(G)$ and $\Pi\mathcal{F}_{\mathsf{Top}_1}^{\mathcal{R}}(G)$ are equivalent to the group freely generated by the set of nontrivial edges/arrows of G.

If a reflexive graph G is not a reflexive tree and it has more than one object, we can choose a maximal reflexive tree G_{mtree} of G. Then, if we denote by $\mathcal{L}_1\mathcal{F}_1^{\mathcal{R}}(G)/\mathcal{L}_1\mathcal{F}_1^{\mathcal{R}}(G_{\mathrm{mtree}})$ the pushout of the inclusion $\mathcal{L}_1\mathcal{F}_1^{\mathcal{R}}(G_{\mathrm{mtree}}) \to \mathcal{L}_1\mathcal{F}_1^{\mathcal{R}}(G)$ along the unique functor between $\mathcal{L}_1\mathcal{F}_1^{\mathcal{R}}(G_{\mathrm{mtree}})$ and the terminal groupoid, we get:

$$\mathcal{L}_1\mathcal{F}_1^{\mathcal{R}}(G) \simeq \mathcal{L}_1\mathcal{F}_1^{\mathcal{R}}(G)/\mathcal{L}_1\mathcal{F}_1^{\mathcal{R}}(G_{\mathrm{mtree}}) \cong \mathcal{L}_1\mathcal{F}_1^{\mathcal{R}}(G/G_{\mathrm{mtree}}),$$

in which, analogously, G/G_{mtree} denotes the pushout of the morphism induced by the inclusion $G_{\mathrm{mtree}} \to G$ along the unique morphism $G_{\mathrm{mtree}} \to \bullet$ in the category of reflexive graphs Rgrph.

Since the reflexive graph G/G_{mtree} has only one object, we have that

$$\mathcal{L}_1\mathcal{F}_1^{\mathcal{R}}(G/G_{\mathrm{mtree}}) \simeq \Pi\mathcal{F}_{\mathsf{Top}_1}^{\mathcal{R}}(G/G_{\mathrm{mtree}}) \cong \Pi\left(\mathcal{F}_{\mathsf{Top}_1}^{\mathcal{R}}(G)/\mathcal{F}_{\mathsf{Top}_1}^{\mathcal{R}}(G_{\mathrm{mtree}})\right)$$

in which the last isomorphism follows from the fact that $\mathcal{F}_{\mathsf{Top}_1}^{\mathcal{R}}$ is left adjoint. Since $\Pi\left(\mathcal{F}_{\mathsf{Top}_1}^{\mathcal{R}}(G)/\mathcal{F}_{\mathsf{Top}_1}^{\mathcal{R}}(G_{\mathrm{mtree}})\right) \simeq \Pi\mathcal{F}_{\mathsf{Top}_1}^{\mathcal{R}}(G)$, the proof is complete. This actually can be done in a pseudonatural equivalence as we show below.

Theorem 5.11. *There is a natural transformation $\mathcal{L}_1\mathcal{F}_1 \longrightarrow \Pi\mathcal{F}_{\mathsf{Top}_1}$ which is an objectwise equivalence.*

Proof: Consider the unit of the adjunction $\mathcal{F}_{\mathsf{Top}_1} \dashv \mathcal{C}_{\mathsf{Top}_1}$, denoted in this proof by η. We have that the horizontal composition $\mathrm{Id}_{\mathcal{P}_1\mathfrak{i}_2} * \eta$ gives a natural transformation $\mathcal{P}_1\mathfrak{i}_2 \longrightarrow \mathcal{P}_1\mathfrak{i}_2\overline{\mathcal{F}_{\mathsf{Top}_1}}$. We, then, compose this natural transformation with the obvious isomorphism $\mathcal{P}_1\mathfrak{i}_2\overline{\mathcal{F}_{\mathsf{Top}_1}} \longrightarrow \mathcal{P}_1\mathfrak{i}_2\mathfrak{u}_2\mathcal{C}_{\mathsf{Top}_2}\mathcal{F}_{\mathsf{Top}_1}$ obtained from the isomorphism of Proposition 5.9

Now, we suitably past this natural transformation with the counit of $\mathfrak{i}_2 \dashv \mathfrak{u}_2$ and get a natural transformation $\mathcal{P}_1\mathfrak{i}_2 \longrightarrow \mathcal{P}_1\mathcal{C}_{\mathsf{Top}_2}\mathcal{F}_{\mathsf{Top}_1}$, which, after composing with the isomorphism of Proposition 4.17, gives $\mathcal{F}_1 \longrightarrow \mathcal{P}_1\mathcal{C}_{\mathsf{Top}_2}\mathcal{F}_{\mathsf{Top}_1}$.

The horizontal composition of this natural isomorphism with $\mathrm{Id}_{\mathcal{L}_1}$ gives our natural transformation $\mathcal{L}_1\mathcal{F}_1 \longrightarrow \mathcal{L}_1\mathcal{P}_1\mathcal{C}_{\mathsf{Top}_2}\mathcal{F}_{\mathsf{Top}_1} \cong \Pi\mathcal{F}_{\mathsf{Top}_1}$. It is an exercise of basic algebraic topology to show that, as a consequence of the considerations of Remark 5.10, this natural transformation is an objectwise equivalence. ■

5.1. Further on Topology. To get the relation between small computads and topological spaces, we use the isomorphism of Theorem 4.3. In particular, an object $(\mathfrak{g}, \mathfrak{g}_2, G)$ of cmp is a diagram $\mathfrak{g} : \mathsf{2} \to \mathsf{grph}$

$$\widehat{\mathfrak{G}} \times \mathfrak{i}_1(\mathfrak{g}_2) \to \overline{\mathcal{F}_1}(G),$$

in which $\mathfrak{g}_2$ is a set and G is a small graph. We also fix the homeomorphism $\mathrm{cir}^{-1} : \mathcal{F}_{\mathsf{Top}_1}(\widehat{\mathfrak{G}}) \to S^1$ which is the mate of the morphism of graphs $\mathrm{cir}' : \widehat{\mathfrak{G}} \to \mathcal{C}_{\mathsf{Top}_1}(S^1)$ which takes the edges of $\widehat{\mathfrak{G}}$ to the continuous maps $\mathrm{h}_1', \mathrm{h}_0' : I \to S^1$, $\mathrm{h}_1'(t) := \mathrm{h}_1(t)$, $\mathrm{h}_0'(t) := \mathrm{h}_0(t)$ (which are edges between 0 and 1 in $\mathcal{C}_{\mathsf{Top}_1}(S^1)$). More generally, for each set $\mathfrak{g}_2$, we fix the homeomorphism

$$\mathrm{cir} \times \mathfrak{g}_2 : S^1 \times D(\mathfrak{g}_2) \to \mathcal{F}_{\mathsf{Top}_1}(\widehat{\mathfrak{G}} \times \mathfrak{i}_1(\mathfrak{g}_2)).$$

Analogously to the case of graphs, we can associate each computad with a "topological computad", which is a CW-complex of dimension 2. The functor $\mathcal{C}_{\mathsf{Top}_2} : \mathsf{Top} \to \mathsf{cmp}$ is actually right adjoint to the functor $\mathcal{F}_{\mathsf{Top}_2} : \mathsf{cmp} \to \mathsf{Top}$ defined as follows: if $(\mathfrak{g}, \mathfrak{g}_2, G)$ is a small computad $\mathfrak{g} : \widehat{\mathfrak{G}} \times \mathfrak{i}_1(\mathfrak{g}_2) \to \overline{\mathcal{F}_1}(G)$, then $\mathcal{F}_{\mathsf{Top}_2}(\mathfrak{g}, \mathfrak{g}_2, G)$ is the pushout of $\mathrm{h} \times D(\mathfrak{g}_2) : S^1 \times D(\mathfrak{g}_2) \to B^2 \times D(\mathfrak{g}_2)$ along the composition of the morphisms

$$S^1 \times D(\mathfrak{g}_2) \xrightarrow{(\mathrm{cir}\times\mathfrak{g}_2)} \mathcal{F}_{\mathsf{Top}_1}(\widehat{\mathfrak{G}} \times \mathfrak{i}_1(\mathfrak{g}_2)) \xrightarrow{(\mathcal{F}_{\mathsf{Top}_1}\mathfrak{g})} \mathcal{F}_{\mathsf{Top}_1}\overline{\mathcal{F}_1}(G) \xrightarrow{\lceil\cdot\rceil_G} \mathcal{F}_{\mathsf{Top}_1}(G)$$

in which $\lceil\cdot\rceil : \mathcal{F}_{\mathsf{Top}_1}\overline{\mathcal{F}_1} \longrightarrow \mathcal{F}_{\mathsf{Top}_1}$ is the natural transformation of Remark 5.7.

Lemma 5.12. *A small computad $(\mathfrak{g}, \mathfrak{g}_2, G)$ is connected if and only if $\mathcal{F}_{\mathsf{Top}_2}(\mathfrak{g}, \mathfrak{g}_2, G)$ is a path connected topological space.*

Let $\mathfrak{g} = (\mathfrak{g}, \mathfrak{g}_2, G)$ be a small connected computad. We denote by $\mathtt{T}$ the maximal tree of $\mathfrak{u}_2(\mathfrak{g}, \mathfrak{g}_2, G)$. Consider the pushout of $\mathcal{F}_{\mathsf{Top}_2}\mathfrak{i}_2(\mathtt{T}) \to *$ along the composition

$$\mathcal{F}_{\mathsf{Top}_2}\mathfrak{i}_2(\mathtt{T}) \to \mathcal{F}_{\mathsf{Top}_2}\mathfrak{i}_2\mathfrak{u}_2(\mathfrak{g}, \mathfrak{g}_2, G) \to \mathcal{F}_{\mathsf{Top}_2}(\mathfrak{g}, \mathfrak{g}_2, G)$$

in which $\mathcal{F}_{\mathsf{Top}_2}\mathfrak{i}_2(\mathtt{T}) \to \mathcal{F}_{\mathsf{Top}_2}(\mathfrak{i}_2\mathfrak{u}_2(\mathfrak{g}, \mathfrak{g}_2, G))$ is induced by the inclusion of the maximal tree of the graph $\mathfrak{u}_2(\mathfrak{g}, \mathfrak{g}_2, G)$ and $\mathcal{F}_{\mathsf{Top}_2}(\mathfrak{i}_2\mathfrak{u}_2(\mathfrak{g}, \mathfrak{g}_2, G)) \to \mathcal{F}_{\mathsf{Top}_2}(\mathfrak{g}, \mathfrak{g}_2, G)$

is induced by the counit of $\mathfrak{i}_2 \dashv \mathfrak{u}_2$. Since this is actually a pushout of a homotopy equivalence along a cofibration (that is to say, this is a homotopy pushout along a homotopy equivalence), we get that $\mathcal{F}_{\mathsf{Top}_2}(\mathfrak{g}, \mathfrak{g}_2, G)$ has the same homotopy type of the obtained pushout which is a wedge of spheres, balls and circumferences.

Theorem 5.13. *For each small computad* $(\mathfrak{g}, \mathfrak{g}_2, G)$*, there is an equivalence*

$$\Pi\mathcal{F}_{\mathsf{Top}_2}(\mathfrak{g}, \mathfrak{g}_2, G) \simeq \mathcal{L}_1\mathcal{P}_1(\mathfrak{g}, \mathfrak{g}_2, G).$$

Remark 5.14. It is clear that the adjunction $\mathcal{F}_{\mathsf{Top}_2} \dashv \mathcal{C}_{\mathsf{Top}_2}$ can be lifted to an adjunction $\mathcal{F}^{\mathsf{Gr}}_{\mathsf{Top}_2} \dashv \mathcal{C}^{\mathsf{Gr}}_{\mathsf{Top}_2}$ in which $\mathcal{F}^{\mathsf{Gr}}_{\mathsf{Top}_2} : \mathsf{cmp}_{\mathsf{Gr}} \to \mathsf{Top}$ is defined as follows: if $(\mathfrak{g}, \mathfrak{g}_2, G)$ is a small groupoidal computad,

$$\mathfrak{g} : \widehat{\mathfrak{G}} \times \mathfrak{i}_1(\mathfrak{g}_2) \to \overline{\mathcal{L}_1\mathcal{F}_1}(G),$$

then $\mathcal{F}^{\mathsf{Gr}}_{\mathsf{Top}_2}(\mathfrak{g}, \mathfrak{g}_2, G)$ is the pushout of $\mathrm{h} \times D(\mathfrak{g}_2) : S^1 \times D(\mathfrak{g}_2) \to B^2 \times D(\mathfrak{g}_2)$ along $\lceil \cdot \rceil^{\mathsf{Gr}}_G \cdot (\mathcal{F}_{\mathsf{Top}_1}\mathfrak{g}) \cdot (\mathrm{cir} \times \mathfrak{g}_2)$. We have an isomorphism $\mathcal{F}^{\mathsf{Gr}}_{\mathsf{Top}_2}\mathcal{L}_1^{\mathsf{cmp}} \cong \mathcal{F}_{\mathsf{Top}_2}$.

Theorem 5.15. *For each small groupoidal computad* $(\mathfrak{g}, \mathfrak{g}_2, G)$*, there is an equivalence*

$$\Pi\mathcal{F}^{\mathsf{Gr}}_{\mathsf{Top}_2}(\mathfrak{g}, \mathfrak{g}_2, G) \simeq \mathcal{P}_{(1,0)}(\mathfrak{g}, \mathfrak{g}_2, G).$$

6. Deficiency

In this section, we study presentations of small categories/groupoids, focusing on thin groupoids and categories. Roughly, the main result of this section computes the minimum of equations/2-cells necessary to get a presentation of a groupoid generated by a given graph G with finite Euler characteristic. This result motivates our definition of deficiency of a (finitely presented) groupoid/category. We start by giving the basic definitions of deficiency of algebras over Set.

6.1. Algebras over Set. Let $\mathcal{T} = (\mathcal{T}, m, \eta)$ be a monad on Set. We denote a $\mathcal{T}$-presentation $R : \mathfrak{G}^{\mathrm{op}} \to \mathsf{Set}$,

$$R(2) \rightrightarrows \mathcal{T}(S),$$

by $\langle S, R\rangle$. If S and $R(2)$ are finite, the presentation $\langle S, R\rangle$ is called *finite*. If a $\mathcal{T}$-algebra $(A, \mathcal{T}(A) \to A)$ has a finite presentation $\langle S, R\rangle$, it is called *finitely* ($\mathcal{T}$-)*presented*.

In this context, the *($\mathcal{T}$-)deficiency of a $\mathcal{T}$-presentation* $\langle S, R\rangle$ is defined by

$$\mathrm{def}_{\mathcal{T}}(\langle S, R\rangle) := |S| - |R(2)|$$

in which $|-|$ gives the cardinality of the set. The *($\mathcal{T}$-)deficiency of a finitely presented $\mathcal{T}$-algebra* $(A, \mathcal{T}(A) \to A)$, denoted by $\mathrm{def}_{\mathcal{T}}(A, \mathcal{T}(A) \to A)$, is the maximum of the set

$$\{\mathrm{def}_{\mathcal{T}}(\langle S, R\rangle) : \langle S, R\rangle \text{ presents } (A, \mathcal{T}(A) \to A)\}.$$

Remark 6.1. Consider the *free real vector space monad* and the notion of presentation of vector spaces induced by it. In this context, the notion of finitely presented vector space coincides with the notion of finite dimensional vector space and it is a consequence of the rank-nulity theorem that the deficiency of a finite dimensional vector space is its dimension.

The notion of deficiency and finite presentations induced by the free group monad $\overline{\mathcal{L}_0\mathcal{F}_0}$ coincide with the usual notions (see [15]). Analogously, the respective usual notions of deficiency and finite presentations are induced by the free monoid monad and free abelian group monad.

It is well known that, if a (finitely presented) group has positive deficiency, then this group is nontrivial (actually, it is not finite). Indeed, if H is a group which has a presentation with positive deficiency, then $\mathsf{Group}(H, \mathbb{R})$ is a vector space with a presentation with positive deficiency. This implies that $\mathsf{Group}(H, \mathbb{R})$ has positive dimension and, then, H is not trivial. In particular, we conclude that the trivial group has deficiency 0.

We present a suitable definition of deficiency of groupoids and, then, we prove that thin groupoids have deficiency 0. Before doing so, we recall elementary aspects of Euler characteristics and define what we mean by finitely presented category.

6.2. Euler characteristic. If X is a topological space, we denote by $\mathrm{H}^i(X)$ its ordinary i-th cohomology group with coefficients in $\mathbb{R}$. Assuming that the dimensions of the cohomology groups of a topological space X are finite, recall that the *Euler characteristic of a topological space* X is given by

$$\chi(X) := \sum_{i=0}^{\infty} (-1)^i \dim \mathrm{H}^i(X)$$

whenever all but a finite number of terms of this sum are 0.

If G is a small graph, it is known that $\chi(\mathcal{F}_{\mathsf{Top}_1}(G)) = |G(1)| - |G(2)|$ whenever the cardinality of the sets $G(1)$, $G(2)$ are finite. Also, a connected small graph G is a tree if and only if $\chi(\mathcal{F}_{\mathsf{Top}_1}(G)) = 1$. As a corollary of Theorem 2.13, we get:

Corollary 6.2. *Let G be a connected small graph. If $\chi(\mathcal{F}_{\mathsf{Top}_1}(G)) = 1$, $\mathcal{L}_1\mathcal{F}_1(G)$ and $\mathcal{F}_1(G)$ are thin.*

If $(\mathfrak{g}, \mathfrak{g}_2, G)$ is a connected small computad, since $\mathcal{F}_{\mathsf{Top}_2}(\mathfrak{g}, \mathfrak{g}_2, G)$ has the same homotopy type of a wedge of spheres, closed balls and circumferences, $\mathrm{H}^0\left(\mathcal{F}_{\mathsf{Top}_2}(\mathfrak{g})\right) = \mathbb{R}$ and $\mathrm{H}^i\left(\mathcal{F}_{\mathsf{Top}_2}(\mathfrak{g})\right) = 0$ for all $i > 2$. Furthermore, assuming that $\chi(\mathcal{F}_{\mathsf{Top}_1}\mathfrak{u}_2(\mathfrak{g}, \mathfrak{g}_2, G)) = \chi(\mathcal{F}_{\mathsf{Top}_1}(G))$ and $\mathfrak{g}_2$ are finite, we have that:

$$\chi\left(\mathcal{F}_{\mathsf{Top}_2}(\mathfrak{g}, \mathfrak{g}_2, G)\right) = \chi(\mathcal{F}_{\mathsf{Top}_1}(G)) + |\mathfrak{g}_2|.$$

Remark 6.3. All considerations about $\mathcal{F}_{\mathsf{Top}_2}$ have analogues for $\mathcal{F}^{\mathsf{Gr}}_{\mathsf{Top}_2}$. In particular, if $(\mathfrak{g}, \mathfrak{g}_2, G)$ is a connected small groupoidal computad $\mathcal{F}^{\mathsf{Gr}}_{\mathsf{Top}_2}(\mathfrak{g}, \mathfrak{g}_2, G)$ is a CW-complex and has the same homotopy type of a wedge of spheres, closed balls and circumferences. Moreover, $\chi\left(\mathcal{F}^{\mathsf{Gr}}_{\mathsf{Top}_2}(\mathfrak{g}, \mathfrak{g}_2, G)\right) = \chi(\mathcal{F}_{\mathsf{Top}_1}(G)) + |\mathfrak{g}_2|$ provided that $\chi(\mathcal{F}_{\mathsf{Top}_1}(G))$ and $\mathfrak{g}_2$ are finite.

6.3. Deficiency of a Groupoid. Observe that $\sigma_2(G)$ gives a (natural) presentation of the thin category freely generated by G. More precisely, $\mathcal{P}_1\sigma_2 \cong \overline{\mathcal{M}_1}\mathcal{F}_1$. Yet, $\sigma_2(G)$ gives always a presentation of $\overline{\mathcal{M}_1}\mathcal{F}_1(G)$ with more equations than necessary.

Remark 6.4. Let G be the graph below. In this case, the set of 2-cells $\sigma_2(G)_2$ is given by $\{(w, w) : w \in \mathcal{F}_1(G)(2)\} \cup \{(yx, b), (yxa, ba), (ba, yxa), (b, yx)\}$ with obvious projections.

$$\begin{tikzcd}
\cdot \arrow[r, "a"] & \cdot \arrow[rr, "b"] \arrow[dr, "x"] & & \cdot \\
& & \cdot \arrow[ur, "y"] &
\end{tikzcd}$$

On one hand, the computad $\sigma_2(G)$ induces the presentation of $\overline{\mathcal{M}_1}\mathcal{F}_1(G)$ with the equations:

$$\begin{cases} w = w \text{ if } w \in \mathcal{F}_1(G)(\mathsf{2}) \\ yx = b \\ yxa = ba \\ b = yx \\ ba = yxa. \end{cases}$$

On the other hand, the computad

$$\mathsf{2} \rightrightarrows \mathcal{F}_1(G),$$

in which the image of one functor is the arrow yx while the image of the other functor is b, gives a presentation of $\overline{\mathcal{M}_1}\mathcal{F}_1(G)$ with less equations than $\sigma_2(G)$.

The main theorem about presentations of thin groupoids in low dimension is Theorem 6.7. This result gives a lower bound to the number of equations we need to present a thin groupoid. We start with our first result, which is a direct corollary of Theorem 5.15:

Corollary 6.5. *Let $(\mathfrak{g}, \mathfrak{g}_2, G)$ be a small connected groupoidal computad. $\mathcal{P}_{(1,0)}(\mathfrak{g}, \mathfrak{g}_2, G)$ is thin if and only if $\mathcal{F}^{\mathsf{Gr}}_{\mathsf{Top}_2}(\mathfrak{g}, \mathfrak{g}_2, G)$ is 1-connected which means that the fundamental group $\pi_1(\mathcal{F}^{\mathsf{Gr}}_{\mathsf{Top}_2}(\mathfrak{g}, \mathfrak{g}_2, G))$ is trivial.*

Proof: The fundamental group $\pi_1(\mathcal{F}_{\mathsf{Top}_2}(\mathfrak{g}, \mathfrak{g}_2, G))$ is trivial if and only if $\Pi(\mathcal{F}^{\mathsf{Gr}}_{\mathsf{Top}_2}(\mathfrak{g}, \mathfrak{g}_2, G))$ is thin. By Theorem 5.15, we conclude that $\pi_1(\mathcal{F}^{\mathsf{Gr}}_{\mathsf{Top}_2}(\mathfrak{g}, \mathfrak{g}_2, G))$ is trivial if and only if $\mathcal{P}_{(1,0)}(\mathfrak{g}, \mathfrak{g}_2, G)$ is thin. ∎

Remark 6.6. Of course, last corollary applies also to the case of presentation of groupoids via computads. More precisely, if $(\mathfrak{g}, \mathfrak{g}_2, G)$ is a small connected computad,

$$\mathcal{L}_1\mathcal{P}_1(\mathfrak{g}, \mathfrak{g}_2, G) \cong \mathcal{P}_{(1,0)}\mathcal{L}_1^{\mathsf{Cmp}}(\mathfrak{g}, \mathfrak{g}_2, G)$$

is thin if and only if the fundamental group of

$$\mathcal{F}^{\mathsf{Gr}}_{\mathsf{Top}_2}\mathcal{L}_1^{\mathsf{Cmp}}(\mathfrak{g}, \mathfrak{g}_2, G) \cong \mathcal{F}_{\mathsf{Top}_2}(\mathfrak{g}, \mathfrak{g}_2, G)$$

is trivial.

Theorem 6.7. *If* $(\mathfrak{g}, \mathfrak{g}_2, G)$ *is a small connected groupoidal computad and*

$$\mathbb{Z} \ni \chi\left(\mathcal{F}^{\mathsf{Gr}}_{\mathsf{Top}_2}(\mathfrak{g}, \mathfrak{g}_2, G)\right) < 1,$$

then $\mathcal{P}_{(1,0)}(\mathfrak{g}, \mathfrak{g}_2, G)$ *is not thin.*

Proof: Recall that

$$\chi\left(\mathcal{F}^{\mathsf{Gr}}_{\mathsf{Top}_2}(\mathfrak{g}, \mathfrak{g}_2, G)\right) = 1 - \dim \mathrm{H}^1\left(\mathcal{F}^{\mathsf{Gr}}_{\mathsf{Top}_2}(\mathfrak{g}, \mathfrak{g}_2, G)\right) + \dim \mathrm{H}^2\left(\mathcal{F}^{\mathsf{Gr}}_{\mathsf{Top}_2}(\mathfrak{g}, \mathfrak{g}_2, G)\right).$$

Therefore, by hypothesis,

$$\dim \mathrm{H}^1\left(\mathcal{F}^{\mathsf{Gr}}_{\mathsf{Top}_2}(\mathfrak{g}, \mathfrak{g}_2, G)\right) > \dim \mathrm{H}^2\left(\mathcal{F}^{\mathsf{Gr}}_{\mathsf{Top}_2}(\mathfrak{g}, \mathfrak{g}_2, G)\right).$$

In particular, we conclude that $\dim \mathrm{H}^1\left(\mathcal{F}^{\mathsf{Gr}}_{\mathsf{Top}_2}(\mathfrak{g}, \mathfrak{g}_2, G)\right) > 0$. By the Hurewicz isomorphism theorem and by the universal coefficient theorem, this fact implies that the fundamental group $\pi_1\left(\mathcal{F}^{\mathsf{Gr}}_{\mathsf{Top}_2}(\mathfrak{g}, \mathfrak{g}_2, G)\right)$ is not trivial. By Corollary 6.5, we get that $\mathcal{P}_{(1,0)}(\mathfrak{g}, \mathfrak{g}_2, G)$ is not thin. ∎

Corollary 6.8. *If* $(\mathfrak{g}, \mathfrak{g}_2, G)$ *is a small connected groupoidal computad which presents a thin groupoid and* $\chi(\mathcal{F}_{\mathsf{Top}_1}(G))$ *is finite, then*

$$\chi(\mathcal{F}_{\mathsf{Top}_1}(G)) + |\mathfrak{g}_2| - 1 \geqslant 0.$$

In particular, Corollary 6.8 implies that, if G is such that $\chi(\mathcal{F}_{\mathsf{Top}_1}(G))$ is finite, we need at least $1 - \chi(\mathcal{F}_{\mathsf{Top}_1}(G))$ equations to get a presentation of $\overline{\mathcal{M}_1}\,\overline{\mathcal{L}_1}\mathcal{F}_1(G)$.

Definition 6.9. [Finitely Presented Groupoids and Categories] A groupoid/category X is *finitely presented* if there is a small connected groupoidal computad/small connected computad $(\mathfrak{g}, \mathfrak{g}_2, G)$ which presents X, such that $\chi(\mathcal{F}_{\mathsf{Top}_1}(G))$ and $|\mathfrak{g}_2|$ are finite.

Recall the definition of deficiency of groups w.r.t. the free group monad $\overline{\mathcal{L}_0\mathcal{F}_0}$ given in 6.1. Definition 6.9 agrees with the definition of finitely $\overline{\mathcal{L}_0\mathcal{F}_0}$-presented groups. Moreover, as explained in Proposition 6.11, Definition 6.10 also agrees with the definition of $\overline{\mathcal{L}_0\mathcal{F}_0}$-deficiency of groups.

Definition 6.10. [Deficiency of a Groupoid] Let X be a finitely presented groupoid. The *deficiency of a presentation* of X by a small connected groupoidal computad $(\mathfrak{g}, \mathfrak{g}_2, G)$ is defined by $\operatorname{def}(\mathfrak{g}, \mathfrak{g}_2, G) := 1 - |\mathfrak{g}_2| - \chi(\mathcal{F}_{\mathsf{Top}_1}(G))$, provided that $|\mathfrak{g}_2|$ and $\chi(\mathcal{F}_{\mathsf{Top}_1}(G))$ are finite.

Moreover, the *deficiency of the groupoid* X is the maximum of the set
$$\left\{\left(1-\chi\left(\mathcal{F}^{\mathsf{gr}}_{\mathsf{Top}_2}(\mathfrak{g},\mathfrak{g}_2,G)\right)\right)\in\mathbb{Z} : \mathcal{P}_{(1,0)}(\mathfrak{g},\mathfrak{g}_2,G)\cong X \text{ and } \chi(\mathcal{F}_{\mathsf{Top}_1}(G))\in\mathbb{Z}\right\}.$$

Proposition 6.11. *If X is a finitely presented group, the deficiency of $\Sigma(X)$ w.r.t. presentations by groupoidal computads is equal to* def $_{\overline{\mathcal{L}_0\mathcal{F}_0}}(X)$.

Proof: This result follows from Remark 4.19. ∎

Theorem 6.8 is the first part of Corollary 6.15. The second part is Theorem 6.13 which is easy to prove: but we need to give some explicit constructions to give further consequences in 6.4. To do that, we need the terminology introduced in:

Remark 6.12. Given a small reflexive graph G, a morphism f of $\mathcal{F}^{\mathcal{R}}_1(G)$ determines a subgraph of G, namely, the smallest (reflexive) subgraph G' of G, called the image of f, such that f is a morphism of $\mathcal{F}^{\mathcal{R}}_1(G')$. More generally, given a small computad $\mathfrak{g}=(\mathfrak{g},\mathfrak{g}_2,G)$ of Rcmp, it determines a subgraph G' of G, called the *image of the computad* $\mathfrak{g}$ in G, which is the smallest graph G' satisfying the following: there is a computad $\mathfrak{g}':\mathfrak{g}_2\times\mathfrak{G}\to\mathcal{F}^{\mathcal{R}}_1(G')$ such that

$$\begin{tikzcd}
\mathfrak{g}_2\times\mathfrak{G} \arrow[r, "\mathfrak{g}'"] \arrow[dr, "\mathfrak{g}"'] & \mathcal{F}^{\mathcal{R}}_1(G') \arrow[d] \\
 & \mathcal{F}^{\mathcal{R}}_1(G)
\end{tikzcd}$$

commutes. We also can consider the *graph domain* and the *graph codomain* of a small computad $\mathfrak{g}=(\mathfrak{g},\mathfrak{g}_2,G)$, $\mathfrak{g}:\mathfrak{G}^{\mathrm{op}}\to\mathsf{Cat}$, which are respectively the smallest subgraphs $\mathfrak{g}^{d^1}$ and $\mathfrak{g}^{d^0}$ of G such that $\mathfrak{g}(d^1)(\mathfrak{g}_2\times 2)$ and $\mathfrak{g}(d^0)(\mathfrak{g}_2\times 2)$ are respectively in $\mathcal{F}^{\mathcal{R}}_1(\mathfrak{g}^{d^1})$ and $\mathcal{F}^{\mathcal{R}}_1(\mathfrak{g}^{d^0})$.

Of course, we can consider the notions introduced above in the category of computads or groupoidal computads as well.

Theorem 6.13. *Let G be a small connected graph such that $\chi(\mathcal{F}_{\mathsf{Top}_1}(G))\in\mathbb{Z}$ (equivalently, $\pi_1(\mathcal{F}_{\mathsf{Top}_1}(G))$ is finitely generated). There is a groupoidal computad $(\widehat{\mathfrak{g}},\mathfrak{g}_2,G)$ which presents $\overline{\mathcal{M}_1}\,\overline{\mathcal{L}_1}\mathcal{F}_1(G)$ such that $|\mathfrak{g}_2|=1-\chi(\mathcal{F}_{\mathsf{Top}_1}(G))$.*

Proof: Without losing generality, in this proof we consider reflexive graphs, and computads over reflexive graphs. Let G be a small reflexive connected graph such that its fundamental group is finitely generated. If G_{mtree} is the maximal (reflexive) tree of G, we know that the image of the (natural) morphism of reflexive graphs $G \to G/G_{\mathrm{mtree}}$ by the functor $\mathcal{L}_1\mathfrak{F}_1^{\mathcal{R}}$ is an equivalence. That is to say, we have a natural equivalence $\mathcal{L}_1\mathfrak{F}_1^{\mathcal{R}}(G) \to \mathcal{L}_1\mathfrak{F}_1^{\mathcal{R}}(G/G_{\mathrm{mtree}})$ which is in the image of $\mathcal{L}_1\mathfrak{F}_1^{\mathcal{R}}$.

In particular, each arrow f of G/G_{mtree} corresponds to a unique arrow $\widehat{f}$ of G such that $\widehat{f}$ is not an arrow of G_{mtree} and the image of $\widehat{f}$ by $G \to G/G_{\mathrm{mtree}}$ is f.

Recall that, since G/G_{mtree} has only one object, $\mathcal{L}_1\mathfrak{F}_1^{\mathcal{R}}(G/G_{\mathrm{mtree}})$ is the suspension of the group freely generated by the set $G/G_{\mathrm{mtree}}(2)$ of arrows. By hypothesis, this set is finite and has $1 - \chi(\mathfrak{F}_{\mathsf{Top}_1}(G)) \in \mathbb{N}$ elements. Thereby we have a computad $\mathfrak{g} : \mathfrak{G}^{\mathrm{op}} \to \mathsf{cat}$,

$$\mathfrak{g}_2 \times 2 \rightrightarrows \mathfrak{F}_1^{\mathcal{R}}(G/G_{\mathrm{mtree}}),$$

in which $\mathfrak{g}_2 := G/G_{\mathrm{mtree}}(2)$, $\mathfrak{g}(d^0)(f, 0 \to 1) = f$ and $\mathfrak{g}(d^0)(f, 0 \to 1) = \mathrm{id}$. The computad $\mathcal{L}_1^{\mathsf{Rcmp}}(\mathfrak{g}) : \mathfrak{G}^{\mathrm{op}} \to \mathsf{gr}$ gives a presentation of the trivial group.

The computad $\mathfrak{g}$ lifts through $G \to G/G_{\mathrm{mtree}}$ to a (small) groupoidal computad $\widehat{\mathfrak{g}} : \mathfrak{G}^{\mathrm{op}} \to \mathsf{cat}$ over G. More precisely, we define $\widehat{\mathfrak{g}} = (\mathfrak{g}, \mathfrak{g}_2, G)$,

$$\widehat{\mathfrak{g}}(1) = \mathcal{L}_1\mathfrak{F}_1^{\mathcal{R}}(G),\ \widehat{\mathfrak{g}}(2) = \mathfrak{g}_2 \times 2,\ \widehat{\mathfrak{g}}(d^0)(f, 0 \to 1) = \widehat{f} \ \text{ and } \ \widehat{\mathfrak{g}}(d^1)(f, 0 \to 1) = \dot{f}$$

in which $\dot{f}$ is the unique morphism of the (thin) subgroupoid $\mathcal{L}_1\mathfrak{F}_1^{\mathcal{R}}(G_{\mathrm{mtree}})$ of $\mathcal{L}_1\mathfrak{F}_1^{\mathcal{R}}(G)$ such that the domain and codomain of $\dot{f}$ coincide respectively with the domain and codomain of $\widehat{f}$.

Of course, this construction provides a 2-natural transformation which is pointwise an equivalence $\widehat{\mathfrak{g}} \longrightarrow \mathcal{L}_1^{\mathsf{Rcmp}}(\mathfrak{g})$,

$$\begin{tikzcd}
\mathfrak{g}_2 \times 2 \arrow[r, shift left] \arrow[r, shift right] \arrow[d, equal] & \mathcal{L}_1\mathfrak{F}_1^{\mathcal{R}}(G) \arrow[d] \\
\mathfrak{g}_2 \times 2 \arrow[r, shift left] \arrow[r, shift right] & \mathcal{L}_1\mathfrak{F}_1^{\mathcal{R}}(G/G_{\mathrm{mtree}}).
\end{tikzcd}$$

It is easy to see that, in this case, it induces an equivalence between the coequalizers. Thereby $\widehat{\mathfrak{g}}$ presents a thin groupoid, which is $\mathcal{L}_1\mathcal{P}_1(\sigma_2^{\mathsf{Gr}}(G))$. This completes the proof. ∎

Remark 6.14. The graph domain and the graph codomain of the computad $\widehat{\mathfrak{g}}$ constructed in the proof above are, respectively, inside and outside the maximal tree G_{mtree}. More precisely, for every $\alpha \in \widehat{\mathfrak{g}}_2 = \mathfrak{g}_2$, the $\widehat{\mathfrak{g}}(d^1)(\alpha, 0 \to$

$1)$ is a morphism of $\mathcal{L}_1\mathcal{F}_1^{\mathcal{R}}(G_{\mathrm{mtree}})$ and $\widehat{\mathfrak{g}}(d^0)(\alpha, \mathsf{0} \to \mathsf{1})$ is a morphism of length one which is not an arrow of G_{mtree}.

By Theorem 6.13 and Theorem 6.7 we get:

Corollary 6.15. *The deficiency of a finitely presented thin groupoid is* 0. *In particular, this result generalizes the fact that the deficiency of the trivial group is* 0.

Remark 6.16. [Finite measure and deficiency] Let $\mathbb{R}^+_\infty$ be the category whose structure comes from the totally ordered set of the non-negative real numbers with a top element ∞ with the usual order. The initial object is of course 0, while the terminal object is ∞.

Let $\mathfrak{X}'$ be the subcategory of monomorphisms of a category $\mathfrak{X}$. A *finite (strong/naive) measure* on $\mathfrak{X}$ is a functor $\mu : \mathfrak{X}' \to \mathbb{R}^+_\infty$ that preserves finite coproducts (including the empty coproduct, which is the initial object).

A pair $(\mathfrak{X}, \mu)$ together with a monad $\mathcal{T}$ on $\mathfrak{X}$ give rise to a notion of *finite $\mathcal{T}$-presentation*: a presentation as in the $\mathcal{T}$-presentation diagram is μ-finite if $\mu(G_1)$ and $\mu(G_2)$ are finite. In this case, we define the *$(\mathcal{T}, \mu)$-deficiency* of such a μ-finite $\mathcal{T}$-presentation by $\mathrm{def}_{(\mathcal{T},\mu)} := \mu(G_1) - \mu(G_2)$. If X is a $\mathcal{T}$-algebra which admits a finite presentation, X is called finitely $\mathcal{T}$- presented.

For instance, *cardinality* is a measure on the category of sets Set which induces the notions of finite $\mathcal{T}$-presentation and $\mathcal{T}$-deficiency of algebras over sets given in 6.1.

Finally, consider the category of graphs $\mathsf{Grph}_{\mathsf{finEu}}$ with finite Euler characteristic: the measure *Euler characteristic* χ and the monad $\overline{\mathcal{L}_1\mathcal{F}_1}$ induce the notion of $(\overline{\mathcal{L}_1\mathcal{F}_1}, \chi)$-deficiency of an $\overline{\mathcal{L}_1\mathcal{F}_1}$-presentation. If we consider the inclusion of Theorem 4.5, this notion of deficiency coincides with the notion of deficiency of a presentation via groupoidal computad given in 6.3.

6.4. Presentation of Thin Categories. The results on presentations of thin groupoids can be used to study presentations of thin categories. For instance, if a presentation of a thin groupoid can be lifted to a presentation of a category, then this category is thin provided that the lifting presents a category that satisfies the cancellation law. To make this statement precise (which is Proposition 6.18), we need:

Definition 6.17. [Lifting Groupoidal Computads] We denote by $\mathsf{cmp}_{\mathsf{lift}}$ the pseudopullback (iso-comma category) of $\mathcal{P}_{(1,0)} : \mathsf{cmp}_{\mathsf{gr}} \to \mathsf{gr}$ along $\mathcal{L}_1\mathcal{P}_1$:

$\mathsf{cmp} \to \mathsf{gr}$. A small computad $\mathfrak{g}$ is called a *lifting of the small groupoidal computad* $\mathfrak{g}'$ if there is an object $\zeta^{\mathfrak{g}}_{\mathfrak{g}'}$ of $\mathsf{cmp}_{\mathsf{lift}}$ such that the images of this object by the functors

$$\mathsf{cmp}_{\mathsf{lift}} \to \mathsf{cmp}_{\mathsf{gr}}, \qquad \mathsf{cmp}_{\mathsf{lift}} \to \mathsf{cmp}$$

are respectively $\mathfrak{g}'$ and $\mathfrak{g}$.

Proposition 6.18. *If $\mathfrak{g}'$ is a groupoidal computad that presents a thin groupoid and $\mathcal{P}_1(\mathfrak{g})$ satisfies the cancellation law, then $\mathfrak{g}$ presents a thin category.*

Proof: By hypothesis, $\mathcal{P}_{(1,0)}(\mathfrak{g}') \cong \mathcal{L}_1\mathcal{P}_1(\mathfrak{g})$ is a thin groupoid and $\mathcal{P}_1(\mathfrak{g})$ satisfies the cancellation law. Hence $\mathcal{P}_1(\mathfrak{g})$ is a thin category. ■

Theorem 6.19. *If G is small connected fair graph such that $\chi(\mathcal{F}_{\mathsf{Top}_1}(G)) \in \mathbb{Z}$, then there is a computad $(\mathfrak{g}, \mathfrak{g}_2, G)$ of cmp such that $|\mathfrak{g}_2| = 1 - \chi(\mathcal{F}_{\mathsf{Top}_1}(G))$ which presents the groupoid $\overline{\mathcal{M}_1}\mathcal{L}_1\mathcal{F}_1(G)$.*

Proof: Let G_{mtree} be a maximal weak tree of G which is also the maximal tree. Let $(\widehat{\mathfrak{g}}, \mathfrak{g}_2, G)$ be the groupoidal computad constructed in the proof of Theorem 6.13 using the maximal tree G_{mtree}.

We will prove that the groupoidal computad $(\widehat{\mathfrak{g}}, \mathfrak{g}_2, G)$ can be lifted to a computad. In order to do so, we need to prove that, for each $\alpha \in \mathfrak{g}_2$, the restriction $\widehat{\mathfrak{g}}|_\alpha : \mathfrak{G}^{\mathrm{op}} \to \mathsf{gr}$,

$$\{\alpha\} \times \mathsf{2} \rightrightarrows \mathcal{L}_1\mathcal{F}_1(G),$$

can be lifted to a small computad. By Remark 6.14, we know that $\widehat{\mathfrak{g}}(d^1)(\alpha, \mathsf{0} \to \mathsf{1})$ is a morphism of $\mathcal{L}_1\mathcal{F}_1(G_{\mathrm{mtree}})$ and $\widehat{\mathfrak{g}}(d^0)(\alpha, \mathsf{0} \to \mathsf{1})$ is a morphism of length one which is not an arrow of G_{mtree}. Since G_{mtree} is a maximal weak tree, we conclude that the image of $\widehat{\mathfrak{g}}|_\alpha$ is not a weak tree. Hence there are parallel morphisms f_0, f_1 of $\mathcal{F}_1(G)$ that determine the same graph determined by the image of $\widehat{\mathfrak{g}}|_\alpha$ (see Remark 6.12) such that f_0 is a morphism of $\mathcal{F}_1(G_{\mathrm{mtree}})$. Therefore, $\widehat{\mathfrak{g}}$ can be lifted to $(\mathfrak{g}|_\alpha, \{\alpha\}, G)$ given by $\mathfrak{g}|_\alpha : \mathfrak{G}^{\mathrm{op}} \to \mathsf{cat}$, $\mathfrak{g}|_\alpha(d^0)(\alpha, \mathsf{0} \to \mathsf{1}) = f_0$ and $\mathfrak{g}|_\alpha(d^1)(\alpha, \mathsf{0} \to \mathsf{1}) = f_1$. ■

As a corollary of the proof, we get:

Corollary 6.20. *If G is small connected fair graph such that $\chi(\mathcal{F}_{\mathsf{Top}_1}(G)) \in \mathbb{Z}$ and $(\mathfrak{g}, \mathfrak{g}_2, G)$ is a small computad which presents $\overline{\mathcal{M}_1}\mathcal{F}_1(G)$, then $|\mathfrak{g}_2| \geqslant 1 - \chi(\mathcal{F}_{\mathsf{Top}_1}(G))$.*

Proof: As consequence of the constructions involved in the last proof, for each 2-cell of the computad $\mathfrak{g}$ of Theorem 6.19, there are parallel morphisms in $\mathcal{F}_1(G)$ such that they can be represented by (completely) different lists of arrows of G. ∎

However, in the conditions of the result above, often we need more than $1 - \chi(\mathcal{F}_{\mathsf{Top}_1}(G)))$ equations. The point is that the lifting given in Theorem 6.19 often does not present a category that satisfies the cancellation law. As consequence of the proof of Theorem 6.19, we get a generalization. More precisely:

Corollary 6.21. *Let G be a small connected graph such that $\chi(\mathcal{F}_{\mathsf{Top}_1}(G)) \in \mathbb{Z}$. Consider the groupoidal computad $(\widehat{\mathfrak{g}}, \mathfrak{g}_2, G)$ constructed in Theorem 6.13.*

There is a largest groupoidal computad of the type $(\widehat{\mathfrak{h}}, \mathfrak{h}_2, G)$ which is a subcomputad of $\widehat{\mathfrak{g}}$ and can be lifted to a computad $(\mathfrak{h}, \mathfrak{h}_2, G)$ in the sense of Definition 6.17. We have that $\min\left\{|\mathfrak{x}_2| : \mathcal{P}_1(\mathfrak{x}, \mathfrak{x}_2, G) \cong \overline{\mathcal{M}_1}\mathcal{F}_1(G)\right\} \geqslant |\mathfrak{h}_2|$.

Definition 6.22. A pair (G, G_{mtree}) is called a *monotone graph* if G is a small connected graph, $\chi(\mathcal{F}_{\mathsf{Top}_1}(G))) \in \mathbb{Z}$, G_{mtree} is a maximal weak tree of G and, whenever there exists an arrow $f : x \to y$ in G, either $x \leqslant y$ or $y \leqslant x$ in which $\leqslant$ is the partial order of the poset $\mathcal{F}_1(G_{\mathrm{mtree}})$.

If (G, G_{mtree}) is a *monotone graph* and $f : x \to y$ is an arrow such that $y \leqslant x$, f is called a *nonincreasing arrow* of the monotone graph. Finally, if (G, G_{mtree}) does not have nonincreasing arrows, (G, G_{mtree}) is called a *strictly increasing graph.*

Theorem 6.23. *Let (G, G_{mtree}) be a strictly increasing graph. There is a computad $(\mathfrak{g}, \mathfrak{g}_2, G)$ such that $|\mathfrak{g}_2| = 1 - \chi(\mathcal{F}_{\mathsf{Top}_1}(G)))$ which presents $\overline{\mathcal{M}_1}\mathcal{F}_1(G, G_{\mathrm{mtree}})$.*

Proof: For each arrow $f : x \to y$ outside the maximal weak tree G_{mtree}, there is a unique morphism $\dot{f} : x \to y$ in $\mathcal{F}_1(G_{\mathrm{mtree}})$. It is enough, hence, to define $\mathfrak{g}_2 := \{\alpha_f : f \in G(2) - G_{\mathrm{mtree}}(2)\}$, $\mathfrak{g}(d^0)(\alpha_f, \mathsf{0} \to \mathsf{1}) := \dot{f}$ and $\mathfrak{g}(d^1)(\alpha_f, \mathsf{0} \to \mathsf{1}) := f$. It is clear that this is a lifting of the groupoidal computad $\widehat{\mathfrak{g}}$ of Theorem 6.13. Actually, $\mathfrak{g}$ is precisely the lifting given by Theorem 6.19. Moreover, it is also easy to see that $\mathcal{P}_1(\mathfrak{g})$ satisfies the cancellation law. Therefore the category presented by $\mathfrak{g}$ is thin. ∎

As a consequence of Corollary 6.20 and Theorem 6.23, we get:

Corollary 6.24. *Let* (G, G_{mtree}) *be a strictly increasing graph. The minimum of the set* $\left\{|\mathfrak{g}_2| : \mathcal{P}_1(\mathfrak{g}, \mathfrak{g}_2, G) \cong \overline{\mathcal{M}_1}\mathcal{F}_1(G)\right\}$ *is equal to* $1 - \chi(\mathcal{F}_{\mathsf{Top}_1}(G))$.

Theorem 6.25. *Let* (G, G_{mtree}) *be a monotone graph with precisely* n *nonincreasing arrows. There is a computad* $(\mathfrak{g}, \mathfrak{g}_2, G)$ *such that* $|\mathfrak{g}_2| = 1 - \chi(\mathcal{F}_{\mathsf{Top}_1}(G))) + \mathsf{n}$ *which presents* $\overline{\mathcal{M}_1}\mathcal{F}_1(G, G_{\mathrm{mtree}})$.

Proof: For each nonincreasing arrow $f : x \to y$ outside the maximal weak tree G_{mtree}, either there is a unique morphism $\dot{f} : y \to x$ in $\mathcal{F}_1(G_{\mathrm{mtree}})$ or there is a unique $\dot{f} : x \to y$ in $\mathcal{F}_1(G_{\mathrm{mtree}})$. We define A^* the set of the nonincreasing arrows of G outside G_{mtree} and $\mathsf{A} := G(2) - G_{\mathrm{mtree}}(2) - \mathsf{A}^*$. We define

$$\mathfrak{g}_2 := \{\alpha_f : f \in \mathsf{A}\} \cup \left\{\beta_{(f,j)} : f \in \mathsf{A}^* \text{ and } j \in \{-1, 1\}\right\},$$

$$\mathfrak{g}(d^0)(\alpha_f, 0 \to 1) := \dot{f}, \quad \mathfrak{g}(d^1)(\alpha_f, 0 \to 1) := f,$$

$$\mathfrak{g}(d^0)(\beta_{(f,1)}, 0 \to 1) := \dot{f}f, \quad \mathfrak{g}(d^1)(\beta_{(f,1)}, 0 \to 1) := \mathrm{id},$$

$$\mathfrak{g}(d^0)(\beta_{(f,-1)}, 0 \to 1) := f\dot{f}, \quad \mathfrak{g}(d^1)(\beta_{(f,-1)}, 0 \to 1) := \mathrm{id}.$$

It is clear that is a lifting of the groupoidal computad $\widehat{\mathfrak{g}}$ of Theorem 6.13. Actually, the lifting given by Theorem 6.19 is a subcomputad of $\mathfrak{g}$. Moreover, it is also easy to see that $\mathcal{P}_1(\mathfrak{g})$ satisfies the cancellation law. Therefore the category presented by $\mathfrak{g}$ is thin. ■

Remark 6.26. If we generalize the notion of deficiency of a groupoid and define: the *deficiency of a finitely presented category* X (by presentations via computads) is, if it exists, the maximum of the set

$$\left\{\left(1 - \chi\left(\mathcal{F}_{\mathsf{Top}_2}(\mathfrak{g}, \mathfrak{g}_2, G)\right)\right) \in \mathbb{Z} : \mathcal{P}_1(\mathfrak{g}, \mathfrak{g}_2, G) \cong X \text{ and } \chi(\mathcal{F}_{\mathsf{Top}_1}(G)) \in \mathbb{Z}\right\},$$

then, given a strictly increasing graph (G, G_{mtree}), the deficiency of $\overline{\mathcal{M}_1}\mathcal{F}_1(G, G_{\mathrm{mtree}})$ is 0. However, for instance, the deficiency of the thin category (by presentation of computads) $\nabla 2$ is not 0: it is -1. More generally, by Corollary 6.20 the deficiency of category freely generated by a tree (characterized in Theorem 2.22 and Corollary 2.23) is 0, while the deficiency of category freely generated by a weak tree G is $\chi(G) - 1$. Furthermore, if (G, G_{mtree}) is a monotone graph, the deficiency (by presentations via computads) of $\overline{\mathcal{M}_1}\mathcal{F}_1(G, G_{\mathrm{mtree}})$ is $-\mathsf{n}$ in which n is the number of nontrivial isomorphisms of X (see Theorem 6.25).

7. Higher Dimensional Icons

Icons were originally defined in [23]. They were introduced as a way of organizing bicategories in a 2-category, recovering information of the tricategory of bicategories, pseudofunctors, pseudonatural/oplax natural transformations and modifications. Thereby, icons allow us to study aspects of these 2-categories of 2-categories/bicategories via 2-dimensional universal algebra.

There are examples of applications of this concept in [23, 22]. In this setting, on one hand, we get a 2-category $2\mathsf{Cat}$ which is the 2-category of 2-categories, 2-functors and icons. On the other hand, we have the 2-category Bicat of bicategories, pseudofunctors and icons.

The inclusion $2\mathsf{Cat} \to \mathsf{Bicat}$ can be seen as an inclusion of strict algebras in the pseudoalgebras of a 2-monad. Therefore, we can apply 2-monad theory to get results about these categories of algebras. The 2-monadic coherence theorem [4, 21, 24] can be applied to this case and we get, in particular, the celebrated result that states that "every bicategory is biequivalent to a 2-category".

In Section 8, we show that the 2-categories $2\mathsf{Cat}$ and Bicat provide a concise way of constructing freely generated 2-categories as coinserters. We also show analogous descriptions for n-categories. In order to do so, we give a definition of higher dimensional icon and construct 2-categories $n\mathsf{Cat}$ of n-categories in this section. It is important to note that there are many higher dimensional versions of icons and, of course, the best choice depends on the context. The definition of 3-dimensional icon presented herein is similar to that of "ico-icon" introduced in [11], but the scope herein is limited to strict n-categories.

Definition 7.1. [V-graphs] Let V be a 2-category. An object G of the 2-category $V\mathsf{Grph}$ is a discrete category $G(\mathsf{1}) = G_0$ of Cat with a hom-object $G(A,B)$ of V for each ordered pair (A,B) of objects of $G(\mathsf{1})$.

A 1-cell $F : G \to H$ of $V\mathsf{Grph}$ is a functor $F_0 : G(\mathsf{1}) \to H(\mathsf{1})$ with a collection of 1-cells $\left\{F_{(A,B)} : G(A,B) \to H(F_0(A), F_0(B))\right\}_{(A,B)\in G_0\times G_0}$ of V. The composition of 1-cells in $V\mathsf{Grph}$ is defined in the obvious way.

A 2-cell $\alpha : F \Rightarrow G$ is a collection of 2-cells $\left\{\alpha_{(A,B)} : F_{(A,B)} \Rightarrow G_{(A,B)}\right\}_{(A,B)\in G_0^2}$ in V. It should be noted that the existence of such a 2-cell implies, in particular, that $F_0 = G_0$. The horizontal and vertical compositions of 2-cells in $V\mathsf{Grph}$ come naturally from the horizontal and vertical compositions in V.

Let V be a 2-category with finite products and large coproducts (indexed in discrete categories of Cat). Assume that V is distributive w.r.t. these large coproducts. We can define a 2-monad $\mathcal{T}_V$ on $V\mathsf{Grph}$ such that $\mathcal{T}_V(G)_0 = G_0$ and

$$\mathcal{T}_V(G)(A,B) = \sum_{j\in\mathbb{N}} \sum_{(C_1,\dots,C_j)\in G_0^j} G(C_j,B)\times\cdots\times G(C_1,C_2)\times G(A,C_1),$$

in which $\sum$ denotes coproduct and this coproduct includes the term for $j=0$ which is $G(A,B)$. The actions of $\mathcal{T}_V$ on the 1-cells and 2-cells are defined in the natural way. The component $m_{_G} : \mathcal{T}_V^2(G) \longrightarrow \mathcal{T}_V(G)$ of the multiplication is identity on objects, while the 1-cells between the hom-objects are induced by the isomorphisms given by the distributivity and identities $G(C_j,B)\times\cdots\times G(A,C_1) = G(C_j,B)\times\cdots\times G(A,C_1)$. The component $\eta_{_G} : G \longrightarrow \mathcal{T}_V(G)$ of the unit is identity on objects and the 1-cells between the hom-objects are given by the natural morphisms $G(A,B) \to \sum_{j\in\mathbb{N}}\sum_{(C_1,\dots,C_j)\in G_0^j} G(C_j,B)\times\cdots\times G(A,C_1)$ which correspond to the "natural inclusions" for $j=0$.

In this context, we denote by V-Cat the category of V-enriched categories (described in Section 1) w.r.t. the underlying cartesian category of V.

Lemma 7.2. *Let V be a 2-category satisfying the properties above. The underlying category of the 2-category of strict 2-algebras $\mathcal{T}_V$-$\mathsf{Alg}_{\mathsf{s}}$ is equivalent to V-Cat.*

Proof: This follows from a classical result that states that the enriched categories are the Eilenberg-Moore algebras of the underlying monad of $\mathcal{T}_V$. See, for instance, [3]. ∎

Remark 7.3. We could consider the general setting of a 2-category V with a monoidal structure which preserves (large) coproducts (see, for instance, [30]), but this is not in our scope.

Corollary 7.4. *The underlying category of the 2-category of strict 2-algebras $\mathcal{T}_{\mathsf{Cat}}$-$\mathsf{Alg}_{\mathsf{s}}$ is equivalent to 2-Cat.*

Definition 7.5. [$n\mathsf{Cat}$] We define $2\mathsf{Cat} := \mathcal{T}_{\mathsf{Cat}}$-$\mathsf{Alg}_{\mathsf{s}}$ and $\mathsf{Bicat} :=$ Ps-$\mathcal{T}$-Alg. An icon is just a 2-cell of Bicat. More generally, we define

$$n\mathsf{Cat} := \mathcal{T}_{(n-1)\mathsf{Cat}}\text{-}\mathsf{Alg}_{\mathsf{s}}.$$

The 2-cells of $n\mathsf{Cat}$ are called *n-icons*. Following this definition, *icons* are also called 2-*icons* and 1-*icons* are just natural transformations between functors.

Proposition 7.6. *The underlying category of $n\mathsf{Cat}$ is the category of n-categories and n-functors n-Cat.*

Remark 7.7. We say that an internal graph $\mathfrak{d} : \mathfrak{G}^{\mathrm{op}} \to n\text{-}\mathsf{Cat}$ satisfies the *n-coincidence property* if, whenever $\mathbb{N} \ni m \leqslant n$, $\mathfrak{d}(d^1)(\kappa) = \mathfrak{d}(d^0)(\kappa)$ for every m-cell κ of X.

If $F, G : X \to Y$ are n-functors, $n > 1$ and there is an n-icon $\alpha : F \Rightarrow G$, then, in particular, the pair (F, G) defines an internal graph

$$X \overset{F}{\underset{G}{\rightrightarrows}} Y$$

in $n\mathsf{Cat}$ (or n-Cat) that satisfies the $(n-2)$-coincidence property. For instance, if there is an icon $\alpha : F \Rightarrow G$ between 2-functors (or pseudofunctors), then the internal graph defined by (F, G) satisfies the 0-coincidence property: this means that $F(\kappa) = G(\kappa)$ for any 0-cell (object) κ of X.

Definition 7.8. [Universal n-cell] For each $n \in \mathbb{N}$, we denote by $\mathsf{2}_n$ the n-category with a nontrivial n-cell $\widehat{\kappa}$ with the following universal property: *if κ is an n-cell of an n-category X, then there is a unique n-functor $F : \mathsf{2}_n \to X$ such that $F(\widehat{\kappa}) = \kappa$.*

Remark 7.9. We have isomorphisms $\mathsf{2}_1 \cong \mathsf{2}$ and $\mathsf{2}_0 \cong \mathsf{1}$. Moreover, in general, $\mathsf{2}_n$ is an n-category but we also denote by $\mathsf{2}_n$ the image of this n-category by the inclusion $n\text{-}\mathsf{Cat} \to (n+m)\text{-}\mathsf{Cat}$ for $m \geqslant 1$. Therefore, for instance, we can consider inclusions $\mathsf{2}_n \to \mathsf{2}_{n+m}$ which are $(n+m)$-functors, *i.e.* morphisms of $(n+m)$-Cat.

Of course, $\mathsf{2}_n$ has a unique nontrivial n-cell. This n-cell is *denoted herein by $\widehat{\kappa}_n$*, or just $\widehat{\kappa}$ whenever it does not cause confusion.

Theorem 7.10. *Let $F, G : \mathsf{2}_n \to Y$ be $(n+1)$-functors such that $F(\kappa) = G(\kappa)$ for all m-cell κ, provided that $m < n$. There is a one-to-one correspondence between the $(n+1)$-cells $F(\widehat{\kappa}) \Longrightarrow G(\widehat{\kappa})$ of Y and $(n+1)$-icons $F \Rightarrow G$.*

8. Higher Computads

Recall that a derivation scheme is a pair $(\mathfrak{d}, \mathfrak{d}_2)$ in which $\mathfrak{d}_2$ is a discrete category and $\mathfrak{d} : \mathfrak{G}^{\mathrm{op}} \to \mathsf{Cat}$ is an internal graph with the same format of $\mathfrak{d}$-diagram (described in Section 4) satisfying the 0-coincidence property. Roughly, the 2-category freely generated by a derivation scheme is the category $\mathfrak{d}(\mathsf{1})$ freely added with the 2-cells of $\mathfrak{d}_2$ in the following way, for each $\alpha \in \mathfrak{d}_2$, we freely add a 2-cell

$$\alpha : \mathfrak{d}(d^1)(\alpha, \mathsf{0} \to \mathsf{1}) \Rightarrow \mathfrak{d}(d^0)(\alpha, \mathsf{0} \to \mathsf{1}).$$

This construction is described in [34]. More precisely, it is constructed a 2-category $\mathcal{F}_{\textsf{2-Der}}(\mathfrak{d})$ with the following universal property: a 2-functor $G : \mathcal{F}_{\textsf{2-Der}}(\mathfrak{d}) \to X$ is uniquely determined by a pair (G_1, G_2) in which $G_1 : \mathfrak{d}(1) \to X$ is a 2-functor (between categories) and $G_2 : \mathfrak{d}_2 \to \textsf{2-Cat}(2_2, X)$ is a 2-functor (between discrete categories) satisfying the codomain and domain conditions, which means that, given $\alpha \in \mathfrak{d}_2$, the 1-cell domain of $G_2(\alpha)$ is equal to $\mathfrak{d}(d^1)(\alpha, \mathsf{0} \to \mathsf{1})$ and the codomain of $G_2(\alpha)$ is equal to $\mathfrak{d}(d^0)(\alpha, \mathsf{0} \to \mathsf{1})$.

Theorem 8.1. *There is a functor* $\mathcal{F}_{\textsf{2-Der}} : \textsf{Der} \to \textsf{2-Cat}$ *which gives the 2-category freely generated by each derivation scheme. Furthermore, for each derivation scheme* $(\mathfrak{d}, \mathfrak{d}_2)$*,*

$$\mathcal{F}_{\textsf{2-Der}}(\mathfrak{d}) \cong \mathfrak{I} \ast \mathfrak{d},$$

in which, by abuse of language, $\mathfrak{I} \ast \mathfrak{d}$ *denotes the coinserter in* $2\textsf{Cat}$ *of the internal graph* $\mathfrak{d}$ *composed with the inclusion* $\textsf{Cat} \to 2\textsf{Cat}$.

Proof: An object of the inserter

$$2\textsf{Cat}(\mathfrak{d}(1), X) \rightrightarrows 2\textsf{Cat}(\mathfrak{d}_2 \times 2, X).$$

is a 2-functor $G_1 : \mathfrak{d}(1) \to X$ and an icon $G_1\left(\mathfrak{d}(d^1)\right) \Rightarrow G_1\left(\mathfrak{d}(d^0)\right)$ which means a 2-cell $G_2(\alpha)$ for each $\alpha \in \mathfrak{d}_2$ by Theorem 7.10 such that the 1-cell domain of $G_2(\alpha)$ is equal to $\mathfrak{d}(d^1)(\alpha, \mathsf{0} \to \mathsf{1})$ and the codomain of $G_2(\alpha)$ is equal to $\mathfrak{d}(d^0)(\alpha, \mathsf{0} \to \mathsf{1})$. This proves that the coinserter is determined by the universal properties of the 2-category freely generated by the derivation scheme of $\mathfrak{d}$. ■

We already can construct the 2-category freely generated by a computad. This is precisely the 2-category freely generated by its underlying derivation scheme. More precisely, there is an obvious forgetful functor $\textsf{Cmp} \to \textsf{Der}$ and the functor $\mathcal{F}_2 : \textsf{Cmp} \to \textsf{2-Cat}$ is obtained from the composition of such forgetful functor with $\mathcal{F}_{\textsf{2-Der}}$.

Definition 8.2. $[2_n]$ For each $n \in \mathbb{N}$, of course, there are precisely two inclusions $2_{(n-1)} \to 2_n$. This gives an n-functor

$$\mathfrak{I}_n : \mathfrak{G} \to n\text{-}\textsf{Cat}, \qquad 2_{(n-1)} \rightrightarrows 2_n.$$

Definition 8.3. $[\mathfrak{G}_n]$ Consider the usual forgetful functor $(n+1)\text{-}\textsf{Cat} \to n\text{-}\textsf{Cat}$. The image of $2_{(n+1)}$ by this forgetful functor is denoted by $\mathfrak{G}_n$.

Lemma 8.4. *The internal graph of Definition 8.2 induces an n-functor $\mathsf{2}_{(n-1)}\coprod\mathsf{2}_{(n-1)}\to\mathsf{2}_n$. The pushout in n-Cat of this n-functor along itself is isomorphic to $\mathfrak{G}_n$.*

Furthermore, there is an inclusion n-functor $\mathfrak{G}_{(n-1)}\to\mathsf{2}_n$ induced by the counit of the adjunction with right adjoint being n-$\mathsf{Cat}\to(n-1)$-Cat. The pushout in n-Cat of this inclusion along itself is isomorphic to $\mathfrak{G}_n$.

Definition 8.5. [Higher Derivation Schemes] Consider the functor $(-\times\mathfrak{G}_{n-1}):\mathsf{SET}\to(n-1)$-$\mathsf{Cat}, Y\mapsto Y\times\mathfrak{G}$. The *category of derivation n-schemes* is the comma category n-$\mathsf{Der}:=(-\times\mathfrak{G}_{n-1}/\mathrm{Id}_{\mathsf{Cat}})$.

Remark 8.6. Of course, $\mathsf{Der}=2$-Der. Also, it is clear that the derivation n-scheme is just a pair $(\mathfrak{d},\mathfrak{d}_2)$ in which $\mathfrak{d}_2$ is a discrete category and $\mathfrak{d}:\mathfrak{G}^{\mathrm{op}}\to(n-1)$-$\mathsf{Cat}$ is an internal graph

$$\mathfrak{d}_2\times\mathsf{2}_{(n-1)}\rightrightarrows\mathfrak{d}(1)$$

satisfying the $(n-2)$-coincidence property.

We can define a forgetful functor $\mathfrak{C}_{n\text{-}\mathsf{Der}}:n$-$\mathsf{Cat}\to n$-$\mathsf{Der}$ where $\mathfrak{C}_{n\text{-}\mathsf{Der}}(X):\mathfrak{G}^{\mathrm{op}}\to(n-1)$-$\mathsf{Cat}$ is an internal graph (derivation scheme)

$$n\text{-}\mathsf{Cat}(\mathsf{2}_n,X)\times\mathsf{2}_{(n-1)}\rightrightarrows X$$

in which n-$\mathsf{Cat}(\mathsf{2}_n,X)$ denotes the set of n-functors $\mathsf{2}_n\to X$ and, by abuse of language, X is the underlying $(n-1)$-category of X. This graph is obtained from the graph n-$\mathsf{Cat}[\mathfrak{I}_n-,X]:\mathfrak{G}^{\mathrm{op}}\to n$-$\mathsf{Cat}$: firstly, we compose each nontrivial morphism in the image of this graph with the inclusion n-$\mathsf{Cat}(\mathsf{2}_n,X)\to n$-$\mathsf{Cat}[\mathsf{2}_n,X]$ (induced by the counit of the adjunction given by the inclusion and underlying set) as follows:

$$\mathsf{Cat}(\mathsf{2}_n,X)\longrightarrow n\text{-}\mathsf{Cat}[\mathsf{2}_n,X]\rightrightarrows n\text{-}\mathsf{Cat}[\mathsf{2}_{(n-1)},X]$$

and, then, we take the mates:

$$n\text{-}\mathsf{Cat}(\mathsf{2}_n,X)\times\mathsf{2}_{(n-1)}\rightrightarrows X. \qquad (\mathfrak{C}_{n\text{-}\mathsf{Der}}(X)\text{-diagram})$$

Finally, we compose this internal graph $\mathfrak{G}^{\mathrm{op}}\to n$-$\mathsf{Cat}$ with the underlying functor n-$\mathsf{Cat}\to(n-1)$-Cat.

The universal property that defines $\mathfrak{F}_{2\text{-}\mathsf{Der}}$ is precisely the universal property of being left adjoint to $\mathfrak{C}_{2\text{-}\mathsf{Der}}$, namely a morphism of derivation schemes $G:\mathfrak{d}\to\mathfrak{C}_{2\text{-}\mathsf{Der}}(X)$ corresponds to a pair of 2-functors (G_1,G_2) with the universal property described in the proof of Theorem 8.1.

Theorem 8.7. *There is an adjunction $\mathcal{F}_{2\text{-}\mathsf{Der}} \dashv \mathcal{C}_{2\text{-}\mathsf{Der}}$. More generally, there is an adjunction $\mathcal{F}_{n\text{-}\mathsf{Der}} \dashv \mathcal{C}_{n\text{-}\mathsf{Der}}$ in which $\mathcal{F}_{n\text{-}\mathsf{Der}}(\mathfrak{d}) := \mathfrak{I} \circledast \mathfrak{d}$ where, by abuse of language, $\mathfrak{I} \circledast \mathfrak{d}$ denotes the coinserter in $n\mathsf{Cat}$ of the derivation n-scheme $\mathfrak{d} : \mathfrak{G}^{\mathrm{op}} \to (n-1)\text{-}\mathsf{Cat}$ with the inclusion $(n-1)\text{-}\mathsf{Cat} \to n\mathsf{Cat}$.*

Proof: Similarly to the proof of Theorem 8.1, this result follows from the universal property of the coinserter and from Theorem 7.10. ■

Proposition 8.8. *In this proposition, we denote by $\mathfrak{I}_n$ the functor $\mathfrak{I}_n : \mathfrak{G} \to n\text{-}\mathsf{Cat}$ composed with the isomorphism $\mathfrak{G}^{\mathrm{op}} \to \mathfrak{G}$. In this case, $\mathfrak{I}_n$ is itself a higher derivation scheme. Then $\mathcal{F}_{(n+1)\text{-}\mathsf{Der}}(\mathfrak{I}_n)$ is isomorphic to $\mathsf{2}_{(n+1)}$.*

Remark 8.9. The inclusion $\mathsf{Cmp} \to \mathsf{Der}$ has a right adjoint $(-)_{\mathsf{Cmp}} : \mathsf{Der} \to \mathsf{Cmp}$ such that, given a derivation scheme $\mathfrak{d} : \mathfrak{d}_2 \times \mathfrak{G} \to X$, $(\mathfrak{d})_{\mathsf{Cmp}}$ is the pullback $\mathrm{comp}_X^*(\mathfrak{d})$ in Cat of the morphism $\mathfrak{d}$ along comp_X. It is clear that this adjunction is induced by the adjunction $\mathcal{F}_1 \dashv \mathcal{C}_1$.

Theorem 8.10. *There is an adjunction $\mathcal{F}_2 \dashv \mathcal{C}_2$ such that $\mathcal{F}_2 : \mathsf{Cmp}\ 2\text{-}\mathsf{Cat}$ gives the 2-category freely generated by each computad. More precisely, given a computad $\mathfrak{g} : \mathfrak{G}^{\mathrm{op}} \to \mathsf{Cat}$ in the format of the $\mathfrak{d}$-diagram, $\mathcal{F}_2(\mathfrak{g})$ is the coinserter in $2\mathsf{Cat}$ of $\mathfrak{g}$ composed with the inclusion $\mathsf{Cat} \to 2\mathsf{Cat}$.*

Proof: It is enough to define the adjunction $\mathcal{F}_2 \dashv \mathcal{C}_2$ as the composition of the adjunctions $- \dashv (-)_{\mathsf{Cmp}}$ and $\mathcal{F}_{2\text{-}\mathsf{Der}} \dashv \mathcal{C}_{2\text{-}\mathsf{Der}}$. ■

Definition 8.11. [n-computads] For each $n \in \mathbb{N}$, consider the functor $(- \times \mathfrak{G}_n) : \mathsf{SET} \to n\text{-}\mathsf{Cat}, Y \mapsto Y \times \mathfrak{G}_n$. The *category of $(n+1)$-computads* is defined by the comma category

$$(n+1)\text{-}\mathsf{Cmp} := (- \times \mathfrak{G}_n / \mathcal{F}_n)$$

in which $\mathcal{F}_n$ is the composition of the inclusion $n\text{-}\mathsf{Cmp} \to n\text{-}\mathsf{Der}$ with $\mathcal{F}_{n\text{-}\mathsf{Der}}$.

Remark 8.12. By Lemma 8.4, it is easy to see that an n-computad is just a triple $(\mathfrak{g}, \mathfrak{g}_2, G)$ in which $\mathfrak{g}_2$ is a discrete category, G is a $(n-1)$-computad and $\mathfrak{g} : \mathfrak{G}^{\mathrm{op}} \to (n-1)\text{-}\mathsf{Cat}$ is an internal graph

$$\mathfrak{g}_2 \times \mathsf{2}_{(n-1)} \rightrightarrows \mathcal{F}_{(n-1)}(G) \qquad (n\text{-computad diagram})$$

satisfying the $(n-2)$-coincidence property. Or, more concisely, by Remark 8.6, an n-computad is just a derivation n-scheme $(\mathfrak{g}, \mathfrak{g}_2)$ with a $(n-1)$-computad G such that $\mathfrak{g}(1) = \mathcal{F}_{(n-1)}(G)$.

Theorem 8.13 (Freely Generated n-Categories). *For each $n \in \mathbb{N}$, there is a functor $\mathfrak{F}_n : n\text{-}\mathsf{Cmp} \to n\text{-}\mathsf{Cat}$ such that, given an n-computad as in the n-computad diagram, $\mathfrak{F}_n(\mathfrak{g})$ is given by the coinserter in $n\mathsf{Cat}$ of $\mathfrak{g} : \mathfrak{G}^{\mathrm{op}} \to (n-1)\text{-}\mathsf{Cat}$ composed with the inclusion $(n-1)\text{-}\mathsf{Cat} \to n\mathsf{Cat}$. This functor is left adjoint to a functor $\mathfrak{C}_n : n\text{-}\mathsf{Cat} \to n\text{-}\mathsf{Cmp}$ which gives the underlying n-computad of each n-category.*

Proof: Of course, $\mathfrak{F}_n$ coincides with the functor $\mathfrak{F}_n$ of Definition 8.11. We prove by induction that $\mathfrak{F}_n$ is left adjoint. It is clear that $\mathfrak{F}_1 \dashv \mathfrak{C}_1$. We assume by induction that we have an adjunction $\mathfrak{F}_m \dashv \mathfrak{C}_m$.

We have that $\mathfrak{F}_m \dashv \mathfrak{C}_m$ induces an adjunction $(-) \dashv (-)_{(m+1)\text{-}\mathsf{Cmp}}$ in which the left adjoint is the inclusion $m\text{-}\mathsf{Cmp} \to m\text{-}\mathsf{Der}$ similarly to what is described in Remark 8.9. That is to say, $(\mathfrak{d})_{(m+1)\text{-}\mathsf{Cmp}}$ is the pullback of $\mathfrak{d}$ along the component of the counit of $\mathfrak{F}_m \dashv \mathfrak{C}_m$ on $\mathfrak{d}(1)$.

Finally, we compose the adjunction $\mathfrak{F}_{(m+1)\text{-}\mathsf{Der}} \dashv \mathfrak{C}_{(m+1)\text{-}\mathsf{Der}}$ with the adjunction $(-) \dashv (-)_{(m+1)\text{-}\mathsf{Cmp}}$ to get the desired adjunction $\mathfrak{F}_{(m+1)} \dashv \mathfrak{C}_{(m+1)}$. ■

Remark 8.14. Recall that Bicat is herein the 2-category of bicategories, pseudofunctors and icons. In the 2-dimensional case, the coinserter successfully gives the bicategory freely generated by a 2-computad. Namely, the functor $\mathfrak{F}_{\mathsf{Bicat}} : \mathsf{Cmp} \to \mathsf{Bicat}$ is given by $\mathfrak{F}_{\mathsf{Bicat}}(\mathfrak{g})$ is the coninserter of $\mathfrak{g}$ composed with the inclusion $\mathsf{Cat} \to \mathsf{Bicat}$.

An n-category X is a *free n-category* if there is an n-computad $\mathfrak{g} : \mathfrak{G}^{\mathrm{op}} \to (n-1)\text{-}\mathsf{Cat}$ such that $\mathfrak{F}_n(\mathfrak{g}) \cong X$.

Definition 8.15. Let $\mathfrak{g} = (\mathfrak{g}, \mathfrak{g}_2, G)$ be an n-computad. The objects of $\mathfrak{g}_2$ are called n-cells of $\mathfrak{g}$, while, whenever $n \geqslant m > 0$, an $(n-m)$-cell of $\mathfrak{g}$ is an $(n-m)$-cell of the $(n-1)$-computad G. In this context, we use the following terminology for graphs in Grph: the 0-cells of a graph are the objects and its 1-cells are the arrows.

Similarly to the 2-dimensional case, we denote an n-cell by $\iota : \alpha \Longrightarrow \alpha'$ if $\mathfrak{g}(d^0)(\alpha, \widehat{\kappa}) = \alpha'$ and $\mathfrak{g}(d^1)(\alpha, \widehat{\kappa}) = \alpha$.

Remark 8.16. For each $n \in \mathbb{N}$ such that $n > 1$, there is a forgetful functor $\mathfrak{u}_n : n\text{-}\mathsf{Cmp} \to (n-1)\text{-}\mathsf{Cmp}$, $(\mathfrak{g}, \mathfrak{g}_2, G) \mapsto G$. This forgetful functor has a left adjoint $\mathfrak{i}_n : (n-1)\text{-}\mathsf{Cmp} \to n\text{-}\mathsf{Cmp}$ such that

$$\mathfrak{i}_n(\mathfrak{g}) : \quad \varnothing \times \mathsf{2}_{(n-1)} \rightrightarrows \mathfrak{F}_{(n-1)}(\mathfrak{g})$$

and a right adjoint $\sigma_n : (n-1)\text{-}\mathsf{Cmp} \to n\text{-}\mathsf{Cmp}$, defined by $\sigma_n(G) = (G^{\sigma_n}, G_2^{\sigma_n}, G)$ in which there is precisely one n-cell $\iota_{(\alpha,\alpha')} : \alpha \Longrightarrow \alpha'$ for each ordered pair (α, α') with same domain and codomain of $\mathcal{F}_{(n-1)}(G)$. Actually, it should be observed that the description of these functors are similar to those given in Remark 4.15.

9. Freely Generated 2-Categories

Recall the adjunction $\mathcal{E}_{\mathsf{Cmp}} \dashv \mathcal{R}_{\mathsf{Cmp}}$ in which $\mathcal{E}_{\mathsf{Cmp}} : \mathsf{Cmp} \to \mathsf{RCmp}$ is the inclusion (see Definition 4.1). We also can consider the 2-category freely generated by computad over a reflexive graph. More precisely, given a computad $\mathfrak{g}$ of RCmp, $\mathcal{F}_2^{\mathcal{R}}(\mathfrak{g})$ is the coinserter of $\mathfrak{g} : \mathfrak{G}^{\mathrm{op}} \to 2\mathsf{Cat}$. It is clear that $\mathcal{F}_2^{\mathcal{R}}$ is left adjoint to a forgetful functor $\mathcal{C}_2^{\mathcal{R}}$. Moreover, $\mathcal{R}_{\mathsf{Cmp}}\mathcal{C}_2^{\mathcal{R}} \cong \mathcal{C}_2$ and $\mathcal{F}_2^{\mathcal{R}}\mathcal{E}_{\mathsf{Cmp}} \cong \mathcal{F}_2$.

In this section, following our approach of the 1-dimensional case, we give some results relating free 2-categories with locally thin categories and locally groupoidal categories. In order to do so, we also consider the (strict) concept of $(2,0)$-category given in Definition 9.6 and the $(2,0)$-category freely generated by a computad which provides a way of studying some elementary aspects of free 2-categories. We start by giving some sufficient conditions to conclude that a 2-category is not free.

Remark 9.1. [Length [34]] Recall that $\sigma_2 : \mathsf{Grph} \to \mathsf{Cmp}$ is right adjoint and $\bigcirc$ is the terminal graph in Grph. Therefore $\sigma_2(\bigcirc) : \mathfrak{G}^{\mathrm{op}} \to \mathsf{Cat}$ is the terminal computad. If $\mathfrak{g}$ is a computad, the length 2-functor is defined by $\ell^{\mathfrak{g}} := \mathcal{F}_2(\mathfrak{g} \to \sigma_2(\bigcirc))$. It should be noted that $\ell^{\mathfrak{g}}$ reflects identity 2-cells.

The 2-category $\mathcal{F}_2\sigma_2(\bigcirc)$ is described in [34]. The unit of the adjunction $\mathcal{F}_2 \dashv \mathcal{C}_2$ induces a morphism of computads $\sigma_2(\bigcirc) \to \mathcal{C}_2\mathcal{F}_2\sigma_2(\bigcirc)$. The image of the 2-cells of $\sigma_2(\bigcirc)$ are called herein *simple 2-cells*. If α is a composition in $\mathcal{F}_2\sigma_2(\bigcirc)$ of a simple 2-cell with (only) 1-cells (identity 2-cells), α is called a *whiskering of a simple 2-cell*. It is clear that every 2-cell of $\sigma_2(\bigcirc)$ is given by successive vertical compositions of whiskering of simple 2-cells. It is also easy to see that $\sigma_2(\bigcirc)$ does not have nontrivial invertible 2-cells.

The counit of the adjunction $\mathcal{F}_2 \dashv \mathcal{C}_2$ induces a 2-functor $\mathrm{past}_X : \mathcal{F}_2\mathcal{C}_2(X) \to X$ for each 2-category X, called *pasting*.

Remark 9.2. Similarly to the 1-dimensional case, the terminal reflexive computad of Rcmp is the computad with only one 0-cell, the trivial 1-cell and only one 2-cell. That is to say, the computad $\mathfrak{G} \to \mathcal{F}_1^{\mathcal{R}}(\bullet)$ which is the unique functor between $\mathfrak{G}$ and the terminal category $\mathcal{F}_1^{\mathcal{R}}(\bullet)$. If $\mathfrak{h}$ is a subcomputad of $\mathfrak{g}$ in Rcmp, we denote by $\mathfrak{g}/\mathfrak{h}$ the pushout of the inclusion $\mathfrak{h} \to \mathfrak{g}$ along the unique morphism of reflexive computads between $\mathfrak{h}$ and the terminal reflexive computad in Rcmp.

As a particular case of Proposition 9.3, if a 2-category X has a nontrivial invertible 2-cell, then X is not a free 2-category. Consequently, any locally thin 2-category that has a nontrivial invertible 2-cell is not a free 2-category.

Proposition 9.3. *Let α be an invertible 2-cell of a 2-category X in $2\text{-}\mathsf{Cat}$. If we can write α as pasting of 2-cells in which at least one of the 2-cells is nontrivial, then X is not free.*

Proof: Let α be a pasting of 2-cells in $\mathcal{F}_2(\mathfrak{g})$. We have that $\ell^{\mathfrak{g}}(\alpha)$ is a pasting of 2-cells of $\mathcal{F}_2\sigma_2(\bigcirc)$ with at least one nontrivial 2-cell. Therefore $\ell^{\mathfrak{g}}(\alpha)$ is not identity and, hence, α is not invertible. ∎

Recall that there is an adjunction $\mathcal{M}_2 \dashv M_2$ which induces a monad $\overline{\mathcal{M}_2}$, in which $M_2 : \mathsf{Prd\text{-}Cat} \to 2\text{-}\mathsf{Cat}$ is the inclusion.

Corollary 9.4. *Let X be a 2-category in $2\text{-}\mathsf{Cat}$. Assume that $\beta : f \Rightarrow g$ is a 2-cell of X such that $f \neq g$. If the pasting of β with another 2-cell is a 2-cell $\alpha : h \Rightarrow h$, then $\overline{\mathcal{M}_2}(X)$ is not a free 2-category.*

Proof: The unit of the monad $\overline{\mathcal{M}_2}$ gives, in particular, a 2-functor $X \to \overline{\mathcal{M}_2}(X)$. Therefore, the image of $\alpha : h \Rightarrow h$ by this 2-functor is also the pasting of a nontrivial 2-cell with other 2-cells, but, since $\overline{\mathcal{M}_2}(X)$ is locally thin, α is the identity. Therefore $\overline{\mathcal{M}_2}(X)$ is not free by Proposition 9.3. ∎

Proposition 9.5. *Consider the computad $\mathfrak{g}^{\dot{\Delta}_2} : \mathfrak{G}^{\mathrm{op}} \to \mathsf{Cat}$ defined in Example 4.7. The locally thin 2-category $\overline{\mathcal{M}_2}\mathcal{F}_2(\mathfrak{g}^{\dot{\Delta}_2})$ is not a free 2-category. In particular, $\mathcal{F}_2(\mathfrak{g}^{\dot{\Delta}_2})$ and $\overline{\mathcal{L}_2}\mathcal{F}_2(\mathfrak{g}^{\dot{\Delta}_2})$ are not locally thin.*

Proof: Since $\mathcal{M}_2\mathcal{F}_2(\mathfrak{g}^{\dot{\Delta}_2})$ is locally thin, we conclude that:

$$\begin{tikzcd}
0 \arrow[r, "d"] \arrow[d, "d"'] \arrow[dr, phantom, "\overset{\vartheta}{\Longleftarrow}"] & 1 \arrow[d, "d^1"] \arrow[ddr, equal, bend left] \arrow[ddr, phantom, "\overset{n_1}{\Longleftarrow}"] & \\
1 \arrow[r, "d^0"'] \arrow[drr, equal, bend right] \arrow[drr, phantom, "\overset{n_0}{\Longleftarrow}"] & 2 \arrow[dr, "s^0"] & \\
 & & 1
\end{tikzcd}
=
\begin{tikzcd}
0 \arrow[d, bend right, "d"'] \arrow[d, bend left, "d"] \arrow[d, phantom, "="] \\
1
\end{tikzcd}
\qquad \text{(identity descent diagram)}$$

Therefore, by Corollary 9.4, the proof is complete. ■

If a 2-category X is locally groupoidal and free, then every 2-cell of X is identity. Hence, in this case, X is locally discrete, that is to say, it is a free 1-category.

We call $\mathcal{M}_2\mathcal{F}_2(\mathfrak{g})$ the *locally thin 2-category freely generated by* $\mathfrak{g}$. But we often consider such a 2-category as an object of 2-Cat, that is to say, we often consider $\overline{\mathcal{M}_2}\mathcal{F}_2(\mathfrak{g})$.

Definition 9.6. [(n,m)-Categories] If $m < n$, an *(n,m)-category* X is an n-category of n-Cat such that, whenever $n \geqslant r > m$, all r-cells of X are invertible. The full subcategory of n-Cat consisting of the (n,m)-categories is denoted by (n,m)-Cat.

For instance, groupoids are $(1,0)$-categories and locally groupoidal categories are $(2,1)$-categories. The adjunction $\mathcal{L}_1 \dashv \mathcal{U}_1$ also induces an adjunction $\mathcal{L}_{(2,0)} \dashv \mathcal{U}_{(2,0)}$ in which $\mathcal{U}_{(2,0)} : (2,0)\text{-}\mathsf{Cat} \to 2\text{-}\mathsf{Cat}$ is the inclusion. Thereby, given a computad $\mathfrak{g}$ of Cmp, we can consider the *locally groupoidal 2-category* $\mathcal{L}_2\mathcal{F}_2(\mathfrak{g})$ *freely generated by the computad* $\mathfrak{g}$, as well as the *$(2,0)$-category* $\mathcal{L}_{(2,0)}\mathcal{F}_2(\mathfrak{g})$ *freely generated by* $\mathfrak{g}$.

Sometimes, we denote $\mathcal{L}_{(2,1)} := \mathcal{L}_2$ and $\mathcal{U}_{(2,1)} := \mathcal{U}_{(2,1)}$.

Remark 9.7. Let X be a $(2,0)$-category of $(2,0)$-Cat and assume that Y is a sub-2-category of X. We denote by X/Y the pushout of the inclusion $Y \to X$ along the unique 2-functor between Y and the terminal 2-category. If Y is locally discrete and thin (that is to say, a thin category), then X/Y is isomorphic to X.

Definition 9.8. A 2-category X satisfies the $(2,1)$-*cancellation law* if it satisfies the cancellation law w.r.t. the vertical composition of 2-cells (that is to say, it satisfies the cancellation law locally).

A 2-category X satisfies the $(2,0)$-*cancellation law* if it satisfies the $(2,1)$-*cancellation law* and, whenever X has 1-cells f, g and 2-cells α, β such that $\mathrm{id}_f * \alpha * \mathrm{id}_g = \mathrm{id}_f * \beta * \mathrm{id}_g$, $\alpha = \beta$.

It is clear that, if a 2-category X satisfies the $(2,0)$-cancellation law, in particular, the underlying category of X satisfies the cancellation law. Moreover, every $(2,1)$-category satisfies the $(2,1)$-cancellation law and every $(2,0)$-category satisfies the $(2,0)$-cancellation law.

Finally, the components of the units of the adjunctions $\mathcal{L}_2 \dashv \mathcal{U}_2$ and $\mathcal{L}_{(2,0)} \dashv \mathcal{U}_{(2,0)}$ are locally faithful on 2-categories satisfying respectively the $(2,1)$-cancellation law and the $(2,0)$-cancellation law. Thereby:

Theorem 9.9. *Let X be a 2-category. If X satisfies the $(2,1)$-cancellation law and $\mathcal{L}_2(X)$ is locally thin, then X is locally thin as well. Analogously, if X satisfies the $(2,0)$-cancellation law and $\mathcal{L}_{(2,0)}(X)$ is locally thin, then X is locally thin as well*

Corollary 9.10. *Let $\mathfrak{g}$ be an object of* Cmp. *Consider the following statements:*

- **(a):** $\mathcal{L}_{(2,0)}\mathcal{F}_2(\mathfrak{g})$ *is locally thin;*
- **(b):** $\mathcal{L}_2\mathcal{F}_2(\mathfrak{g})$ *is locally thin;*
- **(c):** $\mathcal{F}_2(\mathfrak{g})$ *is locally thin.*

We have that (a) implies (b) implies (c).

Proof: It is clear that $\mathcal{F}_2(\mathfrak{g})$ and $\mathcal{L}_2\mathcal{F}_2(\mathfrak{g})$ satisfies the $(2,0)$-cancellation law. Therefore we get the result by Theorem 9.9. ■

Definition 9.11. A 2-category X satisfies the *underlying terminal property* or *u.t.p.* if the underlying category of X is the terminal category.

On one hand, by the Eckman-Hilton argument, given any small 2-category X with only one object $*$, the vertical composition of 2-cells $\mathrm{id} \Rightarrow \mathrm{id}$ coincides with the horizontal one and they are commutative. Therefore, in this context, the set of 2-cells $\mathrm{id} \Rightarrow \mathrm{id}$ endowed with the vertical composition is a commutative monoid, denoted by $\Omega^2(X) := X(*,*)(\mathrm{id},\mathrm{id})$.

On the other hand, given a commutative monoid Y, the suspension $\Sigma(Y)$ is naturally a monoidal category (in which the monoidal structure coincides with the composition). This allows us to consider the *double suspension* $\Sigma^2(Y)$ which is a 2-category satisfying *u.t.p.* and the set of 2-cells $\mathrm{id} \Rightarrow \mathrm{id}$ is the underlying set of Y, while the vertical and horizontal compositions of $\Sigma^2(Y)$ are given by the operation of Y. More precisely, there is a fully faithful functor

$$\Sigma^2 : \mathsf{AbGroup} \to (2,1)\text{-}\mathsf{cat}$$

between the category of abelian groups and the category of small locally groupoidal 2-categories which is essentially surjective on the full subcategory of 2-categories satisfying *u.t.p.* such that $\Omega^2\Sigma^2 \cong \mathrm{Id}_{\mathsf{AbGroup}}$.

If $(\mathfrak{g}, \mathfrak{g}_2, G)$ is a small 2-computad in which $G = \bullet^{\mathcal{E}}$ is the connected graph without arrows, then $\mathcal{L}_{(2,0)}\mathcal{F}_2(\mathfrak{g}, \mathfrak{g}_2, G)$ is isomorphic to the double suspension of a free abelian group.

Theorem 9.12. *If* $(\mathfrak{g}, \mathfrak{g}_2, \bullet)$ *is a small reflexive computad, then* $\mathcal{L}_{(2,0)}\mathcal{F}_2^{\mathcal{R}}(\mathfrak{g}) \cong \mathcal{L}_2\mathcal{F}_2^{\mathcal{R}}(\mathfrak{g}) \cong \Sigma^2\pi_2(\mathcal{F}_{\mathsf{Top}_2}(\mathfrak{g}))$.

Proof: Since $\mathcal{F}_1^{\mathcal{R}}(\bullet)$ is the terminal category, $\Omega^2\left(\mathcal{L}_2\mathcal{F}_2^{\mathcal{R}}(\mathfrak{g})\right)$ is the abelian group freely generated by the set $\mathfrak{g}_2$ that is also isomorhic to $\pi_2(\mathcal{F}_{\mathsf{Top}_2}(\mathfrak{g}))$.

To complete the proof, it is enough to observe that $\mathcal{L}_{(2,0)}(X) \cong \mathcal{L}_2(X)$ whenever X does not have nontrivial 1-cells. ∎

We say that a computad $\mathfrak{g}$ is 1-*connected* if $\mathcal{F}_{\mathsf{Top}_2}(\mathfrak{g})$ is simply connected. By Corollary 6.5, a computad $\mathfrak{g}$ is 1-connected if and only if $\mathcal{L}_1\mathcal{P}_1(\mathfrak{g})$ is connected and thin.

Definition 9.13. [*f.c.s.*] Let $\mathfrak{g} = (\mathfrak{g}, \mathfrak{g}_2, G)$ be a computad of Rcmp with only one 0-cell and let $\mathfrak{h}$ be a subcomputad of $\mathfrak{g}$.

We call $\mathfrak{g}^{\mathfrak{b}} := \mathfrak{h}$ a *full contractible subcomputad of* $\mathfrak{g}$ or, for short, *f.c.s.* of $\mathfrak{g}$, if $\mathcal{L}_{(2,0)}\mathcal{F}_2^{\mathcal{R}}(\mathfrak{g}^{\mathfrak{b}})$ has a unique 2-cell $f \Rightarrow \mathrm{id}$ or a 2-cell $\mathrm{id} \Rightarrow f$ for each 1-cell f of $\mathfrak{g}$. In particular, if $\mathfrak{g}^{\mathfrak{b}}$ is an *f.c.s.* of $\mathfrak{g}$, $\mathfrak{g}^{\mathfrak{b}}$ has every 1-cell of $\mathfrak{g}$.

It should be noted that, if $\mathfrak{g}^{\mathfrak{b}}$ is an *f.c.s.* of $\mathfrak{g}$, we are already assuming that $\mathfrak{g}$ is an object of Rcmp.

There are small (reflexive) computads with only one 0-cell and no full contractible subcomputad. For instance, consider the computad $\mathfrak{x}$ with two 1-cells f, g and with 2-cells $\alpha : gf \Rightarrow \mathrm{id}$ and $\beta : \mathrm{id} \Rightarrow g$. The number of 2-cells of any subcomputad belongs to $\{0, 1, 2\}$. It is clear that the subcomputads with only one 2-cell are not full contractible subcomputads. It remains to prove that the whole computad is not an *f.c.s.* of itself. Indeed, the 2-cells $\mathrm{id}_{g^{-1}} * (\beta \cdot \alpha)$ and $\alpha \cdot \left(\beta * \mathrm{id}_f\right)$ below are both 2-cells $f \Rightarrow \mathrm{id}$ of $\mathcal{L}_{(2,0)}\mathcal{F}_2^{\mathcal{R}}(\mathfrak{x})$.

$$
\begin{tikzcd}
* \arrow[r, "f"] & * \\
* \arrow[u, bend left=60, equal, ""{name=E, right}] \arrow[u, bend right=40, "g"', ""{name=G, left}] \arrow[r, "\mathrm{id}"'] & * \arrow[u, "\mathrm{id}"']
\arrow[from=E, to=G, phantom, "\overset{\beta}{\Longrightarrow}"]
\end{tikzcd}
\quad
\begin{tikzcd}
* \arrow[d, bend right=70, "gf"'] \arrow[d, equal] \arrow[d, bend left=70, "g"] \arrow[d, phantom, shift right=5ex, "\overset{\alpha}{\Longrightarrow}"] \arrow[d, phantom, shift left=5ex, "\overset{\beta}{\Longrightarrow}"] \\
* \arrow[d, "g^{-1}"'] \\
*
\end{tikzcd}
$$

Theorem 9.14. *If $\mathfrak{g}^{\mathfrak{b}}$ is an f.c.s., then the 2-categories $\mathcal{F}_2^{\mathfrak{R}}(\mathfrak{g}^{\mathfrak{b}})$, $\mathcal{L}_2\mathcal{F}_2^{\mathfrak{R}}(\mathfrak{g}^{\mathfrak{b}})$ and $\mathcal{L}_{(2,0)}\mathcal{F}_2^{\mathfrak{R}}(\mathfrak{g}^{\mathfrak{b}})$ are locally thin.*

Proposition 9.15. *If $\mathfrak{g}^{\mathfrak{b}} = (\mathfrak{g}^{\mathfrak{b}}, \mathfrak{g}_2^{\mathfrak{b}}, G)$ is an f.c.s., then $\mathcal{F}_{\mathsf{Top}_2}^{\mathfrak{R}}(\mathfrak{g}^{\mathfrak{b}})$ is contractible. In particular, it is simply connected and, hence, $\mathfrak{g}^{\mathfrak{b}}$ is 1-connected.*

Proof: It is enough to see that $\mathcal{F}_{\mathsf{Top}_2}^{\mathfrak{R}}(\mathfrak{g}^{\mathfrak{b}})$ is a wedge of (closed) balls. ∎

Theorem 9.16. *Assume that $\mathfrak{g}^{\mathfrak{b}}$ is an f.c.s. of $(\mathfrak{g}, \mathfrak{g}_2, G)$. The following statements are equivalent:*

– $\mathcal{L}_{(2,0)}\mathcal{F}_2^{\mathfrak{R}}\left(\mathfrak{g}/\mathfrak{g}^{\mathfrak{b}}\right)$ *is locally thin;*
– $\mathcal{L}_{(2,0)}\mathcal{F}_2^{\mathfrak{R}}(\mathfrak{g})$ *is locally thin;*
– $\mathcal{L}_2\mathcal{F}_2^{\mathfrak{R}}(\mathfrak{g})$ *is locally thin;*
– $\mathcal{F}_2^{\mathfrak{R}}(\mathfrak{g})$ *is locally thin.*

Proof: $\mathfrak{g}/\mathfrak{g}^{\mathfrak{b}}$ is the computad $(\mathfrak{h}, \mathfrak{h}_2, \bullet)$ in which $\mathfrak{h}_2 = \mathfrak{g}_2 - \mathfrak{g}_2^{\mathfrak{b}}$. Therefore $\mathcal{L}_{(2,0)}\mathcal{F}_2^{\mathfrak{R}}\left(\mathfrak{g}/\mathfrak{g}^{\mathfrak{b}}\right)$ is locally thin if and only if $\mathfrak{g}_2 = \mathfrak{g}_2^{\mathfrak{b}}$, which means that $\mathfrak{g} = \mathfrak{g}^{\mathfrak{b}}$. Since $\mathcal{L}_{(2,0)}\mathcal{F}_2^{\mathfrak{R}}(\mathfrak{g}^{\mathfrak{b}})$ is locally thin, the proof is complete. ∎

Let $\mathfrak{g} = (\mathfrak{g}, \mathfrak{g}_2, G)$ be a small connected computad of Rcmp. Assume that G_{mtree} is a maximal tree of G. We have that the computad

$$\mathfrak{g}_2 \times \mathsf{2} \rightrightarrows \mathcal{F}_1^{\mathfrak{R}}(G) \longrightarrow \mathcal{F}_1^{\mathfrak{R}}(G/G_{\mathrm{mtree}})$$

obtained from the composition of the morphisms in the image of $\mathfrak{g}$ with the natural morphism $\mathcal{F}_1^{\mathfrak{R}}(G) \to \mathcal{F}_1^{\mathfrak{R}}(G/G_{\mathrm{mtree}})$ is the pushout of the mate of the inclusion $G_{\mathrm{mtree}} \to \mathfrak{u}_2^{\mathfrak{R}}(\mathfrak{g})$ under the adjunction $\mathfrak{i}_2^{\mathfrak{R}} \dashv \mathfrak{u}_2^{\mathfrak{R}}$ along the unique functor between $\mathfrak{i}_2^{\mathfrak{R}}(G_{\mathrm{mtree}})$ and the terminal reflexive computad. That is to say, it is the quotient $\mathfrak{g}/\mathfrak{i}_2^{\mathfrak{R}}(G_{\mathrm{mtree}})$.

Definition 9.17. [*f.c.s.* triple] We say that $(\mathfrak{g}, G_{\mathrm{mtree}}, \mathfrak{h}^{\mathfrak{b}})$ is an *f.c.s. triple* if $\mathfrak{g}$ is a small connected reflexive computad, G_{mtree} is a maximal tree of the underlying graph of $\mathfrak{g}$ and $\mathfrak{h}^{\mathfrak{b}}$ is an *f.c.s.* of $\mathfrak{g}/\mathfrak{i}_2^{\mathcal{R}}(G_{\mathrm{mtree}})$. In this case, we denote by $\check{\mathfrak{h}^{\mathfrak{b}}}$ the reflexive computad

$$\left(\mathfrak{g}/\mathfrak{i}_2^{\mathcal{R}}(G_{\mathrm{mtree}})\right)/\mathfrak{h}^{\mathfrak{b}}.$$

Corollary 9.18. *Let* $(\mathfrak{g}, G_{\mathrm{mtree}}, \mathfrak{h}^{\mathfrak{b}})$ *be an f.c.s. triple. The* $(2,0)$*-category* $\mathcal{L}_{(2,0)}\mathcal{F}_2^{\mathcal{R}}(\mathfrak{g})$ *is locally thin if and only if* $\mathcal{L}_{(2,0)}\mathcal{F}_2^{\mathcal{R}}(\check{\mathfrak{h}^{\mathfrak{b}}})$ *is locally thin.*

Proof: By Remark 9.7, $\mathcal{L}_{(2,0)}\mathcal{F}_2^{\mathcal{R}}(\mathfrak{g}/\mathfrak{i}_2^{\mathcal{R}}(G_{\mathrm{mtree}}))$ is locally thin if and only if $\mathcal{L}_{(2,0)}\mathcal{F}_2^{\mathcal{R}}(\mathfrak{g})$ is locally thin. By Theorem 9.16, the former is locally thin if and only if $\mathcal{L}_{(2,0)}\mathcal{F}_2^{\mathcal{R}}(\check{\mathfrak{h}^{\mathfrak{b}}})$ is locally thin. ∎

As a consequence of Corollary 9.18 and Theorem 9.12, we get:

Corollary 9.19. *Let* $(\mathfrak{g}, G_{\mathrm{mtree}}, \mathfrak{h}^{\mathfrak{b}})$ *be an f.c.s. triple. The* $(2,0)$*-category* $\mathcal{L}_{(2,0)}\mathcal{F}_2^{\mathcal{R}}(\mathfrak{g})$ *is locally thin if and only if* $\pi_2\mathcal{F}_{\mathsf{Top}_2}^{\mathcal{R}}(\mathfrak{g})$ *is trivial.*

Proof: By Theorem 9.12, $\mathcal{L}_{(2,0)}\mathcal{F}_2^{\mathcal{R}}(\check{\mathfrak{h}^{\mathfrak{b}}})$ is isomorphic to $\Sigma^2\pi_2\mathcal{F}_{\mathsf{Top}_2}^{\mathcal{R}}(\check{\mathfrak{h}^{\mathfrak{b}}})$. Therefore, by Corollary 9.18 we conclude that $\mathcal{L}_{(2,0)}\mathcal{F}_2^{\mathcal{R}}(\mathfrak{g})$ is locally thin if and only if $\Sigma^2\pi_2\mathcal{F}_{\mathsf{Top}_2}^{\mathcal{R}}(\check{\mathfrak{h}^{\mathfrak{b}}})$ is trivial.

To complete the proof, it remains to prove that $\pi_2\mathcal{F}_{\mathsf{Top}_2}^{\mathcal{R}}(\check{\mathfrak{h}^{\mathfrak{b}}}) \cong \pi_2\mathcal{F}_{\mathsf{Top}_2}^{\mathcal{R}}(\mathfrak{h})$. Indeed, since $\mathcal{F}_{\mathsf{Top}_2}^{\mathcal{R}}$ preserves colimits and the terminal reflexive computad, we get that

$$\mathcal{F}_{\mathsf{Top}_2}^{\mathcal{R}}(\mathfrak{g}/\mathfrak{i}_2^{\mathcal{R}}(G_{\mathrm{mtree}})) \cong \mathcal{F}_{\mathsf{Top}_2}^{\mathcal{R}}(\mathfrak{g})/\mathcal{F}_{\mathsf{Top}_2}^{\mathcal{R}}\mathfrak{i}_2^{\mathcal{R}}(G_{\mathrm{mtree}})$$

and, since $\mathcal{F}_{\mathsf{Top}_2}^{\mathcal{R}}\mathfrak{i}_2^{\mathcal{R}}(G_{\mathrm{mtree}}) \to \mathcal{F}_{\mathsf{Top}_2}^{\mathcal{R}}(\mathfrak{g})$ is a cofibration which is an inclusion of a contractible space, we conclude that $\mathcal{F}_{\mathsf{Top}_2}^{\mathcal{R}}(\mathfrak{g}/\mathfrak{i}_2^{\mathcal{R}}(G_{\mathrm{mtree}}))$ has the same homotopy type of $\mathcal{F}_{\mathsf{Top}_2}^{\mathcal{R}}(\mathfrak{g})$. Analogously, we conclude that $\mathcal{F}_{\mathsf{Top}_2}^{\mathcal{R}}(\check{\mathfrak{h}^{\mathfrak{b}}})$ has the same homotopy type of $\mathcal{F}_{\mathsf{Top}_2}^{\mathcal{R}}(\mathfrak{g}/\mathfrak{i}_2^{\mathcal{R}}(G_{\mathrm{mtree}}))$, since $\mathcal{F}_{\mathsf{Top}_2}^{\mathcal{R}}(\mathfrak{h}^{\mathfrak{b}}) \to \mathcal{F}_{\mathsf{Top}_2}^{\mathcal{R}}(\mathfrak{g}/\mathfrak{i}_2^{\mathcal{R}}(G_{\mathrm{mtree}}))$ is a cofibration which is an inclusion of a contractible space. ∎

Remark 9.20. The study of possible higher dimensional analogues of the isomorphisms given in Remark 5.10 and in Theorem 5.11 would depend on the study of notions of higher fundamental groupoids, higher homotopy groupoids and higher Van Kampen theorems [6, 7, 12]. This is outside of the scope of this paper.

10. Presentations of 2-categories

As 2-computads give presentations of categories with equations between 1-cells, $(n+1)$-computads give presentations of n-categories with equations between n-cells. Contrarily to the case of presentations of categories via computads, it is clear that, for $n > 1$, there are n-categories that do not admit presentations via $(n+1)$-computads.

Definition 10.1. [Presentation of n-categories via $(n+1)$-computads] Given $n \in \mathbb{N}$, an $(n+1)$-computad $\mathfrak{g} : \mathfrak{G}^{\mathrm{op}} \to n\text{-}\mathsf{Cat}$ as the n-computad diagram of 8.12 *presents the n-category X* if the coequalizer of $\mathfrak{g}$ in $n\text{-}\mathsf{Cat}$ is isomorphic to X. There is a functor $\mathcal{P}_n : (n+1)\text{-}\mathsf{Cmp} \to n\text{-}\mathsf{Cat}$ which, for each $(n+1)$-computad $\mathfrak{g}$, gives the category $\mathcal{P}_n(\mathfrak{g})$ presented by $\mathfrak{g}$.

The underlying $(n-1)$-category of every n-category that admits a presentation via a $(n+1)$-computad is a free $(n-1)$-category. Thereby:

Proposition 10.2. *Let X be an n-category in $n\text{-}\mathsf{Cat}$. If the underlying $(n-1)$-category of X is not free, then X does not admit a presentation via an $(n+1)$-computad.*

In this section, as the title suggests, our scope is restricted to presentations of 2-categories. Similarly to the 1-dimensional case, we are mainly interested on presentations of locally thin 2-categories, $(2,1)$-categories or $(2,0)$-categories.

We consider (reflexive) small $(2,0)$-categorical and $(2,1)$-categorical (reflexive) small 3-computads which are 3-dimensional analogues of groupoidal computads, called respectively $(3,0,\mathcal{R})$-computads and $(3,1,\mathcal{R})$-computads. More precisely, for each $m \in \{0,1\}$, we define the category of $(3,m,\mathcal{R})$-computads by the comma category $(3,2,m)\text{-}\mathsf{Rcmp} := (-\times\mathcal{L}_{(2,m)}(\mathfrak{G}_2)/\mathcal{L}_{(2,m)}\mathcal{F}_2^{\mathcal{R}})$ in which

$$(-\times\mathcal{L}_{(2,m)}(\mathfrak{G}_2)) : \mathsf{Set} \to (2,m)\text{-}\mathsf{cat}, \quad Y \mapsto Y \times \mathcal{L}_{(2,m)}(\mathfrak{G}_2).$$

Whenever $2 > m \geqslant 0$, we have a functor $\mathcal{P}^{\mathcal{R}}_{(2,m)} : (3,2,m)\text{-}\mathsf{Rcmp} \to (2,m)\text{-}\mathsf{cat}$ that gives the $(2,m)$-category presented by each $(3,2,m,\mathcal{R})$-computad. More precisely, for each $2 > m \geqslant 0$, a $(3,2,m,\mathcal{R})$-computad is a functor $\mathfrak{g} : \mathfrak{G}^{\mathrm{op}} \to (2,m)\text{-}\mathsf{cat}$

$$\mathfrak{g}_2 \times \mathcal{L}_{(2,m)}(\mathsf{2}_2) \rightrightarrows \mathcal{L}_{(2,m)}\mathcal{F}_2^{\mathcal{R}}(G) \qquad ((3,2,m,\mathcal{R})\text{-computad diagram})$$

and $\mathcal{P}^{\mathfrak{R}}_{(2,m)}(\mathfrak{g})$ is the coequalizer of $\mathfrak{g}$ in $(2,m)$-cat. For short, by abuse of language, by $\mathfrak{i}_3$ the functors $\mathsf{Rcmp} \to (3,2,m)$-$\mathsf{Rcmp}$ induced by $\mathfrak{i}_3$.

Theorem 10.3. *Assume that $G^{\mathfrak{b}}$ is an f.c.s. of the small reflexive 2-computad G. If $(\mathfrak{g}, \mathfrak{g}_2, G)$ is a small $(3,2,0,\mathfrak{R})$-computad, then the following statements are equivalent:*

- $\mathcal{P}^{\mathfrak{R}}_{(2,0)}\left(\mathfrak{g}/\mathfrak{i}_3(G^{\mathfrak{b}})\right)$ *is locally thin;*
- $\mathcal{P}^{\mathfrak{R}}_{(2,0)}\left(\mathfrak{g}\right)$ *is locally thin.*

Proof: We have that $\mathcal{P}^{\mathfrak{R}}_{(2,0)}\mathfrak{i}_3(G^{\mathfrak{b}}) \cong \mathcal{L}_{(2,0)}\mathcal{F}_2(G^{\mathfrak{b}})$ is locally thin. Therefore

$$\mathcal{P}^{\mathfrak{R}}_{(2,0)}\left(\mathfrak{g}\right) \text{ and } \mathcal{P}^{\mathfrak{R}}_{(2,0)}\left(\mathfrak{g}/\mathfrak{i}_3(G^{\mathfrak{b}})\right) \cong \mathcal{P}^{\mathfrak{R}}_{(2,0)}\left(\mathfrak{g}\right)/\mathcal{P}^{\mathfrak{R}}_{(2,0)}\left(\mathfrak{i}_3(G^{\mathfrak{b}})\right)$$

are biequivalent. Thereby the result follows. ∎

In the setting of the result above, since we are assuming that the 2-computad G has only one 0-cell, we get that there is a $(3,2,0,\mathfrak{R})$-computad $(\mathfrak{g}, \mathfrak{g}_2, G)$ such that $|\mathfrak{g}_2|$ is precisely the number of 2-cells of $G/G^{\mathfrak{b}}$ and $\mathcal{P}^{\mathfrak{R}}_{(2,0)}\left(\mathfrak{g}\right)$ is locally thin.

Theorem 10.4. *Assume that $G^{\mathfrak{b}}$ is an f.c.s. of a 2-computad G in* Rcmp. *There is a $(3,2,0,\mathfrak{R})$-computad $(\mathfrak{g}, \mathfrak{g}_2, G)$ such that $\mathfrak{g}_2 = G_2 - G^{\mathfrak{b}}_2$ and $\mathcal{P}^{\mathfrak{R}}_{(2,0)}\left(\mathfrak{g}\right)$ is locally thin. In other words, $\mathfrak{g}$ presents the locally thin $(2,0)$-category $\overline{\mathcal{M}_2}\,\overline{\mathcal{L}_{(2,0)}}\mathcal{F}^{\mathfrak{R}}_2\left(G\right)$ freely generated by G.*

Proof: Recall, by Theorem 9.16, that we can consider that $(G/G^{\mathfrak{b}})_2 = G_2 - G^{\mathfrak{b}}_2$. Also, by hypothesis, for each nontrivial 1-cell f of G, $\mathcal{L}_{(2,0)}\mathcal{F}^{\mathfrak{R}}_2(G^{\mathfrak{b}})$ has a unique 2-cell $\beta_f : f \Rightarrow \mathrm{id}$ or $\beta_f : \mathrm{id} \Rightarrow f$.

We define the $(3,2,0,\mathfrak{R})$-computad $(\mathfrak{g}, \mathfrak{g}_2, G)$

$$\mathfrak{g}_2 \times \mathcal{L}_{(2,0)}(2_2) \rightrightarrows \mathcal{L}_{(2,0)}\mathcal{F}^{\mathfrak{R}}_2(G).$$

For each $\alpha \in \mathfrak{g}_2 = G_2 - G^{\mathfrak{b}}_2$, we put $\mathfrak{g}(d^1)(\alpha, \widehat{\kappa}) := \alpha : f \Rightarrow g$ and $\mathfrak{g}(d^0)(\alpha, \widehat{\kappa}) := \dot{\alpha}$ in which $\dot{\alpha}$ is the composition of (possibly the inverse) of β_f and (possibly the inverse) of β_g in $\mathcal{L}_{(2,0)}\mathcal{F}^{\mathfrak{R}}_2(G)$, that is to say, in other words, $\dot{\alpha}$ is the unique 2-cell with same domain and codomain of α in $\mathcal{L}_{(2,0)}\mathcal{F}^{\mathfrak{R}}_2(G^{\mathfrak{b}})$.

It is clear that $\mathcal{P}^{\mathfrak{R}}_{(2,0)}\left(\mathfrak{g}/\mathfrak{i}_3(G^{\mathfrak{b}})\right)$ is locally thin. Therefore the result follows from Theorem 10.3. ∎

Corollary 10.5. *Let $(G, \mathtt{T}, H^{\mathfrak{b}})$ be an f.c.s. triple. There is a $(3,2,0,\mathcal{R})$-computad $(\mathfrak{h}, \mathfrak{h}_2, G)$ such that $\mathfrak{h}_2 = \check{H}^{\mathfrak{b}}{}_2 = G_2 - H_2^{\mathfrak{b}} = (G/\mathtt{T})_2 - H_2^{\mathfrak{b}}$ and $\mathcal{P}^{\mathcal{R}}_{(2,0)}(\mathfrak{h})$ is locally thin.*

Proof: We denote $G/\mathfrak{i}_2^{\mathcal{R}}(\mathtt{T})$ by H. Consider the $(3,2,0,\mathcal{R})$-computad $\mathfrak{g} = (\mathfrak{g}, \mathfrak{g}_2, H)$ as constructed in Theorem 10.4. Since each 2-cell of H corresponds to a unique 2-cell of G, we can lift $\mathfrak{g}$ to a $(3,2,0,\mathcal{R})$-computad $(\mathfrak{g}, \mathfrak{g}_2, G)$. We get this lifting $(\mathfrak{g}, \mathfrak{g}_2, G)$

$$\mathfrak{g}_2 \times \mathcal{L}_{(2,0)}(2_2) \rightrightarrows \mathcal{L}_{(2,0)}\mathcal{F}_2^{\mathcal{R}}(H) \longrightarrow \mathcal{L}_{(2,0)}\mathcal{F}_2^{\mathcal{R}}(G)$$

after composing each morphism in the image of $\mathfrak{g}$ with $\overline{\mathcal{L}_{(2,0)}}\mathcal{F}_2^{\mathcal{R}}(H) \simeq \overline{\mathcal{L}_{(2,0)}}\mathcal{F}_2^{\mathcal{R}}(G)$. Moreover, since

$$\mathcal{P}^{\mathcal{R}}_{(2,0)}(\mathfrak{g}) \cong \mathcal{P}^{\mathcal{R}}_{(2,0)}\left(\mathfrak{h}/\mathfrak{i}_3\mathfrak{i}_2^{\mathcal{R}}(\mathtt{T})\right) \cong \mathcal{P}^{\mathcal{R}}_{(2,0)}(\mathfrak{h})\,/\mathcal{P}^{\mathcal{R}}_{(2,0)}\left(\mathfrak{i}_3\mathfrak{i}_2^{\mathcal{R}}(\mathtt{T})\right)$$

is locally thin, the result follows from Remark 9.7. ∎

Analogously to Definition 6.17, we have:

Definition 10.6. [Lifting of 3-Computads] We denote by $(3,2,1)$-$\mathsf{Rcmp}_{\mathsf{lift}}$ the pseudopullback (iso-comma category) of $\mathcal{P}^{\mathcal{R}}_{(2,0)}$ along $\mathcal{L}_{(2,0)}\mathcal{U}_{(2,1)}\mathcal{P}^{\mathcal{R}}_{(2,1)}$. A $(3,2,1,\mathcal{R})$-computad $\mathfrak{g}$ is called a *lifting of the* $(3,2,0,\mathcal{R})$-*computad* $\mathfrak{g}'$ if there is an object $\zeta^{\mathfrak{g}}_{\mathfrak{g}'}$ of $(3,2,1)$-$\mathsf{Rcmp}_{\mathsf{lift}}$ such that the images of this object by the functors

$$(3,2,1)\text{-}\mathsf{Rcmp}_{\mathsf{lift}} \to (3,2,0)\text{-}\mathsf{Rcmp}, \qquad (3,2,1)\text{-}\mathsf{Rcmp}_{\mathsf{lift}} \to (3,2,1)\text{-}\mathsf{Rcmp}$$

are respectively $\mathfrak{g}'$ and $\mathfrak{g}$. Analogously, we say that a (reflexive) 3-computad $\mathfrak{h}$ is a lifting of a $(3,2,m,\mathcal{R})$-computad $\mathfrak{h}'$ if $\mathcal{L}_{(2,m)}\mathcal{U}_{(2,m)}\mathcal{P}^{\mathcal{R}}_{(2,m)}(\mathfrak{h}') \cong \mathcal{P}^{\mathcal{R}}_2(\mathfrak{h})$.

Proposition 10.7. *If a $(3,2,1,\mathcal{R})$-computad $\mathfrak{g}$ is a lifting of a $(3,2,0,\mathcal{R})$-computad $\mathfrak{g}'$ such that $\mathcal{P}^{\mathcal{R}}_{(2,0)}(\mathfrak{g}')$ is locally thin, then $\mathcal{P}^{\mathcal{R}}_{(2,1)}(\mathfrak{g})$ is locally thin provided that $\mathcal{P}^{\mathcal{R}}_{(2,1)}(\mathfrak{g})$ satisfies the $(2,0)$-cancellation law.*

Analogously, if a 3-computad $\mathfrak{h}$ is a lifting of a $(3,2,m,\mathcal{R})$-computad $\mathfrak{h}'$ and $\mathcal{P}^{\mathcal{R}}_{(2,m)}(\mathfrak{h}')$ is locally thin, then $\mathcal{P}^{\mathcal{R}}_2(\mathfrak{h})$ is locally thin provided that $\mathcal{P}^{\mathcal{R}}_2(\mathfrak{g})$ satisfies the $(2,m)$-cancellation law.

Proof: By hypothesis, $\mathcal{P}^{\mathcal{R}}_{(2,1)}(\mathfrak{g}) \cong \mathcal{L}_{(2,0)}\mathcal{U}_{(2,1)}\mathcal{P}^{\mathcal{R}}_{(2,1)}(\mathfrak{g})$ and $\mathcal{U}_{(2,1)}\mathcal{P}^{\mathcal{R}}_{(2,1)}$ satisfies the $(2,0)$-cancellation law. Therefore $\mathcal{P}^{\mathcal{R}}_{(2,1)}(\mathfrak{g})$ is locally thin. ∎

10.1. The 2-category $\dot{\Delta}_{\mathrm{Str}}$. In [24, 25, 26], we consider 2-dimensional versions of a subcategory $\dot{\Delta}_3'$ of $\dot{\Delta}_3$. For instance, the bicategorical replacement of the category $\dot{\Delta}_3'$. Here, we study the presentations of this locally thin $(2,1)$-category, including the application of our results to the presentation of the bicategorical replacement of $\dot{\Delta}_2$. Following the terminology of [25], we have:

Definition 10.8. The 2-computad $\dot{\mathfrak{d}}_{\mathrm{Str}} = (\mathfrak{g}^{\dot{\Delta}_3}, \mathfrak{g}_2^{\dot{\Delta}_3}, G_{\dot{\Delta}_3})$ is defined by the graph

$$\begin{tikzcd} 0 \arrow[r, "d"] & 1 \arrow[r, shift left=6, "d^0"] \arrow[r, shift right=6, "d^1"'] & 2 \arrow[l, "s^0" description] \arrow[r, shift left=6, "\partial^0"] \arrow[r, "\partial^1" description] \arrow[r, shift right=6, "\partial^2"'] & 3 \end{tikzcd}$$

with the 2-cells:

$$\begin{aligned} \sigma_{01} &: \partial^1 d^0 \Rightarrow \partial^0 d^0 & n_0 &: s^0 d^0 \Rightarrow \mathrm{id}_1 \\ \sigma_{02} &: \partial^2 d^0 \Rightarrow \partial^0 d^1 & n_1 &: \mathrm{id}_1 \Rightarrow s^0 d^1 \\ \sigma_{12} &: \partial^2 d^1 \Rightarrow \partial^1 d^1 & \vartheta &: d^1 d \Rightarrow d^0 d \end{aligned}$$

We denote by $\dot{\Delta}_{\mathrm{Str}}$ the locally thin $(2,1)$-category $\overline{\mathcal{M}_2}\,\overline{\mathcal{L}_2}\mathcal{F}_2(\dot{\mathfrak{d}}_{\mathrm{Str}})$ freely generated by the 2-computad $\dot{\mathfrak{d}}_{\mathrm{Str}}$. We also define the subcomputad $\mathfrak{d}_{\mathrm{Str}}$ of $\dot{\mathfrak{d}}_{\mathrm{Str}}$ such that $\Delta_{\mathrm{Str}} = \overline{\mathcal{M}_2}\,\overline{\mathcal{L}_2}\mathcal{F}_2(\mathfrak{d}_{\mathrm{Str}})$ is the full sub-2-category of $\dot{\Delta}_{\mathrm{Str}}$ and $\mathrm{obj}(\Delta_{\mathrm{Str}}) = \{1,2,3\}$.

Lemma 10.9. *Let $\mathfrak{g}^{\Delta_2} = (\mathfrak{g}^{\Delta_2}, \mathfrak{g}_2^{\Delta_2}, G_{\Delta_2})$ be the full subcomputad of $\dot{\mathfrak{d}}_{\mathrm{Str}}$ defined by*

$$\begin{tikzcd} 1 \arrow[r, shift left=6, "d^0"] \arrow[r, shift right=6, "d^1"'] & 2 \arrow[l, "s^0" description] \end{tikzcd}$$

with the 2-cells: $n_0 : s^0 d^0 \Rightarrow \mathrm{id}_1$, $n_1 : \mathrm{id}_1 \Rightarrow s^0 d^1$. The $(2,0)$-category freely generated by $\mathfrak{g}^{\Delta_2}$ is locally thin. In particular, the full sub-2-category $\Delta_{\mathrm{Str}_2} := \overline{\mathcal{M}_2}\,\overline{\mathcal{L}_2}\mathcal{F}_2\left(\mathfrak{g}^{\Delta_2}\right)$ of the 2-category $\dot{\Delta}_{\mathrm{Str}}$ is isomorphic to $\overline{\mathcal{L}_2}\mathcal{F}_2\left(\mathfrak{g}^{\Delta_2}\right)$.

Proof: We should prove that $\overline{\mathcal{L}_{(2,0)}}\mathcal{F}_2(\mathfrak{g}^{\Delta_2})$ is locally thin. By abuse of language, we denote by $\mathcal{E}_{\mathsf{cmp}}(\mathfrak{g}^{\Delta_2})$ the 2-computad $\mathfrak{g}^{\Delta_2}$. We, then, take the maximal tree of the underlying graph of $\mathfrak{g}^{\Delta_2}$ defined by $2 \xrightarrow{s^0} 1$ and denote it by G_{s^0}.

By Remark 9.7, $\overline{\mathcal{L}_{(2,0)}}\mathcal{F}_2^{\mathfrak{R}}\left(\mathfrak{g}^{\Delta_2}/\mathfrak{i}_2^{\mathfrak{R}}(G_{s^0})\right)$ is locally thin if and only if $\overline{\mathcal{L}_{(2,0)}}\mathcal{F}_2^{\mathfrak{R}}\left(\mathfrak{g}^{\Delta_2}\right)$ is locally thin. The quotient $\mathfrak{g}^{\Delta_2}/\mathfrak{i}_2^{\mathfrak{R}}(G_{s^0})$ is a computad with 1-cells $\widetilde{d}^0$, $\widetilde{d}^1$ and 2-cells $\widetilde{n}_0 : \widetilde{d}^0 \Rightarrow \mathrm{id}$ and $\widetilde{n}_1 : \mathrm{id} \Rightarrow \widetilde{d}^1$. It is clear, then, that $\mathfrak{g}^{\Delta_2}/\mathfrak{i}_2^{\mathfrak{R}}(G_{s^0})$ is an *f.c.s* of itself. Thereby the proof is complete. ■

Furthermore, the full sub-2-category Δ_{Str} of $\dot{\Delta}_{\mathrm{Str}}$ is a free $(2,1)$-category as proved in:

Theorem 10.10 (Δ_{Str}). *There is an isomorphism of 2-categories* $\Delta_{\mathrm{Str}} \cong \overline{\mathcal{L}_2}\mathcal{F}_2(\mathfrak{d}_{\mathrm{Str}})$.

Proof: Since Δ_{Str_2} is a full sub-2-category and locally thin, it is enough to prove that $\Delta_{\mathrm{Str}}(\mathsf{1},\mathsf{3})$ and $\Delta_{\mathrm{Str}}(\mathsf{2},\mathsf{3})$ are thin.

It is clear that the nontrivial 2-cells of $\Delta_{\mathrm{Str}}(\mathsf{1},\mathsf{3})$ are horizontal compositions of 2-cells of $\Delta_{\mathrm{Str}}(\mathsf{1},\mathsf{1})$ with σ_{01}, σ_{02} and σ_{12}. More precisely, the set of nontrivial 2-cells of $\Delta_{\mathrm{Str}}(\mathsf{1},\mathsf{3})$ is equal to

$$\left\{\sigma_{01} * \alpha, \sigma_{02} * \alpha, \sigma_{12} * \alpha \middle| \, (\alpha : f \Rightarrow g : \mathsf{1} \to \mathsf{1}) \in \Delta_{\mathrm{Str}}(\mathsf{1},\mathsf{1})\right\}.$$

This proves that $\Delta_{\mathrm{Str}}(\mathsf{1},\mathsf{3})$ is thin. Moreover, since the set of 2-cells of $\Delta_{\mathrm{Str}}(\mathsf{2},\mathsf{1}) = \Delta_{\mathrm{Str}_2}(\mathsf{2},\mathsf{1})$ is equal to $\left\{\alpha * \mathrm{id}_{s^0} \middle| \, (\alpha : f \Rightarrow g : \mathsf{1} \to \mathsf{1}) \in \Delta_{\mathrm{Str}}(\mathsf{1},\mathsf{1})\right\}$, it follows that the set of 2-cells of $\Delta_{\mathrm{Str}}(\mathsf{2},\mathsf{3})$ is equal to

$$\left\{\beta * \mathrm{id}_{s^0} \middle| \, (\beta : f \Rightarrow g : \mathsf{1} \to \mathsf{3}) \in \Delta_{\mathrm{Str}}(\mathsf{1},\mathsf{3})\right\}.$$

Since we already proved that $\Delta_{\mathrm{Str}}(\mathsf{1},\mathsf{3})$ is thin, we conclude that $\Delta_{\mathrm{Str}}(\mathsf{2},\mathsf{3})$ is thin. Hence, as $\Delta_{\mathrm{Str}}(\mathsf{3},\mathsf{2})$ is the initial (empty) category, the proof is complete. ■

As proved in Proposition 9.5, $\overline{\mathcal{L}_2}\mathcal{F}_2(\mathfrak{g}^{\dot{\Delta}_2})$ is not locally thin. We prove below that $\dot{\Delta}_{\mathrm{Str}_2} := \overline{\mathcal{M}_2}\,\overline{\mathcal{L}_2}\mathcal{F}_2(\mathfrak{g}^{\dot{\Delta}_2})$ can be presented by a 3-computad with only one 3-cell that corresponds to the equation given in the identity descent diagram.

Theorem 10.11 ($\dot{\Delta}_{\mathrm{Str}_2}$)**.** *The* 3*-computad* $\mathfrak{h}^{\dot{\Delta}_2}$ *defined by the* 2*-computad* $\mathfrak{g}^{\dot{\Delta}_2}$ *with only the* 3*-cell*

$$\begin{tikzcd}
0 \arrow[r, "d"] \arrow[d, "d"'] \arrow[dr, phantom, "\overset{\vartheta}{\Leftarrow}"] & 1 \arrow[d, "d^1"] \arrow[ddr, equal, bend left] \arrow[dr, phantom, "\overset{n_1}{\Leftarrow}"] & & & 0 \arrow[dd, bend right, "d"'] \arrow[dd, bend left, "d"] \arrow[dd, phantom, "="] \\
1 \arrow[r, "d^0"'] \arrow[drr, equal, bend right] \arrow[dr, phantom, "\overset{n_0}{\Leftarrow}"] & 2 \arrow[dr, "s^0"] & & \Longrightarrow & \\
& & 1 & & 1
\end{tikzcd}$$

(identity descent 3-cell)

presents the locally thin $(2,1)$*-category* $\dot{\Delta}_{\mathrm{Str}_2}$. *In other words,* $\overline{\mathcal{L}_2}\,\mathcal{P}_2(\mathfrak{h}^{\dot{\Delta}_2}) \cong \dot{\Delta}_{\mathrm{Str}_2}$.

Proof: By abuse of language, we denote $\mathcal{E}_{\mathsf{cmp}}(\mathfrak{g}^{\dot{\Delta}_2})$ by $\mathfrak{g}^{\dot{\Delta}_2}$. We denote by $\mathtt{T}$ the maximal tree

$$0 \xrightarrow{\;d\;} 1 \xleftarrow[\;s^0\;]{} 2$$

of the underlying graph of $\mathfrak{g}^{\dot{\Delta}_2}$.

The (reflexive) 2-computad $\mathfrak{g}^{\dot{\Delta}_2}/\mathfrak{i}_2^{\mathcal{R}}(\mathtt{T})$ is defined by the 1-cells $\widetilde{d^0}$, $\widetilde{d^1}$ and 2-cells $\widetilde{\vartheta} : \widetilde{d^1} \Rightarrow \widetilde{d^0}$, $\widetilde{n_0} : \widetilde{d^0} \Rightarrow \mathrm{id}$ and $\widetilde{n_1} : \mathrm{id} \Rightarrow \widetilde{d^1}$, while the 2-computad $\mathfrak{g}^{\dot{\Delta}_2}_{fcs} := \mathfrak{g}^{\dot{\Delta}_2}/\mathfrak{i}_2^{\mathcal{R}}(G_{s^0})$, defined in the proof of Lemma 10.9, is an $\underline{f.c.s.}$ of $\mathfrak{g}^{\dot{\Delta}_2}/\mathfrak{i}_2^{\mathcal{R}}(\mathtt{T})$.

By the proof of Theorem 10.4, we get a presentation of $\overline{\mathcal{M}_2}\,\overline{\mathcal{L}_{(2,0)}}\mathcal{F}_2^{\mathcal{R}}(\mathfrak{g}^{\dot{\Delta}_2}/\mathfrak{i}_2^{\mathcal{R}}(\mathtt{T}))$ by a $(3,2,0,\mathcal{R})$-computad $\mathfrak{j}'$ such that $\mathfrak{j}'_2 = \mathfrak{g}_2^{\dot{\Delta}_2} - \mathfrak{g}^{\dot{\Delta}_2}_{fcs_2}$. This $(3,2,0,\mathcal{R})$-computad is defined by the 2-computad $\mathfrak{g}^{\dot{\Delta}_2}/\mathfrak{i}_2^{\mathcal{R}}(\mathtt{T})$ with the 3-cell $\widetilde{\vartheta} \Longrightarrow \widetilde{n_0}^{-1} \cdot \widetilde{n_1}^{-1}$.

Thereby $\mathcal{U}_{(2,0)}\mathcal{P}^{\mathcal{R}}_{(2,0)}(\mathfrak{h}') \cong \overline{\mathcal{M}_2}\,\overline{\mathcal{L}_{(2,0)}}\mathcal{F}_2^{\mathcal{R}}(\mathfrak{g}^{\dot{\Delta}_2}/\mathfrak{i}_2^{\mathcal{R}}(\mathtt{T}))$. Furthermore, by Corollary 10.5, composing each morphism in the image of $\mathfrak{j}'$ with the equivalence

$$\mathcal{L}_{(2,0)}\mathcal{F}_2^{\mathcal{R}}(\mathfrak{g}^{\dot{\Delta}_2}) \cong \mathcal{L}_{(2,0)}\mathcal{F}_2^{\mathcal{R}}(\mathfrak{g}^{\dot{\Delta}_2}/\mathfrak{i}_2^{\mathcal{R}}(\mathtt{T})),$$

we get a $(3,2,0,\mathcal{R})$-computad $\mathfrak{j}$ which presents $\overline{\mathcal{M}_2}\,\overline{\mathcal{L}_{(2,0)}}\mathcal{F}_2^{\mathcal{R}}(\mathfrak{g}^{\dot{\Delta}_2})$. This $(3,2,0,\mathcal{R})$-computad $\mathfrak{j}$ is defined by the 2-computad $\mathfrak{g}^{\dot{\Delta}_2}$ with the 3-cell

$$\mathrm{id}_{s^0} * \vartheta \Longrightarrow \left(n_0^{-1} \cdot n_1^{-1}\right) * \mathrm{id}_d.$$

It is clear that the (reflexive) computad $\mathfrak{g}^{\dot{\Delta}_2}$ together with the identity descent 3-cell define a (reflexive) 3-computad $\mathfrak{h}'$ which is a lifting of $\mathfrak{j}$. Since $\mathcal{L}_2\mathcal{P}_2^{\mathcal{R}}(\mathfrak{h})$ clearly satisfies the $(2,0)$-cancellation law, this completes the proof. ∎

Theorem 10.12 ($\dot{\Delta}_{\mathrm{Str}}$)**.** *The 3-computad $\mathfrak{h}^{\dot{\Delta}}$ defined by the 2-computad $\mathfrak{g}^{\dot{\Delta}_2}$ with the 3-cell identity descent 3-cell and the 3-cell below*

$$\begin{tikzcd}
0 \arrow[r, "d"] \arrow[d, "d"'] \arrow[dr, phantom, "\overset{\vartheta}{\Rightarrow}"] & 1 \arrow[r, "d^0"] \arrow[d, "d^0"] \arrow[dr, phantom, "\overset{\sigma_{01}}{\Rightarrow}"] & 2 \arrow[d, "\partial^0"] & & 3 & 2 \arrow[l, "\partial^0"'] \arrow[r, equal] & 2 \\
1 \arrow[r, "d^1"] \arrow[d, "d^1"'] \arrow[drr, phantom, "\overset{\sigma_{12}}{\Rightarrow}"] & 2 \arrow[r, "\partial^1"] & 3 \arrow[d, "\mathrm{id}_3"] \arrow[rr, Rightarrow] & & 2 \arrow[u, "\partial^2"] \arrow[ur, phantom, "\overset{\sigma_{02}}{\Rightarrow}"] & 1 \arrow[l, "d^0"'] \arrow[u, "d^1"'] \arrow[r, phantom, "\overset{\vartheta}{\Rightarrow}"] & \\
2 \arrow[rr, "\partial^2"'] & & 3 & & 1 \arrow[u, "d^1"] \arrow[ur, phantom, "\overset{\vartheta}{\Rightarrow}"] & 0 \arrow[l, "d"] \arrow[r, "d"'] \arrow[u, "d"'] & 1 \arrow[uu, "d^0"']
\end{tikzcd}$$

(associativity descent 3-cell)

presents the locally thin $(2,1)$-category $\dot{\Delta}_{\mathrm{Str}}$. That is to say, $\overline{\mathcal{L}_2}\,\mathcal{P}_2(\mathfrak{h}^{\dot{\Delta}}) \cong \dot{\Delta}_{\mathrm{Str}}$.

Proof: Recall that $\dot{\Delta}_{\mathrm{Str}_2} \to \dot{\Delta}_{\mathrm{Str}}$ is a full inclusion of a locally thin 2-category and $\dot{\Delta}_{\mathrm{Str}}(\mathsf{3},\mathsf{n})$ is thin for any object n of $\dot{\Delta}_{\mathrm{Str}}$. Hence it only remains to prove that $\dot{\Delta}_{\mathrm{Str}}(\mathsf{0},\mathsf{3})$ is thin.

Since the set of 2-cells of $\dot{\Delta}_{\mathrm{Str}}(\mathsf{0},\mathsf{2})$ is given by

$$\{\vartheta\} \cup \left\{\mathrm{id}_{d^i} * \alpha | i \in \{0,1\} \text{ and } (\alpha : f \Rightarrow g : \mathsf{0} \to \mathsf{1}) \in \Delta_{\mathrm{Str}}(\mathsf{0},\mathsf{1})\right\},$$

we conclude that $\dot{\Delta}_{\mathrm{Str}}(\mathsf{0},\mathsf{3})$ is the thin groupoid freely generated by the graph $\mathbb{S}$ defined by the morphisms $\mathsf{0} \to \mathsf{3}$ as objects and the set of arrows (2-cells) $\mathbb{T} \cup \mathbb{T}'$ in which

$$\mathbb{T}' := \{\sigma_{ij} * \alpha | i,j \in \{0,1\}, i < j \text{ and } (\alpha : f \Rightarrow g : \mathsf{0} \to \mathsf{1}) \in \Delta_{\mathrm{Str}}(\mathsf{0},\mathsf{1})\}$$

and $\mathbb{T} := \{\sigma_{ij} * \mathrm{id}_d | i,j \in \{0,1\} \text{ and } i < j\} \cup \{\mathrm{id}_{\partial^i} * \vartheta | i \in \{0,1,2\}\}$.

We consider the full subgraph of $\mathbb{S}$ with objects in the set

$$\mathbb{O} = \left\{\partial^i \cdot d^j \cdot d | i,j \in \{0,1,2\} \text{ and } j \neq 2\right\}.$$

The set of arrows of $\mathbb{S}$ is precisely $\mathbb{T}$ and, by abuse of language, we also denote the graph by $\mathbb{T}$.

The set of the arrows (2-cells) $\mathbb{T}'$ defines a subgroupoid of $\dot{\Delta}_{\mathrm{Str}}(\mathsf{0},\mathsf{3})$, also denoted by $\mathbb{T}'$. Since $\dot{\Delta}_{\mathrm{Str}}(\mathsf{0},\mathsf{1})$ is thin, it is clear that $\mathbb{T}'$ is thin. Moreover, it is clear that $\mathbb{T}'$ is the coproduct of $\mathbb{T}'_{12}$, $\mathbb{T}'_{02}$ and $\mathbb{T}'_{01}$ which are respectively the subgroupoids defined by the sets of 2-cells $\{\sigma_{12} * \alpha | (\alpha : f \Rightarrow g) \in \Delta_{\mathrm{Str}}(\mathsf{0},\mathsf{1})\}$,

$\{\sigma_{02} * \alpha | \, (\alpha : f \Rightarrow g) \in \Delta_{\mathrm{Str}}(0,1)\}$ and $\{\sigma_{01} * \alpha | \, (\alpha : f \Rightarrow g) \in \Delta_{\mathrm{Str}}(0,1)\}$. In particular, there is not any 2-cell in $\mathbb{T}'$ between any object of $\mathbb{T}'_{ij}$ and any object $\mathbb{T}'_{xy}$ whenever $(i,j) \neq (x,y)$. For instance, there is no arrows (2-cells) $f \Rightarrow \partial^2 \cdot d^1 \cdot d$, $g \Rightarrow \partial^2 \cdot d^0 \cdot d \Rightarrow$ and $h \Rightarrow \partial^1 \cdot d^0 \cdot d \Rightarrow$ in $\mathbb{T}$ for every f, g, h objects of $\mathbb{T}$ such that f is outside $\mathbb{T}'_{12}$, g is outside $\mathbb{T}'_{02}$ and h is outside $\mathbb{T}'_{01}$.

Therefore, it is enough to study the thin groupoid freely generated by $\mathbb{T}$. More precisely, we have only to observe that the equation given by the 3-cell associativity descent 3-cell indeed presents the thin groupoid freely generated by the graph:

$$\begin{tikzcd}
\partial^0 \cdot d^0 \cdot d & \partial^0 \cdot d^1 \cdot d \arrow[l, "\mathrm{id}_{\partial^0} * \vartheta"'] \\
\partial^1 \cdot d^0 \cdot d \arrow[u, "\sigma_{01} * \mathrm{id}_d"] & \partial^2 \cdot d^0 \cdot d \arrow[u, "\sigma_{02} * \mathrm{id}_d"'] \\
\partial^1 \cdot d^1 \cdot d \arrow[u, "\mathrm{id}_{\partial^1} * \vartheta"] & \partial^2 \cdot d^1 \cdot d \arrow[u, "\mathrm{id}_{\partial^2} * \vartheta"'] \arrow[l, "\sigma_{12} * \mathrm{id}_d"]
\end{tikzcd} \qquad (\mathbb{T})$$

■

10.2. Topology. Analogously to the 1-dimensional case, we denote by $\widehat{\mathfrak{G}_2}$ the 2-computad such that $\mathcal{F}_2(\widehat{\mathfrak{G}_2}) \cong \mathfrak{G}_2$. We also have higher dimensional analogues for Theorem 4.3. This isomorphism gives an embedding $(n+1)\text{-}\mathsf{cmp} \to \mathcal{P}\mathrm{re}(\overline{\mathcal{F}_n})$ which shows that $(n+1)$-computads are indeed $\overline{\mathcal{F}_n}$-presentations.

If we denote $\mathfrak{i}_{1!} = \mathfrak{i}_1$ and $\mathfrak{i}_{(n+1)!} = \mathfrak{i}_{(n+1)}\mathfrak{i}_{n!}$, we have:

Theorem 10.13. *More generally, there is an isomorphism* $(n+1)\text{-}\mathsf{cmp} \cong (\mathfrak{i}_{n!}(-) \times \widehat{\mathfrak{G}_n}/\overline{\mathcal{F}_n})$ *in which*

$$\mathfrak{i}_{n!}(-) \times \widehat{\mathfrak{G}_n} : \mathsf{Set} \to \mathsf{cmp}, \quad Y \mapsto \mathfrak{i}_{n!}(Y) \times \widehat{\mathfrak{G}_n}.$$

In particular, there is an isomorphism $3\text{-}\mathsf{cmp} \cong (\mathfrak{i}_2\mathfrak{i}_1(-) \times \widehat{\mathfrak{G}_2}/\overline{\mathcal{F}_2})$.

Observe that, analogously to the 2-dimensional case presented in 5.1, we have a homeomorphism

$$\mathfrak{g}_2 \times \mathrm{cir}_2 : D(\mathfrak{g}_2) \times S^2 \to \mathcal{F}_{\mathsf{Top}_2}(\mathfrak{i}_2\mathfrak{i}_1(\mathfrak{g}_2) \times \widehat{\mathfrak{G}_2}).$$

for each set $\mathfrak{g}_2$.

There are higher dimensional analogues of the association of each small computad with a CW-complex given in 5.1. Nevertheless, again, analogously to Remark 9.20, we do not have higher dimensional analogues of the results given in Remark 5.10, Theorem 5.11 and Theorem 5.13.

We sketch a 2-dimensional version of the natural transformation $[\_] : \overline{\mathcal{F}_1}\mathcal{C}_{\mathsf{Top}_1} \longrightarrow \mathcal{C}_{\mathsf{Top}_1}$ to get the association of each small 3-computad with a 3-dimensional CW-complex.

Given a 2-cell α of $\overline{\mathcal{F}_2}\mathcal{C}_{\mathsf{Top}_2}(E)$, we have that there is a unique way of getting α as pasting of 2-cells of $\mathcal{C}_{\mathsf{Top}_2}(E)$. That is to say, it is a "formal pasting" of homotopies. We can glue these homotopies to get a new homotopy, which is what we define to be $[\underline{\alpha}]^2_E : B^2 \to E$. This defines a natural transformation $[\_]^2 : \overline{\mathcal{F}_2}\mathcal{C}_{\mathsf{Top}_2} \longrightarrow \mathcal{C}_{\mathsf{Top}_2}$. We denote by $\lceil\cdot\rceil^2$ the mate under the adjunction $\mathcal{F}_{\mathsf{Top}_2} \dashv \mathcal{C}_{\mathsf{Top}_2}$ and itself.

Given a small 3-computad, seen as a morphism $\mathfrak{g} : \mathfrak{i}_2\mathfrak{i}_1(\mathfrak{g}_2) \times \widehat{\mathfrak{G}_2} \to \overline{\mathcal{F}_2}(G)$ of small 2-computads, $\mathcal{F}_{\mathsf{Top}_3}(\mathfrak{g}, \mathfrak{g}_2, G)$ is the pushout of the inclusion $S^2 \times D(\mathfrak{g}_2) \to B^3 \times D(\mathfrak{g}_2)$ along the composition of the morphisms

$$D(\mathfrak{g}_2) \times S^2 \xrightarrow{(\mathrm{cir}_2 \times \mathfrak{g}_2)} \mathcal{F}_{\mathsf{Top}_2}(\mathfrak{i}_2\mathfrak{i}_1(\mathfrak{g}_2) \times \widehat{\mathfrak{G}_2}) \xrightarrow{\mathcal{F}_{\mathsf{Top}_2}(\mathfrak{g})} \mathcal{F}_{\mathsf{Top}_2}\overline{\mathcal{F}_2}(G) \xrightarrow{\lceil\cdot\rceil^2_G} \mathcal{F}_{\mathsf{Top}_2}(G) \qquad .$$

The topological space $\mathcal{F}_{\mathsf{Top}_3}(\mathfrak{g}, \mathfrak{g}_2, G)$ is clearly a CW-complex of dimension 3. Furthermore, of course, we have groupoidal and reflexive versions of $\mathcal{F}_{\mathsf{Top}_3}$ as well, such as $\mathcal{F}^{\mathcal{R}}_{\mathsf{Top}_3} : 3\text{-}\mathsf{Rcmp} \to \mathsf{Top}$.

Lemma 10.14. *If $(\mathfrak{g}, \mathfrak{g}_2, G)$ has only one 0-cell and only one 1-cell and $\pi_2\mathcal{F}_{\mathsf{Top}_3}(\mathfrak{g}, \mathfrak{g}_2, G)$ is not trivial, then $\mathcal{L}_{(2,0)}\mathcal{P}_2(\mathfrak{g}, \mathfrak{g}_2, G)$ is not locally thin.*

Thereby, by Theorem 10.3, we get:

Theorem 10.15. *Assume that $G^{\mathfrak{b}}$ is an f.c.s. of the small reflexive 2-computad G. If $(\mathfrak{g}, \mathfrak{g}_2, G)$ is a small (reflexive) 3-computad such that $\pi_2\mathcal{F}^{\mathcal{R}}_{\mathsf{Top}_3}(\mathfrak{g}, \mathfrak{g}_2, G)$ is not trivial, $\mathcal{L}_{(2,0)}\mathcal{P}^{\mathcal{R}}_2(\mathfrak{g})$ is not locally thin.*

Proof: It follows from Theorem 10.3 and from the fact that $\mathcal{F}^{\mathcal{R}}_{\mathsf{Top}_3}\mathfrak{i}_3(G^{\mathfrak{b}})$ is contractible and its inclusion in $\mathcal{F}^{\mathcal{R}}_{\mathsf{Top}_3}(\mathfrak{g}, \mathfrak{g}_2, G)$ is a cofibration. ■

Since $\mathcal{F}_{\mathsf{Top}_3}(\mathfrak{g}, \mathfrak{g}_2, G)$ has the same homotopy type of a wedge of circumferences, 2-dimensional balls, 3-dimensional balls and spheres, we know that Euler characteristic $\chi\left(\mathcal{F}_{\mathsf{Top}_3}(\mathfrak{g}, \mathfrak{g}_2, G)\right)$ is equal to

$$\chi\left(\mathcal{F}_{\mathsf{Top}_2}(G)\right) - |\mathfrak{g}_2|\,,$$

whenever both $\chi\left(\mathcal{F}_{\mathsf{Top}_2}(G)\right)$ and $|\mathfrak{g}_2|$ are finite.

Corollary 10.16. *Assume that* $G^{\mathfrak{d}}$ *is an f.c.s. of the small reflexive* 2-*computad* G. *If* $(\mathfrak{g}, \mathfrak{g}_2, G)$ *is a small (reflexive)* 3-*computad such that*

$$\mathbb{Z} \ni \chi\left(\mathcal{F}^{\mathcal{R}}_{\mathsf{Top}_3}(\mathfrak{g}, \mathfrak{g}_2, G)\right) > 1,$$

then $\mathcal{L}_{(2,0)}\mathcal{P}_2(\mathfrak{g}, \mathfrak{g}_2, G)$ *is not locally thin.*

Proof: Recall that

$$\chi\left(\mathcal{F}^{\mathcal{R}}_{\mathsf{Top}_3}(\mathfrak{g})\right) = 1 - \dim \mathrm{H}^1\left(\mathcal{F}^{\mathcal{R}}_{\mathsf{Top}_3}(\mathfrak{g})\right) + \dim \mathrm{H}^2\left(\mathcal{F}^{\mathcal{R}}_{\mathsf{Top}_3}(\mathfrak{g})\right) - \dim \mathrm{H}^3\left(\mathcal{F}^{\mathcal{R}}_{\mathsf{Top}_3}(\mathfrak{g})\right).$$

Since $\mathcal{F}^{\mathcal{R}}_{\mathsf{Top}_3}(\mathfrak{g}, \mathfrak{g}_2, G)$ is clearly 1-connected, $\dim \mathrm{H}^1\left(\mathcal{F}^{\mathcal{R}}_{\mathsf{Top}_3}(\mathfrak{g}, \mathfrak{g}_2, G)\right) = 0$. Therefore, by hypothesis,

$$\dim \mathrm{H}^2\left(\mathcal{F}^{\mathcal{R}}_{\mathsf{Top}_3}(\mathfrak{g}, \mathfrak{g}_2, G)\right) > \dim \mathrm{H}^3\left(\mathcal{F}^{\mathcal{R}}_{\mathsf{Top}_3}(\mathfrak{g}, \mathfrak{g}_2, G)\right) \geqslant 0.$$

In particular, we conclude that $\dim \mathrm{H}^2\left(\mathcal{F}^{\mathcal{R}}_{\mathsf{Top}_3}(\mathfrak{g}, \mathfrak{g}_2, G)\right) > 0$. By the Hurewicz isomorphism theorem and by the universal coefficient theorem, this fact implies that the fundamental group $\pi_2\left(\mathcal{F}^{\mathcal{R}}_{\mathsf{Top}_3}(\mathfrak{g}, \mathfrak{g}_2, G)\right)$ is not trivial. By Theorem 10.15, we get that $\mathcal{L}_{(2,0)}\mathcal{P}^{\mathcal{R}}_2(\mathfrak{g})$ is not locally thin. ■

Assume that $(\mathfrak{g}, \mathfrak{g}_2, G)$ is a small (reflexive) 3-computad such that there is an *f.c.s.* triple $(G, \mathtt{T}, H^{\mathfrak{d}})$. Then $\mathcal{F}^{\mathcal{R}}_{\mathsf{Top}_3}\mathfrak{i}_3\mathfrak{i}_2(\mathtt{T}) \to \mathcal{F}_{\mathsf{Top}_3}(\mathfrak{g}, \mathfrak{g}_2, G)$ is an cofibrant inclusion of a contractible space. Thereby, $\pi_2\mathcal{F}^{\mathcal{R}}_{\mathsf{Top}_3}(\mathfrak{g}, \mathfrak{g}_2, G)$ is trivial if and only if

$$\pi_2\left(\mathcal{F}^{\mathcal{R}}_{\mathsf{Top}_3}(\mathfrak{g})/\mathcal{F}^{\mathcal{R}}_{\mathsf{Top}_3}\mathfrak{i}_3\mathfrak{i}_2(\mathtt{T})\right) \cong \pi_2\left(\mathcal{F}_{\mathsf{Top}_3}(\mathfrak{g}/\mathfrak{i}_3\mathfrak{i}_2(\mathtt{T}))\right)$$

is trivial. Therefore, since $\mathcal{L}_{(2,0)}\mathcal{P}_2(\mathfrak{g}/\mathfrak{i}_3\mathfrak{i}_2(\mathtt{T}))$ is locally thin if and only if $\mathcal{L}_{(2,0)}\mathcal{P}_2(\mathfrak{g})$ is locally thin, it follows from Theorem 10.15 and Corollary 10.16 the result below:

Corollary 10.17. *Assume that* $(\mathfrak{g}, \mathfrak{g}_2, G)$ *is a small (reflexive)* 3-*computad such that there is an f.c.s. triple* $(G, \mathtt{T}, H^{\mathfrak{d}})$. *If* $\pi_2\mathcal{F}^{\mathcal{R}}_{\mathsf{Top}_3}(\mathfrak{g}, \mathfrak{g}_2, G)$ *is not trivial,* $\mathcal{L}_{(2,0)}\mathcal{P}^{\mathcal{R}}_2(\mathfrak{g})$ *is not locally thin. Furthermore,*

$$\mathbb{Z} \ni \chi\left(\mathcal{F}^{\mathcal{R}}_{\mathsf{Top}_3}(\mathfrak{g}, \mathfrak{g}_2, G)\right) > 1,$$

then $\mathcal{L}_{(2,0)}\mathcal{P}_2(\mathfrak{g}, \mathfrak{g}_2, G)$ *is not locally thin.*

In particular, we get that, whenever such a 3-*computad presents a locally thin* (2, 0)-*category,* $|\mathfrak{g}_2| \geqslant \chi\left(\mathcal{F}^{\mathcal{R}}_{\mathsf{Top}_2}(G)\right) - 1$.

This also works for the $(3,2,0,\mathcal{R})$-version of $\mathcal{F}_{\mathsf{Top}_3}$ which would show that the presentation by $(3,2,0,\mathcal{R})$-computads given in Corollary 10.5 is in a sense the best presentation via $(3,2,0,\mathcal{R})$-computads of the locally thin $(2,0)$-category generated by the reflexive computad G if $\mathcal{F}^{\mathcal{R}}_{\mathsf{Top}_2}(G)$ has finite Euler characteristic. For instance, by Corollary 10.17, since $\chi\left(\mathcal{F}^{\mathcal{R}}_{\mathsf{Top}_2}(\mathfrak{g}^{\dot{\Delta}_2})\right) = 2$, the presentation via 3-computad given in Theorem 10.11 has the least number of 3-cells.

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