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STRONGLY MIXED RANDOM ERRORS IN MANN'S ITERATION ALGORITHM FOR A CONTRACTIVE REAL FUNCTION

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Abstract: This work deals with the Mann's stochastic iteration algorithm under $\alpha-$mixing random errors. We establish the Fuk-Nagaev's inequalities that enable us to prove the almost complete convergence with its corresponding rate of convergence. Moreover, these inequalities give us the possibility of constructing a confidence interval for the unique fixed point. Finally, to check the feasibility and validity of our theoretical results, we consider some numerical examples, namely a classical example from astronomy.

Keywords: Fixed point; Fuk-Nagaev's inequalities; Mann's iteration algorithm; α-mixing; Rate of convergence; confidence domain.

1. Introduction

In many mathematical problems arising from various domains, the existence of a solution is the same as the existence of a fixed point by some appropriate transformation of the problem. The most known problem in that framework is the root existence which can be tackled easily as the existence of a fixed point and vice versa. Therefore, the fixed point theory is of paramount importance in engineering sciences and many areas of mathematics. Fixed point theory provides conditions under which maps have the existence and uniqueness of solutions. Over the last decades, that theory has been revealed as one of the most significant tool in the study of nonlinear problems. In particular, in many fields, equilibria or stability are fundamental concepts that can be described in terms of fixed points. For example, in economics, a Nash equilibrium of a game is a fixed point of the game's best response correspondence. However, in informatics, programming language compilers use fixed point computations for program analysis, for example in data-flow analysis, which is often required for code optimization. The vector

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of PageRank values of all web pages is the fixed point of a linear transformation derived from the World Wide Web's link structure. In astronomy, the eccentric anomaly E of a given planet is related to a fixed point equation that cannot be solved analytically, this will be well described in example (9) and many examples could be found in engineering sciences such as physics, geology, chemistry, biology, mechanical statistics, etc.
Mathematically, a fixed point problem is presented under the following form

$$\text{Find } x \in X \text{ such that } Fx = x \tag{1}$$

Where F is an operator, defined on a given space X. The solutions of that equation if they exist are called "fixed points" of the mapping F. The classical result in fixed point theory is the Banach fixed-point theorem [2]; it ensures the existence and uniqueness of a fixed point of certain self-maps of a metric space. Additionally, it provides a constructive numerical method to approximate the fixed point.

After verifying the existence and uniqueness conditions, it is necessary to find (or approximate) the unique fixed point of the problem (1). This leads to find the unique root of $F - id_X$ where id_X denotes the identity operator on X. Analytically, to find that root, one has to reverse the operator $F - id_X$ and one could immediately think about the difficulty when dealing with inversion and most of the time that task is impossible. Alternatively, numerical methods become the most appropriate tool and have attracted many researches these last decades. The pioneering work after Picard's iterative method was introduced by Mann [15] to remedy the problem of convergence while using the Picard's method for approximating the fixed point of nonexpansive mapping. Later, many modified algorithms were introduced, by considering the stochastic part, i.e., considering the errors generated by the numerical evaluation of the algorithm. For an account of relevant literature on that topic, see ([3, 4, 5, 9, 11, 12, 13, 14]).

In the framework of this paper, we consider the Mann iterative algorithm as described in (3) by taking into account the committed errors at each evaluation of the approximated fixed point x_n. These errors are supposed to be random and modeled by random variables and we suppose them to be strong mixing. Recall that a sequence (ξ_i) is said to be strong mixing or

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$\alpha-$mixing if the following condition is satisfied:

$$\alpha(n) = \sup_{A\in\mathcal{F}_{-\infty}^{k}, C\in\mathcal{F}_{k+n}^{+\infty}} |\mathbb{P}(A\cap C) - \mathbb{P}(A)\mathbb{P}(C)| \underset{n\longrightarrow+\infty}{\longrightarrow} 0 \tag{2}$$

where $\mathcal{F}_l^m$ denotes the σ-algebra engendered by events of the form $\{(\xi_{i_1},\cdots,\xi_{i_k})\in B\}$, where $l\leq i_1<i_2<\cdots<i_k\leq m$ and B is a Borel set.

The notion of α-mixing was firstly introduced by Rosenbaltt in 1956 [17] and the central limit theorem has been established. The strong mixing random variables have many interest in linear processes and find many application in finance, for more motivating examples and properties concerning the mixing notions, see [7, 10].

In this paper, we establish Fuk-Nagaev's inequalities. These inequalities allow us to prove the almost complete convergence of Mann's algorithm to the fixed point, with convergence rate and the possibility of giving a confidence interval. To strengthen the obtained theoretical results, some numerical examples are considered.

The rest of the paper is organized as follows: In section 2, the statement of the problem is described and some known results are recalled. In section 3, some new results were established by using stochastic methods. In section 4, the validity of our approach is checked up by considering some numerical examples.

2. Preliminaries

Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space and $f:\mathbb{R}\to\mathbb{R}$ a non-linear function. We consider the stochastic Mann's iteration algorithm:

$$x_{n+1} = (1-a_n)x_n + b_n f(x_n) + c_n\xi_n, \tag{3}$$

where the sequences of positive numbers $(a_n)_{n\geq1}$, $(b_n)_{n\geq1}$ and $(c_n)_{n\geq1}$ satisfy the following conditions,

$$\lim_{n\to+\infty} b_n = \lim_{n\to+\infty} a_n = 0 \text{ and } \sum_{n=1}^{+\infty} b_n = \sum_{n=1}^{+\infty} a_n = +\infty$$
$$\sum_{n=1}^{+\infty} c_n^2 < +\infty.$$

Without loss of generality, we take

$$a_n = b_n = \frac{a}{n} \text{ and } c_n = \frac{a}{n^2}, a>0.$$

Hence, the stochastic Mann's iteration algorithm (3) takes the following form:

$$x_{n+1} = \left(1 - \frac{a}{n}\right) x_n + \frac{a}{n}\left[f\left(x_n\right) + \frac{1}{n}\xi_n\right]. \tag{4}$$

We now introduce some classical hypothesis that will be useful tools for the proof of established results in the sequel:

(H1) : The fixed point x^* satisfies

$$\exists\ N > 0,\ \left|x_1 - x^*\right| \leq N < +\infty. \tag{5}$$

(H2) : The function f is contractive, i.e, it satisfies the following property:

$$\forall\ x, y \in \mathbb{R}, \left|f\left(x\right) - f\left(y\right)\right| \leq c\left|x - y\right|, c \in (0,1). \tag{6}$$

(H3) : The random variables $(\xi_i)_i$ fulfill the condition of uniform decrease, that is,

$$\exists\ p > 2,\ \forall\ t > 0,\ \mathbb{P}\left\{|\xi_i| > t\right\} \leq t^{-p}. \tag{7}$$

(H4) : The coefficients of the α-mixing sequence $(\xi_n)_n$ satisfy the following arithmetic decay condition:

$$\exists\ d \geq 1,\ \exists\ \beta > 1 : \alpha\left(n\right) \leq d\ n^{-\beta}, \forall\ n \in \mathbb{N}^*. \tag{8}$$

(H5) : The α-mixing coefficients satisfy the following condition

$$\exists\ \rho > 0,\ \rho\frac{\left(\beta + 1\right)p}{\beta + p} > 2. \tag{9}$$

Remark 1. The assumption (H1) is classical. Arbitrary choice of x_1 and the existence of x^* allows us to assume such supposition. The contraction's assumption (H2) ensures the existence and uniqueness of the fixed point x^* of the function f according to Banach's theorem for fixed point. When the function f is differentiable, the condition (H2) is equivalent to the boundness of the derivative f', i.e, $\exists\ c > 0, \sup_x \left|f'\left(x\right)\right| \leq c < 1$. The hypothesis (H3) is satisfied for all bounded random variables and Gaussian ones. Assumption (H4) is used in order to characterize the dependence structure of errors. Moreover, combined to (H3), the assumption (H4) allows the obtention of Fuk-Nagaev's inequalities [16], which ensures the almost complete convergence result. As a particular example, the geometric α-mixing sequence $(\xi_i)_i$ and its mixing coefficients are defined as follows,

$$\exists\ d_0 > 0, \exists\ \kappa \in (0,1) : \alpha\left(n\right) \leq d_0\ \kappa^n, \forall\ n\ \in \mathbb{N}^*.$$

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Let n be the mean angular motion of the Mercury's orbit around the sun, t the elapsed time since the planet was last closet to the sun (this is called perifocus or perihelion in astronomy) and $e = \sqrt{1 - \frac{b^2}{a^2}}$, the eccentricity of the planet's elliptical orbit. Using Kepler's laws of planetary motion, we obtain the location of the planet at time t.

$$\begin{cases} x = a\left(\cos\left(E\right) - e\right) \\ y = a\sqrt{1-e^2}\sin\left(E\right) \end{cases}$$

The quantity E is called the eccentric anomaly and is given by the following equation

$$E = nt + e\sin\left(E\right) = M + e\sin\left(E\right).$$

where M is called the mean anomaly which increases linearly in time at the rate n. Note that E is the fixed point of the function f, where $f\left(x\right) = M + e\sin\left(x\right)$ for a given time t and the frequency of the orbit ω. In this equation, we cannot find an explicit formula of the eccentric anomaly E. It is easy to check that f is a contraction, moreover, we have,

$$\left|f\left(x\right) - f\left(y\right)\right| \leq e\left|x - y\right|,$$

which ensures the existence and uniqueness of the fixed point E.

For our simulation, by choosing the Mercury planet, we have its eccentricity $e = 0.20563069$ and the mean anomaly M= 3.05076572 (The Mann's process is implemented for $a = 0.9, \varphi = 0.7$), and given an initial guess $x_1 = 3$, we obtain the following iterates:

n	x_n	$\left\|x_n - x_{n-1}\right\|$	$\left\|x_n - x_{fp}\right\|$
100	3.066277803444744	5.084582991976561e-06	3.292480386907215e-05
1000	3.066247563732222	2.842158499660741e-07	2.685091347043311e-06
10^4	3.066245125153754	6.291136500635730e-08	2.465128789985727e-07
10^5	3.066244900326525	7.696832948766996e-09	2.168565016447133e-08

Remark 10. Note that the numerical solution of the equation $f(x) = x$ given by Matlab is $x_{fp} = 3.066244878640875$. As we can observe, the used Mann algorithm gives nice approximations of the unique fixed point of the function f. Thus, the complementary numerical examples considered above make the obtained theoretical results of convergence well palpable.

The assumption (H5) will be useful for specifying the rate of almost complete convergence of the stochastic Mann's iteration algorithm. That condition is classical, see [1, 8].

First, we state the following theorem which will be used in the sequel during the proof of the main result.

Theorem 2. *Let $\left(\xi_i\right)_{i\in\mathbb{N}^*}$ be a centered sequence of real-valued random variables and $\left(\alpha_n\right)_{n\in\mathbb{N}^*}$ the corresponding sequence of mixing coefficients as defined in (2) such that the hypothesis (H3) and (H4) are satisfied. Let us set*

$$s_n^2 = \sum_{i=1}^{n}\sum_{k=1}^{n}\left|Cov\left(\xi_i, \xi_k\right)\right|.$$

Then, for every real numbers $r \geq 1$ and $\lambda > 0$, we have

$$\mathbb{P}\left(\sup_{k\in[1,n]}\left|\sum_{i=1}^{k}\xi_i\right| \geq 4\lambda\right) \leq 4\left(1 + \frac{\lambda^2}{rs_n^2}\right)^{\frac{-r}{2}} + 4Cnr^{-1}\left(\frac{r}{\lambda}\right)^{\frac{(\beta+1)p}{(\beta+p)}}.$$

Proof: The proof is well detailed in [16], page 84 to 87. ∎

Lemma 3. *Using the hypothesis (H1), we get the following inequality:*

$$\left|x_{n+1} - x^*\right| \leq N\prod_{i=1}^{n}\left(1 - \frac{a\left(1-c\right)}{i}\right) + \sum_{i=1}^{n}\frac{a}{i^2}\prod_{j=i+1}^{n}\left(1 - \frac{a\left(1-c\right)}{j}\right)\left|\xi_i\right|. \quad (10)$$

Proof: The proof is straightforward by induction on n. ∎

Lemma 4. *For all constants $a < 1$, we have the following inequalities*

$$\prod_{j=i+1}^{n}\left(1 - \frac{a\left(1-c\right)}{j}\right) \leq \left(\frac{i+1}{n+1}\right)^{a(1-c)}, \quad (11)$$

and,

$$\sum_{i=1}^{n}\frac{a}{i^2}\prod_{j=i+1}^{n}\left(1 - \frac{a\left(1-c\right)}{j}\right) \leq \frac{aS}{\left(n+1\right)^{a(1-c)}}. \quad (12)$$

Proof: For the first inequality, we have

$$\prod_{j=i+1}^{n}\left(1 - \frac{a\left(1-c\right)}{j}\right) \leq \exp\left(-a\left(1-c\right)\sum_{j=i+1}^{n}\frac{1}{j}\right) \leq \left(\frac{i+1}{n+1}\right)^{a(1-c)}.$$

The second inequality follows immediately from inequality (11) by setting S the sum of the convergent series $\frac{(i+1)^{a(1-c)}}{i^2}$. ■

3. Convergence of Mann iterative algorithm

The following theorem gives the almost complete convergence by mean of Fuk-Nagaev's inequality that will give also the possibility of constructing a confidence interval.

Theorem 5. *Under the assumptions (H1)–(H5) and for any real positive ρ such that $\frac{2(\beta+p)}{p(\beta+1)} < \rho < a(1-c) < 1$, we have:*

$$x_{n+1} - x^* = O\left(\frac{\sqrt{\ln n}}{n^{a(c-1)-\rho}}\right) \qquad a.co. \tag{13}$$

Proof: Using the inequality (10), we have

$$\begin{aligned}
&\mathbb{P}\left(|x_{n+1} - x^*| > \varepsilon\right) \\
\leq\ &\mathbb{P}\left(N\prod_{i=1}^{n}\left(1-\frac{a(1-c)}{i}\right) + \sum_{i=1}^{n}\frac{a}{i^2}\prod_{j=i+1}^{n}\left(1-\frac{a(1-c)}{j}\right)|\xi_i| > \varepsilon\right) \\
\leq\ &\mathbb{P}\left(N\prod_{i=1}^{n}\left(1-\frac{a(1-c)}{i}\right) + \sum_{i=1}^{n}\frac{a}{i^2}\prod_{j=i+1}^{n}\left(1-\frac{a(1-c)}{j}\right)\mathbb{E}|\xi_i| \geq \frac{\varepsilon}{2}\right) \\
&+\mathbb{P}\left(\sum_{i=1}^{n}\frac{a}{i^2}\prod_{j=i+1}^{n}\left(1-\frac{a(1-c)}{j}\right)(|\xi_i| - \mathbb{E}|\xi_i|) > \frac{\varepsilon}{2}\right).
\end{aligned} \tag{14}$$

Firstly, since the left hand side inside the following probability is not random, we have

$$\mathbb{P}\left(N\prod_{i=1}^{n}\left(1-\frac{a(1-c)}{i}\right) + \sum_{i=1}^{n}\frac{a}{i^2}\prod_{j=i+1}^{n}\left(1-\frac{a(1-c)}{j}\right)\mathbb{E}|\xi_i| > \frac{\varepsilon}{2}\right) \leq K_1 e^{-n^{2a(1-c)}\varepsilon^2}. \tag{15}$$

We set

$$Z_i = \frac{an^{a(1-c)}}{i^2}\prod_{j=i+1}^{n}\left(1-\frac{a(1-c)}{j}\right)(|\xi_i| - \mathbb{E}|\xi_i|).$$

Box-Muller :

$$g_k = \sqrt{-2ln(u_1)}\ cos(2\pi u_2) \tag{33}$$

where u_1 and u_2 are uniform distributed random numbers on (0.1)

Example 8. We consider the following function defined by:

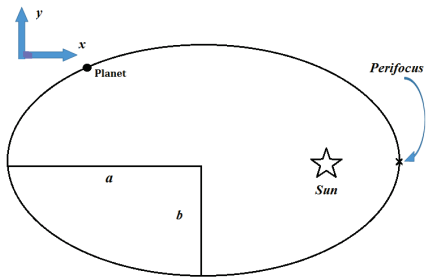
$$\begin{aligned} f &: [0,5] \to [0,5] \\ x &\mapsto \sqrt{x+1} \end{aligned}$$

The function f is a contractive function with $c = \max_{x\in[0,+\infty)}\left|f'(x)\right| = \frac{1}{2}$. Hence f has a unique fixed point $x^* = \frac{1+\sqrt{5}}{2} = 1,618033988749895$, which is known as golden ration. For $x_1 = 1.3, a = \frac{1}{4}$ and $\varphi = 0.8$, the following results are obtained:

n	x_n	$\lvert x_n - x^*\rvert$
10^3	1.614142671526978	0.003891317222917
10^4	1.615420224332314	0.002613764417581
10^5	1.616115916146472	0.001918072603423

Example 9. Most of mathematical problems come from engineering sciences (physics, chemistry, geology, astronomy, etc.). When studying some physical problems using the appropriate mathematical models, we obtain an equation or a set of equations and usually cannot be solved analytically because in general, these equations are corrupted by noise or the known mathematical tools do not allow us to solve them. To illustrate this fact, we consider the following classical example from astronomy.

Consider a planet in an orbit around the sun as described by the following diagram.



Corollary 7. *Under the assumptions (H1)–(H5), for a given level σ, there exists a natural integer n_σ for which the fixed point x^* of the function f belongs to closed interval of center x_{n_σ} and radius ε with a probability greater than or equal to $1-\sigma$.*

$$\forall\, \varepsilon > 0, \forall\, \sigma > 0, \exists\, n_\sigma \in \mathbb{N} : \mathbb{P}\left(|x_{n_\sigma} - x^*| \leq \varepsilon\right) \geq 1 - \sigma. \tag{29}$$

Proof: Indeed, using Kronecker's Lemma, we obtain $\lim\limits_{n\to+\infty} \alpha(n) = 0$ which implies

$$\lim_{n\to+\infty} \frac{K_1}{n^{1+\delta}} + \frac{K_2}{n^{\rho\frac{r}{2}}} + \frac{K_3}{n^{\rho\frac{(\beta+1)p}{\beta+p}} \left(\ln n\right)^{\frac{(\beta+1)p}{\beta+p}}} = 0. \tag{30}$$

Since there exists a natural integer n_σ such that

$$\forall\, n \in \mathbb{N}, n \geq n_\sigma - 1 \Longrightarrow \frac{K_1}{n^{1+\delta}} + \frac{K_2}{n^{\rho\frac{r}{2}}} + \frac{K_3}{n^{\rho\frac{(\beta+1)p}{\beta+p}} \left(\ln n\right)^{\frac{(\beta+1)p}{\beta+p}}} \leq \sigma, \tag{31}$$

thus, (29) arises from (28) and (31). ∎

4. Numerical results

In this section, a simulation study is proposed to check the validity of our obtained theoretical results. We consider two examples. In the first one, a contractive function where its unique fixed point is known exactly and we compare the fixed point with the approximated ones obtained using the Mann's iterative algorithm. In the second example, we consider a classical problem from astronomy, where the mathematical equation cannot be solved to obtain the exact value of the fixed point and we use the Cauchy's criterium to compare two successive iterates to ensure the convergence of the sequence obtained using iterative Mann's algorithm.

$$x_{n+1} = \left(1 - \frac{a}{n}\right) x_n + \frac{a}{n}\left[f\left(x_n\right) + \frac{1}{n}\xi_n\right]$$

$0 < a(1-c) < 1$, $\xi_0 = 0$, $n \in \mathbb{N}^*$.

To characterize the strong mixing random errors (ξ_i), we consider an autoregressive model $(\xi_i)_i$ of order 1 (see [6]), described as follows

$$\xi_{i+1} = \varphi\xi_i + g_i, \tag{32}$$

where g_i is a Gaussian white noise process, φ is a constant such that $|\varphi| < 1$. For the simulation of Gaussian random variables $(g_i)_i$, we use the method of

Note that the random variables $(Z_i)_i$ are centered and according to (7), we show that, there exists a positive constant M such that,

$$\forall\, t > 0\ ,\ \mathbb{P}\left(|Z_i| > t\right) \leq Mt^{-p}. \tag{16}$$

Finally, we notice that if the random errors (ξ_i) are α-mixing, then the random variables (Z_i) remain also with mixing coefficients less than or equal to those of the sequence $(\xi_i)_i$. Thus, applying the Fuk-Nagaev's exponential inequality given by Rio (Theorem 2) to the variables (Z_i), we obtain for any $\varepsilon > 0$ and $r \geq 1$:

$$\begin{aligned}\mathbb{P}\left(|x_{n+1} - x^*| > \varepsilon\right) \ &\leq\ K_1 e^{-n^{2a(1-c)}\varepsilon^2} + 4\left(1 + \frac{\varepsilon^2 n^{2a(1-c)}}{4rs_n^2}\right)^{\frac{-r}{2}} \\ &\quad + 4Cnr^{-1}\left(\frac{2r}{\varepsilon n^{a(1-c)}}\right)^{\frac{(\beta+1)p}{(\beta+p)}}\end{aligned} \tag{17}$$

where,

$$C = 2Mp\left(2p-1\right)^{-1}\left(2^\beta d\right)^{\frac{p-1}{\beta+p}} \text{ and } s_n^2 = \sum_{i=1}^{n}\sum_{k=1}^{n}\left|Cov\left(Z_i, Z_k\right)\right|.$$

Let us bound the double sum of covariances s_n^2,

$$s_n^2 = \sum_{i=1}^{n}\sum_{k=1}^{n}\left|Cov\left(Z_i, Z_k\right)\right| = \sum_{i=1}^{n} Var\left(Z_i\right) + \sum_{i=1}^{n}\sum_{k\neq i}^{n}\left|Cov\left(Z_i, Z_k\right)\right|.$$

We recall that,

$$\sum_{i=1}^{n} Var\left(Z_i\right) \leq \sum_{i=1}^{n} \frac{a^2\left(i+1\right)^{2a(1-c)}}{i^4} Var\left(|\xi_i|\right) \leq S_v \tag{18}$$

since it is a partial sum of a convergent series with positive terms.

On the other hand, for $i \neq k$, we have

$$\left|Cov\left(Z_i, Z_k\right)\right| \leq \frac{a^2\left(i+1\right)^{a(1-c)}}{i^2}\frac{\left(k+1\right)^{a(1-c)}}{k^2}\left|\mathbb{E}\left(|\xi_i| - \mathbb{E}\left|\xi_i\right|\right)\left(|\xi_k| - \mathbb{E}\left|\xi_k\right|\right)\right|. \tag{19}$$

According to the inequality given by Ibragimov [10] (Theorem 17.2.2 page 307), we obtain:

$$\left|\mathbb{E}\left(|\xi_i| - \mathbb{E}\left|\xi_i\right|\right)\left(|\xi_k| - \mathbb{E}\left|\xi_k\right|\right)\right| \leq \left(4 + 6C\right)\left(\alpha\left(|i-k|\right)\right)^{\frac{p-2}{p}}, \tag{20}$$

consequently,

$$|Cov\left(Z_i, Z_k\right)| \leq a^2\left(4+6C\right)\frac{\left(i+1\right)^{a(1-c)}}{i^2}\frac{\left(k+1\right)^{a(1-c)}}{k^2}\left(\alpha\left(|i-k|\right)\right)^{\frac{p-2}{p}}. \quad (21)$$

Since the mixing coefficients of the sequence $\left(|\xi_i| - \mathbb{E}\left|\xi_i\right|\right)_i$ are less than or equal to those of the sequence $\left(\xi_i\right)_i$, we get,

$$\sum_{i=1}^{n}\sum_{k\neq i}^{n}|Cov\left(Z_i, Z_k\right))| \leq \sum_{i=1}^{n}\sum_{k=1}^{n}a^2\left(4+6C\right)\frac{\left(i+1\right)^{a(1-c)}}{i^2}\frac{\left(k+1\right)^{a(1-c)}}{k^2}\left(\alpha(1)\right)^{\frac{p-2}{p}}$$

$$\leq \left(4+6C\right)\sum_{i=1}^{n}\frac{\left(i+1\right)^{a(1-c)}}{i^2}\sum_{k=1}^{n}a^2\frac{\left(k+1\right)^{a(1-c)}}{k^2} \leq S_c. \quad (22)$$

Combining (18) and (22), we obtain:

$$s_n^2 \leq S_v + S_c = S. \quad (23)$$

So, from (23), we have the inequality

$$\mathbb{P}\left(|x_{n+1} - x^*| > \varepsilon\right) \leq T_1 + T_2 + T_3, \quad (24)$$

where

$$T_1 = K_1 e^{-n^{2a(1-c)}\varepsilon^2},\ T_2 = 4\left(1+\frac{n^{2a(1-c)}\varepsilon^2}{4rS}\right)^{\frac{-r}{2}} \text{ and } T_3 = 4Cnr^{-1}\left(\frac{r}{n^{a(1-c)}\varepsilon}\right)^{\frac{(\beta+1)p}{\beta+p}}.$$

For a well chosen positive number r and ε, the quantities T_1, T_2 and T_3 become a general terms of convergent series. Consequently, we obtain,

$$\sum_{n=1}^{+\infty}\mathbb{P}\left(|x_{n+1} - x^*| > \varepsilon\right) < +\infty$$

that ensures the almost complete convergence of the sequence $\left(x_n\right)_n$ to the unique fixed point x^*. The choice of the tuning positive number r will be specified while deriving the corresponding rate of convergence.

Recall that $x_n - x^* = O\left(\epsilon_n\right)$ almost completely $(a.co)$, where $\left(\epsilon_n\right)_n$ is a sequence of real positive numbers tending to zero, if there exists a positive constant k such that,

$$\sum_{n=1}^{+\infty}\mathbb{P}\left(|x_n - x^*| > k\epsilon_n\right) < +\infty.$$

Basing on the inequalities obtained above, we take:

$$\varepsilon = \varepsilon_n = k\epsilon_n, \text{ where } k = \sqrt{1+\delta},\ \delta > 0 \text{ and } \epsilon_n = \frac{\sqrt{\ln n}}{n^{a(1-c)-\rho}}.$$

Hence, we obtain

$$T_1 = K_1 e^{-(n+1)^{2a(1-c)}\varepsilon^2} \leq K_1 e^{-(1+\delta)\ln n} = \frac{K_1}{n^{1+\delta}}. \quad (25)$$

For a suitably chosen r such that $r > \frac{2}{\rho}$, we obtain,

$$T_2 = 4\left(1+\frac{\left(1+\delta\right)n^{\rho}}{rS}\right)^{\frac{-r}{2}} \leq K_2 n^{-\rho\frac{r}{2}} \quad (26)$$

where,

$$K_2 = \left(\frac{rS}{1+\sigma}\right)^{r/2}.$$

With regard to T_3, we have,

$$\begin{aligned} T_3 &\leq 4Cnr^{-1}\left(\frac{r}{\sqrt{1+\delta}n^{\rho}\ln n}\right)^{\frac{(\beta+1)p}{\beta+p}} \\ &= 4Cr^{\frac{(\beta+1)p}{\beta+p}-1}\frac{n}{\left(\sqrt{1+\delta}n^{\rho}\ln n\right)^{\frac{(\beta+1)p}{\beta+p}}}. \end{aligned}$$

With r chosen as in (26), we deduce,

$$T_3 \leq K_3\frac{1}{n^{\rho\frac{(\beta+1)p}{\beta+p}-1}\left(\ln n\right)^{\frac{(\beta+1)p}{\beta+p}}}, \quad (27)$$

which is a general term of Bertrand series, it is convergent because of the hypothesis (H5). It leads that:

$$\mathbb{P}\left(|x_{n+1} - x^*| > \sqrt{1+\delta}\frac{\sqrt{\ln n}}{n^{a(1-c)-\rho}}\right) \leq \frac{K_1}{n^{1+\delta}} + \frac{K_2}{n^{\rho\frac{r}{2}}} + \frac{K_3}{n^{\rho\frac{(\beta+1)p}{\beta+p}}\left(\ln n\right)^{\frac{(\beta+1)p}{\beta+p}}}. \quad (28)$$

The right-hand side of the last inequality is a term of a convergent series. ■

Remark 6. Remark that in the obtained rate of convergence given by the formula (13), more the quantity $\frac{2(\beta+p)}{p(\beta+1)}$ is small, more we have the choice of taking ρ smaller and consequently the rate of convergence becomes more interesting.