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NONLINEAR FOKKER-PLANCK EQUATIONS WITH REACTION AS GRADIENT FLOWS OF THE FREE ENERGY

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Abstract: We interpret a class of nonlinear Fokker-Planck equations with reaction as gradient flows over the space of Radon measures equipped with the recently introduced Hellinger-Kantorovich distance. The driving entropy of the gradient flow is not assumed to be geodesically convex or semi-convex. We prove new general isoperimetric-type functional inequalities, which allow us to control the relative entropy by its production. We establish the entropic exponential convergence of the trajectories of the flow to the equilibrium. Along with other applications, this result has an ecological interpretation as a trend to the ideal free distribution for a class of fitness-driven models of population dynamics. Our existence theorem for weak solutions under mild assumptions on the nonlinearity is new even in the absence of the reaction term.

Keywords: functional inequalities, optimal transport, Hellinger-Kantorovich distance, geodesic non-convexity.
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1.Introduction

1.1.Setting. Let Ω be an open connected bounded domain in $\mathbb{R}^d$ with sufficiently smooth boundary and let ν be the outward unit normal along $\partial\Omega$. We are interested in nonnegative solutions of

$$\partial_t u = -\operatorname{div}(u\nabla f) + f u, \qquad (x,t) \in \Omega \times (0,\infty), \tag{1.1}$$

$$u\frac{\partial f}{\partial \nu} = 0, \qquad (x,t) \in \partial\Omega \times (0,\infty), \tag{1.2}$$

$$u = u^0, \qquad (x,t) \in \Omega \times 0. \tag{1.3}$$

Here u is the unknown function, $f = f(x, u(x))$ is a known nonlinear function of x and u, equation (1.2) is the no-flux boundary condition and the initial data u^0 are nonnegative. We refer to Section 1.3 for the motivation and background.

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When considering problem (1.1)–(1.3), we always make the following assumptions concerning the function $f : \Omega \times (0,\infty) \to \mathbb{R}$:

$$f \in C^2(\overline{\Omega} \times (0,\infty)) \cap L^1_{\mathrm{loc}}(\overline{\Omega} \times [0,\infty)) \tag{1.4}$$

$$uf, uf_x \in C(\overline{\Omega} \times [0,+\infty)) \tag{1.5}$$

$$f_u < 0, \tag{1.6}$$

$$\limsup_{u\to\infty} f(x,u) < 0 \quad \forall x \in \overline{\Omega}, \tag{1.7}$$

$$\liminf_{u\to+0} f(x,u) > 0 \quad \forall x \in \overline{\Omega}, \tag{1.8}$$

$$|f(x,u)| + u|f_u(x,u)| + u|f_{xu}(x,u)|$$

$$\leq g(u) \quad \text{a. a. } u > 0;\ g \in L^1_{\mathrm{loc}}[0,\infty), \tag{1.9}$$

$$(uf_x)\big|_{u=0} = 0. \tag{1.10}$$

When needed, we also assume that

$$\text{either } f_x = 0 \text{ for large } u \quad \text{or} \quad \lim_{u\to\infty} f(x,u) = -\infty\ \forall x \in \overline{\Omega} \tag{1.11}$$

$$\text{either } f_x = 0 \text{ for small } u \quad \text{or} \quad \lim_{u\to+0} f(x,u) = \infty\ \forall x \in \overline{\Omega} \tag{1.12}$$

Remark 1.1. We make comfortable assumptions about the smoothness of f. We do not insist that f should be defined for $u = 0$ so as not to exclude the interesting cases such as $f = -(\log u + V(x))$ (which corresponds to the linear Fokker-Planck equation, cf. [14, 19]) and $f = u^\alpha - 1$, $-1 < \alpha < 0$, (the fast diffusion, cf. [36]). However, we assume in (1.5) that the functions uf and uf_x admit continuous extensions to $\overline{\Omega} \times [0,\infty)$. This ensures that the terms in (1.1) make sense. Moreover, we assume (1.10) to avoid certain complications with the entropy production to be defined below.

Remark 1.2. Assumption (1.6) is essential, it ensures the parabolicity of (1.1). The equation may become degenerate or singular only if $u = 0$ or u is large. The latter does not bother us as we only consider bounded solutions in what follows.

Remark 1.3. Assumptions (1.7), (1.8) ensure the existence of a positive equilibrium, see below.

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On the set $[u = 0]$ we have $u f_x = 0$ by (1.10), $\Phi_x = 0$, and $\Phi = 0$, the last equality implying $\nabla\Phi = 0$ a. e. on $[u = 0]$. Thus, we can write

$$\int_\Omega v_n |\nabla f_n|^2 \, \mathrm{d}x \to \int_{[u>0]} \frac{1}{\max(u,\sigma)} |-\nabla\Phi + \Phi_x + u f_x|^2 \, \mathrm{d}x.$$

To sum up, we have

$$\int_\Omega E \, \mathrm{d}x$$
$$\leq C \left(\int_\Omega \frac{u^2}{\max(u,\sigma)} f^2 \, \mathrm{d}x + \int_{[u>0]} \frac{1}{\max(u,\sigma)} |-\nabla\Phi + \Phi_x + u f_x|^2 \, \mathrm{d}x \right),$$

which is even stronger than (1.22). ■

Proof of Theorem 1.12: Let $U \subset L^\infty_+$ be the set of functions such that for any $v \in U$, we have $\Phi(\cdot, v(\cdot)) \in H^1(\Omega)$ and $\|v\|_{L^\infty(\Omega)} \geq c$. By Theorem 1.7 we have the entropy-entropy production inequality (1.22) for U.

Let u be a weak solution of (1.1)–(1.3) with the initial data satisfying (1.28). It follows from the lower L^1-bound (1.26) that $u(t) \in U$ for a. a. $t > 0$. Combining the entropy dissipation and entropy-entropy production inequalities, we obtain

$$\partial_t \mathcal{E}(u(t)) \leq -C_U \mathcal{E}(u(t)) \quad \text{a. a. } t > 0.$$

Letting $e(t) = \mathcal{E}(u(t)) \mathrm{e}^{C_U t}$, we see that $\partial_t e(t) \leq 0$ in the sense of measures, whence e a. e. coincides with a nonincreasing function. Moreover,

$$\operatorname*{ess\,sup}_{t>0} e(t) = \operatorname*{ess\,lim\,sup}_{t\to 0} e(t) = \operatorname*{ess\,lim\,sup}_{t\to 0} \mathcal{E}(u(t)) \mathrm{e}^{C_U t} \leq \mathcal{E}(u^0)$$

by virtue of (1.25), so $e(t) \leq \mathcal{E}(u^0)$ for a. a. $t > 0$, yielding (1.27) with $\gamma = C_U$. ■

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Remark 1.4. Estimate (1.9) ensures that the entropy and energy of the equation are well-defined and well-behaved. Note that at least some restrictions on the growth of f_u as $u \to 0$ are inevitable, as the related very fast diffusion equation is known to behave abnormally [35].

Remark 1.5. Conditions (1.11) and (1.12) are convenient technical assumptions needed for L^∞-bounds (hence for the existence theorem) and for controlling the energy for large u in the proof of Theorem 1.7. However, they are not necessary everywhere, so we explicitely mention them when the need arises.

It follows from (1.6)–(1.8) that for any $x \in \overline{\Omega}$ there exists a unique $m(x) > 0$ such that

$$f(x, m(x)) = 0.$$

Clearly, $m \in C^2(\overline{\Omega})$. It is a stationary solution of (1.1), (1.2). As we will see, all non-zero solutions of the problem converge to m.

1.2. Energy and entropy. Now we will introduce the energy and entropy functionals for equation (1.1) as well as the notion of weak solution.

Put

$$\Phi(x,u) = -\int_0^u \xi f_u(x,\xi) \, \mathrm{d}\xi, \quad \Psi(x,u) = \int_0^u \Phi(x,\xi) \, \mathrm{d}\xi.$$

It is easy to see that

$$\Phi(x,0) = \Psi(x,0) = 0, \; \Phi_u = -u f_u, \; \Phi_x = -\int_0^u \xi f_{xu}(x,\xi) \, \mathrm{d}\xi, \; \Psi_u = \Phi.$$

Observe that both Φ and Ψ are nonnegative and strictly increase with respect to u.

Note that if u is a nonnegative function of x and possibly of t, an L^∞-bound on u is translated into an L^∞-bound on $\Phi(\cdot, u(\cdot))$, i. e. the superposition operator associated with Φ is L^∞-bounded. The same is true of Ψ.

Let u be a classical solution of (1.1)–(1.3). Equation (1.1) can be cast in the equivalent form

$$\partial_t u = \Delta\Phi - \operatorname{div}(\Phi_x + u f_x) + u f, \tag{1.13}$$

where we write Φ for $\Phi(x,u(x,t))$, etc. Multiplying by $\Phi(x,u(x,t))$ and integrating over Ω, we obtain

$$\partial_t \int_\Omega \Psi \,\mathrm{d}x = -\int_\Omega |\nabla\Phi|^2 \,\mathrm{d}x + \int_\Omega (\Phi_x + u f_x)\cdot\nabla\Phi \,\mathrm{d}x + \int_\Omega u f \Phi \,\mathrm{d}x. \quad (1.14)$$

We call the functional

$$\mathcal{W}(u) = \int_\Omega \Psi(x,u(x))\,\mathrm{d}x$$

the *energy* of problem (1.1)–(1.3) and equation (1.14), the *energy identity*. Thus, *any classical solution of* (1.1)–(1.3) *satisfies the energy identity* (1.14).

For our purposes, the energy identity is useful because it allows us to control the integral $\iint_{Q_T} |\nabla\Phi|^2 \,\mathrm{d}x\,\mathrm{d}t$. In particular, we can define the weak solution of (1.1)–(1.3) in a class of functions u such that $\Phi(\cdot,u(\cdot)) \in L^2(0,T;H^1(\Omega))$. It is easier to exploit this assumption in the case of equation (1.13). Thus, we define the weak solution as follows:

Definition 1.6. Let $u^0 \in L^\infty(\Omega)$. A function $u \in L^\infty(Q_T)$ is called a *weak solution* of (1.1)–(1.3) on $[0,T]$ if $\Phi(\cdot,u(\cdot)) \in L^2(0,T;H^1(\Omega))$ and

$$\int_0^T \int_\Omega (u\partial_t\varphi + (-\nabla\Phi + \Phi_x + u f_x)\cdot\nabla\varphi + f u\varphi)\,\mathrm{d}x\,\mathrm{d}t = \int_\Omega u^0(x)\varphi(x,0)\,\mathrm{d}x \quad (1.15)$$

for any function $\varphi \in C^1(\overline{\Omega}\times[0,T])$ such that $\varphi(x,T)=0$. A function $u \in L^\infty_{\mathrm{loc}}([0,\infty);L^\infty(\Omega))$ is called a *weak solution* of (1.1)–(1.3) on $[0,\infty)$ if for any $T>0$ it is a weak solution on $[0,T]$.

Now, let us address the entropy of the problem. Define

$$E(x,u) = -\int_{m(x)}^u f(x,\xi)\,\mathrm{d}\xi.$$

It follows from (1.9) that E is well-defined and continuous on $\overline{\Omega}\times[0,\infty)$. As f decreases with respect to u and $f(x,m(x))=0$, it is clear that $E \geq 0$ and

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Define

$$\widetilde{U} = \{w \in C^1(\Omega)\colon w>0, \|w\|_{L^1(\Omega)} \geq c\}.$$

By Theorem 2.9, there exist a function

$$v(\xi) = \frac{\xi^2}{\max(\xi,\sigma)},$$

where $\sigma > 0$, and a constant $C > 0$ such that

$$\int_\Omega E(x,w(x))\,\mathrm{d}x \leq C \int_\Omega v(w(x))(f(x,w(x)) + |\nabla f(x,w(x))|^2)\,\mathrm{d}x \quad (w \in \widetilde{U}).$$

In particular, as $u_n \in \widetilde{U}$, we see that

$$\int_\Omega E_n \,\mathrm{d}x \leq C \int_\Omega v_n(f_n + |\nabla f_n|^2)\,\mathrm{d}x, \quad (3.30)$$

where $v_n = v(u_n(x))$.

Let us check that we can pass to the limit in (3.30). First, it follows from (3.29) that

$$\int_\Omega E_n \,\mathrm{d}x \to \int_\Omega E \,\mathrm{d}x.$$

Next, note that we clearly have

$$\frac{1}{\max(u_n,\sigma)} \to \frac{1}{\max(u,\sigma)} \quad \text{a. e. in } \Omega \text{ and weakly* in } L^\infty(\Omega)$$

and thus, again using (3.29), we obtain

$$\int_\Omega v_n f_n^2 \,\mathrm{d}x = \int_\Omega \frac{1}{\max(u_n,\sigma)}(u_n f_n)^2 \,\mathrm{d}x \to \int_\Omega \frac{1}{\max(u,\sigma)}(uf)^2 \,\mathrm{d}x.$$

Finally, as u_n is smooth and positive, we can write

$$\int_\Omega v_n |\nabla f_n|^2 \,\mathrm{d}x = \int_\Omega \frac{1}{\max(u_n,\sigma)} |-\nabla\Phi_n + \Phi_{xn} + u_n f_{xn}|^2 \,\mathrm{d}x \to \int_\Omega \frac{1}{\max(u,\sigma)} |-\nabla\Phi + \Phi_x + u f_x|^2 \,\mathrm{d}x.$$

We have:

$$\begin{aligned}\Phi(x,\xi_0+1)-\widetilde{\Phi}_n(x) &= \Phi(x,\xi_0+1)-\Phi(x,u(x))-\varepsilon_n\\ &\geq \Phi(x,\xi_0+1)-\Phi(x,\xi_0)-\varepsilon_n = a-\varepsilon_n.\end{aligned}$$

Thus, for large n we have

$$\widetilde{\Phi}_n(x) \leq \Phi(x,\xi_0+1)-\frac{a}{2}.$$

Upon mollification,

$$\widetilde{\Phi}^{\varepsilon}_n(x) \leq \Phi^{\varepsilon}(x,\xi_0+1)-\frac{a}{2}.$$

For a fixed n, the function $\Phi(\cdot,\xi_0+1)$ is continuous on $\overline{\Omega}$, so the mollifications $\Phi^{\varepsilon}(\cdot,\xi_0+1)$ converge to it uniformly on $\overline{\Omega}$ as $\varepsilon\to 0$. Consequently,

$$(x,\widetilde{\Phi}^{\varepsilon}_n(x)) \in \{(x,\phi)\in\Omega\times(0,\infty)\colon\ \phi\leq\Phi(x,\xi_0+1)\}\subset D \tag{3.28}$$

for all $x\in\Omega$, proving (3.27).

Taking a sequence $\{\Phi_n\}$ as above, we can set $u_n(x)=\Xi(x,\Phi_n(x))$, so that $\Phi_n(x)=\Phi(x,u_n(x))$. Clearly, $u_n\in C^2(\Omega)$ and $u_n>0$. Further, the sequence $\{u_n\}$ is bounded in $L^\infty(\Omega)$ because so is $\{\Phi_n\}$, and due to the continuity of Ξ we have

$$u_n\to u \quad \text{a. e. in } \Omega.$$

As a consequence, for $f_n=f(x,u_n(x))$ and $E_n=E(x,u_n(x))$ we have

$$\left.\begin{aligned} u_n&\to u\\ u_nf_n&\to uf\\ u_nf_{xn}&\to uf_x\\ \Phi_{xn}&\to\Phi_x\\ E_n&\to E\end{aligned}\right\}\begin{array}{l}\text{a. e. in } \Omega\\ \text{and in any } L^p(\Omega),\ 1\leq p<\infty\,,\end{array} \tag{3.29}$$

where we write f for $f(\cdot,u(\cdot))$, etc. In particular, there is no loss of generality in assuming a lower bound

$$\|u_n\|_{L^1(\Omega)}\geq c:=\frac{1}{2}\inf_{u\in U}\|u\|_{L^1(\Omega)}>0$$

(positivity by virtue of (1.21)), where c is obviously independent not only of u_n but of u as well.

$E(x,u)=0$ if and only if $u=m(x)$. The relative *entropy* of equation (1.1) is the functional

$$\mathcal{E}(u)=\int_\Omega E(x,u(x))\,\mathrm{d}x. \tag{1.16}$$

Observe that it is well-defined at least for $u\in L^\infty(\Omega)_+$ as the superposition operator $u\mapsto E(\cdot,u(\cdot))$ is bounded in the spaces $L^\infty_+\to L^\infty_+$.

A straightforward computation shows that for a positive classical solution of (1.1)–(1.3) we have

$$\partial_t\mathcal{E}(u)=-\int_\Omega u(f^2+|\nabla f|^2)\,\mathrm{d}x. \tag{1.17}$$

Equation (1.17) is called the *entropy dissipation identity* and the integral on the right-hand side of (1.17) is called the *entropy production*. However, the term $\int_\Omega u|\nabla f|^2\,\mathrm{d}x$ may make no sense for vanishing or non-smooth u. In order to generalise the definition of the entropy production, we use the identity

$$u|\nabla f|^2=\frac{1}{u}|-\nabla\Phi+\Phi_x+uf_x|^2 \quad (u>0).$$

Given a function $u\in L^\infty_+(\Omega)$ such that $\Phi(\cdot,u(\cdot))\in H^1(\Omega)$, the right-hand side of the last identity is a nonnegative measurable function on $[u>0]$, so we can define the entropy production for such functions by the formula

$$D\mathcal{E}(u)=\int_\Omega uf^2\,\mathrm{d}x+\int_{[u>0]}\frac{1}{u}|-\nabla\Phi+\Phi_x+uf_x|^2\,\mathrm{d}x,$$

where the second integral on the right-hand side may be infinite. Thus, we see that *any positive classical solution of* (1.1)–(1.3) *satisfies the entropy dissipation identity*

$$\partial_t\mathcal{E}(u)=-D\mathcal{E}(u). \tag{1.18}$$

As usual, in the case of weak solutions we establish not the identities (1.14) and (1.18) but rather corresponding inequalities, viz. the *energy inequality*

$$\partial_t\mathcal{W}(u)\leq\int_\Omega\Big(-|\nabla\Phi|^2+(\Phi_x+uf_x)\cdot\nabla\Phi+uf\Phi\Big)\,\mathrm{d}x \tag{1.19}$$

and the *entropy dissipation inequality*

$$\partial_t\mathcal{E}(u)\leq-D\mathcal{E}(u). \tag{1.20}$$

For functions $u \in L^\infty_+(\Omega)$ such that $\Phi \in L^2(0,T;H^1(\Omega))$ we understand (1.19) and (1.20) in the sense of measures, i. e. that for any smooth nonnegative compactly supported function $\chi\colon (0,T) \to \mathbb{R}$ we respectively have

$$-\int_0^T \chi'(t)\mathcal{W}(u)\,\mathrm{d}t \\ \le \iint_{Q_T} \chi(t)\Big(-|\nabla\Phi|^2 + (\Phi_x + u f_x)\cdot\nabla\Phi + u f \Phi\Big)\,\mathrm{d}x\,\mathrm{d}t,$$

$$\int_0^T \chi'(t)\mathcal{E}(u)\,\mathrm{d}t \ge \int_0^T \chi(t)D\mathcal{E}(u)\,\mathrm{d}t.$$

If (1.20) holds in the sense of measures, the derivative $\partial_t\mathcal{E}(u)$ is a nonpositive distribution and hence a measure, while the entropy $\mathcal{E}(u)$ itself a. e. coincides with a non-increasing function.

An important question is whether the entropy can be controlled by the entropy production, since this would imply the exponential stability of the equilibrium. It turns out that this is the case for sets of functions provided that their L^1-norms are bounded away from 0. Specifically, we have

Theorem 1.7 (Entropy-entropy production inequality)**.** *Suppose that f satisfies (1.4)–(1.10) as well as (1.11). Let $U \subset L^\infty_+(\Omega)$ be a set of functions such that for any $u \in U$, we have $\Phi(\cdot,u(\cdot)) \in H^1(\Omega)$ and*

$$\inf_{u\in U} \|u\|_{L^1(\Omega)} > 0. \tag{1.21}$$

Then there exists C_U such that

$$\mathcal{E}(u) \le C_U D\mathcal{E}(u) \quad (u \in U). \tag{1.22}$$

Theorem 1.7 is a consequence of a fairly general functional inequality established in Section 2.

Theorem 1.8 (Existence of weak solutions)**.** *Suppose that f satisfies (1.4)–(1.10) as well as (1.11) and (1.12). Then for any $u^0 \in L^\infty_+(\Omega)$ there exists a nonnegative weak solution $u \in L^\infty(\Omega\times(0,\infty))$ of problem (1.1)–(1.3) enjoying the following properties:*

Proof of Theorem 1.8: Take a decreasing sequence $\varepsilon_n \to 0$ and set

$$u_n^0 = u^0 + \varepsilon_n.$$

By Theorem 3.9, there exists a weak solution u_n of (1.1)–(1.3) with the initial data u_n^0. Contraction (3.15) ensures the comparison principle for this sequence of solutions, whence $u_{n+1} \le u_n$ a. e. in $\Omega\times(0,\infty)$. Consequently, there exists the monotone limit $u \in L^\infty(\Omega\times(0,\infty))$ and moreover, we obviously have the convergences (3.18). From this moment on, the proof copies that of Theorem 3.9 except that (3.13) and (3.14) hold almost everywhere rather then everywhere. ■

We conclude by proving Theorems 1.7 and 1.12.

Proof of Theorem 1.7: Let $D = \{(x,\Phi(x,u))\colon x\in\Omega, u>0\}$ and consider the function $\Xi\colon D \to [0,\infty)$ implicitly defined by the equation

$$\Phi(x,\Xi(x,\phi)) = \phi.$$

As Φ is monotonous with respect to its second argument, Ξ is uniquely defined. Clearly, Ξ is C^2.

Fix $u \in U$. We claim that there exists a sequence of functions $\Phi_n \in C(\overline{\Omega})\cap C^\infty(\Omega)$ such that

$$(x,\Phi_n(x)) \in D \quad (x\in\Omega),$$
$$\Phi_n \to \Phi(\cdot,u(\cdot)) \quad \text{in } H^1\text{and a. e. in } \Omega$$

Indeed, take a sequence $\{\delta_n\}$, where $\delta_n > 0$ and $\delta_n \to 0$, put $\widetilde{\Phi}_n(x) = \Phi(x,u(x)) + \delta_n$, and let $\widetilde{\Phi}_n^\varepsilon$ be the mollification of $\widetilde{\Phi}_n$. Observe that $\widetilde{\Phi}_n$ is strictly positive and so is $\widetilde{\Phi}_n^\varepsilon$. It suffices to show that for any n sufficiently large there exists $\varepsilon_n > 0$ such that whenever $\varepsilon < \varepsilon_n$, we have

$$\{(x,\widetilde{\Phi}_n^\varepsilon(x))\colon x\in\Omega\} \subset D. \tag{3.27}$$

If the second alternative in (1.11) holds, we clearly have $D = (0,\infty)$, so (3.27) obviously holds with any ε.

Assume the first alternative in (1.11). Take $\xi_0 \ge \|u\|_{L^\infty(\Omega)}$ such that $f(x,\xi)$ does not depend on x if $\xi \ge \xi_0$ and set

$$a = -\int_{\xi_0}^{\xi_0+1} u f_u(x,\xi)\,\mathrm{d}x > 0.$$

(3.22). Extracting a subsequence if needed, we see that there is a vectorfield $v \in L^2(\mathcal{O}, \mathrm{d}\mu)$ verifying (3.25) and (3.26). On the other hand, by (3.22) and (3.23),

$$v_n \frac{\varphi(t)}{\max(u_n,\delta)} \to (-\nabla\Phi + \Phi_x + u f_x)\frac{\varphi(t)}{\max(u,\delta)}$$

weakly in $L^1(Q_T)$. Evoking (3.25), we find that

$$\int_{\mathcal{O}} v \cdot \zeta \,\mathrm{d}\mu = \int_{\mathcal{O}} (-\nabla\Phi + \Phi_x + u f_x)\cdot \zeta \,\mathrm{d}\mu$$

for all test functions ζ. By density, we conclude that $v = -\nabla\Phi + \Phi_x + u f_x$ in $L^2(\mathcal{O}, \mathrm{d}\mu)$, and (3.24) follows from (3.26).

Inequality (1.25) is proved in the same way as (1.24) given that it holds for the approximate solutions.

Inequalities (3.13)–(3.15) follow from correspondent inequalities for approximate solutions (Lemmas 3.2 and 3.6), as we obviously have

$$\left.\begin{aligned} (u_n(t) - u_c)^{\pm} &\to (u(t) - u_c)^{\pm} \\ (u_n(t) - \hat{u}_n(t))^{+} &\to (u(t) - \hat{u}(t))^{+} \end{aligned}\right\} \quad \text{in } L^1(\Omega),\ \forall t \geq 0,$$

where the approximations $\hat{u}_n$ are constructed in the same way as u_n.

Contraction (3.15) implies the uniqueness of u.

To obtain the upper bound (1.23), we define $c \leq 0$ by (3.8) and thus have $u^0 \leq u_c$ on Ω, whence in view of contraction (3.13),

$$u(x,t) \leq u_c(x), \quad (x,t) \in \Omega \times (0,\infty).$$

Recalling the formula (3.7) for the norm of u_c, we obtain the upper bound.

To obtain the lower L^1-bound (1.26), we take $u_c = m$ in (3.14), obtaining

$$\begin{aligned} \|u(t)\|_{L^1(\Omega)} \geq \|\min(u(t),m)\|_{L^1(\Omega)} &= \int_\Omega (m - (u(t)-m)^-)\,\mathrm{d}x \\ &\geq \int_\Omega (m - (u^0 - m)^-)\,\mathrm{d}x = \|\min(u(t),m)\|_{L^1(\Omega)}, \end{aligned}$$

as required. ■

3.3. Nonnegative initial data. If initial data (1.3) is only nonnegative, we approximate it with positive functions and reuse the proof of Theorem 3.9 to establish the existence of solutions to (1.1)–(1.3) as stated in Theorem 1.8 (but not uniqueness, owing to the loss of contraction).

(1) *(upper L^∞-bound)*

$$\|u\|_{L^\infty(\Omega\times(0,\infty))} \leq \inf\left\{\xi \geq 0 \colon \sup_{x\in\Omega} f(x,\xi) \leq -\operatorname{ess\,sup}_{x\in\Omega} f^-(x,u^0(x))\right\}; \quad (1.23)$$

(2) *u satisfies the energy inequality (1.19) in the sense of measures and*

$$\operatorname{ess\,lim\,sup}_{t\to+0} \mathcal{W}(u(t)) \leq \mathcal{W}(u^0); \quad (1.24)$$

(3) *u satisfies the entropy dissipation inequality (1.20) in the sense of measures and*

$$\operatorname{ess\,sup}_{t>0} \mathcal{E}(u(t)) \leq \mathcal{E}(u^0); \quad (1.25)$$

(4) *(lower L^1-bound)*

$$\|u(t)\|_{L^1(\Omega)} \geq \|\min(u^0,m)\|_{L^1(\Omega)} \quad a.\ a.\ t>0. \quad (1.26)$$

Remark 1.9. Theorem 1.8, mutatis mutandis, is also valid in the case of a pure Fokker-Planck equation (1.29). Even in this case, our conditions on the nonlinearity f are more relaxed than the ones available in the literature, see, e.g., [1, 4, 21, 22, 12, 36, 7, 3] and the references therein.

Remark 1.10. In the general case, uniqueness of solutions cannot be expected due to the non-Lipschitz reaction term. However, our weak solutions are unique provided the initial data is bounded away from zero, see Theorem 3.9.

Remark 1.11. Under the hypotheses of Theorem 1.8, the right-hand side of (1.23) is always finite (see Remark 3.5). Moreover, if u^0 satisfies an estimate $\|u^0\|_{L^\infty(\Omega)} \leq a$, inequality (1.23) provides an estimate $\|u\|_{L^\infty(\Omega\times(0,\infty))} \leq C_a$.

The next theorem shows that the solutions that we have constructed exponentially converge to m. Note that assumption (1.12) is not needed for the long-time convergence.

Theorem 1.12 (Convergence to equilibrium)**.** *Assume (1.11) and suppose that a weak solution u of (1.1)–(1.3) with the initial data $u^0 \not\equiv 0$ satisfies the entropy dissipation inequality (1.20), inequality (1.25), and the lower L^1-bound (1.26). Then u exponentially converges to m in the sense of entropy:*

$$\mathcal{E}(u(t)) \leq \mathcal{E}(u^0)e^{-\gamma t} \quad a.\ a.\ t>0, \quad (1.27)$$

where $\gamma > 0$ can be chosen uniformly over initial data satisfying

$$\|\min(u^0, m)\|_{L^1(\Omega)} \geq c \tag{1.28}$$

with some $c > 0$.

Theorems 1.7, 1.8, and 1.12 are proved in Section 3.3.

1.3.Motivation and background. The nonlinear Fokker-Planck equation

$$\partial_t u = -\operatorname{div}(u\nabla(f(x,u))) \tag{1.29}$$

is intended to express the behaviour of stochastic systems coming from various branches of physics, chemistry and biology, see [14, 34, 20, 5]. In order to take into account the creation and annihilation of mass, the general drift-diffusion-reaction equation (1.1) was suggested in [13]. In the considerations of [13] (cf. also [14]), the crucial role is played by the free energy functional which up to an additive constant coincides with our relative entropy functional $\mathcal{E}$ from (1.16). We opt for this change of terminology (though for thermodynamists the free energy involves the (physical) entropy, the internal energy, and the temperature) because in mathematical analysis it is convenient to refer to the basic Lyapunov functional of a system as the entropy, cf. [37, p. 270].

On the other hand, equation (1.1) is a general nonlinear model for the spatial dynamics of a population which is tending to achieve the *ideal free distribution* [16, 15] (the distribution which happens if everybody is free to choose its location) in a heterogeneous environment. The dispersal strategy is determined by a local intrinsic characteristic of organisms called *fitness* (see, e.g., [9, 10]). The fitness manifests itself as a growth rate, and simultaneously affects the dispersal as the species move along its gradient towards the most favorable environment. In (1.1), $u(x,t)$ is the density of organisms, and $f(x,u)$ is the fitness. The equilibrium $u(x) \equiv m(x)$ when the fitness is constantly zero corresponds to the ideal free distribution. The original model [29, 9] assumes a linear logistic fitness

$$f = m(x) - u \tag{1.30}$$

but in general it can be any nonlinear function of the spatial variable and the density, cf. [10]. The assumptions (1.6), (1.7), (1.8) are natural as they simply mean that the fitness is decreasing with respect to the population density (as the resources are limited), being positive for very small densities and negative for very large densities. Our Theorem 1.12 indicates that

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Then, taking into account (3.18), we can pass to the limit in (3.21) obtaining

$$-\iint_{Q_T} E\varphi'(t)\,\mathrm{d}x\,\mathrm{d}t \leq -\iint_{Q_T} \frac{\varphi(t)}{\max(u,\delta)}(uf)^2\,\mathrm{d}x\,\mathrm{d}t \\ -\iint_{Q_T} \frac{\varphi(t)}{\max(u,\delta)}|-\nabla\Phi + \Phi_x + uf_x|^2\,\mathrm{d}x\,\mathrm{d}t.$$

On the set $\{(x,t) \in Q_T \colon u(x,t) = 0\}$ we have $uf_x = 0$ (by virtue of (1.10)), $\Phi_x = 0$ and $\Phi = 0$, whence also $\nabla\Phi = 0$ a. e. on this set. Thus, we can write

$$-\iint_{Q_T} E\varphi'(t)\,\mathrm{d}x\,\mathrm{d}t \leq -\iint_{Q_T} \frac{\varphi(t)}{\max(u,\delta)}(uf)^2\,\mathrm{d}x\,\mathrm{d}t \\ -\iint_{u>0} \frac{\varphi(t)}{\max(u,\delta)}|-\nabla\Phi + \Phi_x + uf_x|^2\,\mathrm{d}x\,\mathrm{d}t$$

Letting $\delta \to 0$, by Beppo Levi's theorem we obtain the energy inequality.

To prove the technical claim (3.24), we use a variant of the Banach-Alaoglu theorem in varying $L^2(\mathrm{d}\mu^n)$ spaces:

Lemma 3.10. *Let $\mathcal{O} \subset \mathbb{R}^N$ be an open set, μ_n a sequence of finite non-negative Radon measures narrowly converging to μ, and v_n a sequence of vector fields on $\mathcal{O}$. If*

$$\|v_n\|_{L^2(\mathcal{O},\mathrm{d}\mu_n)} \leq C,$$

then there exists $v \in L^2(\mathcal{O},\mathrm{d}\mu)$ such that, up to extraction of some subsequence,

$$\forall\, \zeta \in \mathcal{C}_c^\infty(\mathcal{O}): \qquad \lim_{n\to\infty} \int_{\mathcal{O}} v_n \cdot \zeta\,\mathrm{d}\mu_n = \int_{\mathcal{O}} v \cdot \zeta\,\mathrm{d}\mu \tag{3.25}$$

and

$$\|v\|_{L^2(\mathcal{O},\mathrm{d}\mu)} \leq \liminf_{n\to\infty} \|v_n\|_{L^2(\mathcal{O},\mathrm{d}\mu_n)}. \tag{3.26}$$

The proof of this fact by optimal transport techniques can be found in [2]; this lemma also follows from a variant of the Banach-Alaoglu theorem [24, Proposition 5.3]. We will apply this lemma with $\mathcal{O} = Q_T$, v_n from (3.23), and the sequence of measures $\mathrm{d}\mu_n(t,x) := \frac{\varphi(t)}{\max(u_n,\delta)}\,\mathrm{d}x\,\mathrm{d}t$, which converges narrowly to $\mathrm{d}\mu(t,x) = \frac{\varphi(t)}{\max(u,\delta)}\,\mathrm{d}x\,\mathrm{d}t$ due to the strong convergence so we get

$$\begin{aligned}\operatorname*{ess\,sup}_{t\in(0,\varepsilon)} \mathcal{W}(u(t)) &\le \liminf_{n\to\infty} \operatorname*{ess\,sup}_{t\in(0,\varepsilon)} \mathcal{W}(u_n(t)) \\ &\le \lim_{n\to\infty} \mathcal{W}(u_n^0) + C\varepsilon \\ &= \mathcal{W}(u^0) + C\varepsilon.\end{aligned}$$

Now sending $\varepsilon \to 0$ we recover (1.24).

Let us show that u satisfies the entropy dissipation inequality on $[0,T]$ in the sense of measures. Let $\varphi \in C^\infty$ be a smooth nonnegative test function vanishing outside of $[0,T]$. The approximate solutions satisfy the entropy dissipation inequality in the sense of measures, so we have

$$\begin{aligned}-\iint_{Q_T} E_n \varphi'(t)\,\mathrm{d}x\,\mathrm{d}t \le &-\iint_{Q_T} \varphi(t) u_n f_n^2 \,\mathrm{d}x\,\mathrm{d}t \\ &-\iint_{u_n>0} \frac{\varphi(t)}{u_n} |-\nabla\Phi_n + \Phi_{xn} + u_n f_{xn}|^2 \,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Consequently, for any $\delta > 0$ we have

$$\begin{aligned}-\iint_{Q_T} E_n \varphi'(t)\,\mathrm{d}x\,\mathrm{d}t \le &-\iint_{Q_T} \frac{\varphi(t)}{\max(u_n,\delta)} (u_n f_n)^2 \,\mathrm{d}x\,\mathrm{d}t \\ &-\iint_{Q_T} \frac{\varphi(t)}{\max(u_n,\delta)} |-\nabla\Phi_n + \Phi_{xn} + u_n f_{xn}|^2 \,\mathrm{d}x\,\mathrm{d}t. \quad (3.21)\end{aligned}$$

Observe that

$$\frac{\varphi(t)}{\max(u_n,\delta)} \to \frac{\varphi(t)}{\max(u,\delta)} \begin{array}{l}\text{a. e. in } Q_T, \\ \text{strongly in any } L^p,\ 1 \le p < \infty, \\ \text{and weakly* in } L^\infty(Q_T),\end{array} \quad (3.22)$$

$$v_n := -\nabla\Phi_n + \Phi_{xn} + u_n f_{xn} \to -\nabla\Phi + \Phi_x + u f_x \quad \text{weakly in } L^2(\Omega) \quad (3.23)$$

We claim that

$$\begin{aligned}&\iint_{Q_T} \frac{\varphi(t)}{\max(u,\delta)} |-\nabla\Phi + \Phi_x + u f_x|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &\le \liminf_{n\to\infty} \iint_{Q_T} \frac{\varphi(t)}{\max(u_n,\delta)} |-\nabla\Phi_n + \Phi_{xn} + u_n f_{xn}|^2 \,\mathrm{d}x\,\mathrm{d}t. \quad (3.24)\end{aligned}$$

the populations converge to the ideal free distribution with an exponential rate.

The existence of weak solutions for the fitness-driven dispersal model (1.1)–(1.3) with the logistic fitness (1.30) was shown in [11], and the entropic exponential convergence to m was established in [24]. The same kind of results for cross-diffusion systems involving several interacting populations (with logistic fitnesses) can be found in [23]. Related two-species models were investigated in [6, 28], where one population uses the fitness-driven dispersal strategy and the other diffuses freely or does not move at all. A system of two interacting populations with a particular nonlinear fitness function has recently been considered in [39], which is the only existing mathematical treatment of a non-logistic fitness model that we are aware of.

But perhaps our main motivation to study (1.1) is that it is a gradient flow of the entropy functional $\mathcal{E}$ with respect to the intriguing recently introduced distance on the space of Radon measures, which is related to the *unbalanced* optimal transport (i.e., failing to preserve the total transported mass), and which is referred to as the Hellinger-Kantorovich distance or the Wasserstein-Fisher-Rao distance [24, 8, 26, 27]. This distance endows the set of Radon measures with a formal (infinite dimensional) Riemannian metric $\langle\cdot,\cdot\rangle$, and provides first- and second-order differential calculus [24] in the spirit of Otto [31, 37, 38]. In particular, one can compute the metric gradients of the functionals of the form

$$\mathcal{F}(u) = \int_\Omega F(x,u(x))\,\mathrm{d}x$$

by the formula

$$\operatorname{grad}\mathcal{F}(u) = -\operatorname{div}\left(u\nabla\frac{\delta F}{\delta u}\right) + u\frac{\delta F}{\delta u}, \quad (1.31)$$

where $\frac{\delta F}{\delta u} = \partial_u F(x,u)$ stands for the first variation with respect to u and $\nabla = \nabla_x$ is the usual gradient in space. We refer to [24] for further details and explanations. Since $f = -\partial_u E$, we can recast (1.1) as a gradient flow

$$\partial_t u = -\operatorname{grad}\mathcal{E}(u). \quad (1.32)$$

The entropy dissipation identity (1.17), which by the way was already known to Frank [13], is then nothing but the archetypal property of gradient flows

$$\frac{d}{dt}\mathcal{E}(u) = -\langle \operatorname{grad}\mathcal{E}(u), \operatorname{grad}\mathcal{E}(u)\rangle_u.$$

In this connection, we recall that for the metric gradient flows like (1.32), the geodesic convexity of the driving entropy functional (or at least semi-convexity, i.e., λ-convexity with a negative constant λ) makes a difference [31, 2, 37, 38, 32]. The presence of convexity allows one to apply minimizing movement schemes [2, 19] to construct solutions to the gradient flow. Moreover, λ-convexity with λ strictly positive enables the Bakry-Emery procedure which usually yields the exponential convergence of the relative entropy to zero. Minimizing movement schemes for Hellinger-Kantorovich gradient flows of geodesically convex functionals and for related reaction-diffusion equations were suggested in [18, 17].

Our entropy $\mathcal{E}$ is geodesically $(-1/2)$-convex with respect to the Hellinger-Kantorovich structure if $f = 1 - u^\alpha$, $\alpha > 0$, but fails to be semi-convex for $f = u^\alpha - 1$, $\alpha < 0$, and for $f = -\log u$ (the latter option corresponds to the interesting case of the Boltzmann entropy). The spatial heterogeneity further complicates the situation. The quadratic (logistic) multicomponent entropy considered in [23, 25] is not even semi-convex. All this can be observed by computing the Hessian of the entropy, cf. [24, Section 3.4]; the non-convexity of the Boltzmann entropy with respect to the Hellinger-Kantorovich metric was also mentioned in [18, 17, 26, 27]. However, Santambrogio [32] emphasizes that the lack of geodesic convexity is not a universal obstacle for the study of gradient flows; our results in the current paper and in [23, 25, 24, 33] illustrate this idea.

2. An isopetimetric-type inequality

2.1. Setting. Motivated by the expressions for the entropy and entropy production, we forget for a while problem (1.1)–(1.3) and consider the integrals

$$\int_\Omega E(x, u(x))\,\mathrm{d}x, \tag{2.1}$$

$$\int_\Omega (g(x, u(x)) + u|\nabla_x f(x, u(x))|^p)\,\mathrm{d}x \tag{2.2}$$

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solution, we have

$$\int_0^T \int_\Omega (u_n \partial_t \varphi + (-\nabla\Phi_n + \Phi_{xn} + u_n f_{xn}) \cdot \nabla\varphi + f_n u_n \varphi)\,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_\Omega u_n^0(x)\varphi(x,0)\,\mathrm{d}x. \tag{3.20}$$

It follows from (3.17), (3.18), and (3.19) that we can pass to the limit in (3.20) and obtain (1.15) for u. Thus, u is indeed a weak solution.

Let us show that u satisfies the energy inequality on $[0,T]$ in the sense of measures. Taking a smooth nonnegative test function $\varphi \in C^\infty$ vanishing outside of $[0,T]$, we write the energy inequality in the sense of measures for the approximate solutions:

$$-\iint_{Q_T} \Psi_n \varphi'(t)\,\mathrm{d}x\,\mathrm{d}t \le -\iint_{Q_T} |\nabla\Phi_n|^2 \varphi(t)\,\mathrm{d}x\,\mathrm{d}t$$
$$+ \iint_{Q_T} \varphi(t)(\Phi_{xn} + u_n f_{xn}) \cdot \nabla\Phi_n\,\mathrm{d}x\,\mathrm{d}t + \iint_{Q_T} u_n f_n \Phi_n \varphi(t)\,\mathrm{d}xc\mathrm{d}t$$

Convergences (3.18) ensure that we can pass to the limit in all the terms but for the first one on the right-hand side. As for the latter, it follows from (3.19) that $\sqrt{\varphi}\nabla\Phi_n \to \sqrt{\varphi}\nabla\Phi$ weakly in $L^2(Q_T)$, whence

$$\iint_{Q_T} \varphi|\nabla\Phi|^2\,\mathrm{d}x\,\mathrm{d}t \le \liminf_{n\to\infty} \iint_{Q_T} \varphi|\nabla\Phi_n|^2\,\mathrm{d}x\,\mathrm{d}t,$$

and the energy inequality follows.

Let us check (1.24). The approximate solutions satisfy

$$\operatorname{ess}\limsup_{t\to+0} \mathcal{W}(u_n(t)) \le \mathcal{W}(u_n^0)$$

so by virtue of (3.11) we obtain

$$\operatorname{ess}\sup_{t\in(0,\varepsilon)} \mathcal{W}(u_n(t)) \le \mathcal{W}(u_n^0) + C\varepsilon.$$

It follows from (3.17) and (3.18) that

$$\mathcal{W}(u_n) \to \mathcal{W}(u) \quad \text{weakly* in } L^\infty(0,\varepsilon),$$
$$\mathcal{W}(u_n^0) \to \mathcal{W}(u^0),$$

Let u_n be the classical solution of (1.1)–(1.3) on $[0,\infty)$ with the initial data u_n^0. For any $T>0$, it follows from Lemma 3.2 that

$$\|u_n - u_m\|_{C([0,T];L^1(\Omega))} \le e^{L_\kappa T}\|u_n^0 - u_m^0\|_{L^1},$$

so $\{u_n\}$ is a Cauchy sequence in $C([0,T];L^1(\Omega))$. As T is arbitrary, we see that $\{u_n\}$ converges in $C([0,\infty);L^1(\Omega))$ to some function u. We claim that it is the sought-for solution.

By Remark 3.4, there exists u_c ($c \le 0$) such that $u_c \ge 1/\kappa$; then u_c dominates the initial data u_n^0 and thus, the solutions u_n as well, which follows from Lemma 3.6. Consequently, the sequence $\{u_n\}$ is bounded in $L^\infty(\Omega\times(0,\infty))$, so it converges to u weakly* in this space, whence $u \in L^\infty(\Omega\times(0,\infty))$.

Put

$$\begin{aligned} f_n &= f(x,u_n(x,t)), & f_{xn} &= f_x(x,u_n(x,t)),\\ \Phi_n &= \Phi(x,u_n(x,t)), & \Phi_{xn} &= \Phi_x(x,u_n(x,t)),\\ \Psi_n &= \Psi(x,u_n(x,t)), & E_n &= E(x,u_n(x,t)). \end{aligned}$$

Fix $T>0$. As the sequence $\{u_n\}$ is bounded in $L^\infty(Q_T)$, so are the sequences $\{u_nf_n\}$, $\{u_nf_{xn}\}$, $\{\Phi_n\}$, $\{\Phi_{xn}\}$, $\{\Psi_n\}$, and $\{E_n\}$. Thus, there is no loss of generality in assuming

$$\left.\begin{aligned} u_n &\to u\\ u_nf_n &\to uf\\ u_nf_{xn} &\to uf_x\\ \Phi_n &\to \Phi\\ \Phi_{xn} &\to \Phi_x\\ \Psi_n &\to \Psi \end{aligned}\right\} \begin{array}{l} \text{a. e. in } Q_T,\\ \text{strongly in any } L^p(Q_T),\ 1\le p<\infty,\\ \text{weakly* in } L^\infty(Q_T),\\ \text{and in the sense of distributions,} \end{array} \tag{3.18}$$

where we write Φ for $\Phi(\cdot,u(\cdot))$, etc. It follows from (3.18) that $\nabla\Phi_n \to \nabla\Phi$ in the sense of distributions. The approximate solutions satisfy the energy inequality and (1.24) while their initial energy is bounded, so we see from (3.12) that the sequence $\nabla\Phi_n$ is bounded in $L^2(Q_T)$. Consequently, $\Phi \in L^2(0,T;H^1(\Omega))$ and

$$\nabla\Phi_n \to \nabla\Phi \quad \text{weakly in } L^2(Q_T). \tag{3.19}$$

Let us check that u is a weak solution of (1.1)–(1.3) on $[0,T]$. Take an admissible test function φ. Writing the weak setting for the approximate

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on their own right. Here Ω a domain in $\mathbb{R}^d$; $p \ge 1$; the functions

$$\begin{gathered} E,g\colon \Omega\times(0,\infty)\to[0,\infty),\\ f\colon \Omega\times(0,\infty)\to\mathbb{R} \end{gathered}$$

are fixed, and u varies over a set U of functions $\Omega\to(0,\infty)$. Observe that the nonnegativity of E and g ensures the existence of the integrals (2.1) and (2.2), although they need not be finite.

The functions f and E introduced in Section 1.2 are, of course, prototypes for the ones appearing in (2.1) and (2.2), but we assume no formal relationship between them. In particular, in this section we do not suppose that f satisfies (1.4)–(1.12).

We would like to know whether (2.1) can be controlled by (2.2) uniformly with respect to $u \in U$. In general, this is not the case. However, we show that under suitable assumptions on the functions E, f, and g, (2.2) does indeed control (2.1) provided that the set U of admissible u is separated from 0 in some sense. The result can be termed an isoperimetric-type inequality, see [38].

For simplicity, we concentrate on the regular case. Section 2.3 contains a discussion of possible generalisations.

Theorem 2.1. *Let Ω be a bounded, connected, open domain in $\mathbb{R}^d$ admitting the relative isoperimetric inequality. Let $p \ge 1$. Suppose that functions $E,g \in C(\Omega\times\mathbb{R}_+)$, and $f \in C^1(\Omega\times(0,+\infty))$ satisfy*

$$E \ge 0,\ g \ge 0; \tag{2.3}$$

$$\lim_{\varepsilon\to 0}\sup_{\substack{0<u\le\varepsilon\\ x\in\Omega}} E(x,u) < \infty; \tag{2.4}$$

$$\inf_{\substack{u>\varepsilon\\ x\in\Omega\\ E(x,u)\neq 0}} \frac{g(x,u)}{E(x,u)} > 0 \quad \forall\varepsilon>0, \tag{2.5}$$

$$\lim_{\varepsilon\to 0}\inf_{\substack{0<u\le\varepsilon\\ x\in\Omega}} f(x,u) > \lim_{\varepsilon\to 0}\sup_{\substack{u>0\\ E(x,u)<\varepsilon}} f(x,u). \tag{2.6}$$

Finally, suppose that a set $U \subset C^1(\Omega)$ consisting of strictly positive functions contains no sequence $\{u_n\}$ such that $\{E(\cdot,u_n(\cdot))\}$ is bounded in $L^1(\Omega)$ and $\{u_n\}$ converges to 0 in measure. Then there exists a constant $C = C(\Omega,p,E,g,f,U)$

such that

$$\int_\Omega E(x,u(x))\,\mathrm{d}x$$

$$\leq C\left(\int_\Omega \left(g(x,u(x)) + u(x)|\nabla_x f(x,u(x))|^p\right)\mathrm{d}x\right) \quad (u \in U). \quad (2.7)$$

Remark 2.2. The isoperimetric inequality for Ω reads

$$P(A;\Omega) \geq c_\Omega |A|^{\frac{d-1}{d}}, \quad A \subset \Omega,\ |A| \leq \frac{1}{2}|\Omega|, \tag{2.8}$$

where $P(A;\Omega)$ denotes the relative perimeter of a Lebesgue measurable set A of locally finite perimeter with respect to Ω, cf. [30, Remark 12.39]. We recall that the relative perimeter is defined as

$$P(A;\Omega) = |\mu_A|(\Omega),$$

where μ_A is the Gauss-Green measure associated with A, see [30]. We note that the support of μ_A is contained in the topological boundary of A.

Remark 2.3. If $E \in C(\overline{\Omega} \times \mathbb{R}_+)$, condition (2.4) is automatically true. If the set $\{(x,u) \in \overline{\Omega} \times \mathbb{R}_+ :\ E(x,u) = 0\}$ is compact, the right-hand side of (2.6) is simplified to $\max_{E(x,u)=0} f(x,u)$ and likewise, if $f \in C(\overline{\Omega} \times \mathbb{R}_+)$, the left-hand side of (2.6) can be written as $\min_x f(x,0)$. As for (2.5), it is more tricky. In Section 2.3 we show that it always holds in a particular setting relevant for gradient flows (Theorem 2.9).

Remark 2.4. The infimum in (2.5) depends on ε and may tend to zero as $\varepsilon \to 0$, otherwise the claim would be trivial.

2.2. Proof of Theorem 2.1.

Here we prove Theorem 2.1. We start with the following remarks.

Observe that under the hypotheses of Theorem 2.1, integral (2.1) is finite for $u \in U$ whenever so is

$$\int_\Omega g(x,u(x))\,\mathrm{d}x.$$

Indeed, according to (2.4) we can choose $\varepsilon > 0$ such that

$$A := \sup_{\substack{0<u\leq\varepsilon \\ x\in\Omega}} E(x,u) < \infty.$$

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whence

$$\mathcal{W}(u(t_1)) + \frac{1}{2}\int_{t_0}^{t_1}\int_\Omega |\nabla\Phi|^2\,\mathrm{d}x\,\mathrm{d}t \leq \mathcal{W}(u(t_0)) + C(t_1 - t_0).$$

Passing to the essential upper limit as $t_0 \to 0$ and estimating $t_1 - t_0 \leq T$, we obtain

$$\mathcal{W}(u(t_1)) + \frac{1}{2}\int_0^{t_1}\int_\Omega |\nabla\Phi|^2\,\mathrm{d}x\,\mathrm{d}t \leq \operatorname{ess}\limsup_{t\to+0} \mathcal{W}(u(t)) + CT,$$

whence (3.11) and (3.12) follow. ■

Theorem 3.9 (Solvability for positive data)**.** *Suppose that f satisfies (1.4)–(1.10) as well as (1.11) and (1.12). Then for any $u^0 \in L_+^\infty$ such that*

$$\kappa \leq u^0 \leq \frac{1}{\kappa} \quad \text{a. e. in } \Omega$$

with some constant $\kappa > 0$, there exists a unique weak solution

$$u \in L_+^\infty(\Omega \times [0,\infty)) \cap C([0,\infty); L^1(\Omega))$$

satisfying the following properties: i) the upper bound (1.23) and lower bound (1.26); ii) the energy and entropy dissipation inequalities as well as (1.24) and (1.25); iii) the restricted contraction

$$\int_\Omega (u - u_c)^+\,\mathrm{d}x \leq \int_\Omega (u^0 - u_c)^+\,\mathrm{d}x \quad (c \leq 0), \tag{3.13}$$

$$\int_\Omega (u - u_c)^-\,\mathrm{d}x \leq \int_\Omega (u^0 - u_c)^-\,\mathrm{d}x \quad (c \geq 0) \tag{3.14}$$

whenever u_c is defined; iv) if $\hat{u}$ is another such solution with the initial data $\hat{u}^0$, the L^1-contraction holds:

$$\|(u(t) - \hat{u}(t))^+\|_{L_1(\Omega)} \leq \mathrm{e}^{L_\kappa t}\|(u^0 - \hat{u}^0)^+\|_{L_1(\Omega)}, \tag{3.15}$$

where L_κ is defined by (3.4).

Proof: Let $\{u_n^0\}$ be a sequence of smooth functions satisfying the no-flux boundary condition such that

$$\kappa \leq u_n^0(x) \leq \frac{1}{\kappa} \quad \text{in } \Omega \tag{3.16}$$

and

$$u_n^0 \to u^0 \quad \text{in } L^1(\Omega) \text{ and a. e. in } \Omega. \tag{3.17}$$

providing the required bounds. ■

3.2. Positive initial data. If the initial data (1.3) is bounded away from 0, we approximate it with smooth functions and prove the existence and uniqueness of weak solutions to (1.1)–(1.3) stated in Theorem 3.9 below.

Lemma 3.8. *Suppose that* $u \in L^\infty_+(Q_T)$ *satisfies the energy inequality* (1.19) *in the sense of measures; then*

$$\|\mathcal{W}(u)\|_{L^\infty(0,T)} \le \operatorname{ess}\limsup_{t\to+0} \mathcal{W}(u(t)) + CT, \tag{3.11}$$

$$\|\nabla\Phi(\cdot,u(\cdot))\|_{L^2(Q_T)} \le 2(\operatorname{ess}\limsup_{t\to+0} \mathcal{W}(u(t)) + CT), \tag{3.12}$$

where $C > 0$ *is defined by an upper bound on* $\|u\|_{L^\infty(\Omega)}$.

Proof: The function

$$t \mapsto \mathcal{W}(u(t)) - \int_0^t \left(-\int_\Omega |\nabla\Phi|^2 \,\mathrm{d}x + \int_\Omega (\Phi_x + u f_x)\cdot\nabla\Phi \,\mathrm{d}x + \int_\Omega u f \Phi \,\mathrm{d}x \right) \mathrm{d}t$$

has a non-positive derivative in the sense of measures, so it a. e. coincides with a non-increasing function. In other words, for a. a. $t_0, t_1 \in (0,T)$, $t_0 < t_1$, we have

$$\mathcal{W}(u(t_1)) - \mathcal{W}(u(t_0)) - \int_{t_0}^{t_1} \left(-\int_\Omega |\nabla\Phi|^2 \,\mathrm{d}x + \int_\Omega (\Phi_x + u f_x)\cdot\nabla\Phi \,\mathrm{d}x + \int_\Omega u f \Phi \,\mathrm{d}x \right) \mathrm{d}t \le 0.$$

An upper bound on $\|u\|_{L^\infty(Q_T)}$ defines essential upper bounds on uf, $\Phi = \Phi(x,u(x,t))$, Φ_x, and $u f_x$, so for a. a. $t \in (t_0,t_1)$ we can estimate

$$\begin{aligned}\int_\Omega (\Phi_x + u f_x)\cdot\nabla\Phi \,\mathrm{d}x + \int_\Omega u f \Phi \,\mathrm{d}x &\le \frac{1}{2}\int_\Omega |\nabla\Phi|^2 \,\mathrm{d}x + \frac{1}{2}\int_\Omega |\Phi_x + u f_x|^2 \,\mathrm{d}x + \int_\Omega u f \Phi \,\mathrm{d}x \\ &\le \frac{1}{2}\int_\Omega |\nabla\Phi|^2 \,\mathrm{d}x + C,\end{aligned}$$

By (2.5), we have

$$B := \inf_{\substack{u>\varepsilon\\ x\in\Omega\\ E(x,u)\neq 0}} \frac{g(x,u)}{E(x,u)} > 0$$

(possibly $B = \infty$). Then $E(x,u) \le g(x,u)/B$ whenever $u > \varepsilon$, so

$$\begin{aligned}\int_\Omega E(x,u(x))\,\mathrm{d}x &= \int_{[u\le\varepsilon]} E(x,u(x))\,\mathrm{d}x + \int_{[u>\varepsilon]} E(x,u(x))\,\mathrm{d}x \\ &\le A|\Omega| + \frac{1}{B}\int_\Omega g(x,u(x))\,\mathrm{d}x < \infty,\end{aligned}$$

as claimed.

Take sequences $\{\varepsilon_n\}$ and $\{\xi_n\}$ such that $\varepsilon_n > 0$, $\varepsilon_n \to 0$,

$$0 < \xi_n \le \inf_{\substack{u>\varepsilon_n\\ x\in\Omega}} \frac{g(x,u)}{E(x,u)}$$

(this is possible according to (2.5)), and $\xi_n \to 0$.

Assume that Theorem 2.1 is not true. Then there exists a sequence of functions $\{u_n\} \subset U$ such that

$$\int_\Omega (g_n + u_n|\nabla f_n|^p)\,\mathrm{d}x \le \varepsilon_n\xi_n \int_\Omega E_n \,\mathrm{d}x, \tag{2.9}$$

where

$$\begin{aligned}E_n(x) &= E(x,u_n(x)),\\ f_n(x) &= f(x,u_n(x)),\\ g_n(x) &= g(x,u_n(x)).\end{aligned}$$

Clearly, $E_n, g_n \in C(\Omega)$ and $f_n \in C^1(\Omega)$. Moreover, it easily follows from (2.3)–(2.6) that

$$E_n(x) \ge 0,\ g_n(x) \ge 0; \tag{2.10}$$

$$\varlimsup_{n\to\infty} \sup_{[u_n\le\varepsilon_n]} E_n < \infty; \tag{2.11}$$

$$\varliminf_{n\to\infty} \inf_{[u_n\le\varepsilon_n]} f_n > \varlimsup_{\varepsilon\to 0} \sup_{[E_n<\varepsilon]} f_n, \tag{2.12}$$

and according to the choice of ξ_n, we have

$$g_n \ge \xi_n E_n \text{ on } [u_n > \varepsilon_n]. \tag{2.13}$$

We want to show that the sequence $\{E_n\}$ is bounded in $L^1(\Omega)$ and $u_n \to 0$ in measure, thus obtaining a contradiction.

We use (2.9) to estimate

$$\begin{aligned}\frac{1}{\varepsilon_n}\int_\Omega u_n|\nabla f_n|^p\,dx &\le \xi_n\int_\Omega E_n\,dx - \frac{1}{\varepsilon_n}\int_\Omega g_n\,dx\\ &\le \xi_n\int_\Omega E_n\,dx - \frac{1}{\varepsilon_n}\int_{[u_n>\varepsilon_n]} g_n\,dx\\ &\le \xi_n\int_\Omega E_n\,dx - \frac{\xi_n}{\varepsilon_n}\int_{[u_n>\varepsilon_n]} E_n\,dx\\ &= -\xi_n(\varepsilon_n^{-1}-1)\int_{[u_n>\varepsilon_n]} E_n\,dx + \xi_n\int_{[u_n\le\varepsilon_n]} E_n\,dx.\end{aligned}$$

Thus, we have

$$\frac{1}{\varepsilon_n}\int_\Omega u_n|\nabla f_n|^p\,dx \le -\xi_n(\varepsilon_n^{-1}-1)\int_{[u_n>\varepsilon_n]} E_n\,dx + \xi_n\int_{[u_n\le\varepsilon_n]} E_n\,dx. \tag{2.14}$$

For large n, the first term on the right-hand side is negative, so we conclude that

$$\frac{1}{\varepsilon_n}\int u_n|\nabla f_n|^p\,dx \le \xi_n \sup_{[u_n\le\varepsilon_n]} E_n|[u_n\le\varepsilon_n]|. \tag{2.15}$$

From (2.14) we get

$$\int_{[u_n>\varepsilon_n]} E_n\,dx \le \frac{1}{\varepsilon_n^{-1}-1}\int_{[u_n\le\varepsilon_n]} E_n\,dx \le \frac{\sup_{[u_n\le\varepsilon_n]} E_n|[u_n\le\varepsilon_n]|}{\varepsilon_n^{-1}-1} \tag{2.16}$$

and by (2.11), the last expression is bounded uniformly with respect to n. Hence the sequence $\{E_n\}$ is bounded in $L^1(\Omega)$.

Lemma 2.5. *Given $a>0$,*

$$\lim_{n\to\infty}|[u_n>\varepsilon_n]\cap[E_n>a]| = 0. \tag{2.17}$$

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and likewise, for $c \ge 0$ we have

$$\int_\Omega (u-u_c)^-\,dx \le \int_\Omega (u^0-u_c)^-\,dx, \quad t>0 \tag{3.10}$$

provided that u_c exists.

Proof: Let us prove (3.9) for $c \le 0$. Computing the derivative of the left-hand side, for a. a. $t>0$ we have:

$$\begin{aligned}\partial_t\int_\Omega (u-u_c)_+\,dx &= \int_\Omega \theta(u-u_c)\partial_t u\,dx\\ &= -\int_\Omega \theta(u-u_c)\operatorname{div}(u\nabla f)u\,dx\\ &\quad+\int_\Omega \theta(u-u_c)uf\,dx =: -I_1+I_2.\end{aligned}$$

As $\nabla f(x,u_c(x)) \equiv 0$, we can use Lemma 3.1 to get $I_1 \ge 0$. Now, the integrand of I_2 can only be non-zero where $u>u_c$, in which case $f\le c\le 0$ due to the monotonicity of f; consequently, $I_2\le 0$. Thus, we have

$$\partial_t\int_\Omega (u-u_c)_+\,dx \le 0$$

and (3.9) follows. Inequality (3.10) is proved in much the same way. ■

Lemma 3.7. *Suppose that f satisfies (1.11) and (1.12). Then for any smooth $u^0\colon\overline{\Omega}\to(0,\infty)$ satisfying the non-flux boundary condition, problem (1.1)–(1.3) has a classical solution.*

Proof: Equation (1.1) can be cast in the form

$$\partial_t u = -uf_u\Delta u - \nabla u\cdot(f_x+f_u\nabla u) - u(f_{xx}+2f_{xu}\cdot\nabla u + f_{uu}|\nabla u|^2 - f).$$

If we show that a classical solution is *a priori* bounded and stays away from 0, we can ignore the fact that the coefficient $-uf_u$ can be degenerate or singular at $u=0,\infty$ and infer the existence of the solution from the classical theory of quasilinear parabolic equations.

Indeed, according to Remark 3.4, we can find u_{c_1} and u_{c_2} such that $c_2 \le 0 \le c_1$ and

$$u_{c_1}(x) \le u^0(x) \le u_{c_2}(x) \quad (x\in\Omega).$$

Then it follows from Lemma 3.6 that

$$u_{c_1}(x) \le u(x,t) \le u_{c_2}(x,t) \quad (x,t)\in\Omega\times(0,\infty),$$

For $c \in \mathbb{R}$, define $u_c \in C^2(\overline{\Omega})$ by

$$f(x, u_c(x)) = c. \tag{3.6}$$

As f is monotonous in u, we see that the function u_c is unique, but it does not need to exist for a given c. Note that $u_0 = m$.

Remark 3.3. There is a simple formula for the L^∞-norm of u_c:

$$\|u_c\|_{L^\infty(\Omega)} = \inf\left\{\xi \geq 0\colon \sup_{x\in\Omega} f(x,\xi) \leq c\right\}. \tag{3.7}$$

It follows from the fact that due to the monotonicity of f, the inequality $\xi \geq \|u_c\|_{L^\infty(\Omega)}$ or, equivalently, $\xi \geq u_c(x)$ for all $x \in \Omega$, holds if and only if $f(x,\xi) \leq f(x, u_c(x)) \equiv c$ for all $x \in \Omega$, i. e. when

$$\sup_{x\in\Omega} f(x,\xi) \leq c.$$

Remark 3.4. *If* (1.11) *holds, for any $u \in L^\infty_+(\Omega)$ the function u_c with*

$$c = -\operatorname{ess\,sup}_{x\in\overline{\Omega}} f^-(x, u(x)) \tag{3.8}$$

is well-defined and $u \leq u_c$ a. e. in Ω. Indeed, if the second alternative in (1.11) holds, for any $x \in \overline{\Omega}$, the function $f(x,\xi)$ assumes all the values in the interval $(-\infty, 0]$ as ξ varies over $[m(x), \infty)$; in particular, $f(x,\xi)$ attains the value c. If, on the other hand, the first alternative in (1.11) holds, take $\xi_1 \geq \|u\|_{L^\infty}$ such that $c_1 := f(x,\xi_1)$ is independent of x and negative. Clearly, for any fixed $x \in \overline{\Omega}$, the function $f(x,\xi)$ takes all the values in the interval $[c_1, 0]$ as ξ varies over $[m(x), \xi_1]$. Now it suffices to observe that due to the monotonicity of f, we have $c \in [c_1, 0]$. One can prove in the same way that *if* (1.12) *holds, for any function u essentially bounded away from 0 on Ω, there exists u_c such that $u \geq u_c$ a. e. in Ω, and $c \geq 0$.*

Remark 3.5. It follows from Remarks 3.4 and 3.3 that if (1.11) holds, the right-hand side of (1.23) is finite for any $u^0 \in L^\infty_+(\Omega)$.

Lemma 3.6 (Restricted L^1-contraction)**.** *Let u be a classical solution of* (1.1)–(1.3) *on $[0,\infty)$. Then for $c \leq 0$ we have*

$$\int_\Omega (u - u_c)^+ \,\mathrm{d}x \leq \int_\Omega (u^0 - u_c)^+ \,\mathrm{d}x, \quad t > 0 \tag{3.9}$$

Proof: Using (2.16), we have:

$$\begin{aligned}
|[u_n > \varepsilon_n] \cap [E_n > a]| &\leq \frac{1}{a}\int_{[u_n>\varepsilon_n]\cap[E_n>a]} E_n \,\mathrm{d}x \\
&\leq \frac{1}{a}\int_{[u_n>\varepsilon_n]} E_n \,\mathrm{d}x \\
&\leq \frac{|[u_n \leq \varepsilon_n]|}{a(\varepsilon_n^{-1} - 1)} \sup_{[u_n\leq\varepsilon_n]} E_n \to 0 \quad (n \to \infty),
\end{aligned}$$

where we have taken into account (2.11), so (2.17) is proved. ■

Lemma 2.6. *Given $a > 0$, for large n we have*

$$|[E_n > a]| \leq 2|[u_n \leq \varepsilon_n]|. \tag{2.18}$$

Proof: Using the estimate

$$|[u_n > \varepsilon_n] \cap [E_n > a]| \leq \frac{|[u_n \leq \varepsilon_n]|}{a(\varepsilon_n^{-1} - 1)} \sup_{[u_n\leq\varepsilon_n]} E_n$$

obtained in the proof of Lemma 2.5, we get

$$\begin{aligned}
|[E_n > a]| &\leq |[u_n \leq \varepsilon_n]| + |[u_n > \varepsilon_n] \cap [E_n > a]| \\
&\leq \left(1 + \frac{\sup_{[u_n\leq\varepsilon_n]} E_n}{a(\varepsilon_n^{-1} - 1)}\right)|[u_n \leq \varepsilon_n]|,
\end{aligned}$$

and the lemma follows. ■

It follows from (2.12) that we can choose $a > 0$, α, and β, all independent of n, such that for large n we have

$$\sup_{[E_n\leq a]} f_n \leq \alpha < \beta \leq \inf_{[u_n\leq\varepsilon_n]} f_n. \tag{2.19}$$

We can assume that the limit

$$\lim_{n\to\infty} |[u_n \leq \varepsilon_n]|$$

exists. It follows from (2.19) that for large n the sets $[u_n \leq \varepsilon_n]$ and $[E_n \leq a]$ are disjoint, so in view of Lemma 2.5 we have

$$|[u_n \leq \varepsilon_n]| + |[E_n \leq a]| \to |\Omega| \tag{2.20}$$

Thus, we actually face three logical possibilities:

$$\lim_{n\to\infty} |[u_n \le \varepsilon_n]| = |\Omega|; \tag{2.21}$$

$$\lim_{n\to\infty} |[u_n \le \varepsilon_n]| = 0; \tag{2.22}$$

$$\lim_{n\to\infty} |[u_n \le \varepsilon_n]| = \mu_0 \in (0, |\Omega|); \tag{2.23}$$

As $\varepsilon_n \to 0$, (2.21) clearly implies $u_n \to 0$ in measure, a contradiction.

In what follows we show that (2.22) and (2.23) are in fact impossible. The following lemma is crucial.

Lemma 2.7. *We have*

$$\frac{1}{\varepsilon_n}\int_\Omega u_n |\nabla f_n|^p \,\mathrm{d}x \ge \frac{1}{|[E_n > a] \cap [u_n > \varepsilon_n]|^{p-1}} \left(\int_\alpha^\beta P([f_n > t], \Omega)\,\mathrm{d}t \right)^p \tag{2.24}$$

Proof: We have

$$\frac{1}{\varepsilon_n}\int_\Omega u_n |\nabla f_n|^p \,\mathrm{d}x \ge \int_{[E_n>a]\cap[u_n>\varepsilon_n]} |\nabla f_n|^p \,\mathrm{d}x \ge \frac{1}{|[E_n > a] \cap [u_n > \varepsilon_n]|^{p-1}} \left(\int_{[E_n>a]\cap[u_n>\varepsilon_n]} |\nabla f_n| \,\mathrm{d}x \right)^p. \tag{2.25}$$

Using the coarea formula, we get:

$$\int_{[E_n>a]\cap[u_n>\varepsilon_n]} |\nabla f_n| \,\mathrm{d}x = \int_{-\infty}^{\infty} P([f_n > t]; [E_n > a] \cap [u_n > \varepsilon_n])\,\mathrm{d}x \ge \int_\alpha^\beta P([f_n > t]; [E_n > a] \cap [u_n > \varepsilon_n])\,\mathrm{d}x \tag{2.26}$$

Fix $t \in (\alpha, \beta)$. Evoking the definition of the relative perimeter, we have

$$P([f_n > t]; [E_n > a] \cap [u_n > \varepsilon_n]) = \left|\mu_{[f_n>t]}\right| ([E_n > a] \cap [u_n > \varepsilon_n]), \tag{2.27}$$

where $\mu_{[f_n>t]}$ is the Gauss-Green measure. Obviously, we have

$$\operatorname{supp} \mu_{[f_n>t]} \cap \Omega \subset \partial_\Omega [f_n > t] \subset [f_n = t]$$

for any $t \in (\alpha, \beta)$. It follows from (2.19) that

$$[f_n = t] \subset [E_n > a] \cap [u_n > \varepsilon_n],$$

As θ is right-continuous, we have

$$\theta(u_n - \hat{u}) \to \theta(u - \hat{u}) \text{ pointwise in } \Omega;$$

moreover, it is clear that

$$f_n \to f \text{ in } C^2(\overline{\Omega}).$$

Passing to the limit in (3.3), we obtain (3.1). ■

Lemma 3.2 (L^1-contraction for positive classical solutions)**.** *Let u and $\hat{u}$ be classical solutions of* (1.1)–(1.3) *on $[0, T]$ with different initial data. Suppose that u and $\hat{u}$ satisfy*

$$\kappa \le u \le \frac{1}{\kappa}, \ \kappa \le \hat{u} \le \frac{1}{\kappa} \quad \textit{in } Q_T$$

with some $\kappa > 0$ and let $L_\kappa > 0$ be such that

$$|u_1 f(x, u_1) - u_2 f(x, u_2)| \le L_\kappa |u_1 - u_2| \quad x \in \overline{\Omega},\ \forall u_1, u_2 \in \left(\kappa, \frac{1}{\kappa}\right). \tag{3.4}$$

Then for a. a. $t > 0$,

$$\partial_t \int_\Omega (u - \hat{u})^+ \,\mathrm{d}x \le L_\kappa \int_\Omega (u - \hat{u})^+ \,\mathrm{d}x. \tag{3.5}$$

Proof: We have:

$$\begin{aligned} \partial_t \int_\Omega (u - \hat{u})^+ \,\mathrm{d}x &= \int_\Omega \theta(u - \hat{u})(\partial_t u - \partial_t \hat{u})\,\mathrm{d}x \\ &= -\int_\Omega \theta(u - \hat{u}) \operatorname{div}(u \nabla f - \hat{u} \nabla \hat{f})\,\mathrm{d}x \\ &\quad + \int_\Omega \theta(u - \hat{u})(uf - \hat{u}\hat{f})\,\mathrm{d}x =: -I_1 + I_2, \end{aligned}$$

where f and $\hat{f}$ stand for $f(x, u(x,t))$ and $f(x, \hat{u}(x,t))$, respectively. By Lemma 3.1, we have $I_1 \ge 0$. To estimate I_2, we use (3.4) and the observation that the integrand vanishes where $u - \hat{u} < 0$, thus obtaining

$$I_2 \le L_\kappa \int_\Omega (u - \hat{u})^+ \,\mathrm{d}x.$$

Inequality (3.5) follows. ■

Lemma 3.1. *If nonnegative* $u, \hat{u} \in C^\infty(\overline{\Omega})$ *satisfy the no-flux boundary condition* (1.2), *then*

$$\int_\Omega \theta(u-\hat{u})\,\mathrm{div}(u\nabla f - \hat{u}\nabla \hat{f})\,\mathrm{d}x \ge 0, \tag{3.1}$$

where f *and* $\hat{f}$ *stand for* $f(x,u(x))$ *and* $f(x,\hat{u}(x))$, *respectively.*

Proof: Without loss of generality, the functions u and $\hat{u}$ are defined and smooth on $\mathbb{R}^d$. Consider the set $\Upsilon := [u-\hat{u} > 0]$. First let us assume that 0 is a regular value of the function $u-\hat{u}$, then the boundary of Υ is smooth. Employing de Giorgi's Gauss-Green formula [30, Theorem 15.9] and the formula for the Gauss-Green measure of an intersection [30, Theorem 16.3], we compute

$$\begin{aligned}\int_\Omega \theta(u-\hat{u})\,\mathrm{div}(u\nabla f - \hat{u}\nabla \hat{f})\,\mathrm{d}x &= \int_{\Upsilon\cap\Omega} \mathrm{div}(u\nabla f - \hat{u}\nabla \hat{f})\,\mathrm{d}x \\ = \int_{\partial^*(\Upsilon\cap\Omega)} (u\nabla f - \hat{u}\nabla \hat{f})\cdot \nu_{\Upsilon\cap\Omega}\,\mathrm{d}\mathcal{H}^{d-1} &= \int_{\partial\Upsilon\cap\Omega} (u\nabla f - \hat{u}\nabla \hat{f})\cdot \nu_{\Upsilon}\,\mathrm{d}\mathcal{H}^{d-1} \\ + \int_{\Upsilon\cap\partial\Omega} (u\nabla f - \hat{u}\nabla \hat{f})\cdot \nu_{\Omega}\,\mathrm{d}\mathcal{H}^{d-1} &+ \int_{[\nu_\Upsilon = \nu_\Omega]} (u\nabla f - \hat{u}\nabla \hat{f})\cdot \nu_{\Omega}\,\mathrm{d}\mathcal{H}^{d-1},\end{aligned}$$

where $\nu_{\Upsilon\cap\Omega}$ is the measure-theoretic outward unit normal vector along the reduced boundary $\partial^*(\Upsilon\cap\Omega)$ of the intersection [30]. Due to the no-flux boundary condition, the last two integrals vanish. On $\partial\Upsilon\cap\Omega$, we have $u=\hat{u}$ and consequently, $f=\hat{f}$. Thus, we can write

$$\int_\Omega \theta(u-\hat{u})\,\mathrm{div}(u\nabla f - \hat{u}\nabla \hat{f})\,\mathrm{d}x = \int_{\partial\Upsilon\cap\Omega} u\nabla(f-\hat{f})\cdot\nu_\Upsilon\,\mathrm{d}\mathcal{H}^{d-1}. \tag{3.2}$$

Due to the monotonicity of f, we have $\Upsilon = [f-\hat{f} < 0]$. We see then that whenever $\nabla(f-\hat{f}) \ne 0$ on $\partial\Upsilon$, $\nabla(f-\hat{f})$ is an outward normal vector along $\partial\Upsilon$. Thus, $\nabla(f-\hat{f})\cdot\nu_\Upsilon \ge 0$ and equality (3.2) gives (3.1).

In the general case, take a decreasing sequence $\varepsilon_n \to 0$ such that 0 is a regular value of $u+\varepsilon_n-\hat{u}$. Set

$$u_n = u+\varepsilon_n,\ f_n = f(x,u_n(x)).$$

By the above, we have

$$\int_\Omega \theta(u_n-\hat{u})\,\mathrm{div}(u_n\nabla f_n - \hat{u}\nabla \hat{f})\,\mathrm{d}x \ge 0. \tag{3.3}$$

---

so

$$\mathrm{supp}\,\mu_{[f_n>t]}\cap\Omega \subset [E_n>a]\cap[u_n>\varepsilon_n]$$

and continuing (2.27), we obtain

$$\begin{aligned}P([f_n>t];[E_n>a]\cap[u_n>\varepsilon_n]) &= \left|\mu_{[f_n>t]}\right|([E_n>a]\cap[u_n>\varepsilon_n]) \\ &= \left|\mu_{[f_n>t]}\right|(\Omega) \\ &= P([f_n>t];\Omega).\end{aligned}$$

Combining this with (2.25) and (2.26), we obtain (2.24). ■

Let us show that (2.22) is impossible. Assume that it holds.

If at a point x we have $f_n(x) > t$, $t\in(\alpha,\beta)$, it follows from (2.19) that necessarily $E_n(x) > a$, i. e. $[f_n>t]\subset[E_n>a]$. It follows from (2.22) and (2.20) that $|[E_n\le a]| \to |\Omega|$ and thus, $|[E_n>a]|\to 0$, so we conclude that $|[f_n>t]|$ is uniformly in t small when n is large. For such large n we can apply the isoperimetric inequality:

$$P([f_n>t];\Omega) \ge c_\Omega |[f_n>t]|^{\frac{d-1}{d}}.$$

Now it follows from (2.19) that $[u_n\le\varepsilon_n]\subset[f_n>t]$, so we have

$$P([f_n>t];\Omega) \ge c_\Omega |[u_n\le\varepsilon_n]|^{\frac{d-1}{d}}.$$

Plugging this estimate into (2.24), we obtain

$$\frac{1}{\varepsilon_n}\int_\Omega u_n|\nabla f_n|^p\,\mathrm{d}x \ge \frac{c_\Omega^p(\beta-\alpha)^p|[u_n\le\varepsilon_n]|^{p(d-1)/d}}{|[E_n>a]\cap[u_n>\varepsilon_n]|^{p-1}}.$$

Estimating

$$|[E_n>a]\cap[u_n>\varepsilon_n]| \le |[E_n>a]| \le 2|[u_n\le\varepsilon_n]|$$

by virtue of (2.18), we obtain

$$\frac{1}{\varepsilon_n}\int_\Omega u_n|\nabla f_n|^p\,\mathrm{d}x \ge \frac{c_\Omega^p(\beta-\alpha)^p|[u_n\le\varepsilon_n]|^{p(d-1)/d}}{2^{p-1}|[u_n\le\varepsilon_n]|^{p-1}} = C|[u_n\le\varepsilon]|^{1-p/d},$$

where C is independent of n.

Combining obtained estimate with (2.15), we get:

$$C|[u_n\le\varepsilon_n]|^{1-\frac{p}{d}} \le \xi_n \sup_{[u_n\le\varepsilon_n]} E_n|[u_n\le\varepsilon_n]|,$$

whence

$$C \le \xi_n \sup_{[u_n\le\varepsilon_n]} E_n|[u_n\le\varepsilon_n]|^{\frac{p}{d}} \to 0 \quad (n\to\infty),$$

as $\xi_n \to 0$ and the suprema are bounded by (2.11). This contradicts the fact that the left-hand side is a positive constant independent of n. Thus, (2.22) is impossible.

It remains to show that (2.23) is also impossible. Assume that it holds.

It is easy to check that in this case we have

$$P([f_n > t];\Omega) \geq p_0 \quad (\alpha < t < \beta), \tag{2.28}$$

where $p_0 > 0$ is independent of t and n. Indeed, we have the inclusions

$$[u_n \leq \varepsilon_n] \subset [f_n > t] \subset [E_n > a]$$

and as in our case the measure of the first and third terms goes to μ_0 as $n \to \infty$, we also have

$$|[f_n > t]| \to \mu_0 \text{ uniformly in } t \in (\alpha, \beta).$$

Now it suffices to apply the isoperimetric equality to $[f_n > t]$ if $\mu_0 < 1/2$ and to $[f_n \leq t]$ otherwise.

Plugging (2.28) into (2.24), we get

$$\frac{1}{\varepsilon_n} \int_\Omega u_n |\nabla f_n|^p \, \mathrm{d}x \geq \frac{1}{|[u_n > \varepsilon_n] \cap [E_n > a]|^{p-1}} (\beta - \alpha)^p p_0^p.$$

Comparing this with (2.15), we obtain

$$\frac{1}{|[u_n > \varepsilon_n] \cap [E_n > a]|^{p-1}} (\beta - \alpha)^p p_0^p$$
$$\leq \xi_n \sup_{[u_n \leq \varepsilon_n]} E_n |[u_n \leq \varepsilon_n]| \to 0 \quad (n \to \infty).$$

As $n \to \infty$, the left-hand side remains bounded away from 0, while the right-hand side goes to 0, a contradiction.

2.3. Generalisations and specialisations. We start with the remark that Theorem 2.1 can often be applied if U is a subset of a space X of functions defined on Ω provided that $C^1(\Omega)$ is dense in X and the integrals (2.1) and (2.2) are continuous with respect to the topology of X. Indeed, if $U_1 = U \cap C^1(\Omega)$ is dense in U, we apply the theorem to U_1 and proceed by density to make sure that the same constant works for U as well. On the

Then from (2.42) and (2.39) we see that

$$|\xi - m(x)| < \delta \Rightarrow -f(x,\xi) - 2\xi f_u(x,\xi) > \varepsilon. \tag{2.44}$$

Further, using (2.43) and (2.40), we have

$$-f(x, m(x) + \delta) = \int_{m(x)}^{m(x)+\delta} (-f_u(x,u)\,\mathrm{d}u) \geq \varepsilon\delta,$$

whence, recalling that f_u is negative and f is decreasing, we conclude

$$\xi \geq m(x) + \delta \Rightarrow -f(x,\xi) - 2\xi f_u(x,\xi) > \varepsilon\delta. \tag{2.45}$$

Now, if $|u - m(x)| < \delta$, the point $\xi_{x,u}$ also satisfies $|\xi - m(x)| < \delta$, so we use (2.44) to conclude from (2.41) that

$$\frac{g(x,u)}{E(x,u)} > \varepsilon. \tag{2.46}$$

If $u \geq m(x)+\delta$, then either $m(x) < \xi_{x,u} < m(x)+\delta$ and we again obtain (2.46), or $\xi_{x,u} \geq m(x) + \delta$ and then we use (2.45) to get

$$\frac{g(x,u)}{E(x,u)} > \varepsilon\delta.$$

Thus,

$$\inf_{\substack{u > \varepsilon \\ x \in \Omega \\ E(x,u) \neq 0}} \frac{g(x,u)}{E(x,u)} \geq \min\left(\min_{\substack{\varepsilon \leq u \leq m(x) - \delta \\ x \in \overline{\Omega}}} \frac{g(x,u)}{E(x,u)}, \varepsilon, \varepsilon\delta \right) > 0,$$

since the function g/E is continuous and positive on the compact set

$$\{(x,u) \colon x \in \overline{\Omega}, \varepsilon \leq u \leq m(x) - \delta\}.$$

We have showed that (2.5) holds.

Thus, the hypotheses of Theorem 2.8 are fulfilled and the inequality follows. ∎

3. Technicalities

3.1. Positive classical solutions. Let

$$\theta(s) = \begin{cases} 1 & \text{if } s > 0, \\ 0 & \text{if } s \leq 0 \end{cases}$$

be the Heaviside step function.

Recalling Remark 2.3, we see that (2.4) holds.

Let us check (2.6). Fix $x \in \Omega$. The function $E(x,u)$ is strictly convex in u and attains its zero minimum only at $u = m(x)$. As $f(x, m(x)) = 0$, we see that

$$\lim_{\varepsilon\to 0} \sup_{E<\varepsilon} f = \max_{E=0} f = 0.$$

On the other hand, as f decreases with respect to u, we have

$$\begin{aligned}\lim_{\varepsilon\to 0} \inf_{\substack{0<u\le\varepsilon\\ x\in\Omega}} f(x,u) &\ge \inf_{x\in\Omega} f(x,\sigma)\\ &= \inf_{x\in\Omega} \int_\sigma^{m(x)} (-f_u(x,u))\,\mathrm{d}u\\ &\ge \min_{\substack{\sigma\le u\le m(x)\\ x\in\overline{\Omega}}} (-f_u(x,u)) \min_{x\in\overline{\Omega}} (m(x)-\sigma) > 0,\end{aligned}$$

so (2.6) indeed holds.

It remains to check (2.5). Without loss of generality, assume that $\varepsilon > 0$ is such that

$$\varepsilon < \frac{1}{2} \min_{x\in\overline{\Omega}} (-2m(x) f_u(x, m(x))), \tag{2.39}$$

$$\varepsilon < \frac{1}{2} \min_{x\in\overline{\Omega}} (-f_u(x, m(x))). \tag{2.40}$$

By Cauchy's mean value theorem, for any $x \in \Omega$, $u > \sigma$, $u \neq m(x)$, we have

$$\begin{aligned}\frac{g(x,u)}{E(x,u)} &= \frac{g(x,u) - g(x,m(x))}{E(x,u) - E(x,m(x))}\\ &= \frac{g_u(x,\xi_{x,u})}{E_u(x,\xi_{x,u})}\\ &= -f(x,\xi_{x,u}) - 2\xi_{x,u} f_u(x,\xi_{x,u}),\end{aligned} \tag{2.41}$$

where $\xi_{x,u}$ is some point between u and $m(x)$.

By uniform continuity, there exists $\delta \in (0, \min_{\overline{\Omega}} m - \sigma)$ such that

$$|\xi - m(x)| < \delta$$

implies

$$|-f(x,\xi) - 2\xi f_u(x,\xi) + 2m(x) f_u(x,m(x))| < \varepsilon, \tag{2.42}$$

$$|f_u(x,\xi) - f_u(x,m(x))| < \varepsilon. \tag{2.43}$$

other hand, if U_1 is not dense in U, we replace U with its small enlargement $\widetilde{U}$ in the cone of nonnegative functions in X and apply the same reasoning to $\widetilde{U}$. A more complicated density argument is used in the proof of Theorem 1.7 given in Section 3.3.

Another question is whether the constant C can be chosen uniformly with respect to the set of functions (E, g, f) if the latter is allowed to vary over a set $\mathcal{X}$. It turns out that Theorem 2.1 can be easily extended to handle this case. Specifically, if the suprema and infima in (2.4)–(2.6) are additionally taken over $(E, g, f) \in \mathcal{X}$, the constant C can be chosen independently of (E, g, f). The proof remains essentially the same. Assuming the converse, we have violating sequences $\{(\widetilde{E}_n, \tilde{g}_n, \tilde{f}_n)\} \subset \mathcal{X}$ and $\{u_n\} \subset U$ such that (2.9) holds with

$$\begin{aligned}E_n(x) &= \tilde{E}_n(x, u_n(x)),\\ f_n(x) &= \tilde{f}_n(x, u_n(x)),\\ g_n(x) &= \tilde{g}_n(x, u_n(x)).\end{aligned}$$

Moreover, the functions E_n, g_n, and f_n satisfy (2.10)–(2.13). The rest of the proof can be reused verbatim.

It should also be noted that the bare u on the right-hand side of (2.7) can be replaced by a nonnegative function $v(x, u(x))$. Of course, in this case it no longer makes sense to require that U should consist exclusively of positive functions. The separation from 0 should be taken in the sense that no sequence $\{v(\cdot, u_n(\cdot))\}$, where $u_n \in U$ and the sequence $\{E_n(\cdot, u_n(\cdot))\}$ is bounded in $L^1(\Omega)$, converges to 0 in measure. However, if v is, for example, an increasing function vanishing at 0, this new condition is clearly equivalent to the original one.

Again, the proof remains essentially unchanged, the sets $[u_n > \varepsilon_n]$ and $[u_n \le \varepsilon_n]$ being replaced by $[v_n > \varepsilon_n]$ and $[v_n \le \varepsilon_n]$, respectively (here $v_n(x) = v(x, u_n(x))$).

Summarising, we have the following strengthened version of Theorem 2.1:

Theorem 2.8. *Let Ω be a bounded, connected, open domain in $\mathbb{R}^d$ admitting the relative isoperimetric inequality. Let $p \ge 1$ and I be an interval (possibly unbounded). Let $\mathcal{X} = \{(E, g, f, v)\}$ be a set of tuples such that $E, g, v \in C(\Omega \times I)$,*

$f \in C^1(\Omega \times I)$, and

$$E \geq 0,\ g \geq 0,\ v \geq 0 \quad \forall (E,g,f,v) \in \mathcal{X}; \tag{2.29}$$

$$\limsup_{\varepsilon \to 0} \left\{ E(x,u) : \begin{array}{c} (E,f,g,v) \in \mathcal{X}, \\ (x,u) \in \Omega \times I, \\ v(x,u) \leq \varepsilon \end{array} \right\} < \infty \tag{2.30}$$

$$\inf \left\{ \frac{g(x,u)}{E(x,u)} : \begin{array}{c} (E,f,g,v) \in \mathcal{X}, \\ (x,u) \in \Omega \times I, \\ E(x,u) \neq 0, \\ v(x,u) > \varepsilon \end{array} \right\} > 0 \quad \forall \varepsilon > 0 \tag{2.31}$$

$$\liminf_{\varepsilon \to 0} \left\{ f(x,u) : \begin{array}{c} (E,f,g,v) \in \mathcal{X}, \\ (x,u) \in \Omega \times I, \\ v(x,u) \leq \varepsilon \end{array} \right\} > \limsup_{\varepsilon \to 0} \left\{ f(x,u) : \begin{array}{c} (E,f,g,v) \in \mathcal{X}, \\ (x,u) \in \Omega \times I, \\ E(x,u) \leq \varepsilon \end{array} \right\} \tag{2.32}$$

Finally, suppose that a set $U \subset C^1(\Omega; I)$ satisfies the following requirement: for any sequences $\{(E_n, g_n, f_n, v_n)\} \subset \mathcal{X}\}$ and $\{u_n\} \subset U$ such that the sequence $\{E_n(\cdot, u_n(\cdot))\}$ is bounded in $L^1(\Omega)$, the sequence $\{v_n(\cdot, u_n(\cdot))\}$ does not converge to 0 in measure. Then there exists a constant C depending only on Ω, p, U and $\mathcal{X}$ such that

$$\int_\Omega E(x,u(x))\,\mathrm{d}x \leq C \int_\Omega \Big(g(x,u(x)) + v(x,u(x)) |\nabla_x f(x,u(x))|^p \Big)\,\mathrm{d}x \quad ((E,g,f,v) \in \mathcal{X}, u \in U).$$

The proof is left to the reader.

Another option would be to allow for nonnegative instead of strictly positive u in Theorem 2.1. In this case one assumes that $E \in C(\Omega \times [0,\infty))$ and that the supremum in (2.4) is taken over $0 \leq u \leq \varepsilon$ and $x \in \Omega$. The resulting inequality differs from (2.7) in that the integral on the right-hand side is taken over $[u > 0]$. The only modification needed in the proof is that whenever g or $u|\nabla f|^p$ are integrated over Ω, the domain of integration should be changed to $[u > 0]$. Note that this does not fit into the previous theorem because f can be undefined on $[u = 0]$.

We conclude by showing that Theorem 2.1 is applicable in a situation relevant for gradient flows. In the subsequent formulation, f_u and E_u denote the derivatives of the functions f and E, respectively, with respect to their second argument.

Theorem 2.9. *Suppose that functions $E \in C(\overline{\Omega} \times [0,\infty))$, $f \in C^1(\overline{\Omega} \times (0,+\infty))$, and $m \in C(\overline{\Omega})$ satisfy*

$$E(x,u) \geq 0, \quad (x,u) \in \overline{\Omega} \times [0,\infty); \tag{2.33}$$

$$m(x) > 0, \quad x \in \overline{\Omega}; \tag{2.34}$$

$$E(x,m(x)) = 0, \quad x \in \Omega; \tag{2.35}$$

$$E_u(x,u) = -f(x,u), \quad (x,u) \in \Omega \times (0,+\infty); \tag{2.36}$$

$$f_u(x,u) < 0, \quad (x,u) \in \overline{\Omega} \times (0,+\infty) \tag{2.37}$$

and let $U \subset C^1(\Omega)$ be a set of strictly positive functions having the property that no sequence $\{u_n\} \subset U$ such that $\{E(\cdot, u_n(\cdot))\}$ is bounded in $L^1(\Omega)$, converges to 0 in measure. Finally, let $\sigma \in (0, \min_{\overline{\Omega}} m)$ and

$$v_\sigma(\xi) = \frac{\xi^2}{\max(\xi,\sigma)}.$$

Then we have

$$\int_\Omega E(x,u(x))\,\mathrm{d}x \leq C \int_\Omega v_\delta(u(x)) \Big((f(x,u(x)))^2 + |\nabla_x f(x,u(x))|^2 \Big)\,\mathrm{d}x \quad (u \in U), \tag{2.38}$$

where $C > 0$ depends on Ω, f, σ, and U.

Remark 2.10. Observe that under the hypotheses of Theorem 2.9, the functions E and m are uniquely defined by f. Indeed, if $x \in \Omega$ is fixed, $E(x,u)$ as a function of u attains its minimum at $m(x) > 0$, so $E_u(x,m(x)) = 0$, i. e. $f(x,m(x)) = 0$, according to (2.36). This uniquely defines $m(x)$, as it follows from (2.37) that $f(x,u)$ strictly decreases with respect to u. Now, $E(x,u)$ is the antiderivative of $-f(x,u)$ with respect to u vanishing at $m(x)$.

Proof: We check the hypotheses of Theorem 2.8 with $I = (0,\infty)$, $p = 2$, $g(x,u) = v_\sigma(u)(f(x,u))^2$, and the set $\mathcal{X}$ consisting of the single tuple (E, g, f, v_σ). Clearly, we have (2.29), while (2.30)–(2.32) are equivalent to (2.4)–(2.6).