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COMPLETE CONTROLLABILITY OF THE KINEMATIC EQUATIONS DESCRIBING PURE ROLLING OF GRASSMANNIANS

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ABSTRACT: This paper studies the controllability properties of certain nonholonomic control systems, describing the rolling motion of Grassmann manifolds over the affine tangent space at a point. The control functions correspond to the freedom of choosing the rolling curve. The nonholonomic constraints are imposed by the non-slip and non-twist conditions on the rolling. These systems are proved to be controllable in some submanifold of the group of isometries of the space where the two rolling manifolds are embedded. The constructive proof of controllability is also partially addressed.

KEYWORDS: Grassmann manifold, Lie group of isometries, rolling motions, Lie algebras, bracket generating property, complete controllability.

1. Introduction

The present paper addresses a mathematical problem, the controllability properties of a rolling system, but is motivated by the various applications of the Grassmann manifold. These manifolds play an important role in many engineering applications that deal with sets of images, such as face recognition problems under varying illumination conditions, or reconstruction of planar scenes from multiple views (see, for instance, [ST16] for several interesting problems about this thematic). Rolling motions of Riemannian manifolds have lately attracted the attention outside the mathematical community also due to its importance in robotics and computer vision. One recent interesting application concerning the use of rolling manifolds for recognizing human actions from 3D skeletal data is nicely treated in [VC16].

The basis for our work is the article [HL07], where the kinematic equations of rolling the grassmann manifold over its affine tangent space at a point have been derived. Section 2 contains an overview of those results. Our main contribution is contained in Section 3, where we prove that the kinematic equations, which can be seen as a nonlinear nonholonomic control system evolving on a certain Lie group, is controllable. The question about how to

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control the system is partially answered in Section 4, by showing how the forbidden motion of twisting can be accomplished by rolling without breaking the nonholonomic constraints of non-slip ands non-twist.

2. Revisiting Grassmannians

2.1. The geometry of Grassmann manifolds. Grassmann manifolds (or Grassmannians), hereafter denoted by $G_{k,n}$, are smooth manifolds consisting of all k-dimensional subspaces of $\mathbb{R}^n$ $(0 < k < n)$. Its dimension is $k(n-k)$. We use a matrix representation of $G_{k,n}$ in which each point is an $n \times n$ projection matrix. More precisely,

$$G_{k,n} := \{P \in s(n) :\ P^2 = P,\ \mathrm{rank}(P) = k\}, \tag{1}$$

where $s(n)$ denotes de vector space of all $n \times n$ symmetric matrices. For more details concerning this representation we refer to [HM94].

These matrices are isospectral, with eigenvalues 1 (of multiplicity k) and 0 (of multiplicity $n-k$). So, for each $P \in G_{k,n}$, there exists $\Theta \in SO(n)$ such that $P = \Theta^\top \left[\begin{smallmatrix} I_k & 0 \\ 0 & 0 \end{smallmatrix}\right] \Theta$.

Consider $s(n)$ equipped with the metric induced by the Frobenius norm $\langle A, B\rangle := \mathrm{tr}(AB)$, so that $G_{k,n}$ is a Riemannian manifold embedded in $(s(n), \langle ., .\rangle)$. The tangent space at any point $P \in G_{k,n}$ is given by

$$T_P G_{k,n} = \{S \in s(n) :\ PS + SP = S.\} \tag{2}$$

Another useful representation of the tangent space is the following, where $so(n)$ is the set of all skew symmetric matrices, and $[.,.]$ denotes the commutator.

$$T_P G_{k,n} = \{[\Omega, P] :\ \Omega \in so(n) \text{ and } P\Omega + \Omega P = \Omega\}. \tag{3}$$

With respect to the above metric, the normal space at a point $P \in G_{k,n}$ is defined as

$$\begin{aligned} T_P^\perp G_{k,n} &:= \{N \in s(n) :\ \mathrm{tr}(NV) = 0,\ \forall V \in T_P G_{k,n}\} \\ &= \{S - [P,[P,S]],\ S \in s(n)\}. \end{aligned} \tag{4}$$

In particular, for the point $P_0 = \left[\begin{smallmatrix} I_k & 0 \\ 0 & 0 \end{smallmatrix}\right]$, elements in $T_{P_0} G_{k,n}$ and in $T_{P_0}^\perp G_{k,n}$ are respectively represented by matrices with the following structure

$$\begin{bmatrix} 0 & Z \\ Z^\top & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix}, \tag{5}$$

where Z is any real $k \times (n-k)$ matrix, $S_1 \in s(k)$ and $S_2 \in s(n-k)$.

2.2. Rolling Grassmann over the affine tangent space at a point. The rolling motion of a manifold M_1 over another manifold M_0 of the same dimension, both isometrically embedded in a higher dimensional Riemannian manifold $\overline{M}$, is described by the action of the group of isometries of the embedding space. For the sake of completeness, we recall here the formal definition of rolling map. This definition is adapted from that given in [S97]. Depending on the context, there are other variations of this definition in the literature, for instance in [MGML12] and [HML08].

Definition 2.1. *Let M_1 and M_0 be two n-dimensional connected manifolds isometrically embedded in a higher dimensional complete Riemannian manifold $\overline{M}$ and let $\overline{G}$ be the connected component of the group of isometries of $\overline{M}$ that contains the identity. A smooth* rolling map of M_1 over M_0, without slip and without twist*, is a smooth curve $\chi : [0,\tau] \to \overline{G}$, satisfying the following three properties, for all $t \in [0,\tau]$:*

(1) *Rolling conditions:*
There exists a smooth curve $\alpha_1 : [0,\tau] \to M_1$, such that
  (a) $\chi(t)\,\alpha_1(t) \in M_0$;
  (b) $T_{\chi(t)\,\alpha_1(t)}\,\chi(t)\,M_1 = T_{\chi(t)\alpha_1(t)}M_0$.

The curve α_1 is called the rolling curve *and the curve $\alpha_0 : [0,\tau] \to M_0$, defined by $\alpha_0(t) := \chi(t)\,\alpha_1(t)$, is called the* development of α_1 on M_0.

(2) *No-slip condition:*
$$\left(\dot{\chi}(t)\chi(t)^{-1}\right)\ \alpha_0(t) = 0. \tag{6}$$

(3) *No-twist conditions:*
  (a) *(Tangential part)*
$$\left(\dot{\chi}(t)\chi(t)^{-1}\right)_*\, T_{\alpha_0(t)}M_0 \subset (T_{\alpha_0(t)}M_0)^{\perp}, \tag{7}$$
  (b) *(Normal part)*
$$\left(\dot{\chi}(t)\chi(t)^{-1}\right)_*\, (T_{\alpha_0(t)}M_0)^{\perp} \subset T_{\alpha_0(t)}M_0. \tag{8}$$

□

We note that $\dot{\chi}(t)\chi(t)^{-1}$ is a mapping from $\overline{M}$ to $T\overline{M}$ defined by:
$$\left(\dot{\chi}(t)\chi(t)^{-1}\right)(p) := \frac{d}{d\sigma}\Big|_{\sigma=t}\left[\left(\chi(\sigma)\ \left(\chi(t)^{-1}\right)(p)\right)\right]\ ,$$

and $\left(\dot{\chi}(t)\chi(t)^{-1}\right)_*$ denotes its differential map. The previous definition can be adjusted to piecewise-smooth rolling maps and rolling curves, simply replacing "for all t" by "for almost all t". We use the term *pure rolling* for rolling motions subject to the constraints of no-slip and no-twist.

Rolling motions of Grassmann manifolds have been studied in [HL07] and we refer to this paper for details that are not included here. In which concerns the Grassmann manifold rolling over the affine tangent space at a point P_0,

$$M_1 = G_{k,n}, \quad M_0 = T^{\text{aff}}_{P_0} G_{k,n} := P_0 + T_{P_0} G_{k,n}, \quad \overline{M} = s(n),$$

and the isometry group of $\overline{M}$ is the semi-direct product

$$\overline{G} = SO(n) \ltimes s(n).$$

Elements in $\overline{G}$ are represented by pairs (Θ, X), and the action of $\overline{G}$ on $s(n)$ is defined by $(\Theta, X)S = \Theta S \Theta^\top + X$.

Assume, without loss of generality, that $P_0 = \left[\begin{smallmatrix} I_k & 0 \\ 0 & 0 \end{smallmatrix}\right]$. Notice that $G_{k,n} \cap T^{\text{aff}}_{P_0} G_{k,n} = \{P_0\}$. A rolling map consists of a curve in $\overline{G}$ whose velocity vector field is restricted to a certain distribution due to the nonholonomic constraints of no-slip and no-twist. This distribution caracterizes the kinematic equations of the rolling motion. The following result has been derived in [HL07] and will be the starting point for the main results in the next section.

Theorem 2.1 ([HL07]). *If (Θ, X) is the solution of the following coupled system of differential equations (the kinematic equations)*

$$\begin{cases} \dot{\Theta}(t) = \Theta(t) \begin{bmatrix} 0 & -U(t) \\ U^\top(t) & 0 \end{bmatrix} \\ \dot{X}(t) = \begin{bmatrix} 0 & U(t) \\ U^\top(t) & 0 \end{bmatrix} \end{cases} \tag{9}$$

with $t \mapsto U(t) \in \mathbb{R}^{k \times (n-k)}$, and satisfying $(\Theta(0), X(0)) = (I_n, 0_n)$, then $t \mapsto \chi(t) = (\Theta^\top(t), X(t)) \in \overline{G}$ is a rolling map along the curve $t \mapsto \alpha_1(t) = \Theta(t) P_0 \Theta^\top(t) \in G_{k,n}$, with development curve $t \mapsto \alpha_0(t) = P_0 + X(t) \in T^{\mathit{aff}}_{P_0} G_{k,n}$.

It is clear that the choice of the function U completely determines the rolling curve (or equivalently its development, since $\alpha_0(t) = \chi(t)\alpha_1(t)$). So, the kinematic equations (9) may be seen as a control system, with control function U, evolving on the Lie group $G = SO(n) \times V$, where $V = T_{P_0} G_{k,n}$ is an additive Lie group. A natural question to ask is whether or not this system is controllable. The next section addresses this issue, following a procedure that has been used to prove controllability of the rolling sphere in [J97] and [Z05].

3. Controllability of the kinematic equations

In the language of geometric control theory, the kinematic equations (9) form a sub-Riemannian control system without drift, evolving on the connected Lie group $G = SO(n) \times V$, where $V = T_{P_0} G_{k,n}$. The Lie algebra of this group is $\mathcal{L}(G) = so(n) \oplus V$, equipped with the Lie bracket which is the commutator in the first component and the trivial bracket in the second component of the direct sum. Controllability means that every two points in G can be joined by trajectories of the system in finite time.

In which concerns proving controllability of the kinematic equations, the first thing that needs to be checked is the algebraic property, known in the literature as the *bracket generating property*. This is a necessary condition for controllability but, in general, it is not sufficient. We refer to [JS72] and [J97] for details concerning controllability of systems evolving on Lie groups. However, in the present situation, we can use a pioneer result about control systems on Lie groups which guarantees that, under some conditions, the bracket generating property is equivalent to controllability. The following is a paraphrase of Theorem 7.1 in [JS72].

Theorem 3.1. *A left-invariant control system without drift, evolving on a connected Lie group G is controllable if and only if the control vector fields generate the Lie algebra of G, i.e, satisfy the bracket generating property.*

It is not clear from the kinematic equations above that we are in the presence of a control system that fits the conditions of this theorem, namely that it is a left-invariant control system. To convince the readers that this is the case, we rewrite (9) in the form

$$\dot{Z}(t) = Z(t) \underbrace{\left(\sum_{i=1}^{r} u_i(t) W_i \right)}_{W(t)}, \quad Z \in G, \quad W_i \in \mathcal{L}(G),$$

where $r = n(n-k)$.

For that, first define

$$A(t) := \begin{bmatrix} 0 & -U(t) \\ U^\top(t) & 0 \end{bmatrix} \quad \text{and} \quad B(t) := \begin{bmatrix} 0 & U(t) \\ U^\top(t) & 0 \end{bmatrix},$$

so that the kinematic equations are written in the form

$$\begin{cases} \dot{\Theta}(t) = \Theta(t)A(t) \\ \dot{X}(t) = B(t) \end{cases} \tag{10}$$

Now, define Z in terms of Θ and X, and W in terms of A and B, as block diagonal matrices:

$$\begin{aligned} Z &:= \operatorname{diag}\left(\Theta, \left[\begin{array}{c|c} I_n & X \\ \hline 0_n & I_n \end{array}\right]\right), \\ W &:= \operatorname{diag}\left(A, \left[\begin{array}{c|c} 0_n & B \\ \hline 0_n & 0_n \end{array}\right]\right). \end{aligned} \tag{11}$$

With these identifications, the kinematic equations (9) can be written as

$$\dot{Z}(t) = Z(t)W(t), \tag{12}$$

which is a left-invariant control system without drift, evolving on the connected Lie group $G = SO(n) \times V$. So, according to Theorem 3.1, what remains to be proved is that the control vector fields generate the Lie algebra of G. To do this without getting too wired notations, we work with the first representation of the kinematic equations, i.e, with equations (9).

Let E_{ij} denote the square matrix with entry (i,j) equal to 1 and all other entries equal to 0. Define $A_{ij} := E_{ij} - E_{ji}$ and $B_{ij} := E_{ij} + E_{ji}$. To simplify notations, whenever one of the two indices is the sum of two integers we put a comma between the indices (for instance, $A_{i(k+l)}$ will be written as $A_{i,k+l}$). With these notations, a canonical basis for the Lie algebra $\mathcal{L}(G)$ is defined as:

$$\{(A_{ij}, 0),\ 1 \le i < j \le n\} \cup \{(0, B_{i,k+j}),\ i = 1, \cdots k;\ j = 1, \cdots n-k\}. \tag{13}$$

The left-invariant control vector fields in (9) can be identified with the following elements in $L(G)$:

$$\{(A_{i,k+j}, B_{i,k+j}),\ i = 1, \cdots k;\ j = 1, \cdots n-k\}. \tag{14}$$

Note that $\dim(\mathcal{L}(G)) = k(n-k) + n(n-1)/2$, while the system has only $k(n-k)$ control functions, which are the entries of the matrix function U. So, we are in the presence of an underactuated control system. The following commutator properties, where δ_{ij} denotes the Kronecker delta (which is 1 if $i=j$ and 0 if $i \neq j$), will be important:

$$[A_{ij}, A_{fl}] \;=\; \delta_{il}A_{jf} + \delta_{jf}A_{il} - \delta_{if}A_{jl} - \delta_{jl}A_{if}. \tag{15}$$

In our situation, to prove that the kinematic equations (9) are controllable reduces to showing that every element in the canonical basis (13) can be written as linear combinations of the $k(n-k)$ control vector fields in (14) and their Lie brackets.

Theorem 3.2. *The control vector fields in (9) are bracket generating, except when $k(n-k) = 1$.*

Proof: It is enough to show that every element in the canonical basis (13) can be written as linear combinations of the $k(n-k)$ elements in (14) and their Lie brackets. Recall that if (Y_1, Z_1) and (Y_2, Z_2) belong to $\mathcal{L}(G) = so(n) \oplus V$, then

$$[(Y_1, Z_1), (Y_2, Z_2)]_{\mathcal{L}(G)} = ([Y_1, Y_2]_{so(n)}, 0). \tag{16}$$

Using (15) and (16), we show that all the basis elements can be obtained by, at most, second order brackets of elements in (14).
First, we generate basis elements of the form $(A_{ij}, 0)$ and $(A_{i,k+j}, 0)$:

$$\begin{aligned} &\text{For } \; 1 \leq i < j \leq k, \\ &(A_{ij}, 0) = -\left[(A_{i,k+l}, B_{i,k+l}), (A_{j,k+l}, B_{j,k+l})\right]; \end{aligned} \tag{17}$$

$$\begin{aligned} &\text{For } \; 1 \leq i, j \leq n-k, \\ &(A_{k+i,k+j}, 0) = \left[(A_{m,k+j}, B_{m,k+j}), (A_{m,k+i}, B_{m,k+i})\right]. \end{aligned} \tag{18}$$

Second, we generate basis elements of the form $(A_{i,k+j}, 0)$ using elements from (14) and from (17):

$$\begin{aligned} &\text{For } \; i = 1, \cdots, k; \; j = 1, \cdots, n-k, \\ &(A_{i,k+j}, 0) = \left[(A_{il}, 0), (A_{l,k+j}, B_{l,k+j})\right]. \end{aligned} \tag{19}$$

Finally, we generate elements of the form $(0, B_{i,k+j})$ using elements from (14) and from (19):

$$\begin{aligned} &\text{For } \; i = 1, \cdots, k; \; j = 1, \cdots, n-k, \\ &(0, B_{i,k+j}) = (A_{i,k+j}, B_{i,k+j}) - (A_{i,k+j}, 0). \end{aligned} \tag{20}$$

This completes the decoupling and proves the statement. Notice that we have to exclude the situation when $k(n-k) = 1$, or equivalently $k = 1$ and $n = 2$, since in this case $\dim(\mathcal{L}(G)) = 2$ and there is only one control vector field that can't generate a 2-dimensional Lie algebra. ■

Corollary 3.3. *Whenever $k(n-k) \neq 1$, the control system (9), describing the pure rolling motions of the Grassmann manifold $G_{k,n}$ over the affine tangent space at the point P_0, is controllable in $G = SO(n) \times T_{P_0}G_{k,n}$.*

4. How to generate the forbidden motions

Due to the nonholonomic constraints, there are some motions resulting from the action of G on $G_{k,n}$ that are forbidden. These are the slips and the twists that we define next.

Definition 4.1. *A slip is any motion of $G_{k,n}$ that results from the action of elements in G of the form (I_n, X), where $X \in T_{P_0}G_{k,n}$. That is, a slip is a pure translation (in the embedding space) by the vector X.*

Definition 4.2. *A twist is any motion that results from the action of elements in G that keep P_0 invariant.*

Remark 4.1. *Notice that for (Θ, X) to keep P_0 invariant we must have $\Theta P_0 \Theta^\top + X = P_0$, which implies that $P_0 - X \in G_{k,n} \cap T^{aff}_{P_0}G_{k,n}$. But the only point that belongs to this intersection is P_0, so $X = 0$. Moreover, we must have $\Theta P_0 \Theta^\top = P_0$, that is, Θ belongs to the isotropy subgroup of $SO(n)$ at P_0, which is defined as*

$$K := \{\Theta \in SO(n), \text{ such that } \Theta P_0 \Theta^\top = P_0\}.$$

We conclude that twists are generated by elements in G of the form $(\Theta, 0)$, with $\Theta \in K$. Also notice that elements in K have the following structure:

$$K = \left\{ \Theta = \begin{bmatrix} \Theta_1 & 0 \\ 0 & \Theta_2 \end{bmatrix}, \; \Theta_1 \in SO(k), \, \Theta_2 \in SO(n-k) \right\}.$$

Controllability of the kinematic equations also means that the forbidden motions (slips and twists) can be performed using admissible motions only, that is, by rolling without twisting or slipping. So, a natural question to ask at this point is: how can we generate the forbidden motions?

The generation of a twist using only admissible motions appears to be much simpler than the generation of a slip. But, in both cases, and similarly to what happens for the rolling sphere, Cartan decompositions of the Lie algebra $so(n)$ and corresponding decompositions of $SO(n)$ seem to play a crucial role to address these problems. We are inspired by the work [KHL06] to generate forbidden motions.

4.1. How to generate twists. It is known that any element in $SO(n)$ can be written as a finite product of Givens rotations, which are elements of the form $e^{\tau A_{ij}}$, $\tau \in \mathbb{R}$ (see, for instance, [KHL06] for some details). So, every twist can also be decomposed as a finite product of elements of the form

$$\left[\begin{array}{c|c} e^{\tau_1 A_{ij}} & 0 \\ \hline 0 & e^{\tau_2 A_{k+l,k+m}} \end{array}\right], \tag{21}$$

where $1 \leq i < j \leq k$ and $1 \leq l < m \leq n-k$.

In order to generate a twist out of admissible motions it is enough to show that each one of the block diagonal elements in (21) can be decomposed into products of Givens rotations generated by elements of the form $A_{r,k+s}$, for $r = 1, \cdots, k$ and $s = 1, \cdots, n-k$, so that the sum of all angles of rotation add up to zero. Note that these are the elements in the Lie algebra of $SO(n)$ related to the control vector fields.

In order to show that this is indeed possible, we first prove the following result.

Proposition 4.1. *Let A, B and C be any three square matrices of the same arbitrary order that satisfy the following commuting relations:*

$$[A, B] = C \ \text{ and } \ [A, C] = -B. \tag{22}$$

Then, for any real parameter τ,

$$\begin{aligned} e^{\tau C} &= e^{(\pi/2)A} e^{\tau B} e^{-(\pi/2)A} \\ &= e^{-(\pi/2)A}\, e^{-\tau B}\, e^{(\pi/2)A}, \end{aligned} \tag{23}$$

and, consequently,

$$e^{\tau C} = e^{(\pi/2)A}\, e^{(\tau/2)B}\, e^{-\pi A}\, e^{-(\tau/2)B}\, e^{(\pi/2)A}. \tag{24}$$

Proof: The proof of the identities (24) is based on the commuting relations above and on properties of the exponential mapping, including the Campbell-Hausdorff formula

$$e^{tA}\, B\, e^{-tA} = e^{t ad_A}\, B = \sum_{i=0}^{+\infty} \frac{t^i}{i!} ad_A^i\, B,$$

where ad_A is the adjoint operator defined by $ad_A\, B := [A, B]$, $ad_A^i\, B := ad_A^{i-1}\,(ad_A\, B)$. Indeed, in this case,

$$e^{tA}\, B\, e^{-tA} = B\, \cos t + C\, \sin t,$$

so, choosing $t = \pm\pi/2$, we get

$$C = e^{\pm(\pi/2)A}\,(\pm B)\, e^{\mp(\pi/2)A},$$

which implies

$$e^{\tau C} = e^{\pm(\pi/2)A}\, e^{\pm\tau B}\, e^{\mp(\pi/2)A}.$$

The identity (24) is obtained from the previous, with a convenient choice of the signals, as follows.

$$\begin{aligned} e^{\tau C} &= e^{(\tau/2)C}\, e^{(\tau/2)C} \\ &= e^{(\pi/2)A}\, e^{(\tau/2)B}\, e^{-\pi A}\, e^{-(\tau/2)B}\, e^{(\pi/2)A}. \quad \blacksquare \end{aligned}$$

Corollary 4.2. *The following identities hold:*

a) *For* $1 \leq i < j \leq k$, $l \in \{1, \cdots, n-k\}$ *and* $\tau_1 \in \mathbb{R}$,

$$\begin{aligned} e^{\tau_1 A_{ij}} &= e^{(\pi/2)A_{j,k+l}}\, e^{(\tau_1/2)A_{i,k+l}}\, e^{-\pi A_{j,k+l}} \\ & \quad e^{-(\tau_1/2)A_{i,k+l}}\, e^{(\pi/2)A_{j,k+l}}. \end{aligned} \tag{25}$$

b) *For* $1 \leq l < m \leq n-k$, $i \in \{1, \cdots, k\}$ *and* $\tau_2 \in \mathbb{R}$,

$$\begin{aligned} e^{\tau_2 A_{k+l,k+m}} &= e^{(\pi/2)A_{i,k+m}}\, e^{(\tau_2/2)A_{i,k+l}}\, e^{-\pi A_{i,k+m}} \\ & \quad e^{-(\tau_2/2)A_{i,k+l}}\, e^{(\pi/2)A_{i,k+m}}. \end{aligned} \tag{26}$$

Proof: This is an immediate consequence of the fact that the triples

$$\{A_{ij}, A_{j,k+l}, A_{i,k+l}\} \text{ and } \{A_{k+l,k+m}, A_{i,k+m}, A_{i,k+l}\}$$

satisfy the commuting relations (22), as the triple $\{A, B, C\}$ does. $\blacksquare$

4.2. How to generate slips. A complete description of a slip in terms of the admissible motions is still under investigation. We expect to obtain concrete results soon, at least for small dimensions.

5. Conclusion

We proved that the kinematic equations that describe the pure rolling motion of the Grassmann manifold $G_{k,n}$ (with $k(n-k) \neq 1$) over the affine tangent space at a point, is controllable in

$$G = SO(n) \times T_{P_0} G_{k,n}.$$

This is the counterpart of a result about controllability of the rolling n-sphere that can be found for instance in [J97] and [Z05]. We have also shown how the forbidden motions of twisting can be generated just by rolling without twisting or slipping.

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