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UNIFORMLY COMPRESSING MEAN CURVATURE FLOW

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Abstract: Michor and Mumford showed that the mean curvature flow is a gradient flow on a Riemannian structure with a degenerate geodesic distance. It is also known to destroy the uniform density of gridpoints on the evolving surfaces. We introduce a related geometric flow which is free of these drawbacks, and study its analytical properties.

Keywords: evolving surface, volume, gradient flow, infinite-dimensional Riemannian manifold.
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1. Introduction

The mean curvature flow [38, 14] is a geometric flow which describes the behaviour of a k-dimensional submanifold $M_t \subset \mathbb{R}^d$, $1 \leq k < d$, which evolves over time t according to the law

$$\frac{dx}{dt} = \vec{H}(x), \tag{1}$$

where x is an arbitrary point of M_t, and $\vec{H}(x)$ is the mean curvature vector of the submanifold in x. It has huge variety of applications ranging from formation of grain boundaries in metals to image processing. The mean curvature flow (MCF) is the formal negative gradient flow of the volume functional $\mathrm{vol} \simeq \mathcal{H}^k$,

$$\partial_t M_t = -\operatorname{grad} \mathrm{vol}(M_t), \tag{2}$$

where the "manifold of k-dimensional submanifolds" is equipped with the L^2 Riemannian structure (see [8, 25] or our formula (17)), and the gradient is understood in the sense of this structure. Hence, as usual in the context of gradient flows [40, 4], the volume functional, being the driving entropy of the

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gradient flow, decays with time in the following way:

$$\frac{d}{dt}\operatorname{vol}(M_t) = -\langle \operatorname{grad}\operatorname{vol}(M_t), \operatorname{grad}\operatorname{vol}(M_t)\rangle_{M_t} = -\int_{M_t} |\vec{H}|^2 \, d\mathcal{H}^k. \quad (3)$$

There is an avalanche of works about the theoretical and numerical aspects of the mean curvature flow. One can introduce minimizing movement schemes à la de Giorgi which exploit the gradient flow structure [2, 24]. The solutions should a priori collapse in finite time, but singular behaviour before the breakdown does not permit existence of smooth solutions up to the final moment. There exist various strategies to go beyond the first singularity. The pioneering work [11] relaxed the notion of solution so that the evolving objects were barely varifolds. The level-set approach [30] led to solutions in the Crandall-Lions viscosity sense [17, 13, 5]. For the curve-shortening flow, i.e., when $k = 1$, one can define and construct a weak solution as a limit of certain curves which live in an ambient space of higher dimension and which are called ramps [3, 32]. Other types of weak solutions for the curve-shortening flow were introduced and studied in [6, 7, 15, 20].

There are two issues which mar the overall harmony. The first one is that the underlying Riemannian distance is degenerate (i.e., one can connect any two surfaces with a path of arbitrarily small Riemannian length), cf. [25, 26, 27, 9, 8], which is unpleasant since a non-degenerate metric space structure is an important precondition in treatment of gradient flows, see [4, 35]. The second issue is that the Hausdorff measure $\mathcal{H}^k$ is not uniformly contracted by the flow, that is, if $T_t : x(0) \mapsto x(t)$ is the flow operator describing the trajectories of material particles forming M_t in the ambient space $\mathbb{R}^d$, then the property

$$(T_t)_{\#}\left(\frac{1}{\operatorname{vol}(M_0)}\mathcal{H}^k \llcorner M_0\right) = \frac{1}{\operatorname{vol}(M_t)}\mathcal{H}^k \llcorner M_t \quad (4)$$

is violated except for some very special scenarios as a shrinking sphere. From the numerical perspective, this means that the flow destroys the uniform density of gridpoints on M_t, which is unwelcome and may cause computational instabilities [28, 39]. For the curve-shortening flow in the plane this can be fixed [1, 28] by adding a certain tangential motion to the right-hand side of (1) in order to conserve the uniform density of the moving particles without affecting the evolution of the shapes.

In this paper, we suggest a different approach which simultaneously eliminates the two above mentioned drawbacks of the mean curvature flow, and

which is applicable for any dimensions k and d. The idea is to consider the flow which is the closest possible to the original MCF (1) in a certain least-squares sense among the flows which uniformly contract the k-Hausdorff measure (in other words, which respect the uniform density of gridpoints). We employ the infinite-dimensional manifold $\mathcal{A}_k$ (consisting of objects of the form $\eta = \lambda\xi$, where $\lambda > 0$ is a scalar which quantifies the volume of M, and ξ is a volume-preserving immersion in the sense of [19]) endowed with the parametrization-invariant L^2 metric. Our flow is driven by the orthogonal projection of the mean curvature onto $T\mathcal{A}_k$. We dub the resulting object the uniformly compressing mean curvature flow (UCMCF) because the evolving surfaces can be thought of as being constituted by fluid particles whose density depends merely on time (the surfaces in question up to a time-dependent constant are *incompressible membranes* [16, 18, 29]). The UCMCF is by construction the negative gradient flow of the volume functional on $\mathcal{A}_k$. It is a genuinely geometric flow in the sense that the evolution of submanifolds $M_t = \eta(t)(M)$ does not depend on their parametrization.

Unlike the tangentially-corrected MCF [1, 28], our flow differs from the classical MCF in the normal direction and thus the geometrical evolution of the submanifolds along the two flows do not coincide in general (although they do coincide for a shrinking sphere). Nevertheless, we show that the qualitative behavior of UCMCF is quite similar to the one of the usual MCF, thus it may be used as a substitute for the MCF in applications.

We will observe that the our flow collapses in finite time, and in order to study the evolution and stability of the shapes before the breakdown we need to renormalize the problem both in time and in space. Surprisingly enough, our normalized flow is also a gradient flow: namely, the positive gradient flow of the L^2-mass on the space of volume-preserving immersions. Our recent work [37] studies the gradient flow of a different functional (potential energy) on a similar Riemannian structure, which turns out to be a model for an overdamped fall of an inextensible string in a gravitational field. A similar mechanical interpretation for our normalized UCMCF is an overdamped motion of an inextensible loop ($k = 1$) or an incompressible membrane ($k > 1$) repelled from the origin with the force field identically equal to the radius vector.

Both the original and normalized flows can be viewed as formal gradient flows on certain submanifolds of the Wasserstein space [31, 40, 41] of probability measures endowed with Otto's Riemannian structure.

We will mostly work with the immersed curves $1 = k < d$. However, Section 2.3 and a substantial part of Section 3.1 are devoted to the general case $1 \leq k < d$. It turns out to be more convenient to analyze the normalized flow, which allows us to descry the asymptotic behaviour of the shapes near the breakdown. We show local strong well-posedness of the problem. We characterize the steady states, and prove global existence of strong solutions for the initial data which are close to the steady states which maximize the driving L^2-mass, i.e., to simple circles. We establish the exponential decay of such a global solution to a steady state. We address the global solvability for any Lipschitz initial curve which does not need to be close to the equilibria, and we prove existence of suitably defined weak solutions. Our approach is based on approximation of the gradient flow on the manifold of volume-preserving immersions by Hilbertian gradient flows; in particular, we do not use the ramps. Unlike [3, 32, 6, 7, 15, 20], our weak solutions are H^1-regular in time.

In this paper, we make a technical and geometrically non-restrictive assumption that the center of mass is fixed at the origin; otherwise the center of mass would fly away to infinity.

2. The flow

2.1. Uniformly compressing curve-shortening flow. Let $\mathcal{S}^1 \simeq \mathbb{R}/\mathbb{Z}$ denote the circle of length 1. For $d \geq 2$ let $\mathcal{K}$ be the space of closed curves $\mathcal{K} := \{\eta : \eta \in H^2(\mathcal{S}^1;\mathbb{R}^d), \int_0^1 \eta(s)ds = 0\}$. Let $L : \mathcal{K} \to \mathbb{R}$, $L(\eta) := \int_0^1 |\partial_s \eta| \, ds$ be the length functional. We consider the space of immersed curves with the constant speed parametrization, i.e.,

$$\mathcal{A} := \{\eta \in \mathcal{K} : |\partial_s \eta(s)| = L(\eta) > 0 \text{ for all } s \in \mathcal{S}^1, \ \int_0^1 \eta(s)ds = 0\}.$$

Arguing as Theorem A.1 in [34] we see that $\mathcal{A}$ is a smooth Hilbert submanifold of $\mathcal{K}$, with the tangent space

$$T_\eta \mathcal{A} = \{w \in H^2(\mathcal{S}^1;\mathbb{R}^d) : \frac{d}{ds}\left(\partial_s w(s) \cdot \partial_s \eta(s)\right) = 0 \text{ for a.e. } s \in \mathcal{S}^1 \text{ and } \int_0^1 w(s) \, ds = 0\}, \ \eta \in \mathcal{A}.$$

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We endow $\mathcal{K}$ with the Riemannian metric

$$\langle v, w\rangle_{T_\eta\mathcal{K}} := \int_0^1 v(s)\cdot w(s)|\partial_s\eta(s)|\, ds,$$

which is invariant under reparametrization, cf. [26]. It induces a metric on $\mathcal{A}$:

$$\langle v, w\rangle_{T_\eta\mathcal{A}} = \int_0^1 v(s)\cdot w(s)L(\eta)\, ds. \tag{5}$$

Proposition 2.1. *The Riemannian distance $d_{\mathcal{A}}$ is non-degenerate.*

Proof: Take any two closed curves $\eta_0, \eta_1 \in \mathcal{A}$. Renormalizing if needed, we can suppose that $\|\eta_0 - \eta_1\|_{L^2(\mathcal{S}^1)} = 1$. We claim that

$$d_{\mathcal{A}}(\eta_0, \eta_1) \geq m := \frac{1}{2}\min(L(\eta_0), \|\eta_0\|^2_{L^2(\mathcal{S}^1)}, 2). \tag{6}$$

If not, there exists a C^1 path $\eta : [0,1] \to \mathcal{A}$, $\eta(0) = \eta_0$, $\eta(1) = \eta_1$, with Riemannian length

$$\begin{aligned}\mathcal{L}(\eta) &:= \int_0^1 \sqrt{\langle\dot\eta(t), \dot\eta(t)\rangle_{T_{\eta(t)}\mathcal{A}}}\, dt \\ &= \int_0^1 \|\dot\eta(t)\|_{L^2(\mathcal{S}^1)}\sqrt{L(\eta(t))}\, dt < m.\end{aligned} \tag{7}$$

Since by the Minkowski inequality and integration by parts

$$\int_0^1 \|\dot\eta(t)\|_{L^2(\mathcal{S}^1)}\, dt \geq \|\eta_1 - \eta_0\|_{L^2(\mathcal{S}^1)} \geq 1,$$

we have $L(\eta(t)) < m^2$ for some t. Due to continuity of $L(\eta(t))$ (and recall that $L(\eta_0) \geq 2m$ from (6)), there exists $t_* \in (0,1)$ such that $L(\eta(t_*)) = m$ and $L(\eta(t)) > m$ for $t < t_*$. Then

$$\mathcal{L}(\eta) \geq \mathcal{L}(\eta|_{(0,t_*)}) \geq \sqrt{m}\|\eta_0 - \eta(t_*)\|_{L^2(\mathcal{S}^1)}. \tag{8}$$

By Wirtinger's inequality,

$$\|\eta(t_*)\|_{L^2(\mathcal{S}^1)} \leq \frac{1}{2\pi}\|\partial_s\eta(t_*)\|_{L^2(\mathcal{S}^1)} = \frac{m}{2\pi}. \tag{9}$$

Combining (7)—(9), we infer

$$\sqrt{2m} \leq \|\eta_0\|_{L^2(\mathcal{S}^1)} \leq \|\eta_0 - \eta(t_*)\|_{L^2(\mathcal{S}^1)} + \|\eta(t_*)\|_{L^2(\mathcal{S}^1)} \leq \sqrt{m}\left(1 + \frac{1}{2\pi}\right),$$

which is a contradiction. ∎

We are interested in the gradient flow of the length functional L in the space $\mathcal{A}$ under the metric (5). We first compute (formally) the orthogonal projection from $T_\eta\mathcal{K}$ (which can be identified with $\mathcal{K}$) onto $T_\eta\mathcal{A}$ with respect to the metric (5) (cf. Proposition 3.2 in [34]).

Lemma 2.2. *Let $\eta \in \mathcal{A}\cap H^4(\mathcal{S}^1;\mathbb{R}^d)$. The orthogonal projection $P_\eta : T_\eta\mathcal{K} \to T_\eta\mathcal{A}$ is given by*

$$P_\eta(z) = z - \partial_s(\sigma\partial_s\eta), \text{ where } \sigma : \mathcal{S}^1 \to \mathbb{R} \text{ solves}$$

$$L^2\partial_{ss}\sigma - |\partial_{ss}\eta|^2\sigma = \partial_s z\cdot\partial_s\eta + const, \tag{10}$$

$$\int_0^1 \sigma(s)ds = 0. \tag{11}$$

Proof: (1) We first show that for any σ satisfying (11), the vector field $\partial_s(\sigma\partial_s\eta)$ is orthogonal to $T_\eta\mathcal{A}$. Indeed, given any $w \in T_\eta\mathcal{A}$,

$$\begin{aligned}\langle\partial_s(\sigma\partial_s\eta), w\rangle_{T_\eta\mathcal{K}} &= L(\eta)\int_0^1 \partial_s(\sigma\partial_s\eta)\cdot w\ ds\\ &= L(\eta)\sigma\partial_s\eta\cdot w\big|_{s=0}^{s=1} - L(\eta)\int_0^1 \sigma\partial_s\eta\cdot\partial_s w\ ds\\ &= 0.\end{aligned}$$

The first term is zero since we work on $\mathcal{S}^1$. The second term is zero because $\partial_s\eta\cdot\partial_s w = const$ and σ has mean zero.

(2) Next we show $w = z - \partial_s(\sigma\partial_s\eta) \in T_\eta\mathcal{A}$. We will mainly check the condition $\partial_s w\cdot\partial_s\eta = const$. Indeed,

$$\partial_s w\cdot\partial_s\eta = \partial_s z\cdot\partial_s\eta - \partial_{ss}\sigma|\partial_s\eta|^2 - 2\partial_s\sigma\partial_{ss}\eta\cdot\partial_s\eta - \sigma\partial_{sss}\eta\cdot\partial_s\eta.$$

The constant speed parametrization $|\partial_s\eta| = L = const$ yields $\partial_{ss}\eta\cdot\partial_s\eta = 0$ and $\partial_{sss}\eta\cdot\partial_s\eta = -|\partial_{ss}\eta|^2$. Thus

$$\partial_s w\cdot\partial_s\eta = \partial_s z\cdot\partial_s\eta - \partial_{ss}\sigma L^2 + \sigma|\partial_{ss}\eta|^2,$$

which is constant by (10).

■

Now we derive the negative gradient flow of the length functional $L(\eta) = \int_0^1 |\partial_s\eta|\ ds$, $\eta \in \mathcal{A}$, under the metric (5). First we observe that the full

Remark 3.22. One can adapt the approach of [37, Section 6] to construct backward generalized solutions to the normalized UCMCF. It seems however that all one can get in this way is the trivial solution $(\xi,\sigma)(t) = (e^t\xi_0, 0)$, $t \le 0$. It satisfies (85)–(88), and is smooth provided ξ_0 is smooth, but obviously violates the strong constraint $|\partial_s\xi| = 1$. This contrasts with Remark 3.21 and with [37] where smoothness implied the strong constraint. Consequently, the method of [37, Section 6] for constructing two different solutions emanating from an initial datum ξ_0 with $|\partial_s\xi_0| = 1$ is not applicable. This leads us to conjecture the uniqueness of the generalized solutions to the normalized UCMCF.

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due to our assumption about ξ_0, and if $t_1 > 0$ then (89) also holds by the continuity of $\partial_s\xi$. From $\partial_t(|\partial_s\xi|^2) \geq 0$ in U we immediately deduce that

$$|\partial_s\xi(t_0, s_0)|^2 \geq |\partial_s\xi(t_1, s_0)|^2 = 1,$$

arriving at a contradiction.

Theorem 4. *For every $\xi_0 \in W^{1,\infty}(\mathcal{S}^1;\mathbb{R}^d)$ with $|\partial_s\xi_0(s)| \leq 1$ for a.e. $s \in \mathcal{S}^1$, there exists a (global in time) generalized solution (ξ, σ) to the normalized UCMCF, and $\sigma(t, s) \geq 0$ for almost every $(t, s) \in Q_\infty$.*

We will not give here the proof since it is rather lengthy, and follows the corresponding steps in [37, Sections 3 and 4], mutatis mutandis. The idea is to rewrite (84) as a first-order system, and to approximate it by Hilbertian gradient flows. We just present here the first-order system and the approximation. Let $\kappa := \sigma\partial_s\xi$, then (ξ, κ, σ) solves

$$\begin{cases} \partial_t\xi = \partial_s\kappa + \xi \\ \kappa = \sigma\partial_s\xi \\ \sigma = \kappa \cdot \partial_s\xi. \end{cases} \tag{90}$$

For $\epsilon > 0$, let

$$F^\epsilon : \mathbb{R}^d \to \mathbb{R}^d, \quad F^\epsilon(\kappa) := \epsilon\kappa + \frac{\kappa}{\sqrt{\epsilon + |\kappa|^2}},$$

$$G^\epsilon(\tau) := (F^\epsilon)^{-1}(\tau),$$

and consider the problem

$$\partial_t\xi^\epsilon = \partial_s(G^\epsilon(\partial_s\xi^\epsilon)) + \xi^\epsilon \text{ in } Q_\infty. \tag{91}$$

Let us introduce the functional

$$\mathcal{E}^\epsilon(\xi) := \int_{\mathcal{S}^1} \epsilon\left(\frac{|G^\epsilon(\partial_s\xi)|^2}{2} - \frac{1}{\sqrt{\epsilon + |G^\epsilon(\partial_s\xi)|^2}}\right) - \frac{1}{2}|\xi|^2.$$

Then (91) can be interpreted as a negative gradient flow of $\mathcal{E}^\epsilon$ with respect to the flat Hilbertian structure of $L^2(\mathcal{S}^1;\mathbb{R}^d)$. We can moreover approximate κ and σ in (90) by

$$\kappa^\epsilon := G^\epsilon(\partial_s\xi^\epsilon), \quad \sigma^\epsilon := G^\epsilon(\partial_s\xi^\epsilon) \cdot \partial_s\xi^\epsilon \geq 0.$$

negative gradient of $L(\eta)$ in $\mathcal{K}$ is

$$-\operatorname{grad}_{\mathcal{K}} L(\eta) = \frac{1}{|\partial_s\eta|}\partial_s\left(\frac{\partial_s\eta}{|\partial_s\eta|}\right) = \frac{\partial_{ss}\eta}{L^2}.$$

By Lemma 2.2 the orthogonal projection of the full gradient to the tangent space $T_\eta\mathcal{A}$ is

$$P_\eta(-\operatorname{grad}_{\mathcal{K}} L(\eta)) = \frac{\partial_{ss}\eta - \partial_s(\sigma\partial_s\eta)}{L^2},$$

where $\sigma : \mathcal{S}^1 \to \mathbb{R}$ satisfies

$$L^2\partial_{ss}\sigma - \sigma|\partial_{ss}\eta|^2 = \partial_{sss}\eta \cdot \partial_s\eta + const = -|\partial_{ss}\eta|^2 + const,$$
$$\int_0^1 \sigma ds = 0.$$

To determine the constant, we integrate in s and thus obtain $const = \int_0^1 (1-\sigma)|\partial_{ss}\eta|^2 ds$. Letting $\tilde{\sigma} := 1 - \sigma$ we get the equation for the gradient flow $\partial_t\eta = -\operatorname{grad}_{\mathcal{A}} L(\eta(t))$:

$$\partial_t\eta(t, s) = L(t)^{-2}\partial_s(\tilde{\sigma}(t, s)\partial_s\eta(t, s)), \tag{12}$$

where $L(t) := L(\eta(t))$, and the Lagrange multiplier $\tilde{\sigma}(t, s)$ satisfies

$$\partial_{ss}\tilde{\sigma}(t, s) - L(t)^{-1}\tilde{\sigma}(t, s)|\partial_{ss}\eta(t, s)|^2$$
$$= -L(t)^{-2}\int_0^1 \tilde{\sigma}(t, s)|\partial_{ss}\eta(t, s)|^2 ds \text{ for all } (t, s)$$
$$\text{and } \int_0^1 \tilde{\sigma}(t, s)\, ds = 1 \text{ for all } t.$$

Remark 2.3 (Sign of $\tilde{\sigma}$)**.** From $\|\partial_{ss}\eta - P_\eta(\partial_{ss}\eta)\|_{L^2(\mathcal{S}^1)} \leq \|\partial_{ss}\eta\|_{L^2(\mathcal{S}^1)}$ as well as the orthogonality $\partial_s\eta \cdot \partial_{ss}\eta = 0$ everywhere it follows that

$$\|\partial_{ss}\eta\|^2_{L^2(\mathcal{S}^1)} \geq \|\partial_s\sigma\|^2_{L^2(\mathcal{S}^1)} + \|\sigma\partial_{ss}\eta\|^2_{L^2(\mathcal{S}^1)}.$$

Using $\tilde{\sigma} = 1 - \sigma$ we can rewrite the above inequality as

$$2\int_{\mathcal{S}^1} \tilde{\sigma}|\partial_{ss}\eta|^2 ds \geq \int_{\mathcal{S}^1} \tilde{\sigma}^2|\partial_{ss}\eta|^2 ds + \int_{\mathcal{S}^1} (\partial_s\sigma)^2 ds.$$

Thus $\int_{\mathcal{S}^1} \tilde{\sigma}|\partial_{ss}\eta|^2 ds \geq 0$ and the equality holds if and only if $\tilde{\sigma} \equiv 0$ on $\mathcal{S}^1$, which cannot happen since the mean of $\tilde{\sigma}$ is 1. Then by (12) $\tilde{\sigma}$ satisfies an inhomogeneous elliptic equation with a negative inhomogeneity, and provided $\eta(t) \in C^2(\mathcal{S}^1)$ by the strong maximum principle we infer that $\tilde{\sigma} > 0$.

Remark 2.4. Assume $\eta(t,s)$ is a classical solution to equation (12) emanating from η_0 which satisfies the constraints $|\partial_s\eta_0(s)| \equiv L(\eta_0)$ and $\int_0^1 \eta_0(s)ds = 0$. Then the constraints are preserved along the flow, i.e.,

$$|\partial_s\eta(t,s)| \equiv L(t) \text{ and } \int_0^1 \eta(t,s)ds = 0 \text{ for each } t.$$

To show the first equality we let $Z(t,s) := |\partial_s\eta(t,s)|^2 - L(t)^2$. Then a direct computation shows that Z satisfies

$$\begin{aligned}\partial_t Z &= L^{-2}\tilde{\sigma}\partial_{ss}Z + 2L^{-2}\partial_s\tilde{\sigma}\partial_s Z + 2\left(\partial_{ss}\tilde{\sigma} - L^{-2}\tilde{\sigma}|\partial_{ss}\eta|^2 - L\partial_t L\right)\\ &= L^{-2}\tilde{\sigma}\partial_{ss}Z + 2L^{-2}\partial_s\tilde{\sigma}\partial_s Z \quad \text{with } Z(0,s) = 0.\end{aligned}$$

Here we have used the equation of $\tilde{\sigma}$ (note that by (i) of Proposition 2.5 below, the right hand side of the equation of $\tilde{\sigma}$ is equal to $L\partial_t L$). By Remark 2.3 above $\tilde{\sigma}(t,s) > 0$ for the classical solutions η. Thus $Z(t,s) = 0$ by the strong maximum principle for parabolic equations. To show that $\int_0^1 \eta = 0$ is preserved along the flow, we integrate the equation along $\mathcal{S}^1$ to obtain $\partial_t \int_0^1 \eta(t,s)ds = 0$.

2.2. Evolution of the length and the L^2-mass. We now exploit the gradient flow structure to derive some evolution properties of the length and L^2-mass of the curves. Throughout this subsection we assume the existence of the H^2 solutions to the gradient flow (12).

The first proposition is about the evolution of the length functional.

Proposition 2.5. *Let η be a solution to* (12)*. Then*

(i) $\partial_t L(\eta) = -L^{-3}\int_0^1 \tilde{\sigma}|\partial_{ss}\eta|^2 ds$.
(ii) $\partial_{tt} L(\eta)^2 \geq 0$.

Proof: 1. Since $|\partial_s\eta(s,t)| = L(t)$, then

$$\begin{aligned}\partial_t L &= \partial_t|\partial_s\eta(t,s)|\\ &= \frac{\partial_s\eta(t,s)}{|\partial_s\eta(t,s)|}\cdot\partial_{ts}\eta(t,s)\\ &= \frac{\partial_s\eta(t,s)}{L}\cdot\partial_s\left(\frac{\tilde{\sigma}\partial_{ss}\eta}{L^2} + \frac{\partial_s\tilde{\sigma}\partial_s\eta}{L^2}\right).\end{aligned}$$

Note that the LHS does not depend on s. Thus an integration in s from 0 to 1 and an integration by parts yield the desired equality.

We begin by rewriting our flow in a form which explicitly involves the arc length parametrization constraint (cf. the beginning of the Section 3):

$$\begin{cases}\partial_t\xi &= \partial_s(\sigma\partial_s\xi) + \xi\\ |\partial_s\xi| &= 1\end{cases} \text{ for } (t,s) \in Q_\infty := (0,\infty)\times\mathcal{S}^1. \tag{84}$$

Definition 3.20. Given an initial datum $\xi_0 \in W^{1,\infty}(\mathcal{S}^1;\mathbb{R}^d)$ with $|\partial_s\xi_0(s)| \leq 1$ for a.e. $s \in \mathcal{S}^1$, we call a pair (ξ,σ) a generalized solution to the normalized UCMCF if

(i) $\xi \in L^\infty_{loc}([0,\infty);W^{1,\infty}(\mathcal{S}^1))^d$, $\partial_t\xi \in L^2_{loc}([0,\infty);L^2(\mathcal{S}^1))^d$, $\sigma \in L^2_{loc}([0,\infty);H^1(\mathcal{S}^1))$ and $\sigma\partial_s\xi \in L^2_{loc}([0,\infty);H^1(\mathcal{S}^1))^d$.
(ii) The pair (ξ,σ) satisfies for a.e. $(t,s) \in Q_\infty$

$$\partial_t\xi(t,s) = \partial_s(\sigma(t,s)\partial_s\xi(t,s)) + \xi, \tag{85}$$

$$\sigma(t,s)\left(|\partial_s\xi(t,s)|^2 - 1\right) = 0, \tag{86}$$

$$|\partial_s\xi(t,s)| \leq 1, \tag{87}$$

and the initial condition

$$\xi(0,s) = \xi_0(s) \text{ for a.e. } s.$$

(iii) The solution ξ satisfies the energy dissipation inequality

$$\int_{\mathcal{S}^1} |\partial_t\xi(t,s)|^2 ds \leq \int_{\mathcal{S}^1} \xi\cdot\partial_t\xi(t,s)ds \tag{88}$$

for a.e. $t \in (0,\infty)$.

Remark 3.21. It is not hard to see that if (ξ,σ) is a C^2 regular solution to (84), then it is also a generalized solution in the sense of Definition 3.20; in particular, (87) and (88) become strict equalities. On the other hand we claim that any generalized solution (ξ,σ) with $\xi \in C^1(\overline{Q_\infty}) \cap C^2(Q_\infty)$ and $|\partial_s\xi_0| = 1$ solves (84).

Formally, this claim is a trivial consequence of (86) since σ is expected to be strictly positive by the strong maximum principle, cf. Remark 2.3. However, we cannot guarantee the strict positivity of σ for the generalized solutions. Nevertheless, to prove the claim it suffices to show that the open set $U := \{(t,s) \in Q_\infty : |\partial_s\xi(t,s)| < 1\}$ is empty. Suppose not, then $\sigma = 0$ a.e. in U due to (86). This implies that $\partial_t\xi = \xi$ in U, whence $\partial_t(|\partial_s\xi|^2) = 2|\partial_s\xi|^2$. For each $(t_0,s_0) \in U$, let $t_1 = \inf\{t \geq 0 : (t,t_0)\times\{s_0\} \subset U\}$. If $t_1 = 0$ then

$$|\partial_s\xi(t_1,s_0)| = 1 \tag{89}$$

after an integration from t to ∞. Thus by the triangle inequality

$$\begin{aligned}|\xi(t,s) - w_0(s+\theta_\infty)| &\le |\xi(t,s) - c(t)w_0(s+\theta(t))| \\ &\quad + |(c(t)-1)w_0(s+\theta(t))| \\ &\quad + |w_0(s+\theta(t)) - w_0(s+\theta_\infty)| \\ &\le |\tilde{\xi}(t,s)| + \frac{1}{2\pi}|c(t)-1| + |\theta(t)-\theta_\infty| \\ &\le Ce^{-\frac{1}{16}t}\epsilon_0.\end{aligned}$$

■

Remark 3.19. The Hessian (37) of the L^2-mass is strictly negative-definite on the tangent vectors $\tilde{\xi} \in T_{w_0}\tilde{\mathcal{A}}$ which satisfy (72), i.e., on those which are orthogonal to the pure rotations:

$$\begin{aligned}\langle Hess\ M(w_0)\tilde{\xi},\tilde{\xi}\rangle_{T_{w_0}\tilde{\mathcal{A}}} &= \int_0^1 |\tilde{\xi}|^2(s)\,ds - \int_0^1 \varsigma(s)\,ds \\ &\le -3\int_0^1 |\tilde{\xi}|^2(s)\,ds = -3\langle\tilde{\xi},\tilde{\xi}\rangle_{T_{w_0}\tilde{\mathcal{A}}}.\end{aligned} \tag{82}$$

Here ς is the initial tension of a geodesic emanating from w_0 at the direction $\tilde{\xi}$, which satisfies

$$\partial_{ss}\varsigma - |\partial_{ss}w_0|^2\varsigma + |\partial_s\tilde{\xi}|^2 = 0. \tag{83}$$

(see [33, 36]). Indeed, since $|\partial_{ss}w_0| \equiv 2\pi$, an integration of (83) together with (72) imply

$$\int_0^1 \varsigma(s)\,ds = \frac{1}{4\pi^2}\int_0^1 |\partial_s\tilde{\xi}|^2(s)\,ds \ge 4\int_0^1 |\tilde{\xi}|^2(s)\,ds,$$

which gives (82). Then one can anticipate the exponential decay (Theorem 3) of the gradient flow in a neighbourhood of w_0 via a Bakry-Émery argument, cf. [40]. However, such argument is not applicablie in our situation since the Riemannian connection of $\tilde{\mathcal{A}}$ is not smooth and $(\tilde{\mathcal{A}}, d_{\tilde{\mathcal{A}}})$ is not a geodesic metric space, cf. Theorem 4.2 in [34] and [29, 10].

3.5. Global existence without restrictions on the initial data. We conclude by showing global solvability of the normalized flow without any restrictions on the initial data. It is an adaptation of the approach we recently developed in [37] for a different gradient flow.

2. Let $\lambda(t) := -L(t)\partial_t L(t) = L^{-2}\int_0^1 \tilde{\sigma}|\partial_{ss}\eta|^2 ds$. It suffices to show that $\partial_t\lambda(t) \le 0$. Note that one can rewrite the equation of $\tilde{\sigma}$ as

$$\partial_{ss}\tilde{\sigma} - L^{-2}\tilde{\sigma}|\partial_{ss}\eta|^2 = -\lambda.$$

Let N being the unit inner normal to the curve η, and let k be such that $\partial_{ss}\eta = kN$. Differentiation of the equation of $\tilde{\sigma}$ in time yields

$$\partial_{ss}\dot{\tilde{\sigma}} - \dot{\tilde{\sigma}}k^2L^{-2} - 2\tilde{\sigma}k\dot{k}L^{-2} + 2L^{-3}\partial_t L\tilde{\sigma}k^2 = -\dot{\lambda}.$$

We multiply the above equation by $\tilde{\sigma}$ and integrate in s. Integrating by parts, we get

$$\begin{aligned}\dot{\lambda} &= -\int_0^1 \dot{\tilde{\sigma}}\partial_{ss}\tilde{\sigma}ds + L^{-2}\int_0^1 \dot{\tilde{\sigma}}\tilde{\sigma}k^2 ds \\ &\quad + 2L^{-2}\int_0^1 \tilde{\sigma}^2 k\dot{k}ds - 2L^{-3}\partial_t L\int_0^1 \tilde{\sigma}^2k^2 ds \\ &= \lambda\int_0^1 \dot{\tilde{\sigma}}ds + 2L^{-2}\int_0^1 \tilde{\sigma}^2k\dot{k}ds - 2L^{-3}\partial_t L\int_0^1 \tilde{\sigma}^2k^2 ds \\ &= 2L^{-2}\int_0^1 \tilde{\sigma}^2k\dot{k}ds - 2L^{-3}\partial_t L\int_0^1 \tilde{\sigma}^2k^2 ds.\end{aligned}$$

Here in the second equality we have used the equation of $\tilde{\sigma}$. The last equality is due to $\int_0^1 \tilde{\sigma}(t,s)ds \equiv 1$. Now we compute $k\dot{k}$. Differentiating the equation of η in s we infer

$$\partial_{ts}\eta = L^{-2}\left(\partial_{ss}\tilde{\sigma}\partial_s\eta + 2\partial_s\tilde{\sigma}\partial_{ss}\eta + \tilde{\sigma}\partial_{sss}\eta\right).$$

Using that $\partial_{sss}\eta\cdot\partial_s\eta = -k^2$ and $\partial_{sss}\eta\cdot N = \partial_s k$ we get

$$\begin{aligned}L^2\partial_{ts}\eta &= \left(\partial_{ss}\tilde{\sigma} - L^{-2}k^2\tilde{\sigma}\right)\partial_s\eta + \left(2\partial_s\tilde{\sigma}k + \tilde{\sigma}\partial_s k\right)N + \tilde{\sigma}R \\ &= -\lambda\partial_s\eta + \left(2\partial_s\tilde{\sigma}k + \tilde{\sigma}\partial_s k\right)N + \tilde{\sigma}R\end{aligned}$$

where $R := \partial_{sss}\eta - (L^{-2}\partial_{sss}\eta\cdot\partial_s\eta)\partial_s\eta - (\partial_{sss}\eta\cdot N)N$ and we have used the equation of $\tilde{\sigma}$. Since $\partial_s N\cdot N = 0$ and $N\cdot R = 0$, we have

$$L^2\partial_{tss}\eta\cdot\partial_{ss}\eta = -\lambda k^2 + k\partial_s\left(2\partial_s\tilde{\sigma}k + \tilde{\sigma}\partial_s k\right) + \tilde{\sigma}(\partial_s R\cdot\partial_{ss}\eta).$$

But $R\cdot\partial_{ss}\eta = 0$, hence $\partial_s R\cdot\partial_{ss}\eta = -R\cdot\partial_{sss}\eta = -|R|^2$. Thus

$$L^2\partial_{tss}\eta\cdot\partial_{ss}\eta = L^2k\dot{k} = -\lambda k^2 + k\partial_s\left(2\partial_s\tilde{\sigma}k + \tilde{\sigma}\partial_s k\right) - \tilde{\sigma}|R|^2. \tag{13}$$

Until this moment we implicitly assumed that $\partial_{ss}\eta \neq 0$. However, (13) is still valid in the points with $\partial_{ss}\eta = 0$ since we can make an agreement that

$k = |R| = 0$ in those points. Plugging the expression (13) into the equation for $\dot{\lambda}$ we derive

$$\dot{\lambda} = -2L^{-4}(\lambda + L\partial_t L)\int_0^1 \tilde{\sigma}^2 k^2 ds + 2L^{-4}\int_0^1 \tilde{\sigma}^2 k\partial_s(2\partial_s\tilde{\sigma}k + \tilde{\sigma}\partial_s k)ds - 2L^{-4}\int_0^1 \tilde{\sigma}^3|R|^2 ds.$$

The first term on the right hand side vanishes due to the definition of λ. Then an integration by parts yields

$$\begin{aligned} L^4\dot{\lambda} &= -2\int_0^1 \partial_s(\tilde{\sigma}^2 k)(\tilde{\sigma}\partial_s k)ds - 4\int_0^1 \partial_s(\tilde{\sigma}^2 k)\partial_s\tilde{\sigma}k ds - 2\int_0^1 \tilde{\sigma}^3|R|^2 ds \\ &= -2\int_0^1 \tilde{\sigma}\left(\tilde{\sigma}\partial_s k + 2\partial_s\tilde{\sigma}k\right)^2 ds - 2\int_0^1 \tilde{\sigma}^3|R|^2 ds. \end{aligned} \tag{14}$$

Note that $\tilde{\sigma} \geq 0$ by Remark 2.3. Thus $\dot{\lambda} \leq 0$ and we complete the proof for (ii). ∎

In the next proposition we show that the L^2-mass of the solution decays with constant speed.

Proposition 2.6. *Let η be a solution to the gradient flow (12). Let $M(t) := \frac{1}{2}\int_0^1 |\eta(t,s)|^2 ds$ be the L^2-mass. Then $\partial_t M(t) = -1$.*

Proof: We multiply the equation of η in (12) by η and integrate in s. An integration by parts implies

$$\partial_t M(t) = -L^{-2}\int_0^1 \tilde{\sigma}|\partial_s\eta|^2 ds.$$

Using that $|\partial_s\eta| = L$ and $\int_0^1 \tilde{\sigma}ds = 1$ we obtain the desired equality. ∎

An immediate consequence of Proposition 2.6 is that the flow becomes extinct in finite time.

Corollary 2.7. *$M(t) \to 0$ as $t \to t^*$, where $t^* = M(0) = \frac{1}{2}\int_0^1 |\eta_0(s)|^2 ds$.*

It is also possible to obtain the decay rate of $L(t)$ near the extinction time t^*.

Corollary 2.8. *Let $L_0 := L(\eta_0) > 0$ and let t^* be the extinction time as in Corollary 2.7. Then for all $t \in [0, t^*)$,*

$\|\partial_s\tilde{\xi}(t,\cdot)\|_{L^2(\mathcal{S}^1)}$ (Proposition 3.1 (ii), Lemma 3.15 and Lemma 3.16). In particular, $\|\xi\|_{L^\infty([T/4,T]\times\mathcal{S}^1)}$ is bounded by an absolute constant, and thus the members of (81) are uniformly bounded. We apply Theorem 1 starting from $t = T$. The solution ξ extends in the Hölder class $C^{2+\alpha,\alpha/2}$ up to $t = T + T_0$ for some $T_0 > 0$ only depending on C_0. Lemma 3.18 applies and yields $\frac{1}{8\pi^2} \leq \sigma(t,s) \leq \frac{1}{2\pi^2}$ in $[0, T+T_0] \times \mathcal{S}^1$ and a universal bound on $\|\xi\|_{L^\infty([0,T+T_0]\times\mathcal{S}^1)}$. Again by Schauder $\|\xi\|_{C^{2+\alpha,\alpha/2}([(T+T_0)/2,T+T_0]\times\mathcal{S}^1)} \leq C_0\|\xi\|_{L^\infty([0,T+T_0]\times\mathcal{S}^1)}$. Repeating the above arguments from $T+2T_0$, $T+3T_0$ and so on, we obtain the global existence of the Cauchy problem (45) in the Hölder class $C^{2+\alpha,\alpha/2}$. ∎

As a by product we also obtain the exponential decay of our solution to the stationary solution w_0 under the initial smallness assumption.

Theorem 3. *Under the same assumptions as in Theorem 2 we have for all $t > 0$,*

$$\|\xi(t,s) - w_0(s+\theta_\infty)\|_{L^\infty(\mathcal{S}^1)} \leq Ce^{-t/16}\|\xi_0 - w_0\|_{H^2(\mathcal{S}^1)}$$

for some universal constant $C > 0$ and some constant θ_∞.

Proof: By Lemma 3.15, Lemma 3.18 and Corollary 3.17, if $\epsilon_0 := \|\xi_0 - w_0\|_{H^2(\mathcal{S}^1)}$ is sufficiently small (say $\epsilon_0 \leq \frac{1}{(100\pi^2)^2}$), then for all $t > 0$

$$\|\tilde{\xi}(t,\cdot)\|_{L^\infty(\mathcal{S}^1)} + (1 - c^2(t)) \leq Ce^{-\frac{1}{16}t}\epsilon_0$$

for some absolute constant $C > 0$. We still need to estimate the evolution of $\theta(t)$. Note that from the expression of $\dot{\theta}$ in (65)

$$|\dot{\theta}(t)| \leq \frac{4\pi^2}{c(t)}\frac{1}{2\pi}\max_{s\in\mathcal{S}^1}\sigma(t,s)\|\partial_s\tilde{\xi}(t,\cdot)\|_{L^2(\mathcal{S}^1)}, \quad t > 0.$$

By (78) if ϵ_0 in (79) is sufficiently small, then $\sigma(t,s) \leq \int_0^1 \sigma(t,s)ds + 3\sqrt{\epsilon_0} \leq \frac{1}{2\pi^2}$. Thus combining Corollary 3.17 and Lemma 3.16 we infer

$$|\dot{\theta}(t)| \leq e^{-\frac{1}{16}t}\|\partial_s\tilde{\xi}(0,\cdot)\|_{L^2(\mathcal{S}^1)}, \quad t > 0.$$

This implies that $\lim_{t\to\infty}\theta(t)$ exists. Letting θ_∞ denote the limit we obtain

$$|\theta(t) - \theta_\infty| \leq 16e^{-\frac{1}{16}t}\|\partial_s\tilde{\xi}(0,\cdot)\|_{L^2} \leq Ce^{-\frac{1}{16}t}\epsilon_0$$

which is no larger than $\frac{1}{4\pi^2}$ by Wirtinger's inequality. We want to show if ϵ_0 is sufficiently small, then the smallness assumption (76) from Lemma 3.18 is satisfied.

To see this, we first consider $\tilde{\xi}_0 := \xi_0 - cw_0(\cdot+\theta)$, where $cw_0(\cdot+\theta) \in \operatorname{argmin}_{w\in\mathcal{C}} \|\xi_0 - w\|_{L^2}$. Then (79) implies

$$\|\tilde{\xi}_0\|_{L^2(\mathcal{S}^1)} \leq \epsilon_0 \text{ and } \|\partial_s\tilde{\xi}_0\|_{L^2(\mathcal{S}^1)} \leq 2\pi\sqrt{\epsilon_0(1+\pi^{-1})}.$$

Indeed, the first inequality is immediate since by the definition of the projection $\|\tilde{\xi}_0\|_{L^2(\mathcal{S}^1)} \leq \|\xi - w_0\|_{L^2(\mathcal{S}^1)}$. The second inequality follows from (67) having observed that

$$\begin{aligned}\frac{c^2}{4\pi^2} &= \|\xi_0\|^2_{L^2(\mathcal{S}^1)} - \|\tilde{\xi}_0\|^2_{L^2(\mathcal{S}^1)}\\ &\geq \|w_0\|^2_{L^2(\mathcal{S}^1)} - 2\|w_0\|_{L^2(\mathcal{S}^1)}\|\xi - w_0\|_{L^2(\mathcal{S}^1)} - \|\tilde{\xi}_0\|^2_{L^2(\mathcal{S}^1)}\\ &\geq \frac{1}{4\pi^2} - \epsilon_0(1+\pi^{-1}).\end{aligned}$$

Next, by viewing (80) as a perturbation of

$$\partial_{ss}\sigma - |\partial_{ss}w_0|^2\sigma = \partial_{ss}\sigma - 4\pi^2\sigma = -1$$

with periodic boundary conditions, whose solution is the constant function $\frac{1}{4\pi^2}$, we have that the solution to (80) satisfies

$$\left|\frac{1}{4\pi^2} - \int_0^1 \sigma ds\right| \leq C\epsilon_0$$

for some universal constant $C > 0$. Hence if ϵ_0 is sufficiently small but universal, the smallness assumption (76) holds.

Now we apply Lemma 3.18 to ξ and get $\frac{1}{8\pi^2} \leq \sigma(t,s) \leq \frac{1}{2\pi^2}$ in $[0,T]\times\mathcal{S}^1$. Thus the equation of ξ is parabolic with the parabolicity bounded uniformly from above and below. By the regularity theory for the parabolic equations

$$\|\xi\|_{C^{2+\alpha,\alpha/2}([T/2,T]\times\mathcal{S}^1)} \leq C_0\|\xi\|_{L^\infty([T/4,T]\times\mathcal{S}^1)} \tag{81}$$

for some $C_0 > 0$ only depending on d, $\|\xi_0\|_{C^{2+\alpha}(\mathcal{S}^1)}$ (This is the interior Schauder estimate for parabolic equations. Since our solutions satisfy periodic bound conditions, the estimate holds globally in $\mathcal{S}^1$). In the meanwhile, the smallness assumption (76) of Lemma 3.18 is satisfied for all $t \in [0,T]$ due to the monotonicity properties of $\int_0^1 \sigma(t,s)ds$, $\|\tilde{\xi}(t,\cdot)\|_{L^2(\mathcal{S}^1)}$ and

(i) $2\sqrt{2}\pi\sqrt{t^*-t} \leq L(t) \leq L_0\sqrt{\frac{t^*-t}{t^*}}$,
(ii) $4\pi^2 \leq -L(t)\partial_t L(t) \leq -(L\partial_t L)(t)\big|_{t=0}$.

Proof: 1. The lower bound follows directly from the Wirtinger's inequality. Indeed, by Proposition 2.6

$$M(t) = t^* - t \text{ for } t \in [0,t^*).$$

On the other hand, by Wirtinger's inequality

$$M(t) \leq \frac{1}{2}\frac{1}{4\pi^2}\int_0^1 |\partial_s\eta|^2 ds = \frac{L(t)^2}{8\pi^2}.$$

Here we have used the assumption $\int_0^1 \eta = 0$. Combining the above two inequalities together we obtain the lower bound

$$L(t) \geq 2\sqrt{2}\pi\sqrt{t^*-t}.$$

Next we show the upper bound. Using $\partial_t M(t) = \int_0^1 \eta\cdot\partial_t\eta = -1$ and the Hölder's inequality we deduce

$$\int_0^1 |\partial_t\eta|^2 ds \geq \frac{1}{\int_0^1|\eta|^2} = \frac{1}{2M(t)} = \frac{1}{2(t^*-t)}. \tag{15}$$

On the other hand, by the gradient flow structure

$$\partial_t L = -L\int_0^1 |\partial_t\eta|^2 ds. \tag{16}$$

Thus $-\partial_t \ln L \geq \frac{1}{2(t^*-t)}$. Then an integration in t from 0 to t yields

$$L(t) \leq L_0\sqrt{\frac{t^*-t}{t^*}}, \quad t\in[0,t^*).$$

2. By (ii) of Proposition 2.5, $t \mapsto -L\partial_t L$ is monotone decreasing. Thus $-L\partial_t L \leq -L\partial_t L\big|_{t=0}$. To see the lower bound, we note that by (16),(15) and Wirtinger's inequality

$$-L(t)\partial_t L(t) = L(t)^2\int_0^1 |\partial_t\eta|^2 \geq \frac{L(t)^2}{2M(t)} \geq 4\pi^2.$$

■

2.3. Higher dimensions. In this subsection we briefly describe how our approach can be implemented in a more general situation. For the sake of simplicity of the presentation, we work with embeddings of a compact manifold into the ambient space $\mathbb{R}^d$, but this can be generalized in various directions, in particular, one can consider immersions instead of embeddings.

Fix a smooth compact connected k-dimensional submanifold $\mathcal{M}$ of $\mathbb{R}^d$. Without loss of generality in the sequel we assume that $\mathrm{vol}(\mathcal{M}) = 1$. Let $\mathcal{K}_k$ be the space of H^m-regular embeddings $\eta : \mathcal{M} \to \mathbb{R}^d$, $\int_{\mathcal{M}} \eta \, d\mathcal{H}^k = 0$, $m > \frac{n+2}{2}$. Each element $v \in T_\eta \mathcal{K}_k$ can be identified with a vector field $v : \mathcal{M} \to \mathbb{R}^d$. We endow the space $\mathcal{K}_k$ with the parametrization-invariant L^2 Riemannian metric (cf. [25])

$$\langle v, w\rangle_{T_\eta \mathcal{K}_k} := \int_{\eta(\mathcal{M})} (v \cdot w) \circ (\eta^{-1}) \, d\mathcal{H}^k = \int_{\mathcal{M}} v \cdot w \; J\eta \; d\mathcal{H}^k, \tag{17}$$

which has a degenerate Riemannian distance [9]. Here $J\eta(x) := \sqrt{\det(d\eta_x)^* \circ (d\eta_x)}$ for each $x \in \mathcal{M}$ is the Jacobian of η. Let

$$\mathrm{vol} : \mathcal{K}_k \to \mathbb{R}, \;\; \mathrm{vol}(\eta) = \mathcal{H}^k(\eta(\mathcal{M}))$$

be the volume functional. By Sobolev embedding $H^m \subset C^1$, vol is continuous w.r.t. the H^m-topology of $\mathcal{K}_k$. Sometimes we will also use a flat metric $\langle \cdot, \cdot \rangle^*$ on $\mathcal{K}_k$:

$$\langle v, w\rangle^*_{T_\eta \mathcal{K}_k} := \int_{\mathcal{M}} v \cdot w \, d\mathcal{H}^k. \tag{18}$$

We consider the submanifold of $\mathcal{K}_k$ consisting of uniformly dilating embeddings, i.e.,

$$\mathcal{A}_k := \{\eta \in \mathcal{K}_k : \mathrm{vol}(\eta) > 0, \;\; \eta_\# (\mathcal{H}^k \llcorner \mathcal{M}) = \frac{1}{\mathrm{vol}(\eta)} \mathcal{H}^k \llcorner \eta(\mathcal{M}),$$
$$\int_{\mathcal{M}} \eta \; d\mathcal{H}^k = 0\}.$$

Let us also define the submanifold of $\mathcal{A}_k$ consisting of volume-preserving embeddings

$$\tilde{\mathcal{A}}_k := \{\eta \in \mathcal{A}_k, \;\; \mathrm{vol}(\eta) = 1\}. \tag{19}$$

The tangent space at $\eta \in \tilde{\mathcal{A}}_k$ is

$$T_\eta \tilde{\mathcal{A}}_k = \{h \in T_\eta \mathcal{K}_k : \mathrm{div}_{\eta(\mathcal{M})}(h \circ \eta^{-1}) = 0\},$$

Using the fundamental theorem of calculus as well as (76) and (i) one can bound the oscillation of $|\tilde{\xi}|^2$ as

$$\left| |\tilde{\xi}(t,s)|^2 - \int_0^1 |\tilde{\xi}(t,s)|^2 ds \right| \le 2 \int_0^1 |\partial_s \tilde{\xi} \cdot \tilde{\xi}| ds \le 2 \|\partial_s \tilde{\xi}(t,\cdot)\|_{L^2} \|\tilde{\xi}(t,\cdot)\|_{L^2}$$
$$\le 2\epsilon.$$

For the term $c\tilde{\xi} \cdot w_0$ we use $|c| \le 1$, $|w_0| = \frac{1}{2\pi}$ and the above oscillation estimate to get $|c\tilde{\xi} \cdot w_0| \le \frac{1}{2\pi} |\tilde{\xi}(t,s)| \le \frac{1}{2\pi} \left(\int_0^1 |\tilde{\xi}(t,s)|^2 ds + 2\epsilon \right)^{1/2} \le \frac{1}{2\pi} \sqrt{3\epsilon}$. Combining together we have for all $(t,s) \in [0,T] \times \mathcal{S}^1$

$$\left| \frac{1}{2} |\xi(t,s)|^2 - \frac{1}{2} \int_0^1 |\xi(t,s)|^2 ds \right| \le \epsilon + 2 \frac{1}{2\pi} \sqrt{3\epsilon} \le 2\sqrt{\epsilon}.$$

Combining this with (78) we obtain the desired oscillation estimate for σ. Taking $\epsilon \le \frac{1}{(32\pi^2)^2}$ and using (76) as well as (i) we obtain the pointwise lower bound for $\sigma(t,s)$. The upper bound follows from (78) and Proposition 3.1. ∎

At the end of this section we show the global well-posedness of the normalized flow in the Hölder class $C^{2+\alpha,\alpha/2}$, under the assumption that initially the curve is sufficiently close to the stationary solution w_0.

Theorem 2. *Let w_0 be the stationary solution defined in* (61). *Given an initial datum $\xi_0 \in \tilde{\mathcal{A}} \cap C^{2+\alpha}(\mathcal{S}^1)$, which satisfies*

$$\|\xi_0 - w_0\|_{H^2(\mathcal{S}^1)} \le \epsilon_0 \tag{79}$$

for some small universal constant $\epsilon_0 > 0$, the Cauchy problem (45) *has a solution*

$$\xi \in C^{2+\alpha,\alpha/2}([0,\infty) \times \mathcal{S}^1), \partial_t \xi \in C^{\alpha,\alpha/2}((0,\infty) \times \mathcal{S}^1).$$

Proof: Let $\xi \in C^{2+\alpha,\alpha/2}([0,T] \times \mathcal{S}^1)$ be the local solution to the Cauchy problem (45) with the initial datum ξ_0 (cf. Theorem 1). Let σ be the Lagrange multiplier such that $\xi_0 + \partial_s(\sigma \partial_s \xi_0) \in T_{\xi_0} \tilde{\mathcal{A}}$. Similar arguments as in Lemma 2.2 yield

$$\partial_{ss} \sigma - |\partial_{ss} \xi_0|^2 \sigma = -1. \tag{80}$$

Note that from (80) we have

$$\int_0^1 \sigma = \|\partial_s(\sigma \partial_s \xi_0)\|^2_{L^2} \le \|\xi_0\|^2_{L^2},$$

Indeed, the first equality is due to (32) and (33). To see the second inequality, we note that by the assumption (76) and the observation above,

$$\begin{aligned}\int_0^1 |\xi|^2 ds - \bar{\sigma} &= \left(\int_0^1 |\xi|^2 - \frac{1}{4\pi^2}\right) + \left(\frac{1}{4\pi^2} - \bar{\sigma}\right) \\ &= \int_0^1 |\tilde{\xi}|^2 ds + (c^2-1)\frac{1}{4\pi^2} + \left(\frac{1}{4\pi^2} - \bar{\sigma}\right) \\ &= \int_0^1 |\tilde{\xi}|^2 ds - \frac{1}{4\pi^2}\int_0^1 |\partial_s \tilde{\xi}|^2 ds + \left(\frac{1}{4\pi^2} - \bar{\sigma}\right) \leq \epsilon,\end{aligned}$$

where in the second last inequality we have used (67).

3. We estimate the oscillation of σ and $|\xi|^2$. First we show that the oscillation of σ is bounded by the oscillation of $|\xi|^2$: for all $(t,s) \in [0,T]\times \mathcal{S}^1$

$$|\sigma(t,s) - \bar{\sigma}(t)| \leq \sqrt{\epsilon} + \left|\frac{1}{2}|\xi(t,s)|^2 - \frac{1}{2}\int_0^1 |\xi(t,s)|^2 ds\right|. \tag{78}$$

For this we multiply the equation of ξ by $\partial_s \xi$ and get $\partial_t \xi \cdot \partial_s \xi = \partial_s \sigma + \xi \cdot \partial_s \xi = \partial_s\left(\sigma + \frac{1}{2}|\xi|^2\right)$. An integration in s yields

$$\begin{aligned}&\left|\sigma(t,s) + \frac{1}{2}|\xi(t,s)|^2 - \int_0^1 \left(\sigma(t,s) + \frac{1}{2}|\xi(t,s)|^2\right) ds\right| \\ &\leq \int_0^1 |\partial_t \xi(t,s)\cdot \partial_s \xi(t,s)|\, ds.\end{aligned}$$

By Hölder's inequality and (77),

$$\int_0^1 |\partial_t \xi(t,s)\cdot \partial_s \xi(t,s)|\, ds \leq \left(\int_0^1 |\partial_t \xi(t,s)|^2 ds\right)^{1/2} \leq \sqrt{\epsilon}$$

Combining the above two inequalities together we obtain (78).

Next we rewrite the oscillation of $|\xi|^2$ by using $|\tilde{\xi}|^2$:

$$\begin{aligned}&\left|\frac{1}{2}|\xi(t,s)|^2 - \frac{1}{2}\int_0^1 |\xi(t,s)|^2 ds\right| \\ &= \left|\frac{1}{2}|\tilde{\xi}(t,s)|^2 - \frac{1}{2}\int_0^1 |\tilde{\xi}(t,s)|^2 ds + 2c\tilde{\xi}\cdot w_0\right|.\end{aligned}$$

thus it is not hard to verify that $T_\eta \mathcal{A}_k = \{h \in T_\eta \mathcal{K}_k : \mathrm{div}_{\eta(\mathcal{M})}(h\circ\eta^{-1}) = const\}$. The metrics (17) and (18) induce metrics on $\mathcal{A}_k$: for $\eta \in \mathcal{A}_k$

$$\langle v, w\rangle_{T_\eta \mathcal{A}_k} = \int_{\mathcal{M}} v\cdot w \ \mathrm{vol}(\eta)\, d\mathcal{H}^k, \tag{20}$$

$$\langle v, w\rangle^*_{T_\eta \mathcal{A}_k} = \int_{\mathcal{M}} v\cdot w\, d\mathcal{H}^k. \tag{21}$$

The induced Riemannian metric on $\tilde{\mathcal{A}}_k$ (both from $\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle^*$) is then

$$\langle v, w\rangle_{T_\eta \tilde{\mathcal{A}}_k} = \langle v, w\rangle^*_{T_\eta \tilde{\mathcal{A}}_k} = \int_{\mathcal{M}} v\cdot w\, d\mathcal{H}^k. \tag{22}$$

By [10, Theorem 7], the orthogonal projection $\tilde{P}_\eta : T_\eta \mathcal{K}_k \to T_\eta \tilde{\mathcal{A}}_k$ (with respect to the invariant L^2 metric $\langle\cdot,\cdot\rangle_{T_\eta \mathcal{K}_k}$) is

$$\tilde{P}_\eta(z) = z - \sigma \vec{H}(\eta) - d\eta.\, \mathrm{grad}^g\, \sigma, \tag{23}$$

where $\vec{H}(\eta)$ is the vectorial mean curvature (the trace of the second fundamental form) corresponding to the embedding η, the metric $g = \eta^*\cdot$ is the pull-back of the Euclidean metric $\cdot$ on $\mathbb{R}^d$, and $\sigma : \mathcal{M} \to \mathbb{R}$ is a Lagrange multiplier, which is a solution to

$$\Delta_g \sigma - |\vec{H}(\eta)|^2 \sigma = \mathrm{div}_{\eta(\mathcal{M})}(z\circ\eta^{-1}).$$

Employing the methods of [10, 29], one can derive from (23) that the map $P_\eta : T_\eta \mathcal{K}_k \to T_\eta \mathcal{A}_k$,

$$\begin{aligned}P_\eta(z) &= z - \sigma\vec{H}(\eta) - d\eta.\, \mathrm{grad}^g\, \sigma, \ \text{where} \\ \Delta_g \sigma - |\vec{H}(\eta)|^2\sigma &= \mathrm{div}_{\eta(\mathcal{M})}(z\circ\eta^{-1}) + const, \quad \int_{\mathcal{M}} \sigma\, d\mathcal{H}^k = 0,\end{aligned} \tag{24}$$

is an orthogonal projection. The key observations in the proof of this claim are that the volume density for $\eta \in \mathcal{A}_k$ is constant (equal to $\mathrm{vol}(\eta)$) and the identity following from the divergence formula

$$\begin{aligned}&\int_{\mathcal{M}} w\cdot\left(\sigma\vec{H}(\eta) + d\eta.\, \mathrm{grad}^g\, \sigma\right)\mathrm{vol}(\eta)\, d\mathcal{H}^k \\ &= -\int_{\eta(\mathcal{M})} (\sigma\circ\eta^{-1})\, \mathrm{div}_{\eta(\mathcal{M})}(w\circ\eta^{-1})\, d\mathcal{H}^k \\ &= -const\int_{\mathcal{M}} \sigma\, \mathrm{vol}(\eta)\, d\mathcal{H}^k\end{aligned} \tag{25}$$

for any $w \in T_\eta \mathcal{A}_k$. Here in the last equality we have used $w \in T_\eta \mathcal{A}_k$ and the characterization of the tangent space $T_\eta \mathcal{A}_k$.

The UCMCF is the gradient flow

$$\partial_t \eta = -\operatorname{grad}_{\mathcal{A}_k} \operatorname{vol}(\eta) \tag{26}$$

of the volume functional on the space $\mathcal{A}_k$ under the metric (20). By construction, the flow operator

$$T_t : \eta(0,s) \mapsto \eta(t,s), \ s \in \mathcal{M},$$

complies with (4).

By the first variation of area formula, the negative $\mathcal{K}_k$-gradient of the volume functional is simply $\vec{H}$, and

$$\langle \vec{H}(\eta), \eta \rangle_{T_\eta \mathcal{K}_k} = -k \operatorname{vol}(\eta). \tag{27}$$

With the projection (24) at hand, by an argument similar to the one from the Section 2.1, we can express the gradient flow (26) in the form

$$\partial_t \eta = \tilde{\sigma} \vec{H}(\eta) + d\eta . \operatorname{grad}^g \tilde{\sigma}, \quad \int_{\mathcal{M}} \tilde{\sigma} \, d\mathcal{H}^k = 1, \quad \eta \in \mathcal{A}_k. \tag{28}$$

A direct computation yields that if a pair $(\eta(t), \tilde{\sigma}(t))$ solves (28), and $r : \mathcal{M} \to \mathcal{M}$ is a volume-preserving diffeomorphism, then $(\eta(t) \circ r, \tilde{\sigma}(t) \circ r)$ also solves (28). Note that the reparametrizations r which do not preserve the Hausdorff measure on $\mathcal{M}$ are ruled out automatically by our construction. Thus UCMCF is a truly geometric flow since it does not depend on possible reparametrizations of the evolving submanifold $\eta(t)(\mathcal{M})$. This claim will become completely transparent after we recast our flow into a parametrization-free form (44).

Let

$$M(\eta) := \frac{1}{2} \int_{\mathcal{M}} |\eta|^2 \, d\mathcal{H}^k$$

be the L^2-mass functional. We are going to see that this functional decays with a constant speed along our gradient flow, cf. Proposition 2.6. Indeed,

The exponential decay of $t \mapsto \|\partial_s \tilde{\xi}(t,\cdot)\|_{L^2}$ immediately yields the convergence of the multiplicative factor $c(t)$ due to (67).

Corollary 3.17. *Under the same assumptions as in Lemma 3.16 we have*

$$|c(t)^2 - 1| \leq e^{-\frac{1}{8}t} \|\partial_s \tilde{\xi}(0,\cdot)\|_{L^2}^2 \text{ for all } t \in (0,T].$$

In order to show the global existence we need to control the norm $\|\xi(t,s)\|_{2+\alpha,0}$ along the flow. For this it suffices to find a pointwise upper and lower bound on the ellipticity coefficient $\sigma(t,s)$. The next lemma says that if $\frac{1}{4\pi^2} - \bar{\sigma}$, $\|\tilde{\xi}\|_{L^2}$ and $\|\partial_s \tilde{\xi}\|_{L^2}$ are sufficiently small at $t = 0$, then the oscillation $\sigma(t,s) - \bar{\sigma}(t)$ and $|\xi|^2(t,s) - \int_0^1 |\xi|^2(t,s) ds$ are under control along the flow.

Lemma 3.18. *Let $\xi \in C^{2+\alpha,\alpha/2}([0,T] \times \mathcal{S}^1)$, $\partial_t \xi \in C^{\alpha,\alpha/2}([0,T] \times \mathcal{S}^1)$ be a solution to the normalized flow (30)–(31) for some $T > 0$. Then for any $0 < \epsilon \leq \frac{1}{(32\pi^2)^2}$, if*

$$\left(\frac{1}{4\pi^2} - \bar{\sigma}(0) \right) + \left\| \tilde{\xi}(0,\cdot) \right\|^2_{L^2(\mathcal{S}^1)} + \left\| \partial_s \tilde{\xi}(0,\cdot) \right\|^2_{L^2(\mathcal{S}^1)} \leq \epsilon, \tag{76}$$

then for all $(t,s) \in [0,T] \times \mathcal{S}^1$ we have

$$\left| |\xi(t,s)|^2 - \int_0^1 |\xi(t,s)|^2 ds \right| \leq 2\sqrt{\epsilon}, \quad |\sigma(t,s) - \bar{\sigma}(t)| \leq 3\sqrt{\epsilon}.$$

In particular, σ satisfies

$$\frac{1}{4\pi^2} - 4\sqrt{\epsilon} \leq \sigma(t,s) \leq \frac{1}{4\pi^2} + 3\sqrt{\epsilon} \text{ for all } (t,s) \in [0,T] \times \mathcal{S}^1.$$

Proof: 1. We observe that by the monotonicity properties from Proposition 3.1, Lemma 3.15 and Lemma 3.16, the smallness assumption (76) holds for all $t \in [0,T]$. We have used that (76) implies (73).

2. We show that

$$\int_0^1 |\partial_t \xi(t,s)|^2 ds = \int_0^1 |\xi(t,s)|^2 ds - \bar{\sigma}(t) \leq \epsilon \text{ for all } t \in (0,T). \tag{77}$$

Then $t \mapsto \|\partial_s \tilde{\xi}(t,\cdot)\|_{L^2(\mathcal{S}^1)}$ *is monotone decreasing. Moreover,*

$$\|\partial_s \tilde{\xi}(t,\cdot)\|_{L^2} \le e^{-\frac{1}{16}t}\|\partial_s \tilde{\xi}(0,\cdot)\|_{L^2}, \quad t \in (0,T).$$

Proof: By (67) and the expression of $\dot{c}$ in (65),

$$\frac{1}{2}\partial_t \int_0^1 |\partial_s \tilde{\xi}|^2 = -c(t)\dot{c}(t) = -\left(1 - 4\pi^2\bar{\sigma}\right)c^2 + 4\pi^2 c \int_0^1 \sigma \partial_s w_0 \cdot \partial_s \tilde{\xi}.$$

Applying (70) to the last term in the above equation we obtain

$$\begin{aligned}\frac{1}{2}\partial_t \int_0^1 |\partial_s \tilde{\xi}|^2 &= -\left(1 - 4\pi^2\bar{\sigma}\right)c^2 + 4\pi^2\left(\frac{1-c^2}{2}\bar{\sigma} - \frac{1}{2}\int_0^1 \sigma|\partial_s \tilde{\xi}|^2\right)\\ &\le -\left(1 - 4\pi^2\bar{\sigma}\right)c^2 + 4\pi^2\frac{1-c^2}{2}\bar{\sigma}.\end{aligned}$$

Next we note that

$$4\pi^2\bar{\sigma} \le 1 - \frac{3}{4}\int_0^1 |\partial_s \tilde{\xi}|^2. \tag{74}$$

Indeed, by (ii) of Proposition 3.1 $\bar{\sigma} \le \int_0^1 |\xi|^2$, which in terms of $\tilde{\xi}$ and c reads $\bar{\sigma} \le \int_0^1 |\tilde{\xi}|^2 + \frac{c^2}{4\pi^2}$. Applying (72) to $\tilde{\xi}$ we have $\bar{\sigma} \le \frac{1}{16\pi^2}\int_0^1 |\partial_s \tilde{\xi}|^2 + \frac{c^2}{4\pi^2}$. Using (67) we obtain (74).
With (74) at hand

$$\begin{aligned}\frac{1}{2}\partial_t \int_0^1 |\partial_s \tilde{\xi}|^2 \le -c^2 &+ \left(1 - \frac{3}{4}\int_0^1 |\partial_s \tilde{\xi}|^2\right)c^2\\ &+ \left(1 - \frac{3}{4}\int_0^1 |\partial_s \tilde{\xi}|^2\right)\frac{1-c^2}{2}.\end{aligned}$$

Applying (67) again to the right hand side of the above inequality, after some algebraic manipulations we arrive at

$$\frac{1}{2}\partial_t \int_0^1 |\partial_s \tilde{\xi}|^2 ds \le \int_0^1 |\partial_s \tilde{\xi}|^2 ds \left(-\frac{1}{4} + \frac{3}{8}\int_0^1 |\partial_s \tilde{\xi}|^2 ds\right). \tag{75}$$

Thus if $\int_0^1 |\partial_s \tilde{\xi}(0,s)| ds = 1 - c(0)^2 \le \frac{1}{2}$ by (73), then (75) implies that $\int_0^1 |\partial_s \tilde{\xi}(0,s)| ds \le \frac{1}{2}$ for all $t \in (0,T]$. Moreover,

$$\frac{1}{2}\partial_t \int_0^1 |\partial_s \tilde{\xi}|^2 ds \le -\frac{1}{16}\int_0^1 |\partial_s \tilde{\xi}|^2 ds$$

for all $t \in (0,T]$. Solving the above differential inequality we complete the proof. ∎

since $\eta \in T_\eta \mathcal{A}_k$ (since $\operatorname{div}_{\eta(\mathcal{M})}(\eta \circ \eta^{-1}) = k$),

$$\begin{aligned}\partial_t M(\eta) &= \int_{\mathcal{M}} \eta \cdot \partial_t \eta \, d\mathcal{H}^k = \frac{1}{\operatorname{vol}(\eta)}\langle \eta, \partial_t \eta\rangle_{T_\eta \mathcal{K}_k} = \frac{1}{\operatorname{vol}(\eta)}\langle \eta, \partial_t \eta\rangle_{T_\eta \mathcal{A}_k}\\ &= -\frac{1}{\operatorname{vol}(\eta)}\langle \eta, \operatorname{grad}_{\mathcal{A}_k} \operatorname{vol}(\eta)\rangle_{T_\eta \mathcal{A}_k}\\ &= -\frac{1}{\operatorname{vol}(\eta)}\langle \eta, \operatorname{grad}_{\mathcal{K}_k} \operatorname{vol}(\eta)\rangle_{T_\eta \mathcal{K}_k}\\ &= \frac{1}{\operatorname{vol}(\eta)}\langle \eta, \vec{H}(\eta)\rangle_{T_\eta \mathcal{K}_k} = -k\end{aligned}$$

by (27). Thus, our flow collapses in finite time $t_* = \frac{1}{k}M(\eta_0)$.

Remark 2.9. The Riemannian distances $d_{\mathcal{A}_k^*}$ and $d_{\tilde{\mathcal{A}}_k}$ on submanifolds of $\mathcal{K}_k$ are non-degenerate since they are controlled from below by the Hilbertian distance $d_{\mathcal{K}_k^*}$ (which is induced by the Riemannian metric $\langle\cdot,\cdot\rangle^*$ in (21)). We do not know whether the Riemannian distance $d_{\mathcal{A}_k}$ is non-degenerate in general, but as we observed in Proposition 2.1, the conjecture is true for $k = 1$. Nevertheless, we have got backup ways to render UCMCF as a gradient flow with respect to a non-degenerate distance. Namely, the gradient flow

$$\partial_t \eta = -\operatorname{grad}_{\mathcal{A}_k^*}(\ln \operatorname{vol}(\eta)) \tag{29}$$

reproduces (26). This is immediate by observing that

$$\operatorname{grad}_{\mathcal{A}_k} \operatorname{vol}(\eta) = \frac{1}{\operatorname{vol}(\eta)}\operatorname{grad}_{\mathcal{A}_k^*} \operatorname{vol}(\eta) = \operatorname{grad}_{\mathcal{A}_k^*}(\ln \operatorname{vol}(\eta)).$$

Other options are presented in Remarks 3.2, 3.4.

3. Normalized flow

3.1. Renormalization. One can ask as the L^2-mass $M(t)$ goes to zero, whether the curve becomes circular. To study this problem we plan to show that the isoperimetric ratio $\frac{2M(t)}{L^2(t)}$ goes to the optimal constant in the Wirtinger's inequality $\frac{1}{4\pi^2}$ as $t \to t^*$. For that and for many other purposes it is convenient to renormalize the flow.

We first introduce a slow time variable. More precisely, for $t \in [0, t^*)$ let

$$\tau(t) := -\ln L(t).$$

Note that by Proposition 2.5(i) and Corollary 2.8, $\tau(t)$ is monotone increasing in t and $\tau \to +\infty$ iff $t \to t^*$; this is also clear in view of (29). Next we consider the normalization

$$\xi(\tau, s) := \frac{\eta(t(\tau), s)}{L(t(\tau))}$$

One advantage of using such renormalization is that the curve $\xi(\tau, s)$ has the unit speed parametrization, i.e.,

$$L(\tau) = |\partial_s \xi(\tau, s)| = 1 \text{ for all } (\tau, s) \in [0, \infty) \times \mathcal{S}^1.$$

A direct computation shows that ξ satisfies the equation

$$\partial_\tau \xi = \partial_s(\sigma \partial_s \xi) + \xi, \tag{30}$$

where $\sigma(\tau, s) = \frac{\tilde{\sigma}(t(\tau), s)}{-(L\partial_t L)(t(\tau))}$ satisfies

$$\partial_{ss}\sigma - \sigma|\partial_{ss}\xi|^2 = -1, \quad \sigma(\tau, 0) = \sigma(\tau, 1). \tag{31}$$

Indeed, from the change of variable $\frac{d\tau}{dt} = -\frac{\partial_t L(t)}{L(t)}$. Thus

$$\partial_\tau \xi = \frac{\partial_t \eta}{L}\frac{dt}{d\tau} - \eta\frac{\partial_t L}{L^2}\frac{dt}{d\tau} = -\frac{\partial_t \eta}{\partial_t L} + \frac{\eta}{L}.$$

Using the equation of η we have

$$\partial_\tau \xi = -\frac{\partial_s(\tilde{\sigma}\partial_s \eta)}{L^2 \partial_t L} + \frac{\eta}{L}.$$

Letting $\sigma := \frac{\tilde{\sigma}}{-L\partial_t L}$ and writing the above equation in terms of ξ we arrive at (30). To derive (31) we can either use the equation of $\tilde{\sigma}$, or use the above equation of ξ together with the constraint $|\partial_s \xi| \equiv 1$.

With the same argument as in Section 2 it is not hard to see that the normalized flow (30) and (31) can be viewed as the positive gradient flow of the L^2-mass $M(\xi) := \frac{1}{2}\int_0^1 |\xi(s)|^2$ on the manifold of immersed curves with arc-length parametrization, cf. (19),

$$\tilde{\mathcal{A}} := \{\xi \in H^2(\mathcal{S}^1; \mathbb{R}^d) : \ |\partial_s \xi(s)| = 1 \text{ for all } s \in \mathcal{S}^1, \ \int_0^1 \xi(s) ds = 0\}$$

with respect to the $L^2(\mathcal{S}^1; \mathbb{R}^d)$-induced metric (see also Remark 3.2 and our recent work [37] which analyzes the gradient flow of the potential energy on a space very similar to $\tilde{\mathcal{A}}$).

Thus (65) can be rewritten as

$$\begin{aligned}\frac{1}{2}\partial_t \int_0^1 |\tilde{\xi}|^2 &= \int_0^1 |\tilde{\xi}|^2 - \int_0^1 \sigma|\partial_s \tilde{\xi}|^2 - \frac{1-c^2}{2}\bar{\sigma} + \frac{1}{2}\int_0^1 \sigma|\partial_s \tilde{\xi}|^2 ds \\ &= \int_0^1 |\tilde{\xi}|^2 - \frac{1}{2}\int_0^1 \sigma|\partial_s \tilde{\xi}|^2 - \frac{1-c^2}{2}\bar{\sigma}.\end{aligned}$$

Applying (67) to the last term of the above equation we obtain

$$\begin{aligned}\frac{1}{2}\partial_t \int_0^1 |\tilde{\xi}|^2 &= \int_0^1 |\tilde{\xi}|^2 - \frac{1}{2}\int_0^1 \sigma|\partial_s \tilde{\xi}|^2 - \frac{1}{2}\bar{\sigma}\int_0^1 |\partial_s \tilde{\xi}|^2 ds \\ &\le \int_0^1 |\tilde{\xi}|^2 - \frac{1}{2}\bar{\sigma}\int_0^1 |\partial_s \tilde{\xi}|^2 ds.\end{aligned} \tag{71}$$

We claim that (62) implies

$$\int_0^1 |\tilde{\xi}(t, \cdot)|^2 ds \le \frac{1}{16\pi^2}\int_0^1 |\partial_s \tilde{\xi}(t, \cdot)|^2 ds. \tag{72}$$

To see this, one can employ Fourier expansion of $\tilde{\xi}(t, \cdot)$. By $\int_0^1 \tilde{\xi} = 0$ as well as the orthogonality condition (63), one finds that the zero and first order Fourier coefficients of $\tilde{\xi}$ are zero.

Applying (72) to (75) we obtain

$$\frac{1}{2}\partial_t \int_0^1 |\tilde{\xi}|^2 \le -\left(8\pi^2 \bar{\sigma}(t) - 1\right)\int_0^1 |\tilde{\xi}|^2.$$

Since $t \mapsto \bar{\sigma}(t)$ is monotone increasing, cf. Proposition 3.1 (ii), (68) implies $8\pi^2 \int_0^1 \sigma ds - 1 \ge \frac{1}{3}$ for all $t \in (0, T]$. Thus

$$\partial_t \int_0^1 |\tilde{\xi}|^2 \le -\frac{2}{3}\int_0^1 |\tilde{\xi}|^2, \quad t \in [0, T].$$

This implies that $t \mapsto \|\tilde{\xi}(t, \cdot)\|_{L^2(\mathcal{S}^1)}$ is monotone decreasing. An integration in t results in (69), which completes the proof. ∎

The next lemma concerns the decay estimate of $t \mapsto \|\partial_s \tilde{\xi}(t, \cdot)\|_{L^2}^2$.

Lemma 3.16. *Assume that $\xi, \tilde{\xi}$ and $\sigma(0, \cdot)$ satisfy the same assumptions as in Lemma 3.15. Assume furthermore that*

$$c(0)^2 \ge \frac{1}{2}. \tag{73}$$

solution and suppose $c(0) \neq 0$, then by the Picard-Lindelöf theorem there exists a unique solution $(c(t), \theta(t))$ to the ODE (65) with the initial value $(c(0), \theta(0))$ on the interval $[0, t_0)$ for some $t_0 > 0$. Furthermore, it is not hard to check that the orthogonality conditions (63) are preserved along the flow. The condition $c(0) \neq 0$ holds if, for example, the classical solution ξ satisfies $\|\xi(0,\cdot) - w_0(\cdot)\|_{L^2(\mathcal{S}^1)} \leq \frac{1}{4}\frac{1}{4\pi^2}$. We will show later that under further lower bound assumptions on $\int_{\mathcal{S}^1} \sigma$ and c at $t = 0$ (cf. (68) and (73)), the parameter $c(t)$ remains uniformly bounded away from zero for all $t \in (0, t_0)$ (cf. Lemma 3.15–Corollary 3.17). This together with the monotonicity of $\int_{\mathcal{S}^1} \sigma(t,\cdot)$ (cf. Proposition 3.1) implies that the solution to the ODE exists up to time T. Thus in the sequel we can and will directly assume that $t_0 = T$.

In the next two lemmas we derive the decay estimates for $\|\xi(t,\cdot)\|_{L^2}$, $\|\partial_s\xi(t,\cdot)\|_{L^2}$ and $1 - c(t)$, under the assumption that at the initial time $\int_0^1 \sigma(0,s)ds \geq \frac{2}{3}\frac{1}{4\pi^2}$ and $c(0)^2 \geq \frac{1}{2}$. We stress that these assumptions are always satisfied if our solution ξ is sufficiently close to the stationary solution w_0 at $t = 0$ (cf. Theorem 2).

Lemma 3.15. *Let $\xi(t,s) \in C^{2+\alpha,\alpha/2}([0,T] \times \mathcal{S}^1)$, $\partial_t\xi(t,s) \in C^{\alpha,\alpha/2}((0,T] \times \mathcal{S}^1)$ be a solution to the normalized flow* (30)–(31). *Let $\bar{\sigma}(t) := \int_0^1 \sigma(t,s)ds$. Assume that at $t = 0$*

$$\bar{\sigma}(0) \geq \frac{2}{3}\frac{1}{4\pi^2}, \tag{68}$$

and decompose $\xi(0) = \tilde{\xi}(0) + c(0)w_0(\cdot + \theta(0))$ with $c(0) \neq 0$. Let $\tilde{\xi}(t,s) := \xi(t,s) - c(t)w_0(s + \theta(t))$, where $(c(t), \theta(t))$ are solutions to the ODE (65) *with the initial value $(c(0), \theta(0))$. Then $t \mapsto \|\tilde{\xi}(t,\cdot)\|_{L^2(\mathcal{S}^1)}$ is monotone decreasing in $(0,T]$. Moreover,*

$$\|\tilde{\xi}(t,\cdot)\|_{L^2(\mathcal{S}^1)} \leq e^{-\frac{1}{3}t}\|\tilde{\xi}(0,\cdot)\|_{L^2(\mathcal{S}^1)}, \quad t \in (0,T]. \tag{69}$$

Proof: We will derive a differential inequality on $\|\tilde{\xi}(t,\cdot)\|_{L^2(\mathcal{S}^1)}$ using its evolution equation from (65). First we multiply σ to the both sides of (66) and integrate over $\mathcal{S}^1$

$$(1 - c^2)\bar{\sigma} = \int_0^1 \sigma|\partial_s\tilde{\xi}|^2 ds + 2c\int_0^1 \sigma\partial_s\tilde{\xi} \cdot \partial_s w_0 ds. \tag{70}$$

The normalized flow (30)–(31) can be interpreted in the spirit of [37] as an overdamped motion of an inextensible loop whose particles are repelled from the origin with the force equal to the radius vector.

Using either the gradient flow structure for the normalized flow or tracing the change of variables and normalization, one obtains the monotonic quantities along the flow.

Proposition 3.1. *Let ξ be a solution to the normalized flow* (30)–(31). *Then*

(i) *$\tau \mapsto \int_0^1 |\xi(\tau,s)|^2 ds$ is monotone increasing. Moreover, $\int_0^1 |\xi(\tau,s)|^2 ds \leq \frac{1}{4\pi^2}$ for all τ.*

(ii) *$\tau \mapsto \int_0^1 \sigma(\tau,s)ds$ is monotone increasing with $\int_0^1 \sigma(\tau,s)ds \leq \int_0^1 |\xi(\tau,s)|^2 ds$ for all τ.*

(iii) *As $\tau \to \infty$ we have $\int_0^1 \sigma(\infty,s)ds = \int_0^1 |\xi(\infty,s)|^2 ds$.*

Proof: 1. We multiply (30) by $\partial_\tau\xi$ and integrate in s from 0 to 1. After an integration by part and using $\partial_s\xi \cdot \partial_{s\tau}\xi = 0$ (which follows from $|\partial_s\xi| = 1$) we obtain that

$$\int_0^1 |\partial_\tau\xi|^2 ds = \partial_\tau\left(\frac{1}{2}\int_0^1 |\xi|^2 ds\right). \tag{32}$$

Thus $\tau \to \int_0^1 |\xi|^2$ is monotone increasing. The upper bound follows from the Wirtinger's inequality

$$\int_0^1 |\xi|^2 \, ds \leq \frac{1}{4\pi^2}\int_0^1 |\partial_s\xi|^2 ds = \frac{1}{4\pi^2}.$$

Again we have used that $\int_0^1 \xi(\tau,s)ds = 0$.

2. From the definition of σ we have

$$\int_0^1 \sigma(\tau,s)ds = \frac{\int_0^1 \tilde{\sigma}ds}{-L\partial_t L} = \frac{1}{-L\partial_t L(t(\tau))}.$$

By (ii) of Proposition 2.5, $-L\partial_t L$ is monotone decreasing in t thus in τ. Thus $\int_0^1 \sigma(\tau,s)ds$ is monotone increasing in τ. To prove the upper bound, we multiply (30) by ξ and integrate in s. Integrating by parts and using that $|\partial_s\xi| = 1$ we have

$$\frac{1}{2}\partial_t\int_0^1 |\xi|^2 ds = \int_0^1 |\xi|^2 ds - \int_0^1 \sigma ds. \tag{33}$$

By (i) the left hand side is larger than or equal to zero. Thus the upper bound follows.

3. By (i) and (ii), $\lim_{\tau\to\infty}\int_0^1|\xi(\tau,s)|^2ds$ and $\lim_{\tau\to\infty}\int_0^1\sigma(\tau,s)ds$ exist and in the limit $\int_0^1|\xi(\infty,s)|^2ds-\int_0^1\sigma(\infty,s)ds\geq 0$. To see the limit is actually zero, we argue by contradiction. Suppose not, then there exist $\epsilon>0$ and $M_\epsilon>0$ such that $\int_0^1|\xi(\tau,s)|^2ds-\int_0^1\sigma(\tau,s)ds\geq\epsilon$ for all $\tau\geq M_\epsilon$. By (33) for any $\tau_1>\tau_2\geq M_\epsilon$

$$\int_0^1|\xi(\tau_1,s)|^2ds-\int_0^1|\xi(\tau_2,s)|^2ds\geq 2\epsilon(\tau_1-\tau_2).$$

By (i) the left hand side is bounded from above by $\frac{1}{4\pi^2}$. However, the right hand side goes to infinity as $\tau_1\to\infty$, which is a contradiction. Thus we have shown that the limit is zero. ∎

Remark 3.2. The general higher dimensional UCMCF (28) can be renormalized in the same way. Namely, the new time is

$$\tau(t):=-\ln\mathrm{vol}(\eta(t)),$$

and the new unknown functions are

$$\xi(\tau,s):=\frac{\eta(t(\tau),s)}{\mathrm{vol}(\eta(t(\tau)))},\ \sigma(\tau,s)=\frac{\tilde\sigma(t(\tau),s)}{-(\mathrm{vol}\,\partial_t\,\mathrm{vol})(\eta(t(\tau)))}.$$

Then the pair (ξ,σ) solves the equation

$$\partial_\tau\xi=\xi+\sigma\vec H(\xi)+d\xi.\,\mathrm{grad}^{\xi^*\cdot}\sigma,\quad \xi\in\tilde{\mathcal{A}}_k. \tag{34}$$

Employing the characterization of the orthogonal projection $\tilde P_\xi:T_\xi\mathcal{K}_k\to T_\xi\tilde{\mathcal{A}}_k$, we immediately rewrite (34) as a positive gradient flow of the L^2-mass:

$$\partial_\tau\xi=\tilde P_\xi\xi=\tilde P_\xi\,\mathrm{grad}_{\mathcal{K}_k}M(\xi)=\mathrm{grad}_{\tilde{\mathcal{A}}_k}M(\xi). \tag{35}$$

Remark 3.3. In order to illustrate the power of the gradient flow structure (35), we will formally derive a neat formula for the evolution of the mean of σ along the UCMCF trajectories, thereby generalizing (ii) in Proposition 3.1. We first observe that the geodesics in $\tilde{\mathcal{A}}_k$ are determined by the condition $\partial_{\tau\tau}\gamma\perp T_\gamma\tilde{\mathcal{A}}_k$, which can be expressed as

$$\partial_{\tau\tau}\gamma=\varsigma\vec H(\gamma)+d\gamma.\,\mathrm{grad}^{\gamma^*\cdot}\varsigma,\quad \gamma\in\tilde{\mathcal{A}}_k, \tag{36}$$

3.14). By using the equation of ξ we obtain the equation of $\tilde\xi$

$$\begin{aligned}\partial_t\tilde\xi(t,s)=&\,\tilde\xi(t,s)+\partial_s(\sigma(t,s)\partial_s\tilde\xi(t,s))+\\&+c(t)\partial_s\left((\sigma(t,s)-\frac{1}{4\pi^2})\partial_sw_0(s+\theta(t))\right)\\&-\dot c(t)w_0(s+\theta(t))-c(t)\dot\theta(t)\partial_sw_0(s+\theta(t)),\end{aligned} \tag{64}$$

where we have used the relation $w_0+\frac{1}{4\pi^2}\partial_{ss}w_0=0$. We multiply (64) by $\tilde\xi(t,\cdot)$, $w_0(\cdot+\theta(t))$ and $\partial_sw_0(\cdot+\theta(t))$ and integrate over $\mathcal{S}^1$. Using the orthogonality condition (63) we quantify the evolution of $\|\xi(t,\cdot)\|_{L^2}^2$ as well as of the parameters $c(t)$ and $\theta(t)$ for $t<T$:

$$\begin{aligned}\frac12\partial_t\int_0^1|\tilde\xi|^2ds=&\int_0^1|\tilde\xi|^2-\int_0^1\sigma|\partial_s\tilde\xi|^2ds\\&-c(t)\int_0^1\sigma\partial_sw_0(\cdot+\theta(t))\cdot\partial_s\tilde\xi ds,\\\dot c(t)=&\left(1-4\pi^2\int_0^1\sigma ds\right)c(t)\\&-4\pi^2\int_0^1\sigma\partial_sw_0(\cdot+\theta(t))\cdot\partial_s\tilde\xi ds,\\c(t)\dot\theta(t)=&\,4\pi^2\int_0^1\sigma w_0(\cdot+\theta(t))\cdot\partial_s\tilde\xi ds.\end{aligned} \tag{65}$$

From the unit speed constraint $|\partial_s\xi(t,s)|=|\partial_sw_0(s+\theta(t))|=1$ for all $(t,s)\in[0,T]\times\mathcal{S}^1$, we obtain

$$|\partial_s\tilde\xi(t,s)|^2+2c(t)\partial_s\tilde\xi(t,s)\cdot\partial_sw_0(s+\theta(t))+c(t)^2=1. \tag{66}$$

Integrating over $\mathcal{S}^1$ and using (63) (note that $\int_0^1\partial_s\tilde\xi\cdot\partial_sw_0ds=-\int_0^1\tilde\xi\cdot\partial_{ss}w_0=0$ since $\partial_{ss}w_0=-4\pi^2w_0$) yields

$$\int_0^1|\partial_s\tilde\xi|^2ds=1-c(t)^2. \tag{67}$$

Here and in the sequel for brevity we write w_0 and ∂_sw_0 instead of $w_0(s+\theta(t))$ and $\partial_sw_0(s+\theta(t))$, respectively.

Remark 3.14 (Differentibility of the parameters $c(t)$ and $\theta(t)$)**.** Suppose $\xi(t,s)\in C^{2+\alpha,\alpha/2}([0,T]\times\mathcal{S}^1)$, $\partial_t\xi(t,s)\in C^{\alpha,\alpha/2}((0,T]\times\mathcal{S}^1)$ is a classical

3.4. Global well-posedness and exponential stability. In this section we study the global well-posedness of the normalized flow (30)–(31) under the assumption that initially the curve is C^2 close to the circle

$$w_0(s) := \frac{1}{2\pi}(\cos(2\pi s), \sin(2\pi s), 0, \cdots, 0). \tag{61}$$

We will mainly show the uniform (in time) boundedness of curvature $|\partial_{ss}\xi|^2$, which yields that the time T in the local well-posedness result has a uniform lower bound. The main idea of the proof is to show that under some smallness assumption at the initial time, the parabolicity is preserved along the flow, i.e., $\sigma > c_0$ pointwise for any $t < T$, where $c_0 > 0$ is some absolute constant.

The proof of the result is based on a dynamical system approach. We let

$$\mathcal{C} := \{cw_0(s+\theta) : c, \theta \in \mathbb{R}\}$$

denote the manifold generated by w_0 which is invariant under the dilation and rotation. For each $t \in (0, T)$ we decompose

$$\xi(t,s) = \tilde{\xi}(t,s) + c(t)w_0(s+\theta(t)), \tag{62}$$

where $c(t)w_0(\cdot+\theta(t))$ is the L^2 projection of $\xi(t)$ onto $\mathcal{C}$, i.e., for each t fixed

$$c(t)w_0(\cdot+\theta(t)) \in \operatorname{argmin}_{w\in\mathcal{C}} \left\{ \int_{\mathcal{S}^1} |\xi(t,\cdot) - w(\cdot)|^2 d\mathcal{H}^1 \right\}.$$

We remark that minimum are achieved by considering the minimization problem over the finite dimensional parameter space

$$\inf_{c,\theta\in\mathbb{R}} \mathcal{F}_\xi(c,\theta), \quad \mathcal{F}_\xi(c,\theta) := \int_{\mathcal{S}^1} |\xi(\cdot) - cw_0(\cdot+\theta)|^2 d\mathcal{H}^1.$$

The first derivatives $\partial\mathcal{F}_\xi(c,\theta)/\partial c$ and $\partial\mathcal{F}_\xi(c,\theta)/\partial\theta$ vanish at the minimizers $(c(t),\theta(t))$, yielding the following orthogonality conditions

$$\int_0^1 \tilde{\xi}(t,s)\cdot w_0(s+\theta(t))ds = \int_0^1 \tilde{\xi}(t,s)\cdot c(t)\partial_s w_0(s+\theta(t))ds = 0. \tag{63}$$

Now we derive the evolution of the parameters $c(t)$ and $\theta(t)$ (assume for now they are differentiable. For more detailed discussion we refer to Remark

cf. [10, 29]. Then we can calculate the Hessian of the L^2-mass, taking into account (25) (with $w = \gamma$ and $\operatorname{div}_{\gamma(\mathcal{M})}(\gamma\circ\gamma^{-1}) = k$):

$$\begin{aligned}\langle Hess\, M(\gamma)\dot\gamma,\dot\gamma\rangle_{T_\gamma\tilde{\mathcal{A}}_k} = \frac{d^2M(\gamma(t))}{d^2t} &= \int_{\mathcal{M}} \partial_{\tau\tau}\gamma\cdot\gamma + \partial_\tau\gamma\cdot\partial_\tau\gamma\, d\mathcal{H}^k \\ &= -k\int_{\mathcal{M}} \varsigma\, d\mathcal{H}^k + \langle\dot\gamma,\dot\gamma\rangle_{T_\gamma\tilde{\mathcal{A}}_k}.\end{aligned} \tag{37}$$

Now we compute in two different ways the second time derivative of the L^2-mass along a trajectory $\xi(t)$ of the gradient flow (35). On one hand,

$$\begin{aligned}\frac{d^2M(\xi)}{d^2t} &= \frac{d}{dt}\langle \operatorname{grad}_{\tilde{\mathcal{A}}_k} M(\xi), \operatorname{grad}_{\tilde{\mathcal{A}}_k} M(\xi)\rangle_{T_\xi\tilde{\mathcal{A}}_k} \\ &= 2\langle Hess\, M(\xi)\dot\xi,\dot\xi\rangle_{T_\xi\tilde{\mathcal{A}}_k} \\ &= -2k\int_{\mathcal{M}} \varsigma\, d\mathcal{H}^k + 2\langle\dot\xi,\dot\xi\rangle_{T_\xi\tilde{\mathcal{A}}_k} \\ &= -2k\int_{\mathcal{M}} \varsigma\, d\mathcal{H}^k + 2\langle \operatorname{grad}_{\tilde{\mathcal{A}}_k} M(\xi), \operatorname{grad}_{\tilde{\mathcal{A}}_k} M(\xi)\rangle_{T_\xi\tilde{\mathcal{A}}_k} \\ &= -2k\int_{\mathcal{M}} \varsigma\, d\mathcal{H}^k + 2\frac{dM(\xi)}{dt},\end{aligned} \tag{38}$$

where $\varsigma(t)$ is the Lagrange multiplier (which may be referred to as the tension) corresponding to the geodesic passing through $\xi(t)$ at the direction $\dot\xi(t)$, see (36). On the other hand, employing (25) and orthogonality of the projection $\tilde{P}_\xi$, we find that

$$\begin{aligned}&\langle \operatorname{grad}_{\tilde{\mathcal{A}}_k} M(\xi), \operatorname{grad}_{\tilde{\mathcal{A}}_k} M(\xi)\rangle_{T_\xi\tilde{\mathcal{A}}_k} = \langle \tilde{P}_\xi\xi, \tilde{P}_\xi\xi\rangle_{L^2(\mathcal{M})} \\ &= \langle\xi,\xi\rangle_{L^2(\mathcal{M})} + \langle\xi, \tilde{P}_\xi\xi - \xi\rangle_{L^2(\mathcal{M})} \\ &= \langle\xi,\xi\rangle_{L^2(\mathcal{M})} + \left\langle \xi, \sigma\vec{H}(\xi) + d\xi.\operatorname{grad}^{\xi^*\cdot}\sigma \right\rangle_{L^2(\mathcal{M})} \\ &= 2M(\xi) - k\int_{\mathcal{M}} \sigma\, d\mathcal{H}^k.\end{aligned} \tag{39}$$

This yields the upper bound

$$k\int_{\mathcal{M}} \sigma\, d\mathcal{H}^k \le 2M(\xi). \tag{40}$$

Differentiating (39) in time and comparing with (38), we deduce

$$\frac{d}{dt}\int_{\mathcal{M}} \sigma \, d\mathcal{H}^k = 2\int_{\mathcal{M}} \varsigma \, d\mathcal{H}^k. \tag{41}$$

In the particular case $\mathcal{M} = \mathcal{S}^1$, it is known [33, 36] that the Lagrange multipliers ς related to the geodesics are always non-negative, so we infer (ii) in Proposition 3.1.

Remark 3.4. We now establish a link with the optimal transport theory [40, 41] by explaining how our flow (35) may be formally viewed as a gradient flow on a submanifold of the Wasserstein space. We recall [31, 40] that the space $\mathcal{P}_2(\mathbb{R}^d)$ of probability measures with finite second moments admits a formal Riemannian structure so that the 2-Wasserstein distance coincides with the geodesic distance. The mapping

$$\Pi : \tilde{\mathcal{A}}_k \to \mathcal{P}_2(\mathbb{R}^d); \quad \Pi(\xi) = \xi_{\#}(\mathcal{H}^k \llcorner \mathcal{M}) = \mathcal{H}^k \llcorner \xi(\mathcal{M})$$

is a restriction of Otto's Riemannian submersion [31]. We refer to [21] for a basic introduction to Riemannian submersions. Actually, Π is a bijection up to volume-preserving diffeomorphisms on $\mathcal{M}$. Set $\rho = \Pi(\xi)$, where $\xi = \xi(\tau)$ is a solution to the normalized UCMCF. Since the L^2-mass functional M is invariant with respect to volume-preserving changes of variables on $\mathcal{M}$, it is consistent to define $M(\rho) := M(\xi)$. Set $\mathcal{B} = \Pi(\tilde{\mathcal{A}}_k)$. Then $\mathcal{B}$ may be formally viewed as a submanifold of $\mathcal{P}_2(\mathbb{R}^d)$. We claim that the evolution of ρ satisfies

$$\partial_\tau \rho = \operatorname{grad}_{\mathcal{B}} M(\rho). \tag{42}$$

Indeed, fix time τ_0, $\xi = \xi(\tau_0)$, and $\rho = \Pi(\xi(\tau_0))$. We need to show that

$$\langle \partial_\tau \rho, v\rangle_{T_\rho \mathcal{P}_2} = \langle \operatorname{grad}_{\mathcal{B}} M(\rho), v\rangle_{T_\rho \mathcal{P}_2} \tag{43}$$

for an arbitrary $v \in T_\rho \mathcal{B}$. Let $\tilde{\rho}(\tau)$ be a curve in $\mathcal{B}$ satisfying $\tilde{\rho}(\tau_0) = \rho$, $\partial_\tau \tilde{\rho}(\tau_0) = v$. Let $\tilde{\xi}$ be the horizontal lift of the curve $\tilde{\rho}$ with respect to the submersion Π passing through ξ at τ_0. Denote by $\partial_\tau \xi_h$ the horizontal component of $\partial_\tau \xi$. Then at time τ_0

$$\begin{aligned}
&\langle \partial_\tau \rho, v\rangle_\rho = \langle \partial_\tau \Pi(\xi), \partial_\tau \Pi(\tilde{\xi})\rangle_\rho = \langle d\Pi(\xi)\cdot \partial_\tau \xi, d\Pi(\xi)\cdot \partial_\tau \tilde{\xi}\rangle_\rho \\
&= \langle \partial_\tau \xi_h, \partial_\tau \tilde{\xi}\rangle_\xi = \langle \partial_\tau \xi, \partial_\tau \tilde{\xi}\rangle_\xi = \langle \operatorname{grad}_{\tilde{\mathcal{A}}_k} M(\xi), \partial_\tau \tilde{\xi}\rangle_\xi \\
&= \partial_\tau M(\tilde{\xi}) = \partial_\tau M(\tilde{\rho}) = \langle \operatorname{grad}_{\mathcal{B}} M(\rho), \partial_\tau \tilde{\rho}\rangle_\rho = \langle \operatorname{grad}_{\mathcal{B}} M(\rho), v\rangle_\rho.
\end{aligned}$$

Note that the geodesic distance on $\mathcal{B}$ is a priori non-degenerate since it is controlled from below by the 2-Wasserstein distance. The strategy above is

Here we have used that $C_P = \frac{|I|^2}{4\pi^2}$ is the optimal Poincaré constant with respect to the interval I. Combining (59) and (60), we arrive at

$$\begin{aligned}
&|I|^{-1/2}\|\sigma\|_{L^2(I)}\left(\|\sigma\|^2_{L^2(I)} - |I|^{-1}\|\sigma\|^2_{L^1(I)}\right) \\
&\le \frac{27}{2}\frac{|I|^2}{4\pi^2}\left(\|\sigma\|^2_{L^2(I)} - |I|^{-1}\|\sigma\|^2_{L^1(I)}\right).
\end{aligned}$$

Note that if $|I| \le \frac{1}{4}$ and $\bar{\sigma} > \frac{27}{32}\frac{1}{4\pi^2}$, then

$$|I|^{-1/2}\|\sigma\|_{L^2(I)} > \frac{27}{2}\frac{|I|^2}{4\pi^2}.$$

Indeed, this immediately follows from Hölder's inequality and (58):

$$|I|^{-1/2}\|\sigma\|_{L^2(I)} \ge |I|^{-1}\|\sigma\|_{L^1} = \bar{\sigma} > \frac{27}{2}\frac{|I|^2}{4\pi^2}.$$

Thus $\|\sigma\|^2_{L^2(I)} = |I|^{-1}\|\sigma\|^2_{L^1(I)}$. This implies that $\sigma(s) \equiv const = \bar{\sigma}$ in I.

3. In the last step we show that $\sigma(s) = 1/(4\pi^2)$ for all $s \in \mathcal{S}^1$. Indeed, from (ii) there exists an interval, say, $(0, s_0)$ for some small $s_0 > 0$ such that $\sigma \equiv \bar{\sigma}$ there. Let $\tau := \sigma^2$. By the Picard theorem the initial value problem ($\tau = \sigma^2$)

$$\partial_{ss}\tau = 6(\bar{\sigma} - \sqrt{\tau}), \quad \tau(s_0) = \bar{\sigma}^2, \ \partial_s \tau(s_0) = 0.$$

has a unique solution in $(s_0, s_0 + \delta)$ for some $\delta > 0$. Since the constant function $\tau \equiv \bar{\sigma}^2$ is a solution, thus necessarily $\tau = \bar{\sigma}^2$ in $(s_0, s_0 + \delta)$. This shows that τ, thus σ, is identically $\bar{\sigma}$ in the whole circle $\mathcal{S}^1$.

With this at hand (54) yields that $|\xi|^2 = const = \bar{\sigma}$ in $\mathcal{S}^1$. Thus ξ is a circle centered at the origin. Since the length of the curve is equal to 1, then $|\xi| = \frac{1}{2\pi k}$ for $k \in \{1, 2, \cdots\}$. By our assumption $\bar{\sigma} \ge \frac{27}{32}\frac{1}{4\pi^2}$ (or because the curve is simple), we should have $|\xi| = \frac{1}{2\pi}$ and $\bar{\sigma} = \frac{1}{4\pi^2}$. ∎

Remark 3.13. From the proof of Proposition 3.12 one can see that we only need the existence of two critical points s_1, s_2 of the curvature function $k(s)$ with $|s_1 - s_2| \le 1/4$. This can also be achieved by assuming the symmetry property $\xi(s) = -\xi(s + 1/2)$ instead of assuming that the curve is simple.

Proof: 1. By Proposition 3.11 ξ is a plane curve with curvature $k > 0$. Moreover, ξ is simple by assumption. Thus by the four-vertex theorem $k(s)$ has at least four critical points. Let $0 \le s_1 < s_2 < \cdots s_J \le 1$ with $|J| \ge 4$ be the critical points. Since $\sigma^2 k = const$ and $\sigma > 0$, σ has the same critical points at s_i, i.e., $\partial_s \sigma(s_i) = 0$ for $i \in J$. We claim that

$$\frac{1}{|I_i|}\int_{I_i} \sigma(s)ds = \bar{\sigma}, \quad I_i = (s_i, s_{i+1}),\ |I_i| = s_{i+1} - s_i,\ i \in J. \tag{58}$$

Indeed, an integration of (55) in s over I_i together with (54) gives

$$\int_{I_i} (\partial_s \sigma)^2 + \int_{I_i} \sigma^2 k^2 = \int_{I_i} |\xi|^2 = \int_{I_i} (3\bar{\sigma} - 2\sigma).$$

On the other hand, multiplying the equation of σ by σ and an integration by parts yield,

$$\int_{I_i} (\partial_s \sigma)^2 + \int_{I_i} \sigma^2 k^2 = \int_{I_i} \sigma.$$

Here we have used $\partial_s \sigma(s_i) = 0$, hence the boundary term in the integration by parts vanishes. From the above two equalities we conclude

$$\int_{I_i} (3\bar{\sigma} - 2\sigma) = \int_{I_i} \sigma.$$

Thus (58) follows.

2. We show that if $\bar{\sigma} \ge \frac{27}{32}\frac{1}{4\pi^2}$, then $\sigma(s) \equiv \bar{\sigma}$ in the intervals I_i with $|I_i| \le \frac{1}{4}$. In particular, since $|J| \ge 4$ there is always an open interval where $\sigma \equiv \bar{\sigma}$ there.

Take I_i such that $|I_i| \le \frac{1}{4}$. For simplicity we write I instead of I_i in the sequel. We multiply both sides of (56) by σ and integrate from s_i to s_{i+1}. An integration by parts gives

$$\int_I \sigma(\partial_s \sigma)^2 ds = 3\int_I (\sigma^2 - \bar{\sigma}\sigma) ds. \tag{59}$$

On the other side, by a generalized Beckner type inequality (see [12, Lemma 4] with $q = \frac{4}{3}$, $p = \frac{3}{2}$ and $f = \sigma^{3/2}$; see also [22] for a link with "unbalanced optimal transport") we have

$$|I|^{-1/2}\|\sigma\|_{L^2(I)} \left(\|\sigma\|^2_{L^2(I)} - |I|^{-1}\|\sigma\|^2_{L^1(I)}\right) \le \frac{9}{2}\frac{|I|^2}{4\pi^2}\int_I \sigma(\partial_s \sigma)^2 ds. \tag{60}$$

applicable to the unnormalized flow. Indeed, the continuation of Π defined by

$$\Pi : \mathcal{A}_k^* \to \mathcal{P}_2(\mathbb{R}^d); \quad \Pi(\eta) = \eta_\#(\mathcal{H}^k \llcorner \mathcal{M})$$

is still a restricted Otto's submersion. Then $\mathcal{B}^* = \Pi(\mathcal{A}_k^*)$ may be formally viewed as a submanifold of $\mathcal{P}_2(\mathbb{R}^d)$, and we are allowed to set $\mathrm{vol}(\rho) := \mathrm{vol}(\eta)$ for $\rho = \Pi(\eta) \in \mathcal{B}^*$. Then (29) can be recast as

$$\partial_t \rho = -\operatorname{grad}_{\mathcal{B}^*} \ln(\mathrm{vol}(\rho)). \tag{44}$$

In the rest of the paper we will work with the normalized flow (30)-(31). In Section 3.2, we show local well-posedness of the problem in some Hölder class. In Section 3.3, we explore the stationary solutions. In Section 3.4, we address global well-posedness and long time asymptotics of the solution for the initial data which are close to steady states, and show exponential decay of the solutions to a steady state. In Section 3.5 we address the global solvability without restrictions on the initial data but in a generalized sense. We stress that there is a one-to-one correspondence between the normalized flow and the original gradient flow (Sections 2 and 3, resp.). After the backward change of variable and renormalization, we can infer the well-posedness of the original flow and the asymptotics of the flow near the extinction time.

3.2. Local well-posedness. In this section we show the local well-posedness of the normalized flow (30)–(31). First we introduce the function spaces we will work with. Given $\alpha, \beta \in [0, 1)$, $T \in (0, \infty)$ and $k \in \mathbb{N} \cup \{0\}$, let

$$C^{k+\alpha,\beta}([0,T] \times \mathcal{S}^1) := \{\xi : [0,T] \times \mathcal{S}^1 \to \mathbb{R}^d : \|\xi\|_{k+\alpha,\beta} < \infty\},$$

where

$$\|\xi\|_{k+\alpha,\beta} := \sup_t \|\xi(t,\cdot)\|_{C^{k+\alpha}(\mathcal{S}^1)} + \sup_s \sum_{j=0}^{k} \|\partial_s^j \xi(\cdot, s)\|_{C^\beta([0,T])}.$$

Here we use $C^{k+\alpha}(\mathcal{S}^1)$ $(C^\beta([0,T]))$ to denote the usual Hölder spaces for functions only depending on one variable. Similarly, let

$$C^{k+\alpha,1+\beta}([0,T] \times \mathcal{S}^1) := \{\xi : [0,T] \times \mathcal{S}^1 \to \mathbb{R}^d : \|\xi\|_{k+\alpha,\beta} + \|\partial_t \xi\|_{k+\alpha,\beta} < \infty\}.$$

The local well-posedness result we want to prove in this section is as follows.

Theorem 1. *Given any initial datum $\xi_0 \in C^{2+\alpha}(\mathcal{S}^1)$ with $|\partial_s \xi_0(s)| = 1$, $\int_0^1 \xi_0(s)ds = 0$, there exists $T > 0$, which depends on $\|\xi_0\|_{C^{2+\alpha}(\mathcal{S}^1)}$, such that the Cauchy problem*

$$\begin{aligned} \partial_t \xi &= \partial_s(\sigma \partial_s \xi) + \xi, \\ \partial_{ss}\sigma - |\partial_{ss}\xi|^2 \sigma &= -1, \\ \xi(0,s) &= \xi_0(s) \end{aligned} \tag{45}$$

has a unique solution $\xi \in C^{2+\alpha,\alpha/2}([0,T]\times\mathcal{S}^1)$, $\partial_t \xi \in C^{\alpha,\alpha/2}([0,T]\times\mathcal{S}^1)$.

Remark 3.5. As in Remark 2.4, if ξ is the solution emanating from ξ_0 provided by Theorem 1, then $\int_0^1 \xi(t,s)ds = 0$ and $|\partial_s \xi(t,s)| \equiv 1$ for all $t \in [0,T]$.

The proof of Theorem 1 is based on the Banach fix point theorem, where we show the solution map $\xi \mapsto \sigma_\xi \mapsto \tilde{\xi}$ is a contraction in the Banach space $C^{2+\alpha,\alpha/2}([0,T]\times\mathcal{S}^1)$. The proof is divided into several lemmas.

Lemma 3.6. *Given $\xi \in C^{2+\alpha,\alpha/2}([0,T]\times\mathcal{S}^1)$ with $|\partial_{ss}\xi(t,\cdot)|$ not identically zero for each $t \in [0,T]$. There exists a unique solution $\sigma \in C^{2+\alpha/2,\alpha/4}([0,T]\times\mathcal{S}^1)$ to the ODE*

$$\partial_{ss}\sigma(t,s) - \sigma(t,s)|\partial_{ss}\xi(t,s)|^2 = -1. \tag{46}$$

The solution satisfies the estimate

$$\|\sigma\|_{2+\alpha/2,\alpha/4} \le C$$

for some constant C depending on d and $\|\partial_{ss}\xi\|_{\alpha,\alpha/2}$, which is uniformly bounded if $\|\partial_{ss}\xi\|_{\alpha,\alpha/2}$ is uniformly bounded.

Proof: Let $k(t,s) := |\partial_{ss}\xi(t,s)|$. Since $k(t,\cdot) \neq 0$, the inhomogeneous equation has a unique solution $\sigma(t,s)$ for each t. Furthermore, by the regularity theory for the elliptic equations $\sigma(t,\cdot) \in C^{2+\alpha}(\mathcal{S}^1)$ for each $t \in [0,T]$ and one has the estimate

$$\|\sigma\|_{2+\alpha,0} \le C\left(\|k\|_{\alpha,0} + 1\right) \tag{47}$$

for some universal $C > 0$ (cf. Lemma 3.7 below for the estimate of $\|\sigma\|_{L^\infty}$).

To derive the regularity in t we consider the equation for $\sigma(t_1,\cdot) - \sigma(t_2,\cdot)$ for any $0 \le t_1 < t_2 \le T$:

$$\begin{aligned} &\partial_{ss}\left(\sigma(t_1,s) - \sigma(t_2,s)\right) - k(t_1,s)^2\left(\sigma(t_1,s) - \sigma(t_2,s)\right) \\ &= \sigma(t_2,s)\left(k^2(t_1,s) - k^2(t_2,s)\right). \end{aligned}$$

Then we multiply (51) by $\partial_s \xi$ and use $|\partial_s \xi| = 1$ to obtain

$$\partial_s \sigma + \frac{1}{2}\partial_s |\xi|^2 = 0 \text{ for all } s \in \mathcal{S}^1,$$

which together with (53) gives

$$\sigma + \frac{1}{2}|\xi|^2 = \frac{3}{2}\bar{\sigma}, \quad \bar{\sigma} := \int_0^1 \sigma. \tag{54}$$

On the other hand, (51) together with the orthogonality $\partial_s \xi \cdot \partial_{ss}\xi = 0$ yield

$$|\partial_s \sigma|^2 + \sigma^2 k^2 = |\xi|^2. \tag{55}$$

Since $\sigma k^2 = \partial_{ss}\sigma + 1$ by the equation of σ, we obtain from (55)

$$|\partial_s \sigma|^2 + \sigma(\partial_{ss}\sigma + 1) = \frac{1}{2}\partial_{ss}(\sigma^2) + \sigma = |\xi|^2,$$

which together with (54) gives the ODE of σ

$$\frac{1}{2}\partial_{ss}(\sigma^2) = 3\bar{\sigma} - 3\sigma. \tag{56}$$

Remember that we have set $\tau := \sigma^2$. Let us rewrite the above equation in terms of τ as $\partial_{ss}\tau = 6\bar{\tau} - 6\tau^{1/2}$, where $\bar{\tau} = \bar{\sigma} = \int_0^1 \tau^{1/2} ds$. Multiplying both sides by $\partial_s \tau$ and integrating we obtain

$$\frac{1}{2}(\partial_s \tau)^2 + V(\tau) = \lambda, \quad V(\tau) := 4\tau^{3/2} - 6\bar{\tau}\tau. \tag{57}$$

At s s.t. $\tau(s) = \tau_e$ we have $\partial_s \tau(s) = 0$, thus $\lambda = V(\tau_e)$ from (57). The potential V satisfies $V''(\tau) = 3\tau^{-1/2}$, hence it is convex on $(0,\infty)$. We note from (54) and the definition $\tau = \sigma^2$ that $\tau \in (0, 3\bar{\tau}/2)$. This implies $V(\tau) \in [-2\bar{\tau}^3, 0)$ with $\min V(\tau) = V(\bar{\tau}^2) = -2\bar{\tau}^3$. Thus $\lambda = V(\tau_e) \in [-2\bar{\tau}^3, 0)$. ∎

If ξ is an m-covered circle, $m \in \mathbb{N}$, then it is easy to see from (54)–(55) that $\sigma = \frac{1}{4\pi^2 m^2}$. In general it is possible to apply the method for the proof of Theorem A in [1] to classify solutions τ (hence σ) to the ODE (57). In the next proposition we show that if ξ is simple and σ is close to $\frac{1}{4\pi^2}$ ($m = 1$), then ξ is a circle and $\sigma = \frac{1}{4\pi^2}$.

Proposition 3.12. *Let $\xi \in C^2(\mathcal{S}^1;\mathbb{R}^d)$ be a solution to (51). Assume ξ is simple, i.e., $\xi(s_1) \neq \xi(s_2)$ for $s_1 \neq s_2$. Assume $\int_0^1 \sigma\, ds \ge \frac{27}{32}\frac{1}{4\pi^2}$. Then $\sigma \equiv \frac{1}{4\pi^2}$, and ξ is a circle centered at 0 with radius $\frac{1}{2\pi}$.*

We start by recalling the stationary equation $\xi : \mathcal{S}^1 \to \mathbb{R}^d$

$$\partial_s(\sigma\partial_s\xi) + \xi = 0, \ |\partial_s\xi| = 1. \tag{51}$$

Proposition 3.10. *Let $\xi \in C^2(\mathcal{S}^1;\mathbb{R}^d)$ be a solution to (51). Then ξ is a plane curve whose curvature satisfies $k(s) > 0$ for all s.*

Proof: Differentiating the equation in s and using the constraint $|\partial_s\xi| = 1$ it is not hard to see that σ satisfies

$$\partial_{ss}\sigma - \sigma|\partial_{ss}\xi|^2 = -1.$$

We have $\sigma \in C^2(\mathcal{S}^1)$ by the elliptic estimate. Furthermore, by the strong maximum principle $\sigma > 0$. This together with the equality $-\xi = \partial_s\sigma\partial_s\xi + \sigma\partial_{ss}\xi$ implies that ξ is a plane curve, since $\xi, \partial_s\xi$ and $\partial_{ss}\xi$ are in the same plane.

For the arc-length parametrized curve we have $\partial_{ss}\xi = kN$, where N unit is the inner normal along the curve and k is the curvature. Differentiating the equation of ξ and using $\partial_s N = -k\partial_s\xi$ we get

$$\left(\partial_{ss}\sigma - \sigma k^2 + 1\right)\partial_s\xi + \left(2\partial_s\sigma k + \sigma\partial_s k\right)N = 0.$$

Thus $\sigma\partial_s k + 2\partial_s\sigma k = 0$, which implies

$$\sigma^2 k = const. \tag{52}$$

Since $\int_0^1 k(s)ds = 2\pi$ for a regular closed plane curve, and $\sigma > 0$, one has $k(s) > 0$ for all $s \in \mathcal{S}^1$. ∎

In the next proposition we derive a first-order ODE of σ.

Proposition 3.11. *Let $\xi \in C^2(\mathcal{S}^1;\mathbb{R}^d)$ be a solution to (51). Let $\tau := \sigma^2$. Then τ satisfies the first integral*

$$\frac{1}{2}(\partial_s\tau)^2 + V(\tau) = \lambda, \quad V(\tau) := 4\tau^{3/2} - 6\bar{\tau}\tau.$$

Here $\bar{\tau} = \int_0^1 \tau^{1/2}ds$ and $\lambda = V(\tau_e)$ is a fixed constant, where τ_e is any extreme value of τ. Moreover, we have $\lambda \in [-2\bar{\tau}^3, 0)$.

Proof: First multiplying (51) by ξ and an integration yield

$$\int_0^1 \sigma ds = \int_0^1 |\xi|^2 ds. \tag{53}$$

By the Schauder estimate,

$$\|\sigma(t_1,s) - \sigma(t_2,s)\|_{2+\alpha/2,0} \le \tilde{C}\|k(t_1,s) - k(t_2,s)\|_{\alpha/2,0},$$

where $\tilde{C}$ is a constant depending on $\|\sigma\|_{\alpha/2,0}$ and $\|k\|_{\alpha/2,0}$, and it is uniformly bounded if $\|k\|_{\alpha/2,0}$ is uniformly bounded (recall the estimate (47) for σ). Here we have used

$$\begin{aligned}&\|\sigma(t_2,s)\left(k^2(t_1,s) - k^2(t_2,s)\right)\|_{\alpha/2,0}\\ &\le 6\|\sigma\|_{\alpha/2,0}\|k\|_{\alpha/2,0}\|k(t_1,\cdot) - k(t_2,\cdot)\|_{\alpha/2,0}\end{aligned}$$

to estimate the right hand side. Since $k(t,s) \in C^{\alpha,\alpha/2}([0,T]\times\mathcal{S}^1)$, it is not hard to see that

$$\|k(t_1,s) - k(t_2,s)\|_{\alpha/2,0} \le C\|k\|_{\alpha,\alpha/2}|t_1 - t_2|^{\alpha/2-\alpha/4}$$

for some universal $C > 0$. Combining the last two inequalities we obtain the desired estimate of $\sigma(t,s)$. ∎

Next we state the pointwise upper and lower bound on $\sigma(t,\cdot)$ in terms of $\|k(t,\cdot)\|_{L^2(\mathcal{S}^1)}$. This is a direct consequence of the upper and lower bound of the Green's function of the Schrödinger operator $\partial_{ss} - k^2$ with the periodic boundary conditions (cf. Proposition A.3 in [34]).

Lemma 3.7. *Let ξ and σ be the same as in Lemma 3.6. Then*

$$\frac{e^{-\rho/2}}{\rho} \le \sigma(t,s) \le 1 + \frac{1}{\rho}, \ \text{where } \rho = \rho(t) = \int_0^1 |\partial_{ss}\xi(t,s)|^2 ds$$

for all $s \in \mathcal{S}^1$.

Fix an initial datum $\xi_0(s)$ as in Theorem 1. Firstly we note that $\int_0^1 |\partial_{ss}\xi_0(s)|^2 ds \ge 4\pi^2$. To see this let $Z := \partial_s\xi_0$. From the assumptions on ξ_0 and using periodicity we have $|Z(s)| \equiv 1$ and $\int_0^1 Z = 0$. Then the claimed inequality follows from the Wirtinger's inequality $\int_0^1 |\partial_s Z|^2 ds \ge 4\pi^2 \int_0^1 |Z|^2 ds = 4\pi^2$. Next we let $\delta_0 := \frac{1}{2}\|\partial_{ss}\xi_0\|_{L^2(\mathcal{S}^1)} \ge \pi$. Let

$$\begin{aligned}M_{\xi_0} := \{\xi \in C^{2+\alpha,\alpha/2}([0,T]\times\mathcal{S}^1) : &\|\xi(t,s) - \xi_0(s)\|_{2+\alpha,\alpha/2} \le \delta_0\\ &\text{and } \xi(0,s) = \xi_0(s)\}.\end{aligned}$$

By the triangle inequality it is not hard to derive that $\delta_0 \le \|\partial_{ss}\xi(t,s)\|_{L^2(\mathcal{S}^1)} \le 3\delta_0$ for any $\xi \in M_{\xi_0}$ and $t \in [0,T]$.

Lemma 3.8. *Given any $\xi \in M_{\xi_0}$, let $\sigma = \sigma_\xi$ be as in Lemma 3.6. Then there is $T_0 = T_0(\delta_0)$ sufficiently small, such that for any $T \in (0, T_0]$ there exists a unique solution $\tilde{\xi}(t, s) \in M_{\xi_0}$ to the initial value problem*

$$\begin{aligned} \partial_t \tilde{\xi} &= \partial_s(\sigma \partial_s \tilde{\xi}) + \tilde{\xi}, \\ \tilde{\xi}(0, s) &= \xi_0(s). \end{aligned} \tag{48}$$

Proof: First we recall that by Lemma 3.6 for any $\xi \in M_{\xi_0}$ there exists a unique solution $\sigma = \sigma_\xi$ in the class $C^{2+\alpha/2,\alpha/4}([0,T] \times \mathcal{S}^1)$, and $\|\sigma\|_{2+\alpha/2,\alpha/4} \leq C(\delta_0)$, where $C(\delta_0)$ is a constant depending on δ_0 and is uniformly bounded if δ_0 is uniformly bounded. Next we have $c_0 \leq \sigma \leq 1 + 1/4\pi^2$ for some $c_0 = c_0(\delta_0) > 0$ by Lemma 3.7, thus the equation (48) is parabolic. Then by the classical well-posedness results for the parabolic equations, there is a unique solution $\tilde{\xi} \in C^{3+\alpha/2,1+\alpha/4}([0,T] \times \mathcal{S}^1)$ (cf. Section 9.2 and 8.12 in [23]) to the equation (48), and $\|\tilde{\xi}\|_{3+\alpha/2,1+\alpha/4} \leq C(\delta_0)$, where $C(\delta_0)$ might be a different constant from above but has the same uniform dependence on the parameter δ_0. By the definition of the Hölder class and interpolation

$$\|\tilde{\xi} - \xi_0\|_{2+\alpha,\alpha/2} = \|\tilde{\xi} - \tilde{\xi}(0,\cdot)\|_{2+\alpha,\alpha/2} \leq CT^{1-\alpha/4}\|\tilde{\xi}\|_{3+\alpha/2,1+\alpha/4}$$

for some universal $C > 0$. Thus for T sufficiently small depending on δ_0, we have $\tilde{\xi} \in M_{\xi_0}$. ∎

Lemma 3.9. *Let $\xi, \tilde{\xi} \in M_{\xi_0}$ be as in Lemma 3.8. Then for T sufficiently small depending on δ_0, the mapping $\xi \mapsto \tilde{\xi}$ is a contraction.*

Proof: Given $\xi_1, \xi_2 \in M_{\xi_0}$, let σ_1 and σ_2 be the solutions to (46) with respect to ξ_1 and ξ_2 correspondingly. Let $k_i := |\partial_{ss}\xi_i|$, $i = 1, 2$. Then $\sigma_1 - \sigma_2$ satisfies the equation

$$\partial_{ss}(\sigma_1 - \sigma_2) - k_1^2(\sigma_1 - \sigma_2) = \sigma_2(k_1^2 - k_2^2).$$

By the similar arguments as in Lemma 3.6 (with slightly more involved estimates when dealing with the regularity in time due to the more complicated right hand side) we have

$$\|\sigma_1 - \sigma_2\|_{2+\alpha/2,\alpha/4} \leq C\|\xi_1 - \xi_2\|_{2+\alpha,\alpha/2} \tag{49}$$

for some $C = C(\delta_0)$.

Let $\tilde{\xi}_1, \tilde{\xi}_2$ be the solutions to (48) with respect to σ_1, σ_2 correspondingly. Then $\zeta := \tilde{\xi}_1 - \tilde{\xi}_2$ satisfies the equation

$$\partial_t \zeta = \partial_s(\sigma_1 \partial_s \zeta) + \zeta + \partial_s\left((\sigma_1 - \sigma_2)\partial_s \tilde{\xi}_2\right), \quad \zeta(0, s) = 0.$$

By the parabolic Schauder estimate we have

$$\|\zeta\|_{3+\alpha/2,1+\alpha/4} \leq C\|\sigma_1 - \sigma_2\|_{2+\alpha/2,\alpha/4} \tag{50}$$

for some $C = C(\delta_0)$. An interpolation together with (50) and (49) yield

$$\begin{aligned} \|\zeta\|_{2+\alpha,\alpha/2} &\leq CT^{1-\alpha/4}\|\zeta\|_{3+\alpha/2,1+\alpha/4} \\ &\leq CT^{1-\alpha/4}\|\sigma_1 - \sigma_2\|_{2+\alpha/2,\alpha/4} \\ &\leq CT^{1-\alpha/4}\|\xi_1 - \xi_2\|_{2+\alpha,\alpha/2}. \end{aligned}$$

Choosing T to be sufficiently small depending on δ_0 we obtain

$$\|\tilde{\xi}_1 - \tilde{\xi}_2\|_{2+\alpha,\alpha/2} = \|\zeta\|_{2+\alpha,\alpha/2} \leq \frac{1}{2}\|\xi_1 - \xi_2\|_{2+\alpha,\alpha/2},$$

which completes the proof. ∎

In the end we provide a proof for the local-posedness of our problem (45).

Proof of Theorem 1: The assumptions on ξ_0 yield that $\|\xi_0\|_{2+\alpha,\alpha/2} \neq 0$. By Lemma 3.9 the mapping $\xi \mapsto \tilde{\xi}$ is a contraction on M_{ξ_0} provided T is sufficiently small depending on ξ_0. By the Banach fixed point theorem, there is a unique function $\xi \in M_{\xi_0}$ with $\xi = \tilde{\xi}$. It is not hard to see that such fixed point ξ is a solution to the initial value problem (45). The solution is Hölder $\alpha/2$ continuous in t up to $t = 0$. The regularity of $\partial_t \xi$ for $t > 0$ follows immediately from the interior regularity of the parabolic equation. ∎

3.3. Stationary solutions to the normalized flow. Our goal is to study the global well-posedness of the normalized equation (30)–(31), and the long time asymptotics of the solution. Before that we investigate the stationary solutions to the normalized equation, and show in this case the Lagrange multiplier σ satisfies an ODE which has a first integral. Just like as for the conventional curve shortening flow [1], the stationary solutions are not necessarily circles in our case. However we will show that the circle (with $\sigma \equiv \frac{1}{4\pi^2}$) is the only solution to the ODE if one assumes that the curve is simple and $\int_0^1 \sigma ds$ satisfies a lower bound $\int_0^1 \sigma ds \geq \frac{27}{32}\frac{1}{4\pi^2}$.