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HEDGEHOG FRAMES AND A CARDINAL EXTENSION OF NORMALITY

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Abstract: The hedgehog metric topology is presented here in a pointfree form, by specifying its generators and relations. This allows us to deal with the pointfree version of continuous (metric) hedgehog-valued functions that arises from it. We prove that the countable coproduct of the metric hedgehog frame with κ spines is universal in the class of metric frames of weight $\kappa \cdot \aleph_0$. We then study κ-collectionwise normality, a cardinal extension of normality, in frames. We prove that this is the necessary and sufficient condition under which Urysohn separation and Tietze extension-type results hold for continuous hedgehog-valued functions. We show furthermore that κ-collectionwise normality is hereditary with respect to F_σ-sublocales and invariant under closed maps.

Keywords: Frame, locale, frame of reals, metric hedgehog frame, metrizable frame, weight of a frame, separating family of localic maps, universal frame, join cozero κ-family, normal frame, κ-collectionwise normal frame, closed map.

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Contents

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1. Introduction

Topological hedgehogs keep generating interest in point-set topology as they are a rich source of counterexamples and applications (for a comprehensive survey on topological hedgehogs we refer to [1]; see also [9]). They may be described as a set of spines identified at a single point. Specifically, for each set I of cardinality κ, the classical *metric hedgehog* $J(\kappa)$ is the disjoint union $\bigcup_{i\in I}[0,1]\times\{i\}$ of κ copies (the spines) of the real unit interval identified at the origin, with the topology generated by the metric

$$d\colon J(\kappa)\times J(\kappa)\longrightarrow[0,+\infty)$$

given by (see e.g. [1, pp. 28] or [9, pp. 251])

$$d\big([(t,i)],[(s,j)]\big)=\begin{cases}|t-s|, & \text{if } j=i,\\ t+s, & \text{if } j\neq i,\end{cases}\qquad (t,i),(s,j)\in J(\kappa).$$

One of the differences between point-set topology and pointfree topology is that one may present frames by generators and relations (similarly to the presentation of groups by generators and relations). Then, for a frame L defined by generators and relations one may define a morphism with domain L just by specifying its values on the generators; it is a frame homomorphism precisely when it turns the defining relations of L into identities in the codomain frame.

In this paper we present the *frame of the metric hedgehog*, by specifying its generators and relations. This is done just from the rationals, independently of any notion of real number. For that we need to recall first that the *frame of reals* $\mathfrak{L}(\mathbb{R})$ (see e.g. [4]) is the frame specified by generators (p,q) for $p,q\in\mathbb{Q}$ and defining relations

(R1) $(p,q)\wedge(r,s)=(p\vee r,q\wedge s)$,
(R2) $(p,q)\vee(r,s)=(p,s)$ whenever $p\le r<q\le s$,
(R3) $(p,q)=\bigvee\{(r,s)\mid p<r<s<q\}$,
(R4) $\bigvee_{p,q\in\mathbb{Q}}(p,q)=1$.

Equivalently, $\mathfrak{L}(\mathbb{R})$ can be specified (see [18]) by generators $(r,\text{—})$ and $(\text{—},r)$ for $r\in\mathbb{Q}$, subject to relations

(r1) $(r,\text{—})\wedge(\text{—},s)=0$ whenever $r\ge s$,
(r2) $(r,\text{—})\vee(\text{—},s)=1$ whenever $r<s$,
(r3) $(r,\text{—})=\bigvee_{s>r}(s,\text{—})$, for every $r\in\mathbb{Q}$,
(r4) $(\text{—},r)=\bigvee_{s<r}(\text{—},s)$, for every $r\in\mathbb{Q}$,

(r5) $\bigvee_{r\in\mathbb{Q}}(r,\text{—}) = 1$,
(r6) $\bigvee_{r\in\mathbb{Q}}(\text{—},r) = 1$.

By dropping relations (r5) and (r6) one has the *frame of extended reals* $\mathfrak{L}(\overline{\mathbb{R}})$ ([6]).

We introduce the metric hedgehog frame as a cardinal generalization of $\mathfrak{L}(\overline{\mathbb{R}})$. Specifically, let κ be some cardinal and let I be a set of cardinality κ. The *frame of the metric hedgehog with κ spines* is the frame $\mathfrak{L}(J(\kappa))$ presented by generators $(r,\text{—})_i$ and $(\text{—},r)$ for $r\in\mathbb{Q}$ and $i\in I$, subject to the defining relations

(h0) $(r,\text{—})_i \wedge (s,\text{—})_j = 0$ whenever $i\neq j$,
(h1) $(r,\text{—})_i \wedge (\text{—},s) = 0$ whenever $r\geq s$ and $i\in I$,
(h2) $\bigvee_{i\in I}(r_i,\text{—})_i \vee (\text{—},s) = 1$ whenever $r_i < s$ for every $i\in I$,
(h3) $(r,\text{—})_i = \bigvee_{s>r}(s,\text{—})_i$, for every $r\in\mathbb{Q}$ and $i\in I$,
(h4) $(\text{—},r) = \bigvee_{s<r}(\text{—},s)$, for every $r\in\mathbb{Q}$.

The purpose of this paper is to present some of the main properties of the metric hedgehog frame (that from now on we shall mostly refer to as simply the *hedgehog frame*), as well as of the corresponding continuous hedgehog-valued functions. We prove that for each cardinal κ, the hedgehog frame $\mathfrak{L}(J(\kappa))$ is a metric frame of weight $\kappa\cdot\aleph_0$, complete in its metric uniformity. Then we show that the countable coproduct of the hedgehog frame with κ spines is universal in the class of metric frames of weight $\kappa\cdot\aleph_0$, that is, every metrizable frame of weight $\kappa\cdot\aleph_0$ is embeddable into a countable cartesian power of the hedgehog frame. Being the hedgehog frame a fundamental example of a collectionwise normal frame, we take the opportunity to study collectionwise normality in frames, a concept originally introduced by A. Pultr ([22]) in connection with metrizability. First, we show that collectionwise normality is hereditary with respect to F_σ-sublocales and that it is a property invariant under closed maps. Then we present the counterparts of Urysohn's separation and Tietze's extension theorems for continuous hedgehog-valued functions. They both characterize κ-collectionwise normality.

2. Preliminaries and notation

A *frame* (or *locale*) L is a complete lattice (with bottom 0 and top 1) such that $a\wedge\bigvee B = \bigvee\{a\wedge b \mid b\in B\}$ for all $a\in L$ and $B\subseteq L$. A frame is precisely a complete Heyting algebra with Heyting operation $\to$ satisfying the standard equivalence $a \wedge b \leq c$ iff $a \leq b \to c$. The *pseudocomplement* of an $a \in L$ is the element $a^* = a \to 0 = \bigvee\{b \in L \mid a \wedge b = 0\}$. A frame L is *regular* if, for each $x \in L$, $x = \bigvee\{y \in L \mid y \prec x\}$ where $y \prec x$ means that $y^* \vee x = 1$.

A subset B of a frame L is a *base* for L if each element in L is join-generated by some set of elements in B. Given a base B for L, the pseudocomplement of $a \in L$ is obviously given by $a^* = \bigvee\{b \in B \mid a \wedge b = 0\}$.

A *frame homomorphism* is a map $h\colon L \to M$ between frames which preserves finitary meets (including the top element 1) and arbitrary joins (including the bottom element 0). Note that $h(x^*) \leq h(x)^*$ for every $x \in L$. A frame homomorphism h is said to be *dense* if $h(a) = 0$ implies $a = 0$ and it is *codense* if $h(a) = 1$ implies $a = 1$. We denote by Frm the category of frames and frame homomorphisms.

As a frame homomorphism $h\colon L \to M$ preserves arbitrary joins, it has a (unique) right adjoint $h_*\colon M \to L$ determined by

$$h_*(b) = \bigvee\{a \in L \mid h(a) \leq b\}.$$

In particular, $\mathrm{id}_L \leq h_* h$ and $h h_* \leq \mathrm{id}_M$. The frame homomorphism h is a surjection if and only if h_* is an embedding if and only if $h h_* = \mathrm{id}_M$.

An important example of a frame is the lattice $\mathfrak{O}X$ of open subsets of any topological space X. The correspondence $X \mapsto \mathfrak{O}X$ is clearly functorial (by taking inverse images), and consequently one has a contravariant functor $\mathfrak{O}\colon \mathsf{Top} \to \mathsf{Frm}$ with the category Top of topological spaces and continuous maps as domain category. The functor $\mathfrak{O}$ has a right adjoint, the *spectrum functor* $\Sigma\colon \mathsf{Frm} \to \mathsf{Top}$, which assigns to each frame L its *spectrum* ΣL, that is, the space of all homomorphisms $\xi\colon L \to \{0,1\}$ with open sets $\Sigma_a = \{\xi \in \Sigma L \mid \xi(a) = 1\}$ for any $a \in L$, and to each frame homomorphism $h\colon L \to M$ the continuous map $\Sigma h\colon \Sigma M \to \Sigma L$ such that $\Sigma h(\xi) = \xi \circ h$.

The category Frm and its dual category Loc of locales and localic maps are the framework of pointfree topology. The contravariance of the functor $\mathfrak{O}$ indicates that on the extension step from classical topology into pointfree topology, it is in Loc that the classical notions have to be considered. For instance, the generalized pointfree subspaces are subobjects in Loc (the *sublocales*), that is, quotients in Frm. A map of locales (*localic map*) $f\colon L \to M$ is the unique right adjoint h_* of a frame homomorphism $h\colon M \to L$. They are precisely the maps that, besides preserving arbitrary meets, reflect the top element (i.e., $f(a) = 1 \Rightarrow a = 1$) and satisfy $f(h(b) \to a) = b \to f(a)$ for every $a \in L$ and $b \in M$. The reader is referred to [19] for more information on frames and locales.

Regarding the frame of reals mentioned at the Introduction, it is worth pointing out that the assignments

$$(r, \text{–}) \longmapsto \{t \in \mathbb{Q} \mid t > r\} \quad \text{and} \quad (\text{–}, r) \longmapsto \{t \in \mathbb{Q} \mid t < r\}$$

determine a surjective frame homomorphism from $\mathfrak{L}(\mathbb{R})$ to $\mathfrak{O}\mathbb{Q}$ (the usual topology on the rationals) as it obviously turns the defining relations (r1)–(r6) into identities in $\mathfrak{O}\mathbb{Q}$. Consequently, $(r, \text{–}) \neq 0 \neq (\text{–}, r)$ for every $r \in \mathbb{Q}$. Moreover, $(r, \text{–}) \wedge (\text{–}, s) = 0$ if and only if $r \geq s$, and $(r, \text{–}) \vee (\text{–}, s) = 1$ if and only if $r < s$.

For any frame L, a *continuous real-valued function* [4] (resp. *extended continuous real-valued function* [6]) on a frame L is a frame homomorphism $h\colon \mathfrak{L}(\mathbb{R}) \to L$ (resp. $h\colon \mathfrak{L}(\overline{\mathbb{R}}) \to L$). We denote by $\mathsf{C}(L)$ and $\overline{\mathsf{C}}(L)$, respectively, the collections of all continuous real-valued functions and extended continuous real-valued functions on L. For each $r \in \mathbb{Q}$, we denote by $\boldsymbol{r}$ the constant function defined by $\boldsymbol{r}(p, q) = 1$ if $p < r < q$ and $\boldsymbol{r}(p, q) = 0$ otherwise.

There is a useful way of specifying (extended) continuous real-valued functions on a frame L with the help of scales ([12, Section 4]). An *extended scale* in L is a map $\sigma\colon \mathbb{Q} \to L$ such that $\sigma(p) \vee \sigma(q)^* = 1$ whenever $p < q$. An extended scale is a *scale* if

$$\bigvee_{p \in \mathbb{Q}} \sigma(p) = 1 = \bigvee_{p \in \mathbb{Q}} \sigma(p)^*.$$

For each extended scale σ in L, the formulas

$$h(p, \text{–}) = \bigvee_{r > p} \sigma(r) \quad \text{and} \quad h(\text{–}, q) = \bigvee_{r < q} \sigma(r)^*, \quad p, q \in \mathbb{Q}, \tag{1}$$

determine an $h \in \overline{\mathsf{C}}(L)$ ([6, Lemma 1]); then, $h \in \mathsf{C}(L)$ if and only if σ is a scale.

For more about continuous real-valued functions on frames we refer to [4].

3. Metric hedgehog frames

Consider $\mathfrak{L}(J(\kappa))$, the hedgehog frame with κ spines which was introduced earlier. It is obvious that $\mathfrak{L}(J(1)) = \mathfrak{L}(\overline{\mathbb{R}})$ (since condition (h0) is vacuously satisfied in this case). Moreover, $\mathfrak{L}(J(2))$ is also isomorphic to $\mathfrak{L}(\overline{\mathbb{R}})$. The isomorphism is induced by the following correspondences (where φ denotes any increasing bijection between $\mathbb{Q}$ and $\mathbb{Q}^+$):

$$\begin{aligned}
&\mathfrak{L}(J(2)) \to \mathfrak{L}(\overline{\mathbb{R}}) :\\
&(r, \text{—})_1 \longmapsto (\varphi(r), \text{—}), \quad (r, \text{—})_2 \longmapsto (\text{—}, -\varphi(r)),\\
&(\text{—}, r) \longmapsto (-\varphi(r), \text{—}) \wedge (\text{—}, \varphi(r)).
\end{aligned}$$

$$\begin{aligned}
&\mathfrak{L}(\overline{\mathbb{R}}) \to \mathfrak{L}(J(2)) :\\
&r \geq 0 : \ (r, \text{—}) \longmapsto (\varphi^{-1}(r), \text{—})_1, \qquad r < 0 : \ (r, \text{—}) \longmapsto (\text{—}, \varphi^{-1}(-r)) \vee \bigvee_{s \in \mathbb{Q}} (s, \text{—})_1\\
&s \leq 0 : \ (\text{—}, s) \longmapsto (\varphi^{-1}(-s), \text{—})_2, \quad s > 0 : \ (\text{—}, s) \longmapsto (\text{—}, \varphi^{-1}(s)) \vee \bigvee_{r \in \mathbb{Q}} (r, \text{—})_2.
\end{aligned}$$

We now introduce the following notation in $\mathfrak{L}(J(\kappa))$:

$$(r, s)_i \equiv (r, \text{—})_i \wedge (\text{—}, s).$$

The set

$$B_\kappa = \{(\text{—}, r) \mid r \in \mathbb{Q}\} \cup \{(r, \text{—})_i \mid r \in \mathbb{Q}, \ i \in I\} \cup \{(r, s)_i \mid r < s \text{ in } \mathbb{Q}, \ i \in I\}$$

forms a base for $\mathfrak{L}(J(\kappa))$ (since it is closed under finite meets by (h3) and (h4)). Hence $a^* = \bigvee\{b \in B_\kappa \mid a \wedge b = 0\}$ for any $a \in \mathfrak{L}(J(\kappa))$.

Remarks 3.1. (1) For each $i \in I$ the assignments

$$(r, \text{—})_j \longmapsto \begin{cases} (r, \text{—}) & \text{if } j = i, \\ 0 & \text{if } j \neq i \end{cases} \quad \text{and} \quad (\text{—}, r) \longmapsto (\text{—}, r)$$

determine a surjective frame homomorphism h_i from $\mathfrak{L}(J(\kappa))$ to $\mathfrak{L}(\overline{\mathbb{R}})$: they obviously turn the defining relations (h0)–(h4) into identities in $\mathfrak{L}(\overline{\mathbb{R}})$. Consequently, $(r, \text{—})_i \neq 0 \neq (\text{—}, r)$ for every $r \in \mathbb{Q}$. Moreover, we have:

(i) $(r, \text{—})_i \wedge (s, \text{—})_j = 0$ if and only if $i \neq j$.
(ii) $(r, s)_i = (r, \text{—})_i \wedge (\text{—}, s) = 0$ if and only if $r \geq s$.
(iii) $\bigvee_{i \in I} (r_i, \text{—})_i \vee (\text{—}, s) = 1$ if and only if $r_i < s$ for every $i \in I$.

Note that $\mathfrak{L}(J(\kappa))$ is compact if and only if κ is finite. Indeed, if $|I| = \kappa$ is infinite, then

$$C = \{(\text{—}, 1)\} \cup \{(0, \text{—})_i \mid i \in I\}$$

is a cover of $\mathfrak{L}(J(\kappa))$ (by (h2)) with no proper subcover. On the other hand, if $|I| = \kappa$ is finite then, using the compactness of $\mathfrak{L}(\overline{\mathbb{R}})$ and the frame homomorphisms h_i, it can be proved that $\mathfrak{L}(J(\kappa))$ is compact (we omit the details of the proof).

Proposition 3.2. *The spectrum $\Sigma\mathfrak{L}(J(\kappa))$ is homeomorphic to the classical metric hedgehog $J(\kappa)$.*

Proof: For each $h \in \Sigma\mathfrak{L}(J(\kappa))$ define

$$\alpha(h) = \bigvee\{r \in \mathbb{Q} \mid \bigvee_{i\in I} h((r,\text{—})_i) = 1\} \in \overline{\mathbb{R}} \quad \text{and}$$
$$\beta(h) = \bigwedge\{s \in \mathbb{Q} \mid h(\text{—},s) = 1\} \in \overline{\mathbb{R}}.$$

For any $r, s \in \mathbb{Q}$ such that $\bigvee_{i\in I} h(r,\text{—})_i = 1 = h(\text{—},s)$ one has

$$h\big(\bigvee_{i\in I}(r,\text{—})_i \wedge (\text{—},s)\big) = \bigvee_{i\in I} h((r,\text{—})_i) \wedge h(\text{—},s) = 1.$$

Then, by (h0), $r < s$. It follows that $\alpha(h) \le \beta(h)$. By (h2), one has $\alpha(h) = \beta(h)$, as otherwise one could take $r, s \in \mathbb{Q}$ such that $\alpha(h) < r < s < h(\beta)$ from which it would follow that

$$0 = \bigvee_{i\in I} h((r,\text{—})_i) \vee h(\text{—},s) = h\big(\bigvee_{i\in I}(r,\text{—})_i \vee (\text{—},s)\big) = 1,$$

a contradiction.

If $\alpha(h) \neq -\infty$, then there exist $r \in \mathbb{Q}$ and $i_h \in I$ such that $h((r,\text{—})_{i_h}) = 1$. Further, by (h1), we conclude that this $i_h \in I$ is unique and that $h((s,\text{—})_j) = 0$ for all $s \in \mathbb{Q}$ and $j \neq i_h$.

Now, consider an increasing bijection φ between $\overline{\mathbb{Q}}$ and $\mathbb{Q}\cap[0,1]$ and define $\pi\colon \Sigma\mathfrak{L}(J(\kappa)) \to J(\kappa)$ as follows (where we identify equivalence classes $[(t,i)]$ with their representatives (t,i), with $t \neq 0$, and denote by $\mathbf{0}$ the class of $(0,i)$):

$$h \longmapsto \pi(h) = \begin{cases} (\varphi(\alpha(h)), i_h), & \text{if } \alpha(h) \neq -\infty, \\ \mathbf{0}, & \text{otherwise.} \end{cases}$$

In order to check that π is one-one, let $h_1, h_2 \in \Sigma\mathfrak{L}(J(\kappa))$. If there exists $r \in \mathbb{Q}$ such that, say, $h_1(\text{—},r) = 1$ and $h_2(\text{—},r) = 0$, then, by (h4),

$$1 = h_1(\text{—},r) = \bigvee_{s<r} h_1(\text{—},s)$$

and consequently there exists $s < r$ such that $h_1(\text{—},s) = 1$. Then, by (h2), we have $\alpha(h_1) \le s < r \le \alpha(h_2)$. Therefore $\pi(h_1) \neq \pi(h_2)$. On the other hand, if there exist $r \in \mathbb{Q}$ and $i \in I$ such that, say, $h_1((r,\text{—})_i) = 1$ and $h_2((r,\text{—})_i) = 0$, then $\alpha(h_1) > r$ since, by (h3),

$$1 = h_1((r,\text{—})_i) = \bigvee_{s>r} h_1((s,\text{—})_i)$$

and, consequently, there exists $s > r$ such that $h_1((s,\text{—})_i) = 1$. One also has $i_{h_1} = i$. If

$$\bigvee_{j\in I} h_2((r,\text{—})_j) = 0,$$

then $\alpha(h_2) \leq r$. On the other hand, if

$$\bigvee_{j\in I} h_2((r,\text{—})_j) = 1,$$

there exists $j \neq i$ such that $h_2((r,\text{—})_j) = 1$ and, consequently, $i_{h_2} = j \neq i = i_{h_1}$. Therefore $\pi(h_1) \neq \pi(h_2)$. The arguments for the other cases are similar.

Furthermore, π is also surjective. Indeed, given (t, i) in $J(\kappa)$ let us define

$$h_{(t,i)} \colon \mathfrak{L}(J(\kappa)) \to \mathbf{2}$$

by

$$h_{(t,i)}(\text{—}, r) = 0 \text{ iff } \varphi(r) > t$$

and

$$h_{(t,i)}((r,\text{—})_j) = 1 \text{ iff } \varphi(r) < t \text{ and } j = i.$$

It is straightforward to check that $h_{(t,i)}$ turns the defining relations (h0)–(h4) into identities in $\mathbf{2}$ and that $\pi(h_{(t,i)}) = (t, i)$. We conclude that π is a bijection with inverse

$$\rho = \pi^{-1} \colon J(\kappa) \to \Sigma\mathfrak{L}(J(\kappa))$$

given by $\rho(t, i) = h_{(t,i)}$.

It only remains to be checked that π is a homeomorphism. For that purpose let $r, s \in \mathbb{Q} \cap [0, 1]$ and $i \in I$. Then one has

$$\begin{aligned} \rho((r, 1] \times \{i\}) &= \{h_{(t,i)} \in \Sigma\mathfrak{L}(J(\kappa)) \mid t > r\} \\ &= \{h_{(t,i)} \in \Sigma\mathfrak{L}(J(\kappa) \mid h_{(t,i)}((\varphi^{-1}(r),\text{—})_i) = 1\} = \Sigma_{(\varphi^{-1}(r),\text{—})_i} \end{aligned}$$

and

$$\begin{aligned} \rho(B(\mathbf{0}, s)) &= \{h_{(t,j)} \in \Sigma\mathfrak{L}(J(\kappa)) \mid t < s\} \\ &= \{h_{(t,j)} \in \Sigma\mathfrak{L}(J(\kappa) \mid h_{(t,j)}(\text{—}, \varphi^{-1}(s)) = 1\} = \Sigma_{(\text{—},\varphi^{-1}(s))}. \end{aligned}$$

Hence π is continuous. On the other hand, let $r, s \in \overline{\mathbb{Q}}$ and $i \in I$. Then one has

$$\begin{aligned}\pi(\Sigma_{(r,\text{—})_i}) &= \{\pi(h) \mid h \in \Sigma\mathfrak{L}(J(\kappa)) \text{ and } h((r,\text{—})_i) = 1\}\\ &= \{\pi(h_{(t,j)}) \mid h_{(t,j)} \in \Sigma\mathfrak{L}(J(\kappa)) \text{ and } h_{(t,j)}((r,\text{—})_i) = 1\}\\ &= \{(t,j) \in J(\kappa) \mid j = i \text{ and } \varphi(r) < t\}\\ &= (\varphi(r), 1] \times \{i\}\end{aligned}$$

and

$$\begin{aligned}\pi(\Sigma_{(\text{—},s)}) &= \{\pi(h) \mid h \in \Sigma\mathfrak{L}(J(\kappa)) \text{ and } h(\text{—}, s) = 1\}\\ &= \{\pi(h_{(t,j)}) \mid h_{(t,j)} \in \Sigma\mathfrak{L}(J(\kappa)) \text{ and } h_{(t,j)}(\text{—}, s) = 1\}\\ &= \{(t,j) \in J(\kappa) \mid \varphi(s) > t\}\\ &= B(\mathbf{0}, \varphi(s)).\end{aligned}$$

Therefore ρ is also continuous and, consequently, $\Sigma\mathfrak{L}(J(\kappa))$ is homeomorphic to $J(\kappa)$. ■

The *spatial reflection* of a frame L is the unit map of the adjunction $\mathfrak{O} \dashv \Sigma$, that is, the frame homomorphism

$$\eta_L \colon L \to \mathfrak{O}\Sigma L, \quad a \mapsto \Sigma_a = \{\xi \in \Sigma L \mid \xi(a) = 1\}.$$

In the present case, as seen in the previous proof, the homeomorphism $\rho\colon J(\kappa) \to \Sigma\mathfrak{L}(J(\kappa))$ induces an isomorphism $\mathfrak{O}\Sigma\mathfrak{L}(J(\kappa)) \to \mathfrak{O}J(\kappa)$ mapping $\Sigma_{(r,\text{—})_i}$ to $(\varphi(r), 1] \times \{i\}$ and $\Sigma_{(\text{—},r)}$ to $B(\mathbf{0}, \varphi(r))$. Hence, we have the following result:

Corollary 3.3. *The frame homomorphism $\mathfrak{L}(J(\kappa)) \to \mathfrak{O}J(\kappa)$ taking $(\text{—}, r)$ to $B(\mathbf{0}, \varphi(r))$ and $(r, \text{—})_i$ to $(\varphi(r), 1] \times \{i\}$ is the spatial reflection map of the frame $\mathfrak{L}(J(\kappa))$.*

Lemma 3.4. *Let $\mathfrak{L}(J(\kappa))$ be the frame of the hedgehog with κ spines and $r, s \in \mathbb{Q}$. Then:*

(1) $(\text{—}, r)^* = \bigvee_{i \in I} (r, \text{—})_i$.
(2) $(r, \text{—})_i{}^* = \bigvee_{\substack{j \neq i \\ s \in \mathbb{Q}}} (s, \text{—})_j \vee (\text{—}, r)$.
(3) $(r, s)_i{}^* = \bigvee_{\substack{j \neq i \\ t \in \mathbb{Q}}} (t, \text{—})_j \vee (\text{—}, r) \vee (s, \text{—})_i$.
(4) $(\text{—}, r)^{**} = (\text{—}, r)$, $(r, \text{—})_i{}^{**} = (r, \text{—})_i$ *and* $(r, s)_i{}^{**} = (r, s)_i$.
(5) *If* $s < r$ *then* $(\text{—}, r) \vee (s, \text{—})_i = (\text{—}, r) \vee \bigvee_{p \in \mathbb{Q}} (p, \text{—})_i$.
(6) *If* $s < q < r$ *then* $(q, \text{—})_i \vee (s, r)_i = (s, \text{—})_i$.

Proof: We only prove the first equality. The other ones follow in a similar way. By Remark 3.1 (1) we have the following:
(i) $(\text{—}, r) \wedge (\text{—}, s) = (\text{—}, r \wedge s) \neq 0$ for all $r, s \in \mathbb{Q}$,
(ii) for each $i \in I$, $(\text{—}, r) \wedge (s, \text{—})_i = (s, r)_i = 0$ iff $s \geq r$, and
(iii) for each $s < t$ in $\mathbb{Q}$ and $i \in I$, $(\text{—}, r) \wedge (s, t)_i = (s, r \wedge t)_i = 0$ iff $s \geq r \wedge t$, that is, $s \geq r$. Consequently, by (h3),

$$(\text{—}, r)^* = \bigvee_{i \in I} \bigvee_{r \leq s} (s, \text{—})_i \vee \bigvee_{i \in I} \bigvee_{r \leq s < t} (s, t)_i = \bigvee_{i \in I} (r, \text{—})_i. \qquad \blacksquare$$

Corollary 3.5. *$\mathfrak{L}(J(\kappa))$ is a regular frame.*

Proof: Let $r, s \in \mathbb{Q}$. From Lemma 3.4, applying (h2), (h3) and (h4), we get:
(i) $(\text{—}, s)^* \vee (\text{—}, r) = \bigvee_{i \in I} (s, \text{—})_i \vee (\text{—}, r) = 1$, i.e. $(\text{—}, s) \prec (\text{—}, r)$ whenever $s < r$ and $(\text{—}, r) = \bigvee_{s<r} (\text{—}, s)$.
(ii) $(s, \text{—})_i^* \vee (r, \text{—})_i = \bigvee_{\substack{j \neq i \\ t \in \mathbb{Q}}} (t, \text{—})_j \vee (\text{—}, s) \vee (r, \text{—})_i = 1$, i.e. $(s, \text{—})_i \prec (r, \text{—})_i$ whenever $s > r$ and $(r, \text{—})_i = \bigvee_{s>r} (s, \text{—})_i$.
(iii) $(r, s)_i^* \vee (r', s')_i = \bigvee_{\substack{j \neq i \\ t \in \mathbb{Q}}} (t, \text{—})_j \vee (\text{—}, r) \vee (s, \text{—})_i \vee (r', s')_i = 1$, i.e. $(r', s')_i \prec (r, s)_i$ whenever $r < r' < s' < s$ and $(r, s)_i = \bigvee_{r<r'<s'<s} (r', s')_i$.

Since B_κ is a base of $\mathfrak{L}(J(\kappa))$, we may conclude that $\mathfrak{L}(J(\kappa))$ is regular. $\blacksquare$

Remarks 3.6. (1) For each $i \in I$, the map $\sigma_i \colon \mathbb{Q} \to \mathfrak{L}(J(\kappa))$ given by $\sigma_i(r) = (r, \text{—})_i$ is an extended scale in $\mathfrak{L}(J(\kappa))$. Indeed, if $r < s$ then, by Lemma 3.4 (2) and (h2), we have

$$\sigma_i(r) \vee \sigma_i(s)^* = (r, \text{—})_i \vee (s, \text{—})_i^* \geq (r, \text{—})_i \vee \Big(\bigvee_{j \neq i} (r, \text{—})_j\Big) \vee (\text{—}, s) = 1.$$

By (1), (h3), Lemma 3.4 (2) and (h4), the formulas

$$\begin{aligned} \pi_i(p, \text{—}) &= \bigvee_{s>p} (s, \text{—})_i = (p, \text{—})_i \quad \text{and} \\ \pi_i(\text{—}, q) &= \bigvee_{s<q} (s, \text{—})_i^* = \bigvee_{s<q} \Big(\Big(\bigvee_{\substack{j \neq i \\ r \in \mathbb{Q}}} (r, \text{—})_j\Big) \vee (\text{—}, s)\Big) \\ &= \Big(\bigvee_{r, j \neq i} (r, \text{—})_j\Big) \vee (\text{—}, q) = (q, \text{—})_i^* \end{aligned}$$

determine a continuous extended real-valued function $\pi_i \colon \mathfrak{L}(\overline{\mathbb{R}}) \to \mathfrak{L}(J(\kappa))$.

(2) Furthermore, the map $\sigma_\kappa \colon \mathbb{Q} \to \mathfrak{L}(J(\kappa))$ given by $\sigma_\kappa(r) = (\text{—}, r)^*$ is also an extended scale in $\mathfrak{L}(J(\kappa))$. Indeed, if $r < s$ then, by Lemma 3.4 and (h2),

we have that

$$\sigma_\kappa(r) \vee \sigma_\kappa(s)^* = (-, r)^* \vee (-, s)^{**} \geq \Big(\bigvee_{i\in I} (r, -)_i\Big) \vee (-, s) = 1.$$

Hence, by (1), Lemma 3.4, (h3) and (h4) the formulas

$$\pi_\kappa(p, -) = \bigvee_{s>p} (-, s)^* = \bigvee_{s>p} \bigvee_{i\in I} (s, -)_i = \bigvee_{i\in I} (p, -)_i = (-, p)^* \quad \text{and}$$
$$\pi_\kappa(-, q) = \bigvee_{s<q} (-, s)^{**} = \bigvee_{s<q} (-, s) = (-, q)$$

determine a continuous extended real-valued function $\pi_\kappa \colon \mathfrak{L}(\overline{\mathbb{R}}) \to \mathfrak{L}(J(\kappa))$. Note that, for any $p \in \mathbb{Q}$,

$$\bigvee_{i\in I} \pi_i(p, -) = \bigvee_{i\in I} (p, -)_i = (-, p)^* = \pi_\kappa(p, -),$$

that is, $\bigvee_{i\in I} \pi_i = \pi_\kappa$.

Recall that the *weight* $w(L)$ ([10]) of a frame L is the smallest infinite cardinal for which there exists a base B for L of cardinality $|B| \leq w(L)$.

Theorem 3.7. *For each cardinal κ, the frame of the metric hedgehog $\mathfrak{L}(J(\kappa))$ is a metric frame of weight $\kappa \cdot \aleph_0$.*

Proof: First note that B_κ is a base for $\mathfrak{L}(J(\kappa))$ of cardinality $|B_\kappa| = \kappa$ whenever $\kappa \geq \aleph_0$ (otherwise, $|B_\kappa| = \aleph_0$), hence $\mathfrak{L}(J(\kappa))$ has weight $\kappa \cdot \aleph_0$.

We only need to prove that $\mathfrak{L}(J(\kappa))$ is a metrizable frame, and for that it suffices to show it carries a uniformity (its *metric uniformity*) with a countable base.

So, for each $n \in \mathbb{N}$, let $C_n = C_n^1 \cup C_n^2 \cup C_n^3 \subseteq B_\kappa$ with

$$C_n^1 = \{(-, r) \mid r < -n\}, \quad C_n^2 = \{(r, -)_i \mid r > n,\ i \in I\} \quad \text{and}$$
$$C_n^3 = \big\{(r, s)_i \mid 0 < s - r < \tfrac{1}{n},\ i \in I\big\}.$$

These C_n are clearly covers of $\mathfrak{L}(J(\kappa))$ and $C_{n+1} \leq C_n$ for all n, since C_{n+1} is obviously contained in C_n. Thus, we have a countable filter base of covers. Further, for each $n \in \mathbb{N}$, $C_{3n} \cdot C_{3n} \leq C_n$:

(1) For each $(-, r) \in C_{3n}^1$,

$$C_{3n}(-, r) = \bigvee\{x \in C_{3n} \mid x \wedge (-, r) \neq 0\} = (-, -3n + \tfrac{1}{3n}) \in C_n.$$

(2) For each $(r, -)_i \in C_{3n}^2$,

$$C_{3n}((r, -)_i = \bigvee\{x \in C_{3n} \mid x \wedge (r, -)_i \neq 0\} = \bigvee_{i\in I} (3n - \tfrac{1}{3n}, -)_i \in C_n.$$

(3) For each $(r,s)_i \in C^3_{3n}$,

$$C_{3n}(r,s)_i = \begin{cases} (\text{—}, -3n) \vee (r - \frac{1}{3n}, s + \frac{1}{3n})_i \leq (\text{—}, -3n + \frac{2}{3n}), & \text{if } r < -3n,\\ (r - \frac{1}{3n}, s + \frac{1}{3n})_i \vee (3n, \text{—})_i \leq (3n - \frac{2}{3n}, \text{—})_i, & \text{if } s > 3n,\\ (r - \frac{1}{3n}, s + \frac{1}{3n})_i, & \text{otherwise,} \end{cases}$$

and so $C_{3n}(r,s)_i \in C_n$.

Finally, for the uniform below relation $\lhd$ defined by $x \lhd y$ in $\mathfrak{L}(J(\kappa))$ iff $C_n x \leq y$ for some $n \in \mathbb{N}$, we have:

(3.7.1) For each $n \in \mathbb{N}$ and $r \leq n$ we have that $C_n(\text{—}, r) = (\text{—}, r + \frac{1}{n})$. Therefore, if $r < s$ then, $C_n(\text{—}, r) \leq (\text{—}, s)$ for any sufficiently large n and thus $(\text{—}, r) \lhd (\text{—}, s)$.

(3.7.2) For each $i \in I$, $n \in \mathbb{N}$ and $s \geq -n$ we have that $C_n(s, \text{—})_i = (s - \frac{1}{n}, \text{—})_i$. Therefore, if $r < s$ then $C_n(s, \text{—})_i \leq (r, \text{—})_i$ for any sufficiently large n and hence $(s, \text{—})_i \lhd (r, \text{—})_i$.

By (h3) and (h4), this proves the admissibility of the covering uniformity, that is, that $x = \bigvee\{y \in \mathfrak{L}(J(\kappa)) \mid y \lhd x\}$ for every $x \in \mathfrak{L}(J(\kappa))$. ■

It may be worth pointing out that the formula

$$d(x) = \inf\{2^{-n} \mid x \leq c \text{ for some } c \in C_n\}$$

from [21, Theorem 4.6] provides the expression for the metric diameter on $\mathfrak{L}(J(\kappa))$ induced by the metric uniformity. Of course, this is a totally bounded metric frame if and only if $\kappa < \aleph_0$.

Corollary 3.8. *For each cardinal κ, the coproduct $\bigoplus_{n\in\mathbb{N}} \mathfrak{L}(J(\kappa))$ is a metric frame of weight $\kappa \cdot \aleph_0$.*

Proof: Any countable coproduct of metrizable frames is a metrizable frame [15, p. 31], hence $\bigoplus_{n\in\mathbb{N}} \mathfrak{L}(J(\kappa))$ is a metric frame, clearly of weight κ or $\aleph_0$ as the case may be. ■

A uniform frame L is said to be *complete* whenever any dense surjection $h\colon M \to L$ of uniform frames is a frame isomorphism.

In the proof of next result we shall make use of the following well-known fact about surjective frame homomorphisms $h\colon M \to L$:

(3.9.1) If M is regular and the right adjoint h_* is also a frame homomorphism, then h is an isomorphism.

(One first notes that $y \prec x$ in M iff $h_*h(y) \prec x$, since

$$y^* \vee x \geq h_*h(y)^* \vee x \geq h_*h(y^*) \vee x \geq y^* \vee x.$$

Then, by regularity, $h_*h(x) = \bigvee\{h_*h(y) \mid y \prec x\} = \bigvee\{y \mid y \prec x\} = x$, which shows that h is one-one.)

Proposition 3.9. *$\mathfrak{L}(J(\kappa))$ is complete in its metric uniformity.*

Proof: Let $h\colon M \to \mathfrak{L}(J(\kappa))$ be a dense surjection of uniform frames (where $\mathfrak{L}(J(\kappa))$ is equipped with its metric uniformity). We need to show that h is a frame isomorphism. Let h_* be the right adjoint of h. Since h is a surjection, $hh_* = \mathrm{id}_{\mathfrak{L}(J(\kappa))}$. Then, using (3.9.1), it suffices to show that h_* is a frame homomorphism, that is, it turns the conditions (h0)-(h4) into identities in M. For checking that, we need to recall two well-known facts concerning dense surjections $h\colon M \to L$ of uniform frames ([4, pp. 25], see also [3]):

(3.9.2) The uniformity of M is generated by the images $h_*[C]$ of the uniform covers C of L.

(3.9.3) For any $a \in L$, $h_*(a) = \bigvee\{h_*(x) \mid x \lhd a\}$ for the strong inclusion $\lhd$ on L induced by its uniformity.

(h0): Need to check that $h_*(r, \text{—})_i \wedge h_*(s, \text{—})_j = 0$ whenever $i \neq j$. This follows from the facts that h_* preserves arbitrary meets and $h_*(0) = 0$ (by the density of h).

(h1): Similarly as with (h0).

(h2): It follows from (3.9.2) that, for any n, $h_*[C_n]$ is a cover of M.

(h3): By (3.9.3), $h_*(r, \text{—})_i = \bigvee\{h_*(x) \mid x \lhd (r, \text{—})_i\}$. Consequently, by (3.7.2),

$$\bigvee_{s>r} h_*(s, \text{—})_i \leq \bigvee\{h_*(x) \mid x \lhd (r, \text{—})_i\}.$$

On the other hand, it is easy to check that $C_n \cdot x \leq (r, \text{—})_i$ implies $x \leq (r + \frac{1}{n}, \text{—})_i$ for any n so that

$$\bigvee_{s>r} h_*(s, \text{—})_i = \bigvee\{h_*(x) \mid x \lhd (r, \text{—})_i\}.$$

Hence we have $h_*(r, \text{—})_i = \bigvee_{s>r} h_*(s, \text{—})_i$ as required.

(h4): By (3.9.3), $h_*(\text{—}, r) = \bigvee\{h_*(x) \mid x \lhd (\text{—}, r)\}$. Moreover, by (3.7.1),

$$\bigvee_{s<r} h_*(\text{—}, s) \leq \bigvee\{h_*(x) \mid x \lhd (\text{—}, r)\}.$$

It is also easy to check that $C_n \cdot x \leq (\text{—}, r)$ implies $x \leq (\text{—}, r - \frac{1}{n})$ for any n so that

$$\bigvee_{s<r} h_*(\text{—}, s) = \bigvee\{h_*(x) \mid x \lhd (\text{—}, r)\}.$$

Hence we have $h_*(\text{—}, r) = \bigvee_{s<r} h_*(\text{—}, s)$ as required. ■

4. Continuous hedgehog-valued functions and join cozero κ-families

Definition 4.1. Let L be a frame. A *continuous (metric) hedgehog-valued function* on L is a frame homomorphism $h\colon \mathfrak{L}(J(\kappa)) \to L$.

Of course, in order to specify a continuous hedgehog-valued function on L, we only need to define it on the generators of $\mathfrak{L}(J(\kappa))$ and to check that it turns the conditions (h0)-(h4) into identities in L.

Now recall that a *cozero element* of a frame L is an element of the form

$$\operatorname{coz} h = h((\text{—}, 0) \vee (0, \text{—})) = \bigvee\{h(p, 0) \vee h(0, q) \mid p < 0 < q \text{ in } \mathbb{Q}\}$$

for some continuous real-valued function h in L. Equivalently, $a \in L$ is a cozero element if and only if there exists an $h \in \mathsf{C}(L)$ such that $\mathbf{0} \leq h \leq \mathbf{1}$ and $a = h(0, \text{—})$. This is the pointfree counterpart to the notion of a cozero set for ordinary continuous real-valued functions. For more information on the *cozero map* $\operatorname{coz}\colon \mathsf{C}(L) \to L$ we refer to [5]. As usual, $\operatorname{Coz} L$ will denote the *cozero lattice* of all cozero elements of L.

Remark 4.2. Note that an element $a \in L$ is a cozero element if and only if there exists an $h \in \overline{\mathsf{C}}(L)$ such that $a = \bigvee_{r\in\mathbb{Q}} h(r, \text{—})$. This can be easily checked by considering an increasing bijection φ between $\mathbb{Q} \cap (0, 1)$ and $\mathbb{Q}$.

In the sequel, we refer to a *disjoint* $\{x_i\}_{i\in I} \subseteq L$ whenever $x_i \wedge x_j = 0$ for every $i \neq j$. Recall that by a *discrete* $\{y_i\}_{i\in I}$ in L it is meant a collection for which there is a cover C of L such that for each $c \in C$, $c \wedge y_i = 0$ for all i with possibly one exception. Note that any discrete system is clearly disjoint: if there is a cover C such that for every $c \in C$, $c \wedge y_i = 0$ for all i except possibly one, then $c \wedge y_i \wedge y_j = 0$ for every $c \in C$ and $i \neq j$, that is, $y_i \wedge y_j = 0$.

Definition 4.3. Let κ be a cardinal. We say that a disjoint collection $\{a_i\}_{i\in I}$, $|I| = \kappa$, of cozero elements of a frame L is a *join cozero κ-family* whenever $\bigvee_{i\in I} a_i$ is again a cozero element.

Proposition 4.4. *Let $h\colon \mathfrak{L}(J(\kappa)) \to L$ be a continuous hedgehog-valued function and $a_i = \bigvee_{r\in\mathbb{Q}} h((r,\text{—})_i)$ for each $i \in I$. Then $\{a_i\}_{i\in I}$ is a join cozero κ-family in L.*

Conversely, given a join cozero κ-family $\{a_i\}_{i\in I}$, there exists a continuous hedgehog-valued function $h\colon \mathfrak{L}(J(\kappa)) \to L$ such that $a_i = \bigvee_{r\in\mathbb{Q}} h((r,\text{—})_i)$ for every $i \in I$.

Proof: First note that by Remark 3.6 (1), $h \circ \pi_i \in \overline{\mathsf{C}}(L)$ and thus, by Remark 4.2,

$$\bigvee_{r\in\mathbb{Q}} (h \circ \pi_i)(r,\text{—}) = \bigvee_{r\in\mathbb{Q}} h((r,\text{—})_i) = a_i$$

is a cozero element for each $i \in I$.

On the other hand, it follows from (r0) that

$$a_i \wedge a_j = \bigvee_{r,s\in\mathbb{Q}} h((r,\text{—})_i) \wedge h((s,\text{—})_j) = 0$$

whenever $i \neq j$.

Finally, by Remark 3.6 (2), $h \circ \pi_\kappa \in \overline{\mathsf{C}}(L)$ and hence

$$\bigvee_{r\in\mathbb{Q}} (h \circ \pi_\kappa)(r,\text{—}) = \bigvee_{r\in\mathbb{Q}} (h\big(\bigvee_{i\in I} (r,\text{—})_i\big) = \bigvee_{i\in I} \bigvee_{r\in\mathbb{Q}} h((r,\text{—})_i) = \bigvee_{i\in I} a_i$$

is a cozero element (by Remark 4.2 again). This shows that $\{a_i\}_{i\in I}$ is a join cozero κ-family in L.

Conversely, let $\{a_i\}_{i\in I}$ be a join cozero κ-family in L. For each $i \in I$ there exists $h_i \in \mathsf{C}(L)$ such that $\mathbf{0} \le h_i \le \mathbf{1}$ and $h_i(0,\text{—}) = a_i$ and, additionally, there exists $h_0 \in \mathsf{C}(L)$ such that $\mathbf{0} \le h_0 \le \mathbf{1}$ and $h_0(0,\text{—}) = \bigvee_{i\in I} a_i$. Let φ be an increasing bijection φ between $\overline{\mathbb{Q}}$ and $\mathbb{Q} \cap (0, 1$. The formulas

$$\begin{aligned} h((r,\text{—})_i) &= h_0(\varphi(r),\text{—}) \wedge h_i(\varphi(r),\text{—}) \quad \text{and} \\ h(\text{—}, r) &= h_0(0,\varphi(r)) \vee \big(\bigvee_{i\in I} h_i(0,\varphi(r))\big) \end{aligned}$$

($r \in \mathbb{Q}$, $i \in I$) determine a continuous hedgehog-valued function. Indeed:

(h0) If $i \neq j$ then $h((r,\text{—})_i) \wedge h((s,\text{—})_j) \le h_i(0,\text{—}) \wedge h_j(0,\text{—}) \le a_i \wedge a_j = 0$.

(h1) If $r \geq s$ then

$$\begin{aligned}
&h((r,\text{—})_i) \wedge h(\text{—},s)\\
&\qquad = h_0(\varphi(r),\text{—}) \wedge h_i(\varphi(r),\text{—}) \wedge \big(h_0(0,\varphi(s)) \vee \big(\bigvee_{j\in I} h_j(0,\varphi(s))\big)\big)\\
&\qquad = h_0(\varphi(r),\text{—}) \wedge h_i(\varphi(r),\text{—}) \wedge \big(\bigvee_{j\in I} h_j(0,\varphi(s))\big)\\
&\qquad \leq \big(h_i(\varphi(r),\text{—}) \wedge h_i(0,\varphi(s))\big) \vee \big(h_i(\varphi(r),\text{—}) \wedge \big(\bigvee_{j\neq i} h_j(0,\varphi(s))\big)\big)\\
&\qquad \leq a_i \wedge \big(\bigvee_{j\neq i} a_j\big) = 0.
\end{aligned}$$

(h2) If $r_i < s$ for each $i \in I$ then

$$\begin{aligned}
&\bigvee_{i\in I} h((r_i,\text{—})_i) \vee h(\text{—},s)\\
&\quad = \bigvee_{i\in I}\big(h_0(\varphi(r_i),\text{—}) \wedge h_i(\varphi(r_i),\text{—})\big) \vee \big(h_0(0,\varphi(s)) \vee \big(\bigvee_{j\in I} h_j(0,\varphi(s))\big)\big)\\
&\quad \geq \bigvee_{i\in I}\big(h_0(\varphi(r_i),\text{—}) \wedge h_i(\varphi(r_i),\text{—})\big) \vee \big(h_0(0,\varphi(s)) \vee h_i(0,\varphi(s))\big)\\
&\quad \geq \bigvee_{i\in I}\big(h_0(\varphi(r_i),\text{—}) \vee h_0(0,\varphi(s))\big) \wedge \big(h_i(\varphi(r_i),\text{—}) \vee h_i(0,\varphi(s))\big) = 1.
\end{aligned}$$

(h3)

$$\begin{aligned}
h((r,\text{—})_i) &= h_0(\varphi(r),\text{—}) \wedge h_i(\varphi(r),\text{—}) = \big(\bigvee_{s>r} h_0(\varphi(s),\text{—})\big) \wedge \big(\bigvee_{s>r} h_i(\varphi(s),\text{—})\big)\\
&= \bigvee_{s>r} h_0(\varphi(s),\text{—}) \wedge h_i(\varphi(s),\text{—}) = \bigvee_{s>r} h((s,\text{—})_i)
\end{aligned}$$

(h4)

$$\begin{aligned}
h(\text{—},r) &= h_0(0,\varphi(r)) \vee \big(\bigvee_{i\in I} h_i(0,\varphi(r))\big)\\
&= \big(\bigvee_{s<r} h_0(0,\varphi(s))\big) \vee \big(\bigvee_{i\in I}\bigvee_{s<r} h_i(0,\varphi(s))\big)\\
&= \bigvee_{s<r} h_0(0,\varphi(s)) \vee \big(\bigvee_{i\in I} h_i(0,\varphi(s))\big) = \bigvee_{s<r} h(\text{—},s).
\end{aligned}$$

On the other hand, for each $i \in I$, we have

$$\begin{aligned}\bigvee_{r\in\mathbb{Q}} h((r,\text{—})_i) &= \bigvee_{r\in\mathbb{Q}} h_0(\varphi(r),\text{—}) \wedge h_i(\varphi(r),\text{—}) \\ &= \Big(\bigvee_{r\in\mathbb{Q}} h_0(\varphi(r),\text{—})\Big) \wedge \Big(\bigvee_{r\in\mathbb{Q}} h_i(\varphi(r),\text{—})\Big) \\ &= h_0(0,\text{—}) \wedge h_i(0,\text{—}) = \Big(\bigvee_{j\in I} a_j\Big) \wedge x_i = a_i. \qquad \blacksquare\end{aligned}$$

Remarks 4.5. (1) If $\kappa = 1$ then, of course, a join cozero κ-family is precisely a cozero element.

(2) Since any finite or countable suprema of cozero elements is a cozero element, it follows that in the case $\kappa \leq \aleph_0$, a join cozero κ-family is precisely a disjoint collection of cozero elements.

(3) A frame L is perfectly normal ([13]) if for each $a \in L$ there is a countable family $(b_n)_{n\in\mathbb{N}}$ in L such that $a = \bigvee_{n\in\mathbb{N}} b_n$ and $b_n \prec a$ for all $n \in \mathbb{N}$. Perfectly normal frames are precisely those frames in which every element is cozero. Therefore, in any perfectly normal frame a join cozero κ-family is precisely a disjoint collection of elements.

(4) By Lemma 1 of [14], for any locally finite collection $\{a_i\}_{i\in I}$ of cozero elements, its join $\bigvee_{i\in I} a_i$ is still a cozero element. Consequently, any locally finite disjoint collection of cozero elements $\{a_i\}_{i\in I}$ is a join cozero κ-family. In particular, any discrete collection of cozero elements $\{a_i\}_{i\in I}$ is a join cozero κ-family. However, not every join cozero κ-familiy is locally finite; for example, $\{\bigvee_{r\in\mathbb{Q}}((r,\text{—})_i)\}_{i\in I}$ is a join cozero κ-family in $\mathfrak{L}(J(\kappa))$ (by Proposition 4.4), but it is locally finite if and only if κ is finite.

As a consequence of Remarks (3) and (4) above and Proposition 4.4, we get:

Corollary 4.6. *Let L be a perfectly normal frame and $\{a_i\}_{i\in I}$ a disjoint collection of elements. Then there exists a continuous hedgehog-valued function $h\colon \mathfrak{L}(J(\kappa)) \to L$ such that $\bigvee_{r\in\mathbb{Q}} h((r,\text{—})_i) = a_i$ for each $i \in I$.* $\blacksquare$

5. Universality: Kowalsky's Hedgehog Theorem

Recall from [10] that a family of frame homomorphisms $\{h_i\colon M_i \to L\}_{i\in I}$ is said to be *separating* in case

$$a \le \bigvee_{i\in I} h_i((h_i)_*(a))$$

for every $a \in L$.

Remark 5.1. (See [10, Fact 3.3]) A family of standard continuous functions $\{f_i\colon X \to Y_i\}_{i\in I}$ separates points from closed sets if for every closed set $K \subseteq X$ and every $x \in X \setminus K$, there is an i such that $f_i(x) \notin f_i[K]$. The family $\{f_i\colon X \to Y_i\}_{i\in I}$ separates points from closed sets if and only if the corresponding family of frame homomorphisms $\{\mathfrak{O}f_i\colon \mathfrak{O}Y_i \to \mathfrak{O}X\}_{i\in I}$ is separating.

We need now to recall the following result ([10, Theorem 3.7]), stated here in frame-theoretical terms.

Let $\{h_i\colon M_i \to L\}_{i\in I}$ be a family of frame homomorphisms. Then there is a frame homomorphism $e\colon \bigoplus_{i\in I} M_i \to L$ such that, for each i, the diagram

$$\begin{tikzcd} M_i \arrow[rr, "q_i"] \arrow[dr, "h_i"'] & & \bigoplus\limits_{i\in I} M_i \arrow[dl, "e"] \\ & L & \end{tikzcd}$$

commutes, where $q_i\colon M_i \to \bigoplus_{i\in I} M_i$ is the i^{th} injection map. The map e need not be a quotient map, but one has the following:

Theorem 5.2. ([10, Theorem 3.7]) *If $\{h_i\colon M_i \to L\}_{i\in I}$ is separating then e is a quotient map.*

Furthermore, given a fixed class $\mathbb{L}$ of frames, a frame T in $\mathbb{L}$ is said to be *universal* in this class ([8]) if for every $L \in \mathbb{L}$ there exists a frame homomorphism from T onto L. We now have the following:

Theorem 5.3. *For each cardinal κ, the coproduct $\bigoplus_{n\in\mathbb{N}} \mathfrak{L}(J(\kappa))$ is universal in the class of metric frames of weight $\kappa \cdot \aleph_0$.*

Proof: *Step 1.* (Corollary 3.8) $\bigoplus_{n\in\mathbb{N}} \mathfrak{L}(J(\kappa))$ is a metric frame of weight $\kappa \cdot \aleph_0$.

Step 2. Let L be a metric frame of weight κ. Then L has a σ-discrete base (cf.

[19, Theorem 4.7]), i.e. there exists a base $B \subseteq L$ such that $B = \bigcup_{n\in\mathbb{N}} B_n$, where $B_n = \{a_n^i\}_{i\in I_n}$ is a discrete family. We can assume with no loss of generality that the cardinality of $\bigcup_{n\in\mathbb{N}} I_n$ is precisely κ.

Step 3. As it is well known (see e.g. [7]), any metric frame is perfectly normal. Hence, it follows from Corollary 4.6 that for each $n \in \mathbb{N}$ there exists a continuous hedgehog-valued function $h_n \colon \mathfrak{L}(J(\kappa)) \to L$ such that $a_n^i = \bigvee_{r\in\mathbb{Q}} h_n((r, \text{—})_i)$ for every $i \in I$.

Step 4. The family $\{h_n \colon \mathfrak{L}(J(\kappa)) \to L\}_{n\in\mathbb{N}}$ is separating:
Let $a \in L$. Then for each $m \in \mathbb{N}$ there is a $C_m \subseteq B_m$ such that $a = \bigvee_{m\in\mathbb{N}} \bigvee C_m$. For each $a_m^i \in C_m$ we have

$$(h_m)_*(a_m^i) = (h_m)_*\Big(\bigvee_{r\in\mathbb{Q}} h_m((r, \text{—})_i)\Big) \geq \bigvee_{r\in\mathbb{Q}} (h_m)_*(h_m((r, \text{—})_i)) \geq \bigvee_{r\in\mathbb{Q}} (r, \text{—})_i,$$

thus

$$\bigvee_{n\in\mathbb{N}} h_n((h_n)_*(a_m^i)) \geq h_m((h_m)_*(a_m^i)) \geq \bigvee_{r\in\mathbb{Q}} h_m((r, \text{—})_i) = a_m^i.$$

It then follows that $\bigvee_{n\in\mathbb{N}} h_n((h_n)_*(a)) \geq a$.

Step 5. We can now apply Theorem 5.2 to conclude that the frame homomorphism $e \colon \bigoplus_{n\in\mathbb{N}} \mathfrak{L}(J(\kappa)) \to L$ such that, for each $n \in \mathbb{N}$, the diagram

$$\begin{tikzcd} \mathfrak{L}(J(\kappa)) \arrow[rr, "q_n"] \arrow[dr, "h_n"'] & & \bigoplus_{n\in\mathbb{N}} \mathfrak{L}(J(\kappa)) \arrow[dl, "e"] \\ & L & \end{tikzcd}$$

commutes, is a quotient map. ∎

This is the pointfree extension of Kowalsky's Hedgehog Theorem ([17]) that shows that every metrizable space is embeddable into a countable cartesian power of the metric hedgehog space.

6. κ-Collectionwise normality

Being metrizable, the hedgehog frame is collectionwise normal [22, Theorem 2.5]. Recall that collectionwise normality is a stronger variant of normality introduced by A. Pultr in [22]: while a frame L is *normal* whether for any $x, y \in L$ satisfying $x \vee y = 1$ there exist disjoint $u, v \in L$ such that $x \vee u = 1 = y \vee v$, it is *collectionwise normal* if for each co-discrete system

$\{x_i\}_{i\in I}$ there is a discrete $\{u_i\}_{i\in I}$ such that $x_i \vee u_i = 1$ for every $i \in I$. Here, by a *co-discrete* $\{x_i\}_{i\in I}$ it is meant a collection for which there is a cover C such that for each $c \in C$, $c \le x_i$ for all i with possibly one exception.

More generally, for a cardinal $\kappa \ge 2$, we say that L is *κ-collectionwise normal* if it satisfies the definition of collectionwise normality for sets I with cardinality $|I| \le \kappa$. Hence collectionwise normality is κ-collectionwise normality for any cardinality κ. If $\kappa \le \lambda$ are two cardinalities, then λ-collectionwise normality implies κ-collectionwise normality. Hence, κ-collectionwise normality implies normality for every κ.

Remarks 6.1. (1) A pair $\{x, y\}$ is co-discrete if and only if $x\vee y = 1$. However, $\{x_1, x_2, \ldots, x_n\}$ is co-discrete only if $x_1\vee x_2\vee\cdots\vee x_n = 1$ but the converse does not hold. On the other hand, $\{x, y\}$ is discrete if and only if $x^* \vee y^* = 1$. It then follows that 2-collectionwise normality is just normality. More generally, it can be proved that κ-collectionwise normality coincides with normality for every $\kappa \le \aleph_0$.

(2) Note that, since $c \le x_i^*$ if and only if $c \wedge x_i = 0$, a system $\{x_i\}_{i\in I}$ is discrete if and only if $\{x_i^*\}_{i\in I}$ is co-discrete.

(3) Moreover, for any co-discrete system $\{x_i\}_{i\in I}$ and any $y \in L$, $y \vee \bigwedge_{i\in I} x_i = \bigwedge_{i\in I}(y \vee x_i)$ ([22]). It then follows that for any discrete system $\{x_i\}_{i\in I}$ such that $x_i \prec y$ for all $i \in I$, $\bigvee_{i\in I} x_i \prec y$ ([22]).

(4) It is an easy exercise to check that frame homomorphisms preserve co-discrete sets (i.e, the image of any co-discrete set is co-discrete).

(5) Note also that surjective localic maps preserve disjoint sets, just because ontoness implies the preservation of the bottom element.

We start with a characterization of normality that will be useful in our study:

Lemma 6.2. *A frame L is normal if and only if for every $a, b \in L$ satisfying $a \vee b = 1$ there exists a system $\{u_n\}_{n\in\mathbb{N}}$ such that:*

(N1) $\displaystyle\bigvee_{n\in\mathbb{N}} (a \vee u_n) = 1.$

(N2) $\displaystyle\bigwedge_{n\in\mathbb{N}} (b \vee u_n^*) = 1.$

Proof: The implication '$\Rightarrow$' is obvious: by the normality condition, there is some $u \in L$ satisfying $a \vee u = b \vee u^* = 1$; take $u_n = u$ for every n. Conversely, let $a \vee b = 1$. By hypothesis, there are $\{u_n\}_n$ and $\{v_n\}_n$ such that $\bigvee_{n\in\mathbb{N}}(a\vee u_n)=1=\bigwedge_{n\in\mathbb{N}}(b\vee u_n^*)$ and $\bigvee_{n\in\mathbb{N}}(b\vee v_n)=1=\bigwedge_{n\in\mathbb{N}}(a\vee v_n^*)$. Let

$$u:=\bigvee_{n\in\mathbb{N}}\big(u_n\wedge\bigwedge_{m\leq n}v_m^*\big)\quad\text{ and }\quad v:=\bigvee_{n\in\mathbb{N}}\big(v_n\wedge\bigwedge_{m\leq n}u_m^*\big).$$

Then

$$a\vee u=\bigvee_{n\in\mathbb{N}}\big((a\vee u_n)\wedge\big(a\vee\bigwedge_{m\leq n}v_m^*\big)\big)=\bigvee_{n\in\mathbb{N}}(a\vee u_n)=1.$$

Similarly, $b\vee v=1$. Finally,

$$u\wedge v=\bigvee_{n,n'}\big(u_n\wedge\bigwedge_{m\leq n}v_m^*\wedge v_{n'}\wedge\bigwedge_{m'\leq n'}u_{m'}^*\big)=0.\qquad\blacksquare$$

We need now to recall some facts and notation about sublocales ([19]):

An $S\subseteq L$ is a *sublocale* of L iff the embedding $S\hookrightarrow L$ is a localic map. This means that S is closed under arbitrary infima and moreover $x\to s\in S$ for every $x\in L$ and $s\in S$. The set $\mathcal{S}(L)$ of all sublocales of L forms a *coframe* (i.e., the dual of a frame) under inclusion, in which arbitrary infima coincide with intersections, $\{1\}$ is the bottom element and L is the top element. There are two special classes of sublocales: the *closed* ones, defined as $\mathfrak{c}(a)=\uparrow a$ for each $a\in L$, and the *open* ones, defined as $\mathfrak{o}(a)=\{a\to b\mid b\in L\}$ for each $a\in L$. The F_σ-sublocales are the countable joins of closed subocales in $\mathcal{S}(L)$ ([16]).

Any sublocale S of a frame L is a frame with meets and Heyting operation as in L but joins may differ. Denoting by φ_S the left adjoint of the embedding $S\hookrightarrow L$, we have

$$\bigvee^S A=\varphi_S(\bigvee A)=\bigwedge\{s\in S\mid s\geq\bigvee A\}\geq\bigvee A.$$

In the particular case of an F_σ-sublocale $S=\bigvee_{n\in\mathbb{N}}\mathfrak{c}(a_n)$ one gets from the formula for the joins in the coframe of sublocales that

$$S=\bigvee_{n\in\mathbb{N}}\mathfrak{c}(a_n)=\big\{\bigwedge A\mid A\subseteq\bigcup_{n\in\mathbb{N}}\uparrow a_n\big\}.$$

It then follows easily that:

(F_σ1) $\varphi_S(a)=\bigwedge_{n\in\mathbb{N}}(a_n\vee a)$. In particular, $0_S=\varphi_S(0)=\bigwedge_{n\in\mathbb{N}}a_n$.
(F_σ2) $a\wedge b=0\to\varphi_S(a)\wedge\varphi_S(b)=0_S$.
(F_σ3) $\varphi_S(a)^{*_S}$ (i.e. the pseudocomplement of $\varphi_S(a)$ in S) satisfies the following: $\varphi_S(a)^{*_S}\geq\varphi_S(a^*)\geq a^*$.
(F_σ4) $\varphi_{\mathfrak{c}(a)}(x)=x\vee a$ for each $x\in L$.
(F_σ5) $\varphi_{\mathfrak{o}(a)}(x)=a\to x$ for each $x\in L$.

Remark 6.3. Defining a collection $\{S_i\}_{i\in I}$ of sublocales as being *discrete* whenever there is an open cover $\mathcal{C}$ of sublocales such that for each $C \in \mathcal{C}$, $C \cap S_i = \{1\}$ for all i with possibly one exception, it is obvious that a co-discrete system $\{x_i\}_{i\in I}$ in L corresponds precisely to the discrete collection of closed sublocales $\{\mathfrak{c}(x_i)\}_{i\in I}$ whereas a discrete system $\{y_i\}_{i\in I}$ in L corresponds precisely to the discrete collection of open sublocales $\{\mathfrak{o}(y_i)\}_{i\in I}$. Therefore, a locale is collectionwise normal if and only if for every discrete collection $\{\mathfrak{c}(x_i)\}_{i\in I}$ of closed sublocales of L there exists a discrete collection $\{\mathfrak{o}(y_i)\}_{i\in I}$ of open sublocales such that $\mathfrak{c}(x_i) \subseteq \mathfrak{o}(y_i)$ for all $i \in I$. This shows that the pointfree notion of collectionwise normality is precisely the formulation of the classical notion in the category of locales.

The next result is the extension to locales of the classical result, due to Urysohn [24], that any F_σ-subspace of a normal space is normal.

Proposition 6.4. *Any F_σ-sublocale of a normal locale is normal.*

Proof: Let $F = \bigvee_{n\in\mathbb{N}} \mathfrak{c}(a_n)$ be an F_σ-sublocale of a normal locale L. We shall denote joins in F by $\overset{F}{\bigvee}$. Let $a, b \in F$ with

$$1 = a \overset{F}{\vee} b = \varphi_F(a \vee b) = \bigwedge_{n\in\mathbb{N}} (a_n \vee a \vee b).$$

Then $a_n \vee a \vee b = 1$ for every $n \in \mathbb{N}$ and by the normality of L it follows that there exists $u_n \in L$ such that

$$a_n \vee a \vee u_n = 1 = b \vee u_n^*.$$

In order to show that F is normal it suffices to check that the family $\{\varphi_F(u_n)\}_{n\in\mathbb{N}}$ satisfies the conditions (N1) and (N2) of Lemma 6.2:

(N1) Using property (F_σ1) we obtain

$$\begin{aligned} \overset{F}{\bigvee}_{n\in\mathbb{N}} \big(a \overset{F}{\vee} \varphi_F(u_n)\big) &= \varphi_F\Big(\bigvee_{n\in\mathbb{N}} \big(a \overset{F}{\vee} \varphi_F(u_n)\big)\Big) \geq \varphi_F\Big(\bigvee_{n\in\mathbb{N}} (a \vee u_n)\Big) \\ &= \bigwedge_{n\in\mathbb{N}} \Big(a_n \vee \bigvee_{n\in\mathbb{N}} (a \vee u_n)\Big) = 1. \end{aligned}$$

(N2) By (F_σ3) we have

$$b \overset{F}{\vee} \varphi_F(u_n)^{*_F} \geq b \overset{F}{\vee} u_n^* \geq b \vee u_n^* = 1$$

for every $n \in \mathbb{N}$. ∎

The following lemma identifies the difference between normality and κ-collectionwise normality.

Lemma 6.5. *A frame L is κ-collectionwise normal if and only if it is normal and for each co-discrete $\{x_i\}_{i\in I}$, $|I| \leq \kappa$, there is a disjoint $\{u_i\}_{i\in I}$ such that $u_i \vee x_i = 1$ for every $i \in I$.*

Proof: The implication '$\Rightarrow$' is obvious since any discrete system is disjoint as remarked above.

Conversely, let $\{x_i\}_{i\in I}$ be a co-discrete system. By hypothesis, there is a disjoint $\{u_i\}_{i\in I}$ such that $u_i \vee x_i = 1$ for every i. Now let

$$D = \{x \in L \mid x \wedge u_i \neq 0 \text{ for at most one } i\}$$

and $\overline{d} = \bigvee D$. Evidently, $u_i \in D$ and thus $u_i \leq \overline{d}$ for every i. Then, using Remark 6.1 (3), we have

$$\overline{d} \vee \bigwedge_I x_i = \bigwedge_{i\in I}(\overline{d} \vee x_i) \geq \bigwedge_{i\in I}(u_i \vee x_i) = 1.$$

Since L is normal, there are $u, v \in L$ such that $u \vee \bigwedge_{i\in I} x_i = 1 = v \vee \overline{d}$ and $u \wedge v = 0$. The system

$$\{y_i := u_i \wedge u\}_{i\in I}$$

is the required discrete system that shows that L is collectionwise normal. Indeed, $C = D \cup \{v\}$ is a cover of L (since $\bigvee C = \overline{d} \vee v = 1$), each $c \in C$ meets at most one y_i (since $y_i \wedge v \leq u \wedge v = 0$ for every i) and moreover $y_i \vee x_i = (u_i \vee x_i) \wedge (u \vee x_i) = u \vee x_i \geq u \vee \bigwedge_{i\in I} x_i = 1$ for every i. ■

The next lemma is the counterpart of Lemma 6.2 for κ-collectionwise normality.

Lemma 6.6. *A frame L is κ-collectionwise normal if and only if for each co-discrete $\{x_i\}_{i\in I}$, $|I| \leq \kappa$, there exists a system $\{u_i^n\}_{i\in I}^{n\in\mathbb{N}}$ such that:*

(CN1) *For each $i \in I$, $\bigvee\limits_{n\in\mathbb{N}} (x_i \vee u_i^n) = 1$.*

(CN2) *For each $i \in I$, $\bigwedge\limits_{n\in\mathbb{N}} (x_i \vee \bigwedge\limits_{j\neq i} (u_j^n)^*) = 1$.*

Proof: $\Rightarrow$: By Lemma 6.5, there is a disjoint $\{u_i\}_{i\in I}$ such that $u_i \vee x_i = 1$ for every i. Put $u_i^n = u_i$ for every $i \in I$ and $n \in \mathbb{N}$. (CN1) is obvious. Regarding (CN2), since $\{u_i\}_{i\in I}$ is disjoint, $u_i \leq u_j^*$ for every $i \neq j$. Hence

$$\bigwedge_{n\in\mathbb{N}} (x_i \vee \bigwedge_{j\neq i}(u_j^n)^*) = x_i \vee \bigwedge_{j\neq i} u_j^* \geq x_i \vee u_i = 1.$$

$\Leftarrow$: If $a \vee b = 1$ then $\{a, b\}$ is co-discrete and by hypothesis there exists $\{u_n\}_{n\in\mathbb{N}}$ and $\{v_n\}_{n\in\mathbb{N}}$ such that

$$\bigvee_{n\in\mathbb{N}} (a \vee u_n) = 1 = \bigvee_{n\in\mathbb{N}} (b \vee v_n) \quad \text{and} \quad \bigwedge_{n\in\mathbb{N}} (a \vee v_n^*) = 1 = \bigwedge_{n\in\mathbb{N}} (b \vee u_n^*).$$

This shows that L is normal (by the characterization in Lemma 6.2). We shall conclude that L is moreover collectionwise normal by way of Lemma 6.5. Let $\{x_i\}_{i\in I}$, $|I| \leq \kappa$, be a co-discrete subset of L. By hypothesis, there is some $\{u_i^n\}_{i\in I}^{n\in\mathbb{N}} \subseteq L$ satisfying conditions (CN1) and (CN2). Set

$$u_i := \bigvee_{n\in\mathbb{N}} \big(u_i^n \wedge \bigwedge_{m\leq n} \bigwedge_{j\neq i} (u_j^m)^*\big)$$

for each $i \in I$. Clearly, $x_i \vee u_i = 1$ for every $i \in I$:

$$\begin{aligned}
1 &= \big(\bigvee_n (x_i \vee u_i^n)\big) \wedge \big(\bigwedge_m (x_i \vee \bigwedge_{j\neq i} (u_j^m)^*)\big) = \bigvee_n \big((x_i \vee u_i^n) \wedge \bigwedge_m (x_i \vee \bigwedge_{j\neq i} (u_j^m)^*)\big) \\
&\leq \bigvee_n \big((x_i \vee u_i^n) \wedge \bigwedge_{m\leq n} (x_i \vee \bigwedge_{j\neq i} (u_j^m)^*)\big) = \bigvee_n \big((x_i \vee u_i^n) \wedge \big(x_i \vee \bigwedge_{m\leq n} \bigwedge_{j\neq i} (u_j^m)^*\big)\big) \\
&= \bigvee_n \big(x_i \vee \big(u_i^n \wedge \bigwedge_{m\leq n} \bigwedge_{j\neq i} (u_j^m)^*\big)\big) = x_i \vee u_i.
\end{aligned}$$

Moreover, the collection $\{u_i\}_{i\in I}$ is disjoint, since

$$u_i \wedge u_j \leq \bigvee_{n,n'} \big(u_i^n \wedge u_j^{n'} \wedge \bigwedge_{m\leq n} (u_j^m)^* \wedge \bigwedge_{m'\leq n'} (u_i^{m'})^*\big) = 0$$

for every $i \neq j$. Indeed, for $n \leq n'$,

$$u_i^n \wedge u_j^{n'} \wedge \bigwedge_{m\leq n} (u_j^m)^* \wedge \bigwedge_{m'\leq n'} (u_i^{m'})^* \leq u_i^n \wedge \bigwedge_{m'\leq n'} (u_i^n)^* = 0$$

and for $n' \leq n$,

$$u_i^n \wedge u_j^{n'} \wedge \bigwedge_{m\leq n} (u_j^m)^* \wedge \bigwedge_{m'\leq n'} (u_i^{m'})^* \leq u_j^{n'} \wedge (u_j^{n'})^* = 0. \qquad \blacksquare$$

We are now ready to prove the counterpart of Proposition 6.4 for κ-collectionwise normality.

Proposition 6.7. *Any F_σ-sublocale of a κ-collectionwise normal locale is κ-collectionwise normal.*

Proof: Let $F = \bigvee_{n\in\mathbb{N}} \mathfrak{c}(a_n)$ be an F_σ-sublocale of L. Let $\{x_i\}_{i\in I}$ be a co-discrete family in F with $|I| \leq \kappa$. Then there is a cover C of F such that

$$\text{for each } c \in C,\ c \leq x_i \text{ for all } i \text{ except possibly one.} \tag{2}$$

Since each $a_n \vee \bigvee C \in F$ and $\overset{F}{\bigvee} C = 1$, we have $a_n \vee \bigvee C = 1$ for every $n \in \mathbb{N}$. Hence each $C_n := C \cup \{a_n\}$ is a cover of L. From (2) it is obvious that these covers assert that $\{x_i \vee a_n\}_{i\in I}$ is a co-discrete family in L for every n. Since L is κ-collectionwise normal, it follows that for each $n \in \mathbb{N}$ there is a discrete $\{u_i^n\}_{i\in I} \subseteq L$ such that

$$x_i \vee a_n \vee u_i^n = 1 \text{ for every } i \in I. \tag{3}$$

Now, take $\{\varphi_F(u_i^n)\}_{i\in I}^{n\in\mathbb{N}} \subseteq F$. In order to conclude that F is collectionwise normal, it suffices to check that this family satisfies conditions (CN1) and (CN2) of Lemma 6.6:

(CN1) For each $i \in I$ we have, by $(F_\sigma 1)$,

$$\overset{F}{\bigvee}_{n\in\mathbb{N}} \big(x_i \overset{F}{\vee} \varphi_F(u_i^n)\big) = \varphi_F\Big(\bigvee_{n\in\mathbb{N}} \big(x_i \overset{F}{\vee} \varphi_F(u_i^n)\big)\Big) \geq \varphi_F\Big(\bigvee_{n\in\mathbb{N}} (x_i \vee u_i^n)\Big)$$
$$= \bigwedge_{n\in\mathbb{N}} \Big(a_n \vee \bigvee_{n\in\mathbb{N}} (x_i \vee u_i^n)\Big) = 1.$$

This is equal to 1 by (3).

(CN2) Since $\{u_i^n\}_{i\in I}$ is discrete, it is disjoint. Therefore, by $(F_\sigma 3)$, for every $j \neq i$ we have $\varphi_F(u_j^n)^{*_F} \geq (u_j^n)^* \geq u_i^n$. Hence, for each $i \in I$ and each $n \in N$ we have

$$x_i \overset{F}{\vee} \bigwedge_{j\neq i} \varphi_F(u_j^n)^{*_F} \geq x_i \overset{F}{\vee} \bigwedge_{j\neq i} (u_j^n)^* \geq x_i \overset{F}{\vee} u_i^n = \bigwedge_{n\in\mathbb{N}} (a_n \vee x_i \vee u_i^n) = 1. \qquad \blacksquare$$

This is the pointfree counterpart of the classical result, originally proved in [23], that κ-collectionwise normality is hereditary with respect to F_σ-sets. (It may be worth emphasizing that the localic proof is much simpler.) In particular, it follows that any closed sublocale of a collectionwise normal locale is collectionwise normal.

Let $h\colon M \to L$ be a frame homomorphism, with corresponding localic map $h_*\colon L \to M$, right adjoint to h. Recall that h is *closed* if

$$h_*(x \vee h(y)) = h_*(x) \vee y$$

for every $x \in L$ and $y \in M$. Lemma 6.5 has another nice consequence:

Proposition 6.8. *Let $h\colon M \to L$ be a one-to-one closed frame homomorphism and κ a cardinal. If L is κ-collectionwise normal, then so is M.*

Proof: The fact that M is normal follows from the normality of L by [11, Corollary 9.4]. Let $\{y_i\}_{i\in I}$ be a co-discrete subset of M. By Remark 6.1 (4), $\{h(y_i)\}_{i\in I}$ is co-discrete and thus there exists a disjoint $\{u_i\}_{i\in I} \subseteq L$ such that $u_i \vee h(y_i) = 1$ for every $i \in I$. Now, by Remark 6.1 (5), $\{h_*(u_i)\}_{i\in I}$ is disjoint. Finally, using the hypothesis that h is closed, we conclude that $h_*(u_i) \vee y_i = h_*(u_i \vee h(y_i)) = h_*(1) = 1$ for every i. ■

Formulated in terms of locales, this result shows that the image of a collectionwise normal locale under any closed localic map is collectionwise normal.

7. κ-Collectionwise normality and the metric hedgehog

In this section, we characterize κ-collectionwise normality in terms of continuous hedgehog-valued functions. The first theorem extends Urysohn's separation theorem for normal frames ([4, 2]), which corresponds to the particular case $\kappa = 2$, as $\mathfrak{L}(J(2)) \cong \mathfrak{L}([0,1])$.

Theorem 7.1 (Urysohn-type theorem)**.** *A frame L is κ-collectionwise normal if and only if for each co-discrete system $\{x_i\}_{i\in I}$, $|I| \le \kappa$, there exists a frame homomorphism $h\colon \mathfrak{L}(J(\kappa)) \to L$ such that $h((0,\text{—})_i^*) \le x_i$ for each $i \in I$.*

Proof: Let L be a κ-collectionwise normal frame and let $\{x_i\}_{i\in I} \subseteq L$ be a co-discrete system in L. By Lemma 6.5, there is a disjoint $\{u_i\}_{i\in I}$ such that $u_i \vee x_i = 1$ for every $i \in I$. By the well-known pointfree Urysohn's separation lemma (in the formulation of [12]), there is, for each $i \in I$, a frame homomorphism $h_i\colon \mathfrak{L}(\overline{\mathbb{R}}) \to L$ such that

$$\bigvee_{r\in\mathbb{Q}} h_i(\text{—}, r) \le x_i \quad \text{and} \quad \bigvee_{r\in\mathbb{Q}} h_i(r, \text{—}) \le u_i.$$

Let h be a frame homomorphism $\mathfrak{L}(J(\kappa)) \to L$ determined on generators by

$$h(\text{—}, r) = \bigvee_{t<r} \bigwedge_{i\in I} h_i(\text{—}, t) \quad \text{and} \quad h((r, \text{—})_i) = h_i(r, \text{—})$$

for all $r \in \mathbb{Q}$ and $i \in I$. This assignment turns the defining relations (h0)–(h4) into identities in L:

(h0): If $i \ne j$ one has $h((r,\text{—})_i) \wedge h((s,\text{—})_j) \le u_i \wedge u_j = 0$.

(h1): If $r \ge s$ then $h((r,\text{—})_i) \wedge h(\text{—}, s) \le h_i(r,\text{—}) \wedge h_i(\text{—}, s) = 0$ for every $i \in I$.

(h2): Let $r_i < s$ in $\mathbb{Q}$ for every $i \in I$. First notice that, by Remark 6.1 (2), $\{u_i^*\}_{i\in I}$ is co-discrete. Moreover, by (r2),

$$1 = h_i(r, \text{—}) \vee h_i(\text{—}, r+1) \leq h_i(r, \text{—}) \vee u_i,$$

from which it follows that $u_i^* \leq h_i(r, \text{—})$ for all $i \in I$ and $r \in \mathbb{Q}$. Hence $\{h_i(r, \text{—})\}_{i\in I}$ is also co-discrete for all $r \in \mathbb{Q}$. Now, let C be the cover that witnesses the co-discreteness of $\{u_i^*\}_{i\in I}$ and let $c \in C$. We have:

(a) If $c \leq u_i^*$ for every $i \in I$, then $c \leq h_i(\text{—}, t)$ for all $i \in I$ and $t \in \mathbb{Q}$. Consequently,

$$c \leq \bigvee_{t<s} \bigwedge_{i\in I} h_i(\text{—}, t) = h(\text{—}, s) \leq \bigvee_{i\in I} h((r_i, \text{—})_i) \vee h(\text{—}, s).$$

(b) If $c \not\leq u_{i_0}^*$ for some $i_0 \in I$, let $t_0 \in \mathbb{Q}$ such that $r_{i_0} < t_0 < s$. Then $c \leq u_i^* \leq h_i(\text{—}, t_0)$ for all $i \neq i_0$. Since $\{h_i(t_0, \text{—})\}_{i\in I}$ is co-discrete, by Remark 6.1 (3), one has that

$$\begin{aligned}\bigvee_{i\in I} h((r_i, \text{—})_i) \vee h(\text{—}, s) &\geq h_{i_0}(r_{i_0}, \text{—}) \vee \Big(\bigwedge_{j\in I} h_j(\text{—}, t_0)\Big) \\ &= \bigwedge_{j\in I} \big(h_{i_0}(r_{i_0}, \text{—}) \vee h_j(\text{—}, t_0)\big) \\ &\geq \Big(\bigwedge_{j\neq i_0} h_j(\text{—}, t_0)\Big) \wedge \big(h_{i_0}(r_{i_0}, \text{—}) \vee h_{i_0}(\text{—}, t_0)\big) \geq c.\end{aligned}$$

Hence $\bigvee_{i\in I} h((r_i, \text{—})_i) \vee h(\text{—}, s) \geq \bigvee_{c\in C} c = 1$.
(h3) and (h4) are obvious.

Finally, it follows from Lemma 3.4 (2) that, for each $i \in I$,

$$\begin{aligned}h((0, \text{—})_i^*) &= h\Big(\bigvee_{\substack{j\neq i\\ s\in\mathbb{Q}}} (s, \text{—})_j \vee (\text{—}, 0)\Big) = \bigvee_{\substack{j\neq i\\ s\in\mathbb{Q}}} h_j(s, \text{—}) \vee \bigvee_{t<0} \bigwedge_{j\in I} h_j(\text{—}, t) \\ &\leq \Big(\bigvee_{j\neq i} u_j\Big) \vee \Big(\bigwedge_{j\in I} x_j\Big) \leq x_i\end{aligned}$$

as $u_j \leq x_i$ if $i \neq j$, since $x_i = (u_j \wedge u_i) \vee x_i = u_j \vee x_i$.

Conversely, L is clearly normal (by the pointfree Urysohn's lemma). Further, by hypothesis, given a co-discrete system $\{x_i\}_{i\in I}$ there exists a frame homomorphism $h\colon \mathfrak{L}(J(\kappa)) \to L$ such that $h((0, \text{—})_i^*) \leq x_i$ for all $i \in I$. Let $u_i = h((-1, \text{—})_i)$ for each $i \in I$. By (h0) the system $\{u_i\}_{i\in I}$ is disjoint.

Moreover,

$$u_i \vee x_i \geq h((-1,\text{–})_i) \vee h((0,\text{–})_i^*) \geq h\big((-1,\text{–})_i \vee \bigvee_{j\neq i} (-1,\text{–})_j \vee (\text{–},0)\big) = 1$$

for every $i \in I$. Hence L is κ-collectionwise normal by Lemma 6.5. ■

Our second theorem is a Tietze-type extension theorem for continuous hedgehog-valued functions. To prove it we need first to introduce some terminology and to recall, from [20], a glueing result for localic maps defined on closed sublocales (that we reformulate here in terms of frame homomorphisms).

For each sublocale S of a frame M, we say that a frame homomorphism $h\colon L \to S$ *has an extension* to M if there exists a further frame homomorphism $\widetilde{h}\colon L \to M$ such that the diagram

$$\begin{tikzcd} L \arrow[rr, "h"] \arrow[dr, "\widetilde{h}"'] & & S \\ & M \arrow[ur, "\varphi_S"'] & \end{tikzcd}$$

commutes. In that case we say that $\widetilde{h}\colon L \to M$ extends h.

Proposition 7.2 (Cf. Theorem 4.4 in [20])**.** *Let L and M be frames, $a_1, a_2 \in M$, and let $h_i\colon L \to \mathfrak{c}(a_i)$ $(i = 1,2)$ be frame homomorphisms such that*

$$h_1(x) \vee a_2 = h_2(x) \vee a_1$$

for all $x \in L$. Then the map $h\colon L \to \mathfrak{c}(a_1) \vee \mathfrak{c}(a_2)$ given by $h(x) = h_1(x) \wedge h_2(x)$ is a frame homomorphism that extends both h_1 and h_2.

Theorem 7.3 (Tietze-type theorem)**.** *A frame L is κ-collectionwise normal if and only if for every closed sublocale $\mathfrak{c}(a)$ of L, each frame homomorphism $h\colon \mathfrak{L}(J(\kappa)) \to \mathfrak{c}(a)$ has an extension to L.*

Proof: (i)⇒(ii): Let L be a κ-collectionwise normal frame and $|I| \leq \kappa$. Further, let $a \in L$ and let $h\colon \mathfrak{L}(J(\kappa)) \to \mathfrak{c}(a)$ be a frame homomorphism. By Remark 3.6 (2) we have a continuous extended real-valued function $h_\kappa = h \circ \pi_\kappa\colon \mathfrak{L}(\overline{\mathbb{R}}) \to \mathfrak{c}(a)$ given by

$$h_\kappa(r,\text{–}) = h((\text{–},r)^*) \quad \text{and} \quad h_\kappa(\text{–},r) = h(\text{–},r).$$

By the well-known pointfree Tietze's extension theorem (in the formulation of [12]), h_κ has a continuous extension $\widetilde{h_\kappa}\colon \mathfrak{L}(\overline{\mathbb{R}}) \to L$. Let

$$F = \bigvee_{r\in\mathbb{Q}} \mathfrak{c}\big(\widetilde{h_\kappa}(\text{—}, r)\big) = \bigvee_{r\in\mathbb{Q}} \mathfrak{o}\big(\widetilde{h_\kappa}(r, \text{—})\big) = \mathfrak{o}\big(\bigvee_{r\in\mathbb{Q}} \widetilde{h_\kappa}(r, \text{—})\big).$$

This is an open F_σ-sublocale of L, hence κ-collectionwise normal (by Proposition 6.7).

For each $i \in I$, let

$$x_i = \bigwedge_{r\in\mathbb{Q}} h((r, \text{—})_i^*).$$

Since $h((r, \text{—})_i^*) \geq h(\text{—}, r) = h_\kappa(\text{—}, r) = \widetilde{h_\kappa}(\text{—}, r) \vee a \geq \widetilde{h_\kappa}(\text{—}, r)$ for every $r \in \mathbb{Q}$ and $i \in I$, it follows that

$$x_i \in \bigvee_{r\in\mathbb{Q}} \mathfrak{c}\big(\widetilde{h_\kappa}(\text{—}, r)\big) = F$$

for all $i \in I$. Moreover, the system $\{x_i\}_{i\in I}$ is co-discrete in F. Indeed, let

$$c_i = \bigwedge_{r\in\mathbb{Q}} \big(\widetilde{h_\kappa}(\text{—}, r) \vee \bigvee_{s\in\mathbb{Q}} h((s, \text{—})_i)\big) \in F$$

for every $i \in I$ and $C = \{c_i \mid i \in I\}$. Then C is a cover of F: in fact, since

$$\bigvee_{r\in\mathbb{Q}} \widetilde{h_\kappa}(r, \text{—}) \leq \bigvee_{r\in\mathbb{Q}} h_\kappa(r, \text{—}) = \bigvee_{r\in\mathbb{Q}} h((\text{—}, r)^*) = \bigvee_{r\in\mathbb{Q}} \bigvee_{i\in I} h((r, \text{—})_i) \leq \bigvee_{i\in I} c_i$$

and $F = \mathfrak{o}\big(\bigvee_{r\in\mathbb{Q}} \widetilde{h_\kappa}(r, \text{—})\big)$, it follows that

$$\overset{F}{\bigvee_{i\in I}} c_i = \big(\bigvee_{r\in\mathbb{Q}} \widetilde{h_\kappa}(r, \text{—})\big) \to \big(\bigvee_{i\in I} c_i\big) = 1.$$

Furthermore, $c_i \leq x_j$ whenever $i \neq j$, since in that case

$$\widetilde{h_\kappa}(\text{—}, r) \vee \bigvee_{s\in\mathbb{Q}} h((s, \text{—})_i) \leq h(\text{—}, r) \vee \bigvee_{s\in\mathbb{Q}} h((s, \text{—})_i) \leq h((r, \text{—})_j^*)$$

for all $r \in \mathbb{Q}$.

Thus, we can apply Lemma 6.5 to get some disjoint $\{u_i\}_{i\in I} \subseteq F$ such that $u_i \overset{F}{\vee} x_i = 1$ for every $i \in I$. This means that

$$1 = u_i \overset{F}{\vee} x_i = \big(\bigvee_{r\in\mathbb{Q}} \widetilde{h_\kappa}(r, \text{—})\big) \to (u_i \vee x_i),$$

that is, $\widetilde{h_\kappa}(r, \text{—}) \leq u_i \vee x_i$ for every $r \in \mathbb{Q}$ and $i \in I$.

Let $h_0\colon \mathfrak{L}(\overline{\mathbb{R}}) \to \mathfrak{c}\big(\bigvee_{i\in I} u_i\big)$ be the frame homomorphism given by

$$h_0(r,\text{—}) = \bigvee_{i\in I} u_i \quad \text{and} \quad h_0(\text{—},r) = 1.$$

We claim that $h_\kappa(x) \vee \bigvee_{i\in I} u_i = h_0(x) \vee a$ for all $x \in \mathfrak{L}(\overline{\mathbb{R}})$. Indeed, let $r \in \mathbb{Q}$. First note that

$$h((r,\text{—})_i) \le h((\text{—},r)^*) = h_\kappa(r,\text{—}) = a \vee \widetilde{h_\kappa}(r,\text{—}) \le a \vee u_i \vee x_i = u_i \vee x_i.$$

Since $h((r,\text{—})_i) \wedge x_i = a$ for all $r \in \mathbb{Q}$ and $i \in I$ we have

$$h((r,\text{—})_i) = h((r,\text{—})_i) \wedge (u_i \vee x_i) = (h((r,\text{—})_i) \wedge u_i) \vee a \le u_i \vee a$$

and therefore

$$\begin{aligned} h_\kappa(r,\text{—}) \vee \big(\bigvee_{i\in I} u_i\big) &= h((\text{—},r)^*) \vee \big(\bigvee_{i\in I} u_i\big) = \big(\bigvee_{i\in I} h((r,\text{—})_i)\big) \vee \big(\bigvee_{i\in I} u_i\big) \\ &= \big(\bigvee_{i\in I} u_i\big) \vee a = h_0(r,\text{—}) \vee a. \end{aligned}$$

On the other hand, we also have $h((r-1,\text{—})_i) \le u_i \vee a$ for all $i \in I$ and thus

$$h_\kappa(\text{—},r) \vee \big(\bigvee_{i\in I} u_i\big) = h(\text{—},r) \vee \big(\bigvee_{i\in I} u_i\big) \ge h(\text{—},r) \vee \bigvee_{i\in I} h((r-1,\text{—})_i) = 1.$$

Hence $h_\kappa(\text{—},r) \vee \big(\bigvee_{i\in I} u_i\big) = 1 = h_0(\text{—},r) \vee a$.

Since the identity $h_\kappa(x) \vee \bigvee_{i\in I} u_i = h_0(x) \vee a$ holds for any generator of $\mathfrak{L}(\overline{\mathbb{R}})$, it then holds for all $x \in \mathfrak{L}(\overline{\mathbb{R}})$.

It follows from Proposition 7.2 that the map $g\colon \mathfrak{L}(\overline{\mathbb{R}}) \to \mathfrak{c}\big(a \wedge \bigvee_{i\in I} u_i\big)$ defined by

$$g(r,\text{—}) = h_\kappa(r,\text{—}) \wedge h_0(r,\text{—}) = h((\text{—},r)^*) \wedge \bigvee_{i\in I} u_i$$

and

$$g(\text{—},r) = h_\kappa(\text{—},r) \wedge h_0(\text{—},r) = h(\text{—},r)$$

is a frame homomorphism. Then, by the pointfree Tietze's extension theorem again, g has a continuous extension to L, say $\widetilde{g}\colon \mathfrak{L}(\overline{\mathbb{R}}) \to L$.

Finally, let $\widetilde{h}$ be the frame homomorphism $\mathfrak{L}(J(\kappa)) \to L$ determined on generators by

$$\widetilde{h}((r,\text{—})_i) = \widetilde{g}(r,\text{—}) \wedge u_i \quad \text{and} \quad \widetilde{h}(\text{—},r) = \widetilde{g}(\text{—},r).$$

for all $r \in \mathbb{Q}$ and $i \in I$. This assignment turns the defining relations (h0)–(h4) into identities in L:

(h0) If $i \ne j$ one has $\widetilde{h}((r,\text{—})_i) \wedge \widetilde{h}((s,\text{—})_j) = \widetilde{g}(r,\text{—}) \wedge u_i \wedge \widetilde{g}(s,\text{—}) \wedge u_j = 0$.

(h1): If $r \geq s$ then $\widetilde{h}((r, \text{—})_i) \wedge \widetilde{h}((\text{—}, s)) = \widetilde{g}(r, \text{—}) \wedge u_i \wedge \widetilde{g}(\text{—}, s) = 0$ for every $i \in I$.

(h2): Let $r_i < s$ in $\mathbb{Q}$ for every $i \in I$. Then

$$\begin{aligned}
\bigvee_{i\in I} \widetilde{h}((r_i, \text{—})_i) \vee \widetilde{h}(\text{—}, s) &= \bigvee_{i\in I} (\widetilde{g}(r_i, \text{—}) \wedge u_i) \vee \widetilde{g}(\text{—}, s) = \\
&= \bigvee_{i\in I} (\widetilde{g}(r_i, \text{—}) \vee \widetilde{g}(\text{—}, s)) \wedge (u_i \vee \widetilde{g}(\text{—}, s)) = \\
&= \big(\bigvee_{i\in I} u_i\big) \vee \widetilde{g}(\text{—}, s) = \big(\bigvee_{i\in I} u_i\big) \vee h_0(\text{—}, s) = \\
&= \big(\bigvee_{i\in I} u_i\big) \vee h(\text{—}, s) \geq \big(\bigvee_{i\in I} h((r, \text{—})_i)\big) \vee h(\text{—}, s) = 1.
\end{aligned}$$

(h3) and (h4) are obvious.

We conclude the proof of this implication by proving that $\widetilde{h}$ is the required extension of h.

For each $r \in \mathbb{Q}$ we have

$$\widetilde{h}(\text{—}, r) \vee a = \widetilde{g}(\text{—}, r) \vee a = h_0(\text{—}, r) \vee a = h(\text{—}, r) \vee a = h(\text{—}, r).$$

On the other hand, if $j \neq i$ then $h((r, \text{—})_j) \wedge u_i \leq (u_j \vee a) \wedge u_i = a \wedge u_i \leq a$. Consequently,

$$\begin{aligned}
\widetilde{h}((r, \text{—})_i) \vee a &= \big(\widetilde{g}(r, \text{—}) \wedge u_i\big) \vee a = \big(\widetilde{g}(r, \text{—}) \vee a\big) \wedge (u_i \vee a) = \\
&= (h_0(r, \text{—}) \vee a) \wedge (u_i \vee a) = (h_0(r, \text{—}) \wedge u_i) \vee a \\
&= \big(h((\text{—}, r)^*) \wedge u_i\big) \vee a = \big(\bigvee_{j\in I} h((r, \text{—})_j) \wedge u_i\big) \vee a \\
&= \big(h((r, \text{—})_i) \wedge u_i\big) \vee a = h((r, \text{—})_i) \wedge (u_i \vee a) = h((r, \text{—})_i).
\end{aligned}$$

Since the identity $\widetilde{h}(x) \vee a = h(x)$ holds for any generator of $\mathfrak{L}(J(\kappa))$, it then holds for all $x \in \mathfrak{L}(J(\kappa))$. Hence $\varphi_{\mathfrak{c}(a)} \circ \widetilde{h} = h$.

(ii)⇒(i): Let $\{x_i\}_{i\in I}$ be a co-discrete system in L with $|I| = \kappa$. Further, let $a = \bigwedge_{i\in I} x_i$, $a_i = \bigwedge_{j\neq i} x_j$ for each $i \in I$ and let $h\colon \mathfrak{L}(J(\kappa)) \to \mathfrak{c}(a)$ be a frame homomorphism determined on generators by

$$h(\text{—}, r) = a \quad \text{and} \quad h((r, \text{—})_i) = a_i$$

for all $r \in \mathbb{Q}$ and $i \in I$. This assignment turns the defining relations (h0)–(h4) into identities in L:

(h0): If $i \neq j$ one has $h((r, \text{—})_i) \wedge h((s, \text{—})_j) = a_i \wedge a_j = a$.

(h1): If $r \geq s$ then $h((r, \text{—})_i) \wedge h(\text{—}, s) = a$ for every $i \in I$.

(h2): Let $r_i < s$ in $\mathbb{Q}$ for every $i \in I$. Further, let C be the cover that witnesses the co-discreteness of $\{x_i\}_{i\in I}$ and let $c \in C$. We have:

(a) If $c \le x_i$ for every $i \in I$, then $c \le a = h(\text{—}, s) \le \bigvee_{i\in I} h((r_i, \text{—})_i) \vee h(\text{—}, s)$.

(b) If $c \not\le x_{i_0}$ for some $i_0 \in I$, then $c \le a_{i_0} = h((r_{i_0}, \text{—})_{i_0}) \le \bigvee_{i\in I} h((r_i, \text{—})_i) \vee h(\text{—}, s)$.

Hence $\bigvee_{i\in I} h((r_i, \text{—})_i) \vee h(\text{—}, s) \ge \bigvee_{c\in C} c = 1$.

(h3) and (h4) are obvious.

Note also that $h((0, \text{—})_i^*) \le x_i$ for every $i \in I$. Indeed, for each $i \in I$,

$$h((0,\text{—})_i^*) = h\Big(\bigvee_{\substack{j\neq i\\ s\in\mathbb{Q}}} (s,\text{—})_j \vee (\text{—},0)\Big) = \bigvee_{\substack{j\neq i\\ s\in\mathbb{Q}}} h((s,\text{—})_j) \vee h(\text{—},0) =$$
$$= \Big(\bigvee_{j\neq i} \bigwedge_{k\neq j} x_k\Big) \vee \Big(\bigwedge_{j\in I} x_j\Big) \le x_i.$$

Then, by hypothesis, there exists a frame homomorphism $\widetilde{h}\colon \mathfrak{L}(J(\kappa)) \to L$ such that $\varphi_{\mathfrak{c}(a)} \circ \widetilde{h} = h$. In particular,

$$\widetilde{h}((0,\text{—})_i^*) \le \Big(\varphi_{\mathfrak{c}(a)} \circ \widetilde{h}\Big)((0,\text{—})_i^*) = h((0,\text{—})_i^*) \le x_i$$

for each $i \in I$. The conclusion that L is κ-collectionwise normal follows now from Theorem 7.1. ■

Note that, by letting $\kappa = 2$, this theorem yields Tietze's extension theorem for normal frames ([2]) as a particular case since $\mathfrak{L}(J(2)) \cong \mathfrak{L}([0,1])$.

Corollary 7.4. *The following are equivalent for a frame L:*

(i) *L is collectionwise normal.*

(ii) *For every cardinal κ and every closed sublocale $\mathfrak{c}(a)$ of L, each frame homomorphism $h\colon \mathfrak{L}(J(\kappa)) \to \mathfrak{c}(a)$ has an extension to L.* ■

8. An alternate description of metric hedgehogs

We end the paper with a brief description of an alternate description of metric hedgehog frames that provides further evidence for our presentation of the metric hedgehog frame by generators and relations.

Given a frame L and a set I with cardinality κ consider the frame product $L^\kappa = \prod_{i\in I} L$, that is, the cartesian product ordered pointwisely, regarded as the collection of all maps $\varphi\colon I \to L$ ordered by $\varphi \leq \psi$ if and only if $\varphi(i) \leq \psi(i)$ for every $i \in I$, with projections

$$p_j = (\varphi \mapsto \varphi(j))\colon L^\kappa \to L \quad (j \in I).$$

For each $a \in L$ and $i \in I$, let $\varphi_a^\kappa, \varphi_a^i \in L^\kappa$ be given by

$$\varphi_a^\kappa(j) = a \quad \text{and} \quad \varphi_a^i(j) = \begin{cases} a, & \text{if } j = i, \\ 0, & \text{if } j \neq i, \end{cases}$$

for any $j \in I$. Suppose L has a *point*, that is, a *completely prime filter* F (specifically, a proper filter satisfying $\bigvee A \in F \Rightarrow a \in F$ for some $a \in A$). Given such L, κ and F let L_κ^F be the subframe of L^κ generated by

$$B_\kappa^F = \{\varphi_a^\kappa \mid a \in F\} \cup \{\varphi_b^i \mid b \in L \setminus F \text{ and } i \in I\}.$$

Note that B_κ^F forms a base of L_κ^F, since it is obviously closed under finite meets.

Lemma 8.1. *If L is regular, then so is L_κ^F.*

Proof: First, let $a \in F$. By regularity, $a = \bigvee\{c \in L \mid c \prec a\}$. As F is completely prime, there exists some $b_0 \in F$ such that $c_0 \prec a$. If $c \prec a$ then $c \vee c_0 \prec a$ and $c \vee c_0 \in F$. Hence

$$a = \bigvee\{c \vee c_0 \mid b \in L,\ b \prec a\} = \bigvee\{c \in L \mid c_0 \leq c \prec a\}.$$

Further, if $c \in F$ then $c^* \notin F$, (otherwise $0 = c \wedge c^* \in F$, a contradiction). Therefore, for each $c \in L$ such that $c_0 \leq c \prec a$, one has that $\varphi_c^\kappa, \varphi_a^\kappa \in B_\kappa^F$ and $\varphi_{c^*}^i \in B_\kappa^F$ for every $i \in I$ and hence $\bigvee_{i\in I} \varphi_{c^*}^i \in L_\kappa^F$. Moreover,

$$\varphi_c^\kappa \wedge \bigvee_{i\in I} \varphi_{c^*}^i = 0 \quad \text{and} \quad \varphi_a^\kappa \vee \bigvee_{i\in I} \varphi_{c^*}^i = 1,$$

thus $\varphi_c^\kappa \prec \varphi_a^\kappa$ in L_κ^F. Consequently,

$$\varphi_a^\kappa = \bigvee\{\varphi_c^\kappa \mid c_0 \leq c \prec a\} \leq \bigvee\{\psi \in L_\kappa^F \mid \psi \prec \varphi_a^\kappa\} \leq \varphi_a^\kappa.$$

Secondly, let $b \notin F$, $i \in I$ and $c \prec b$. Note that $c \notin F$, since F is a filter, and $c^* \in F$, since $c^* \vee b = 1$ and F is prime. Therefore, $\varphi_c^i \in B_\kappa^F$, $\varphi_b^j \in B_\kappa^F$ for every $j \neq i$ and $\varphi_{c^*}^\kappa \in B_\kappa^F$. Hence

$$\psi = \varphi_{c^*}^\kappa \vee \bigvee_{j\neq i} \varphi_b^j \in L_\kappa^F.$$

Obviously, $\varphi_c^i \wedge \psi = 0$ and $\varphi_b^i \vee \psi = 1$, hence $\varphi_c^i \prec \varphi_b^i$ in L_κ^F. Consequently,

$$\varphi_b^i = \bigvee\{\varphi_c^i \mid c \prec b\} \leq \bigvee\{\psi \in L_\kappa^F \mid \psi \prec \varphi_b^i\} \leq \varphi_b^i.$$

As B^F_κ is a base of L^F_κ and we have shown that every element in B^F_κ is a join of elements rather below it, we may conclude that L^F_κ is regular. ■

Theorem 8.2. *The metric hedgehog frame $\mathfrak{L}(J(\kappa))$ is isomorphic to $\mathfrak{L}(\overline{\mathbb{R}})^\kappa_F$ for the completely prime filter*

$$F = \{a \in \mathfrak{L}(\overline{\mathbb{R}}) \mid \textit{there is some } r \in \mathbb{Q} \textit{ such that } (\text{—}, r) \leq a\}.$$

Proof: Let $\Phi\colon \mathfrak{L}(J(\kappa)) \to \mathfrak{L}(\overline{\mathbb{R}})^\kappa_F$ be defined on generators by

$$(\text{—}, r) \mapsto \varphi^\kappa_{(\text{—},r)} \quad \text{and} \quad (r, \text{—})_i \mapsto \varphi^i_{(r,\text{—})}$$

for all $r \in \mathbb{Q}$ and $i \in I$. It is straightforward to check that these assignments turn the defining relations (r0)–(r4) into identities in $\mathfrak{L}(\overline{\mathbb{R}})^\kappa_F$. Ontoness follows from the fact that the elements $(\text{—}, r)$ and $(s, \text{—})$ with $r, s \in \mathbb{Q}$ form a subbase of $\mathfrak{L}(\overline{\mathbb{R}})$ (that is, $\mathfrak{L}(\overline{\mathbb{R}})$ is the smallest subframe of $\mathfrak{L}(\overline{\mathbb{R}})$ containing all those elements).

In order to show that Φ is one-one it suffices to show that it is codense, since both $\mathfrak{L}(J(\kappa))$ are $\mathfrak{L}(\overline{\mathbb{R}})^\kappa_F$ are regular. For that purpose, note that any element $a \in \mathfrak{L}(J(\kappa))$ is of the form

$$a = \bigvee A_1 \vee \bigvee A_2 \vee \bigvee A_3$$

where

$$A_1 \subseteq \{(\text{—}, r) \mid r \in \mathbb{Q}\}, \quad A_2 \subseteq \{(r, \text{—})_i \mid r \in \mathbb{Q} \text{ and } i \in I\}$$

and

$$A_3 \subseteq \{(r, s)_i \mid r < s \text{ in } \mathbb{Q} \text{ and } i \in I\}.$$

Then, $\Phi(a) = 1$ if and only if, for every $i \in I$,

$$\Phi(a)(i) = \bigvee_{(\text{—},r)\in A_1} (\text{—}, r) \vee \bigvee_{(r,\text{—})_i\in A_2} (r, \text{—}) \vee \bigvee_{(r,s)_i\in A_3} (r, s) = 1.$$

Consequently, for each $i \in I$ there exist $(\text{—}, r^i) \in A_1$, $(s^i, \text{—})_i \in A_2$ and $\{(r^i_n, s^i_n)_i\}^{m^i}_{n=0} \subseteq A_3$ such that

$$r^i_0 < s^i, \; r^i < s^i_{m^i} \quad \text{and} \quad r^i_n < s^i_{n-1} < s^i_n \quad \text{for } n = 1, \ldots, m^i.$$

By repeated application of assertions (6) and (5) in Lemma 3.4 one gets

$$a \geq (\text{—}, s^i) \vee (r^i, \text{—})_i \vee \bigvee_{n=0}^{m^i} (r^i_n, s^i_n)_i = (\text{—}, s^i) \vee (r^i_0, \text{—})_i = (\text{—}, s^i) \vee \bigvee_{p\in\mathbb{Q}} (p, \text{—})_i.$$

As this holds for every $i \in I$, then by (h4) it follows that

$$a \geq \bigvee_{i\in I} \big((-, s^i) \vee \bigvee_{p\in\mathbb{Q}} (p, -)_i\big) = 1.$$

We may conclude that Φ is codense, hence injective. ■

Corollary 8.3. *$\mathfrak{L}(J(\kappa))$ is isomorphic to $\mathfrak{O}J(\kappa)$.*

Proof: This follows from the fact that $\mathfrak{O}J(\kappa)$ is isomorphic to $\mathfrak{O}[0,1]^{\kappa}_{N_0}$, where N_0 is the filter of all open neighborhoods of 0, and the fact that there is an isomorphism from $\mathfrak{O}[0,1]$ to $\mathfrak{L}(\overline{\mathbb{R}})$ that maps N_0 onto F. ■

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