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REGULARITY OF INTERFACES FOR A PUCCI TYPE SEGREGATION PROBLEM

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Abstract: We show the existence of a Lipschitz viscosity solution u in Ω to a system of fully nonlinear equations involving Pucci-type operators. We study the regularity of the interface $\partial\{u>0\}\cap\Omega$ and we show that the viscosity inequalities of the system imply, in the weak sense, the free boundary condition $u^+_{\nu_+}=u^-_{\nu_-}$, and hence u is a solution to a two-phase free boundary problem. We show that we can apply the classical method of sup-convolutions developed by the first author in [5,6], and generalized by Wang [20, 21] and Feldman [11] to fully nonlinear operators, to conclude that the regular points in $\partial\{u>0\}\cap\Omega$ form an open set of class $C^{1,\alpha}$. A novelty in our problem is that we have different operators, $\mathcal{F}^+$ and $\mathcal{F}^-$, on each side of the free boundary. In the particular case when these operators are the Pucci's extremal operators $\mathcal{M}^+$ and $\mathcal{M}^-$, our results provide an alternative approach to obtain the stationary limit of a segregation model of populations with nonlinear diffusion in [19].

Keywords: Fully nonlinear elliptic systems, Pucci operator, Regularity for viscosity solutions, Segregation of populations, Regularity of the free boundary.
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1. Introduction

The work in the present paper is motivated by the study of the regularity of the free boundary for a limit problem obtained from a segregation model with nonlinear diffusion studied by the third author in [19]. In the case of two populations, the model takes the form

$$\begin{cases} \mathcal{M}^-(u_1^\epsilon)=\dfrac{1}{\epsilon}u_1^\epsilon u_2^\epsilon & \text{in } \Omega \\ \mathcal{M}^-(u_2^\epsilon)=\dfrac{1}{\epsilon}u_1^\epsilon u_2^\epsilon & \text{in } \Omega \\ u_i^\epsilon=f_i \quad i=1,2, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

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where Ω is a bounded Lipschitz domain of $\mathbb{R}^n$, f_1 and f_2 are non-negative, non-zero, Hölder continuous function defined on $\partial\Omega$, with disjoint supports, $\mathcal{M}^-$ denotes the negative Pucci's extremal operator that will be described later. The non-negative solution u_i^ϵ, $i = 1, 2$ of (1.1) can be seen as a density of the population i, and the parameter $\frac{1}{\epsilon} > 0$ characterizes the level of competition between species. In [19] it is proven that along a subsequence, u_1^ϵ and u_2^ϵ converge uniformly in Ω, as $\epsilon \to 0^+$, respectively to u_1 and u_2, non-negative locally Lipschitz functions, solutions of the following free boundary problem, for $i, j = 1, 2$,

$$\begin{cases} \mathcal{M}^-(u_i) = 0 & \text{in } \{u_i > 0\} \\ \mathcal{M}^-(u_i - u_j) \leq 0 & \text{in } \Omega \\ u_1 u_2 = 0 & \text{in } \Omega \\ u_i = f_i & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Let $u := u_1 - u_2$, then $u_1 = u^+$, $u_2 = u^-$, where u^+, u^- are respectively the positive and negative part of u, and system (1.2) can be rewritten in terms of u as follows

$$\begin{cases} \mathcal{M}^-(u) = 0 & \text{in } \{u > 0\} \\ \mathcal{M}^+(u) = 0 & \text{in } \{u < 0\} \\ \mathcal{M}^-(u) \leq 0 & \text{in } \Omega \\ \mathcal{M}^+(u) \geq 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $f = f_1 - f_2$ and $\mathcal{M}^+(u) = -\mathcal{M}^-(-u)$ is the positive Pucci's operator.

In the present paper we study problems likewise (1.3) in a more general setting. Precisely, we consider the following free boundary problem,

$$\begin{cases} \mathcal{F}^-(u) = 0 & \text{in } \{u > 0\} \\ \mathcal{F}^+(u) = 0 & \text{in } \{u < 0\} \\ \mathcal{F}^-(u) \leq 0 & \text{in } \Omega \\ \mathcal{F}^+(u) \geq 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega \end{cases} \tag{1.4}$$

in a bounded smooth domain $\Omega \subset \mathbb{R}^n$, where f is a Lipschitz function defined on $\partial\Omega$ and $\mathcal{F}^-$ and $\mathcal{F}^+$ are uniformly elliptic operators belonging to a class of extremal operators that includes the Pucci's operators $\mathcal{M}^-$ and $\mathcal{M}^+$. Therefore the limit problem (1.3) can be seen as a particular case of (1.4).

We first prove the existence of a Lipschitz solution u of (1.4). Then, we study the regularity of the free boundary set

$$\Gamma := \partial\{u > 0\} \cap \Omega.$$

Denote $u_1 = u^+$ and $u_2 = u^-$, and let ν_i be the interior unit normal vector to $\{u_i > 0\}$. At this stage we have no information about the regularity of the free boundary Γ and the vectors ν_i may not be defined at every point of Γ. However, we can prove that any Lipschitz solution of (1.4) satisfies in a weak sense (viscosity sense) the following free boundary condition

$$\frac{\partial u_1}{\partial \nu_1} = \frac{\partial u_2}{\partial \nu_2} \quad \text{on } \Gamma,$$

that is, the normal derivative of u is continuous across the free boundary.

This will allow us to apply the regularity theory developed by Caffarelli in the seminal papers [4–6] for free boundary problems associated to linear operators and then extended by Wang [20, 21] to the case of fully nonlinear uniformly elliptic concave operators, to show that the subset of regular points of the free boundary is relatively open in Γ and locally of class $C^{1,\alpha}$, $0 < \alpha \leq 1$.

Let us describe more in details the results of the present paper and the strategies followed. Let $x_0 \in \Gamma$ and assume that Γ is smooth around x_0, then since u is a viscosity solution of the first and second equation in (1.4), by the Hopf Lemma we have

$$0 < \frac{\partial u_1}{\partial \nu_1}(x_0), \frac{\partial u_2}{\partial \nu_2}(x_0) < +\infty,$$

that is u has linear growth away from the free boundary around x_0. Thus, we expect that at points where the solution u "behaves well", in fact both u_1 and u_2 have locally linear growth away from the free boundary. The linear behavior of u_i at a point x_0 of the free boundary without regularity assumptions on Γ can be defined as follows: there exists $\tilde{r} = \tilde{r}(x_0) > 0$ and $M = M(x_0) > 0$ such that for any $0 < r < \tilde{r}$,

$$\sup_{B_r(x_0)} u_i \geq Mr. \tag{1.5}$$

A barrier argument shows that the function u_i satisfies (1.5) at points of Γ where there is a tangent ball to Γ contained in its support, as we will see. Points with this property are dense in Γ. Thus, we define $x_0 \in \Gamma$ to be regular if (1.5) holds true for at least one among u_1 and u_2, see Definition 6.1. Then by using that u satisfies in the viscosity sense

$$\mathcal{F}^-(u) \leq 0 \leq \mathcal{F}^+(u) \quad \text{in } \Omega \tag{1.6}$$

we can actually prove that both u_1 and u_2 have linear behavior at any regular point, as expected. The viscosity inequalities (1.6) have to be understood as a sort of free boundary conditions since they are satisfied in the whole Ω and thus across the free boundary too.

Now, solutions of (1.4) have the properties that the positive and negative parts are subharmonic in Ω. Therefore, we can perform a blow up analysis by using the monotonicity formula. In particular, we can show that if u is a Lipschitz solution of (1.4), then around any regular point the free boundary is flat, meaning that it can be trapped in a narrow neighborhood in between two Lipschitz graphs. If in addition there is a tangent ball from one side at $x_0 \in \Gamma$, meaning that the ball is contained either in the positivity set of u or in its negativity set, then we prove that u has the asymptotic behavior

$$u(x) = \alpha < x - x_0, \nu >^+ - \beta < x - x_0, \nu >^- + o(|x - x_0|), \tag{1.7}$$

where α, $\beta > 0$ and ν is the normal vector to the tangent ball at x_0 pointing inward $\{u > 0\}$. The viscosity inequalities (1.6) then imply $\alpha = \beta$, that is u is asymptotically a plane at x_0. This shows that any Lipschitz viscosity solution of (1.4) is also a viscosity solution to the following two phase free boundary problem

$$\begin{cases} \mathcal{F}^-(u) = 0 & \text{in } \{u > 0\} \\ \mathcal{F}^+(u) = 0 & \text{in } \{u < 0\} \\ \frac{\partial u_1}{\partial \nu_1} = \frac{\partial u_2}{\partial \nu_2} & \text{on } \partial\{u > 0\} \cap \Omega. \end{cases} \tag{1.8}$$

We refer to [3] for the theory of viscosity solutions to free boundary problems. The regularity of the free boundary for problems of type (1.8) with same concave fully nonlinear operator in both the positivity and the negativity set of u and with more general free boundary conditions, has been investigated, as already mentioned, in [20, 21]. More general operators have been considered in [1, 10, 11].

Even though in (1.8) there are different operators on each side of the free boundary, we can still apply the results of [20, 21] and prove that for any solution u of (1.8) the following holds: if the free boundary is flat around a point $x_0 \in \Gamma$, then in a neighborhood of x_0 it is a $C^{1,\alpha}$ surface. Going back to the original free boundary problem (1.4), this result implies that the set of regular points is an open subset of Γ locally of class $C^{1,\alpha}$. In particular, u has the asymptotic behavior (1.7) with $\alpha = \beta$ at any regular point.

To conclude, let us mention that we provide a simpler proof than in [19] of the existence of a Lipschitz solution of (1.4) that does not involve a segregation problem. Moreover as a byproduct of our results, we prove existence of a Lipschitz solution of (1.8). Existence of solutions to free boundary problems is in general a main issue. For (1.8), with $\mathcal{F}^+$ replaced by $\mathcal{F}^-$ it has been proven in [22]. We believe that our existence proofs could be generalized to a larger class of fully nonlinear operators.

1.1. Organization of the paper. The operators $\mathcal{F}^-$ and $\mathcal{F}^+$ are defined and their properties described in Section 2. Some examples are provided too. Our main results, Theorems 3.1, 3.2 and 3.3, are contained in Section 3. In Section 4 we recall the monotonicity formula and some related results. Existence of a Lipschitz solution of the free boundary problem (1.8), i.e. Theorem 3.1, is proven in Section 5. In Section 6 we introduce the notion of regular points and we prove the non degeneracy of both u_1 and u_2 at regular points. Section 7 is devoted to the proof of Theorem 3.2. In Section 8 we prove that for the solution of (1.8) flat free boundaries are Lipschitz and, as a corollary, Theorem 3.3. Finally, some properties of the fundamental solution for the operator $\mathcal{F}^-$ are proven in the Appendix.

2. The operators $\mathcal{F}^-$ and $\mathcal{F}^+$. Notation

We will start by defining the two general fully nonlinear uniformly elliptic operators $\mathcal{F}^-$ and $\mathcal{F}^+$. Let $\mathcal{S}_n$ be the set of symmetric $n \times n$ real matrices. Given $0 < \lambda \leq 1 < \Lambda$, let us denote by $\mathcal{A}_{\lambda,\Lambda}$ the set of matrices of $\mathcal{S}_n$ with eigenvalues in $[\lambda, \Lambda]$; i.e,

$$\mathcal{A}_{\lambda,\Lambda} := \{A \in \mathcal{S}_n \mid \lambda I_n \leq A \leq \Lambda I_n\},$$

where I_n is the identity matrix. Let $\mathcal{A}^1_{\lambda,\Lambda}$ and $\mathcal{A}^2_{\lambda,\Lambda}$ be two not empty subsets of $\mathcal{A}_{\lambda,\Lambda}$ with the property that

$$\text{if } A \in \mathcal{A}^i_{\lambda,\Lambda},\ i = 1, 2, \text{ and } O \in \mathcal{O}(n), \text{ then } OAO^t \in \mathcal{A}^i_{\lambda,\Lambda}, \tag{2.1}$$

where we denote by $\mathcal{O}(n)$ the set of $n \times n$ orthogonal matrices. Moreover, we assume that the identity matrix belongs to both sets,

$$I_n \in \mathcal{A}^1_{\lambda,\Lambda} \cap \mathcal{A}^2_{\lambda,\Lambda}. \tag{2.2}$$

Let $\mathcal{F}^+$ and $\mathcal{F}^-$ be the following operators defined over matrices M in $\mathcal{S}_n$,

$$\mathcal{F}^-(M) := \inf_{A\in\mathcal{A}^1_{\lambda,\Lambda}} \mathrm{Tr}(AM) \tag{2.3}$$

and

$$\mathcal{F}^+(M) := \sup_{A\in\mathcal{A}^2_{\lambda,\Lambda}} \mathrm{Tr}(AM). \tag{2.4}$$

We remark that when $\mathcal{A}^1_{\lambda,\Lambda} = \mathcal{A}^2_{\lambda,\Lambda} = \mathcal{A}_{\lambda,\Lambda}$, then $\mathcal{F}^- = \mathcal{M}^-$ and $\mathcal{F}^+ = \mathcal{M}^+$, where $\mathcal{M}^-$ and $\mathcal{M}^+$ are the Pucci's extremal operators defined, for $M \in \mathcal{S}_n$, as follows

$$\mathcal{M}^-(M) = \inf_{A\in\mathcal{A}_{\lambda,\Lambda}} \mathrm{Tr}(AM) = \lambda\sum_{e_i>0} e_i + \Lambda\sum_{e_i<0} e_i$$

and

$$\mathcal{M}^+(M) = \sup_{A\in\mathcal{A}_{\lambda,\Lambda}} \mathrm{Tr}(AM) = \Lambda\sum_{e_i>0} e_i + \lambda\sum_{e_i<0} e_i,$$

where e_i, $i = 1,\dots,n$ are the eigenvalues of the matrix M.

Proposition 2.1. *$\mathcal{F}^-$ and $\mathcal{F}^+$ satisfy, for $M, N \in \mathcal{S}_n$*

(a) $\mathcal{F}^\pm(tM) = t\mathcal{F}^\pm(M)$ *for any* $t \geq 0$*;*
(b) $\mathcal{F}^+(M+N) \leq \mathcal{F}^+(M) + \mathcal{F}^+(N)$ *and hence* $\mathcal{F}^+$ *is convex;*
(c) $\mathcal{F}^-(M+N) \geq \mathcal{F}^-(M) + \mathcal{F}^-(N)$ *and hence* $\mathcal{F}^-$ *is concave;*
(d) *For any* $M \in \mathcal{S}_n$*,*

$$\mathcal{M}^-(M) \leq \mathcal{F}^-(M) \leq tr(M) \leq \mathcal{F}^+(M) \leq \mathcal{M}^+(M);$$

(e) (*Uniformly Ellipticity*) $\mathcal{M}^-(N) \leq \mathcal{F}^\pm(M+N) - \mathcal{F}^\pm(M) \leq \mathcal{M}^+(N)$*;*

Proof: Properties (a)-(c) are clear from the definitions (2.3) and (2.4) and the properties of the sup and inf functions.

Since $\mathcal{A}^1_{\lambda,\Lambda}, \mathcal{A}^2_{\lambda,\Lambda} \subset \mathcal{A}_{\lambda,\Lambda}$, we have that $\mathcal{M}^-(M) \leq \mathcal{F}^-(M)$ and $\mathcal{F}^+(M) \leq \mathcal{M}^+(M)$ for any $M \in \mathcal{S}_n$. Moreover, (2.2) implies that

$$\mathcal{F}^-(M) \leq \mathrm{tr}(M) \leq \mathcal{F}^+(M).$$

This proves (d).

By (b) and the last inequality in (d), we have that

$$\mathcal{F}^+(M+N) - \mathcal{F}^+(M) \leq \mathcal{F}^+(N) \leq \mathcal{M}^+(N).$$

On the other hand, by the properties of the sup function,

$$\mathcal{F}^+(M+N) \geq \mathcal{F}^+(M) + \inf_{A\in\mathcal{A}^2_{\lambda,\Lambda}} \mathrm{Tr}(AN) \geq \mathcal{F}^+(M) + \mathcal{M}^-(N).$$

This concludes the proof of (e) for $\mathcal{F}^+$. Similarly, one can prove (e) for $\mathcal{F}^-$. ■

Let D be a domain of $\mathbb{R}^n$. With a slight abuse of notation, we define the differential operators, for $u \in C^2(D)$ and $x \in D$,

$$\mathcal{F}^-(u)(x) := \mathcal{F}^-(D^2u(x))$$

and

$$\mathcal{F}^+(u)(x) := \mathcal{F}^+(D^2u(x)),$$

where D^2u is the Hessian matrix of u. By Proposition 2.1, the differential operators $\mathcal{F}^-$ and $\mathcal{F}^+$ are 1-homogeneous, uniformly elliptic, $\mathcal{F}^-$ is concave and $\mathcal{F}^+$ is convex. Moreover, by (d), for any $u \in C^2(D)$,

$$\mathcal{M}^-(u) \leq \mathcal{F}^-(u) \leq \Delta u \leq \mathcal{F}^+(u) \leq \mathcal{M}^+(u), \tag{2.5}$$

where again here we denote

$$\mathcal{M}^-(u)(x) := \mathcal{M}^-(D^2u(x)), \quad \mathcal{M}^+(u)(x) := \mathcal{M}^+(D^2u(x))$$

and by Δu the Laplacian of u. Furthermore, the operators $\mathcal{F}^-$ and $\mathcal{F}^+$ are invariant under rotations, as stated in the following proposition.

Proposition 2.2. *Let O be an orthogonal matrix. Let u be a C^2-function and let $v(x) = u(Ox)$. Then,*

$$\mathcal{F}^\pm(v)(x) = \mathcal{F}^\pm(u)(Ox).$$

Proof: Since $D^2v(x) = O^tD^2u(Ox)O$, we have that

$$\begin{aligned}\mathcal{F}^-(v)(x) &= \inf_{A\in\mathcal{A}^1_{\lambda,\Lambda}} \operatorname{tr}\left(AO^tD^2u(Ox)O\right) = \inf_{A\in\mathcal{A}^1_{\lambda,\Lambda}} \operatorname{tr}\left(OAO^tD^2u(Ox)\right)\\ &= \inf_{A\in\mathcal{A}^1_{\lambda,\Lambda}} \operatorname{tr}\left(AD^2u(Ox)\right) = \mathcal{F}^-(u)(Ox),\end{aligned}$$

where we have used that by (2.1),

$$\mathcal{A}^1_{\lambda,\Lambda} = \{OAO^t \,|\, A \in \mathcal{A}^1_{\lambda,\Lambda},\, O \in \mathcal{O}(n)\}.$$

Similarly, $\mathcal{F}^+(v)(x) = \mathcal{F}^+(u)(Ox)$. ■

Remark 2.3. *By Proposition 2.1, Harnack inequality holds true for nonnegative viscosity solutions of $\mathcal{F}^-(u) \leq 0 \leq \mathcal{F}^+(u)$, see [2, Theorem 4.3]. Observe also that $\mathcal{F}^-$ and $\mathcal{F}^+$ satisfy the comparison principle: if D is a bounded domain and u is a viscosity subsolution for $\mathcal{F}^+$ in D, meaning $\mathcal{F}^+(u) \geq 0$ in the viscosity sense in D, v is a viscosity supersolution for $\mathcal{F}^+$ in D, meaning*

$\mathcal{F}^+(v) \leq 0$ *in the viscosity sense* D*, and* $u \leq v$ *on* ∂D *then* $u \leq v$ *in* $\bar{D}$*; the same result holds for* $\mathcal{F}^-$*, see [2, 9] for more details. In addition, since* $\mathcal{F}^-$ *and* $\mathcal{F}^+$ *are respectively concave and convex, interior* $C^{2,\alpha}$*-estimates for solutions of* $\mathcal{F}^\pm(u) = 0$ *hold true, see [2].*

Remark 2.4. *If* u *is solution to* (1.4)*, then*

$$\Gamma := \partial\{u_1 > 0\} \cap \Omega = \partial\{u_2 > 0\} \cap \Omega.$$

Indeed, if there was $x_0 \in (\partial\{u_1 > 0\} \cap \Omega)\setminus\partial\{u_2 > 0\}$*, then in a ball of radius* r *around* x_0 *we would have*

$$\mathcal{F}^-(u_1) = \mathcal{F}^-(u) \leq 0,\ u_1 \geq 0,\ u_1 \not\equiv 0,\ u_1(x_0) = 0.$$

This contradicts the strong maximum principle.

2.1. Some examples.

Example 2.5. *As discussed in the Introduction, the free boundary problem* (1.3)*, which is the limit problem of a population model studied in [19] that takes into account diffusion with preferential directions, is a particular case of problem* (1.4)*. Indeed by choosing* $\mathcal{A}^1_{\lambda,\Lambda} = \mathcal{A}^2_{\lambda,\Lambda} = \mathcal{A}_{\lambda,\Lambda}$*, we have that* $\mathcal{F}^- = \mathcal{M}^-$ *and* $\mathcal{F}^+ = \mathcal{M}^+$*.*

Example 2.6. *By choosing* $\mathcal{A}^1_{\lambda,\Lambda} = \{I_n\}$ *and* $\mathcal{A}^2_{\lambda,\Lambda} = \mathcal{A}_{\lambda,\Lambda}$*, problem* (1.4) *becomes*

$$\begin{cases} \Delta u = 0 & \text{in } \{u > 0\} \\ \mathcal{M}^+(u) = 0 & \text{in } \{u < 0\} \\ \Delta u \leq 0 & \text{in } \Omega \\ \mathcal{M}^+(u) \geq 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega. \end{cases} \tag{2.6}$$

Since the Bellman equations are very helpful to solve optimal stopping strategies see [16], this type of models can eventually be used to describe situations with multiple strategies.

Example 2.7. *By the uniformly ellipticity, (e) in Proposition 2.1, the operators* $\mathcal{F}^+$ *and* $\mathcal{F}^-$ *are Lipschitz continuous as functions in the space* $\mathcal{S}(n)$*. This regularity is optimal for the Pucci's operators* $\mathcal{M}^-$ *and* $\mathcal{M}^+$*. Indeed, consider* *for example a family of matrices* $\{M_t \,|\, t \in \mathbb{R}\}$ *with eigenvalues* $e_{1,t} = t$ *and* $e_{2,t} = e_{3,t} = \ldots = e_{n,t} = 0$, *then*

$$\mathcal{M}^-(M_t) = \begin{cases} \lambda t & \textit{if } t \geq 0 \\ \Lambda t & \textit{if } t < 0, \end{cases}$$

which is a no more than Lipschitz function for $\lambda < \Lambda$. *However there are operators in the class of extremal ones that we consider here which are more regular. Consider for example,*

$$\mathcal{F}^-(M) = \inf\{tr(AM) \,:\, A \in \mathcal{S}_p\}$$

$$\mathcal{F}^+(M) = \sup\{tr(AM) \,:\, A \in \mathcal{S}_p\}$$

where for $p > 0$,

$$\mathcal{A}^1_{\lambda,\Lambda} = \mathcal{A}^2_{\lambda,\Lambda} = \mathcal{S}_p := \{A = (a_{ij}) \in \mathcal{S}_n \,:\, \|a_{ij} - \delta_{ij}\|_{l^p} \leq r_0 < 1\}$$

for some $\lambda = \lambda(r_0) < 1 \leq \Lambda$. *Since, for example, for* $p = 2$ *the balls in the* l^2 *norm are smooth, one can get a higher than Lipschitz regularity for* $\mathcal{F}^+$ *and* $\mathcal{F}^-$ *and thus, better than* $C^{2,\alpha}$ *estimates for the solutions* u *of* $\mathcal{F}^\pm(u) = 0$.

2.2. Notation. For a function u, ∇u and $D^2 u$ denote respectively the gradient of u and the Hessian matrix of u. The standard Euclidean inner produt is denoted by $< \cdot, \cdot >$. We define $u^+ := \max(u, 0)$ and $u^- := \max(-u, 0)$ which are the positive and negative part of u. In the rest of the paper, for the solution u of (1.4), we will use the notation

$$u_1 := u^+ \quad \text{and} \quad u_2 := u^-, \tag{2.7}$$

at our convenience. Notice that

$$|u|(x) = \max(u_1(x), u_2(x)) = u_1(x) + u_2(x).$$

Furthermore, we denote by

$$\Omega(u_i) := \{u_i > 0\}$$

$i = 1, 2$, the positivity set of u_i and by

$$\Gamma := \partial\{u > 0\} \cap \Omega,$$

the free boundary set. If u has an asymptotic development around x_0 along the direction ν_1 given by,

$$u(x) = \alpha < x - x_0, \nu_1 >^+ - \beta < x - x_0, \nu_1 >^- + o(|x - x_0|)$$

we write that

$$\frac{\partial u_1}{\partial \nu_1} = \alpha \quad \text{and} \quad \frac{\partial u_2}{\partial \nu_2} = \beta,$$

where $\nu_2 = -\nu_1$. We will consider the Euclidean norm for the distance, $d(x, y) = |x - y|$. Furthermore, we denote

$$J_r(u_i, x_0) := \frac{1}{r^2} \int_{B_r(x_0)} \frac{|\nabla u_i|^2}{|x - x_0|^{n-2}} dx, \tag{2.8}$$

and

$$J_r(u, x_0) := J_r(u_1, x_0) J_r(u_2, x_0). \tag{2.9}$$

When $x_0 = 0$ we simply write $J_r(u)$ instead of $J_r(u, 0)$.

3. Main results

Theorem 3.1. *Let Ω be a bounded smooth domain of $\mathbb{R}^n$ and f be a Lipschitz continuous function on $\partial\Omega$ such that $f^+ \not\equiv 0$ and $f^- \not\equiv 0$. Then there exists a viscosity solution u of* (1.4) *such that $u_1 = u^+ \not\equiv 0$ and $u_2 = u^- \not\equiv 0$. Moreover u is Lipschitz continuous in $\bar{\Omega}$.*

Theorem 3.2. *Any Lipschitz solution u of* (1.4) *such that $u_1 = u^+ \not\equiv 0$ and $u_2 = u^- \not\equiv 0$, satisfies in the viscosity sense the free boundary condition*

$$\frac{\partial u_1}{\partial \nu_1} = \frac{\partial u_2}{\partial \nu_2} \quad \text{on } \Gamma = \partial\{u > 0\} \cap \Omega,$$

meaning that: if there exists a tangent ball B at $x_0 \in \Gamma$, such that either $B \subset \Omega(u_1)$ or $B \subset \Omega(u_2)$, then there exists $\alpha > 0$ such that

$$u(x) = \alpha < x - x_0, \nu_1 > +o(|x - x_0|) \tag{3.1}$$

where ν_1 is the normal vector to ∂B at x_0 pointing inward to $\Omega(u_1)$ (and $\nu_2 = -\nu_1$). In particular, u is a viscosity solution to the free boundary problem (1.8).

Theorem 3.3. *Let u be any Lipschitz solution of* (1.4) *and let $\mathcal{R}$ be the set of regular points of u, according to Definition 6.1. Then $\mathcal{R}$ is an open subset of Γ and locally a surface of class $C^{1,\alpha}$, with $0 < \alpha \leq 1$. In particular, u has the asymptotic behavior* (3.1) *at any $x_0 \in \mathcal{R}$.*

The proofs of Theorems 3.1, 3.2 and 3.3 are given respectively in Sections 5, 7 and 8.

4. Backround: The monotonicity formula

In this section we recall the Alt-Caffarelli-Friedman monotonicity formula and some related results that we will used later on in the paper. A proof can be found in [3, 18]. We have:

Theorem 4.1 (Monotonicity formula)**.** *Let $u_1, u_2 \in C(B_1(0))$ be nonnegative subharmonic functions in $B_1(0)$. Assume $u_1 u_2 = 0$ and $u_1(0)u_2(0) = 0$. Let $u = u_1 - u_2$ and*

$$J_r(u) = \frac{1}{r^4} \int_{B_r(0)} \frac{|\nabla u_1|^2}{|x|^{n-2}}\, dx \int_{B_r(0)} \frac{|\nabla u_2|^2}{|x|^{n-2}}\, dx, \quad 0 < r < 1.$$

Then $J_r(u)$ is finite and is a non-decreasing function of r. Moreover,

$$J_r(u) \leq c(n) \|u_1\|^2_{L^2(B_1)} \|u_2\|^2_{L^2(B_1)}, \quad 0 < r \leq \frac{1}{2}.$$

Theorem 4.1 can be applied to $u = u_1 - u_2$ solution of (1.4). Indeed, by (2.5) we have $\Delta u_1 \geq \mathcal{F}^-(u_1) = 0$ in $\Omega(u_1)$ and $\Delta u_2 \geq -\mathcal{F}^+(-u_2) = 0$ in $\Omega(u_2)$. Therefore, both u_1 and u_2 are subharmonic functions in the viscosity sense, and thus is in the distributional sense, in the whole Ω,

$$\Delta u_1 \geq 0 \quad \text{in } \Omega \tag{4.1}$$

and

$$\Delta u_2 \geq 0 \quad \text{in } \Omega. \tag{4.2}$$

Remark 4.2. *Since $J_r(u)$ is a monotone nonnegative function, there exists*

$$J_0(u) := \lim_{r \to 0^+} J_r(u). \tag{4.3}$$

The following theorem gives information on the case $J_r(u)$ constant. A proof of it can be found in [18].

Theorem 4.3. *Let $u_1, u_2 \in C(B_1(0))$ be nonnegative subharmonic functions in $B_1(0)$. Assume $u_1 u_2 = 0$ and $u_1(0)u_2(0) = 0$ and let $u = u_1 - u_2$. If*

$$J_{r_1}(u) = J_{r_2}(u) =: k$$

for some $0 < r_1 < r_2 < 1$, then, either one or the other of the following holds:

(i) *$u_1 \equiv 0$ in $B_{r_2}(0)$ or $u_2 \equiv 0$ in $B_{r_2}(0)$;*

(ii) *there exist a unit vector ν, positive constants α, β and a universal positive constant c_n, such that*

$$k = c_n \alpha^2 \beta^2$$

and for any $x \in B_{r_2}(0)$,

$$u(x) = \alpha < x, \nu >^{+} - \beta < x, \nu >^{-} .$$

5. Proof of Theorem 3.1

We consider the Heaviside function $H : \mathbb{R} \to \{0, 1\}$,

$$H(x) = \begin{cases} 1 \text{ when } x \geq 0 \\ 0 \text{ when } x < 0, \end{cases}$$

and we let H_ϵ denote a smooth approximation of H, satisfying $H'_\epsilon \geq 0$. Consider the fully nonlinear uniformly elliptic operator G, defined by

$$G(u) := H(u)\, \mathcal{F}^{-}(u) + (1 - H(u))\, \mathcal{F}^{+}(u) \tag{5.1}$$

and its ϵ-approximation G_ϵ, defined by

$$G_\epsilon(u) := H_\epsilon(u)\, \mathcal{F}^{-}(u) + (1 - H_\epsilon(u))\, \mathcal{F}^{+}(u). \tag{5.2}$$

To prove existence of a Lipschitz solution of (1.4), we prove that for any $\epsilon > 0$, there exists u^ϵ viscosity solution of the problem

$$\begin{cases} G_\epsilon(u^\epsilon) = 0 & \text{in } \Omega \\ u^\epsilon = f & \text{on } \partial\Omega, \end{cases} \tag{5.3}$$

and that the functions u^ϵ's are Lipschitz continuous uniformly in ϵ. Existence of a solution of (1.4) will then follow by using the Ascoli-Arzelà Theorem and the stability of the viscosity solutions in the sets $\{u > 0\}$ and $\{u < 0\}$.

Remark 5.1. *By Proposition 2.1, in the viscosity sense*

$$\mathcal{M}^{-}(u^\epsilon) \leq \mathcal{F}^{-}(u^\epsilon) \leq G_\epsilon(u^\epsilon) \leq \mathcal{F}^{+}(u^\epsilon) \leq \mathcal{M}^{+}(u^\epsilon)$$

We start by proving that any viscosity solution of (5.3) is Lipschitz continuous with Lipschitz norm independent of ϵ.

Theorem 5.2. *Let* $\epsilon > 0$, Ω *be a bounded smooth domain and* G_ϵ *the operator defined in* (5.2). *Let* $f \in C^{0,1}(\partial\Omega)$ *satisfy*

$$\|f\|_{C^{0,1}(\partial\Omega)} \leq K_0.$$

Then, any continuous viscosity solution u^ϵ *of problem* (5.3) *is Lipschitz continuous in* $\overline{\Omega}$ *and*

$$\|u^\epsilon\|_{C^{0,1}(\overline{\Omega})} \leq C,$$

where $C = C(n, \Omega, \lambda, \Lambda, K_0)$.

Proof: Before giving the precise proof, we will give an heuristic argument, just to give an idea of the main technic. Assume that u^ϵ has a further regularity, for instance $u^\epsilon \in C^3(\Omega)$. Since $\mathcal{F}^-(M)$ and $\mathcal{F}^+(M)$ are Lipschitz continuous with respect to $M \in \mathcal{A}_n$, we have that $\mathcal{F}^-(u^\epsilon)$ and $\mathcal{F}^-(u^\epsilon)$ are Lipschitz continuous with respect to x, therefore we can differentiate a.e. in Ω both sides of the equation

$$G_\epsilon(u^\epsilon) = 0,$$

in any direction $\sigma \in \partial B_1(0)$. Indeed, if we denote

$$\mathcal{F}^{\pm}_{ij}(M) := \frac{\partial \mathcal{F}^{\pm}}{\partial m_{ij}}(M),$$

where $M = (m_{ij})$, we obtain

$$\begin{aligned} 0 = \partial_\sigma G_\epsilon(u^\epsilon) &= H_\epsilon(u^\epsilon)\mathcal{F}^-{}_{ij}(u^\epsilon)\partial_{ij}(\partial_\sigma u^\epsilon) + (1 - H_\epsilon(u^\epsilon))\mathcal{F}^+{}_{ij}(u^\epsilon)\partial_{ij}(\partial_\sigma u^\epsilon) \\ &\quad + H_\epsilon^{'}(u^\epsilon)\left(\mathcal{F}^-(u^\epsilon) - \mathcal{F}^+(u^\epsilon)\right)\partial_\sigma u^\epsilon. \end{aligned}$$

Then, if L denotes the linear operator, with coefficients that depend on u^ϵ,

$$L(\cdot) := \left(H_\epsilon(u^\epsilon)\,\mathcal{F}^-{}_{ij}(u^\epsilon) + (1 - H_\epsilon(u^\epsilon))\,\mathcal{F}^+{}_{ij}(u^\epsilon)\right)\partial_{ij}(\cdot),$$

we can see that $\partial_\sigma u^\epsilon$ is a solution of

$$L(\partial_\sigma u^\epsilon) + \underbrace{H_\epsilon^{'}(u^\epsilon)\left(\mathcal{F}^-(u^\epsilon) - \mathcal{F}^+(u^\epsilon)\right)}_{\le 0}\partial_\sigma u^\epsilon = 0.$$

Since $H_\epsilon^{'}(u^\epsilon)\left(\mathcal{F}^-(u^\epsilon) - \mathcal{F}^+(u^\epsilon)\right) \le 0$ by the maximum principle,

$$\sup_{\Omega} \partial_\sigma u^\epsilon \le \sup_{\partial\Omega} \partial_\sigma u^\epsilon.$$

Now, if σ is a tangential direction to $\partial\Omega$, then

$$\sup_{\partial\Omega} \partial_\sigma u^\epsilon = \sup_{\partial\Omega} \partial_\sigma f \le \|\nabla f\|_{L^\infty(\partial\Omega)}.$$

If σ is a normal vector, then a barrier argument (as the one in Claim 1 below) shows that

$$\sup_{\partial\Omega} \partial_\sigma u^\epsilon \le C.$$

Thus, $\partial_\sigma u^\epsilon$ is bounded in Ω and the arbitrarily of σ implies the result. To overcome the lack of regularity, we will use standard techniques from the theory of viscosity solutions. In particular, we will discretize and prove that the incremental quotient is bounded, meaning that there exists a constant C_0 independent of ϵ such that, for $\sigma \in \partial B_1(0)$ and $h > 0$:

$$\sup_{x, x+h\sigma \in \Omega} \frac{u^\epsilon(x+\sigma h) - u^\epsilon(x)}{h} \leq C_0. \tag{5.4}$$

Then the result holds true. To prove (5.4) we first prove the following claim:

Claim 1 : There exists $C_0 > 0$ independent of ϵ such that for any $x \in \overline{\Omega}$ and any $y \in \partial\Omega$,

$$|u^\epsilon(x) - u^\epsilon(y)| \leq C_0|x - y|.$$

Proof of Claim 1: Consider the function ψ solution to

$$\begin{cases} \mathcal{M}^-(\psi) = 0 & \text{in } \Omega \\ \psi = f & \text{on } \partial\Omega. \end{cases}$$

Then, $\psi \in C^{2,\alpha}(\Omega)$ and $\psi \in C^{0,1}(\overline{\Omega})$, see [2,12]. Remark 5.1 and comparison principle implies $u^\epsilon \geq \psi$ in $\overline{\Omega}$. In particular, if $y \in \partial\Omega$ and $x \in \Omega$, we have

$$u^\epsilon(x) - u^\epsilon(y) \geq \psi(x) - \psi(y) \geq -C_0|x - y|, \tag{5.5}$$

for some $C_0 > 0$ independent of ϵ. Similarly, the inequality

$$u^\epsilon(x) - u^\epsilon(y) \leq C_0|x - y| \tag{5.6}$$

follows by comparing u^ϵ with the solution φ of

$$\begin{cases} \mathcal{M}^+(\varphi) = 0 & \text{in } \Omega \\ \varphi = f & \text{on } \partial\Omega. \end{cases}$$

Claim 1 follows from estimates (5.5) and (5.6).

Next, to prove (5.4), assume by contradiction that, for some $\delta > 0$,

$$\sup_{x, x+h\sigma \in \Omega} \frac{u^\epsilon(x+h\sigma) - u^\epsilon(x)}{h} \geq C_0 + \delta,$$

where $C_0 > 0$ is given in Claim 1. Then, for $\kappa > 0$, we have that

$$\sup_{x, x+h\sigma \in \Omega} \frac{u^\epsilon(x+h\sigma) + \kappa|x|^2 - u^\epsilon(x)}{h} \geq C_0 + \delta. \tag{5.7}$$

In what follows we will make explicit the dependence of the operator $G_\epsilon(u)$ in u and $M \in \mathcal{S}_n$ by using the notation $G_\epsilon(u, M)$. Denote $w_h^\epsilon(x) := u^\epsilon(x + h\sigma) + \kappa|x|^2$. Note that, by the uniformly ellipticity and the fact that u^ϵ is solution of (5.3), w_h^ϵ is a strict subsolution of $G(w_h^\epsilon - \kappa|x|^2, D^2 w_h^\epsilon) = 0$, in $\Omega - h\sigma := \{x - h\sigma \,|\, x \in \Omega\}$ as it satisfies in the viscosity sense

$$\begin{aligned} 0 &= G(w_h^\epsilon - \kappa|x|^2, D^2 w_h^\epsilon - 2\kappa I_n) \leq G(w_h^\epsilon - \kappa|x|^2, D^2 w_h^\epsilon) - \mathcal{M}^-(2\kappa I_n) \\ &= G(w_h^\epsilon - \kappa|x|^2, D^2 w_h^\epsilon) - 2\kappa\lambda n, \end{aligned}$$

from which

$$G_\epsilon(w_h^\epsilon - \kappa|x|^2, D^2 w_h^\epsilon) \geq 2\kappa\lambda n \quad \text{in } \Omega - h\sigma. \tag{5.8}$$

In order to infer a differential inequality satisfied by $w_h^\epsilon(x) - u^\epsilon(x)$ in the viscosity sense, consider for any fixed $\tau_0 > 0$ and $0 < \tau < \tau_0$, the upper τ-envelope of w_h^ϵ and the lower τ-envelope of u^ϵ defined respectively by

$$w^\tau(x) := \sup_{y \in \Omega_{\tau_0} - h\sigma} \left\{ w_h^\epsilon(y) + \tau - \frac{1}{\tau}|y - x|^2 \right\}, \quad x \in \Omega_{\tau_0} - h\sigma$$

$$u_\tau(x) := \inf_{y \in \Omega_{\tau_0}} \left\{ u^\epsilon(y) - \tau + \frac{1}{\tau}|y - x|^2 \right\}, \quad x \in \Omega_{\tau_0},$$

where $\Omega_{\tau_0} := \{x \in \Omega \,|\, d(x, \partial\Omega) > \tau_0\}$ and, for simplicity of notation, we have dropped the dependence on h and ϵ.

Claim 2 : The upper and lower τ-envelopes have the following properties:

a) $w^\tau \in C(\Omega_{\tau_0} - h\sigma)$, $u_\tau \in C(\Omega_{\tau_0})$, $w^\tau \geq w_h^\epsilon + \tau$, $u_\tau \leq u^\epsilon - \tau$, $w^\tau \to w_h^\epsilon$ and $u_\tau \to u^\epsilon$ as $\tau \to 0$ uniformly in $\Omega_{\tau_0} - h\sigma$ and in Ω_{τ_0} respectively.

b) For any $x \in \Omega_{\tau_0} \cap (\Omega_{\tau_0} - h\sigma)$ there exists a concave (resp., convex) paraboloid of opening $2/\tau$ that touches w^τ (resp., u_τ) by below (resp., above) at x. In particular, w^τ and u_τ are punctually second order differentiable almost everywhere in $\Omega_{\tau_0} \cap (\Omega_{\tau_0} - h\sigma)$, meaning that, for a.e. $x_0 \in \Omega_{\tau_0} \cap (\Omega_{\tau_0} - h\sigma)$ there exist paraboloids P_w and P_u, such that, $w^\tau(x) = P_w(x) + o(|x - x_0|^2)$ and $u_\tau(x) = P_u(x) + o(|x - x_0|^2)$ as $x \to x_0$.

c) If $x^\tau \in \Omega_{\tau_0} - h\sigma$is such that $w^\tau(x) = w_h^\epsilon(x^\tau) + \tau - \frac{1}{\tau}|x - x^\tau|^2$, then

$$\frac{1}{\tau}|x - x^\tau|^2 \leq w_h^\epsilon(x^\tau) - w_h^\epsilon(x).$$

If $x_\tau \in \Omega_{\tau_0}$ is such that $u_\tau(x) = u^\epsilon(x_\tau) - \tau + \frac{1}{\tau}|x - x_\tau|^2$, then

$$\frac{1}{\tau}|x - x_\tau|^2 \leq u^\epsilon(x) - u^\epsilon(x_\tau).$$

d) There exists $\tau_1 > 0$ such that for any $\tau < \tau_1$, w^τ is a viscosity (and therefore a.e.) subsolution to

$$G_\epsilon\left(w^\tau(x) - \tau + \frac{1}{\tau}|x - x^\tau|^2 - \kappa|x^\tau|^2, D^2w^\tau(x)\right) = 2\kappa\lambda n, \quad x \in \Omega_{2\tau_0} - h\sigma$$

and u_τ is a viscosity (and therefore a.e.) supersolution to

$$G_\epsilon\left(u_\tau(x) + \tau - \frac{1}{\tau}|x - x_\tau|^2, D^2u_\tau(x)\right) = 0, \quad x \in \Omega_{2\tau_0}.$$

Proof of Claim 2: For the proofs of (a)-(c) see Theorem 5.1 and Lemma 5.2 in [2]. Note that by (c),

$$\frac{1}{\tau}|x - x^\tau|^2 \le w_h^\epsilon(x_\tau) - w_h^\epsilon(x) \le C.$$

Since w_h^ϵ is continuous this implies that

$$\frac{1}{\tau}|x - x^\tau|^2 \to 0 \quad \text{as } \tau \to 0. \tag{5.9}$$

Similarly,

$$\frac{1}{\tau}|x - x_\tau|^2 \to 0 \quad \text{as } \tau \to 0. \tag{5.10}$$

To prove (d), let $x_0 \in \Omega_{2\tau_0} - h\sigma$ and let $P(x)$ be a paraboloid that touches by above w^τ at x_0. Consider the paraboloid

$$Q(x) = P(x + x_0 - x_0^\tau) + \frac{1}{\tau}|x_0 - x_0^\tau|^2 - \tau.$$

By (5.9), we can pick $\tau_1 > 0$ independent of x_0 such that $x_0^\tau \in \Omega_{\tau_0} - h\sigma$ for any $\tau < \tau_1$. Take any x sufficiently close to x_0^τ, so that $x + x_0 - x_0^\tau \in \Omega_{\tau_0} - h\sigma$, then, by definition of w^τ,

$$w_h^\epsilon(x) \le w^\tau(x + x_0 - x_0^\tau) + \frac{1}{\tau}|x_0 - x_0^\tau|^2 - \tau \le Q(x).$$

Moreover, $w_h^\epsilon(x_0^\tau) = Q(x_0^\tau)$, since $w^\tau(x_0) = P(x_0)$. Hence Q touches w_h^ϵ by above at x_0^τ and by (5.8),

$$\begin{aligned} 2\kappa\lambda n &\le G_\epsilon(w_h^\epsilon(x_0^\tau) - \kappa|x_0^\tau|^2, D^2Q(x_0^\tau)) \\ &= G_\epsilon\left(w^\tau(x_0) - \tau + \frac{1}{\tau}|x_0 - x_0^\tau|^2 - \kappa|x_0^\tau|^2, D^2P(x_0)\right). \end{aligned}$$

Similarly one can prove the second viscosity inequality in (d). This concludes the proof of Claim 2.

Let us continue the proof of the theorem. We have assumed that (5.7) is true. If the supremum in (5.7) is attained at $\bar{x}$, then both $\bar{x}$ and $\bar{x}+\sigma h$ have to be in the interior of Ω, for otherwise we would have $u^\epsilon(\bar{x}+h\sigma)-u^\epsilon(\bar{x})\geq C_0h+\delta h-\kappa|\bar{x}|^2\geq C_0h+\delta h-\kappa C(\Omega)$ which contradicts Claim 1 for $\kappa<\delta h/C(\Omega)$. Thus, there exists $\tau_0>0$ such that

$$\sup_{x+h\sigma,x\in\Omega}\frac{w_h^\epsilon(x)-u^\epsilon(x)}{h}=\sup_{x+h\sigma,x\in\Omega_{3\tau_0}}\frac{w_h^\epsilon(x)-u^\epsilon(x)}{h}\geq C_0+\delta.$$

For τ small enough, by (a) of Claim 2, there exists $x_0\in\Omega_{\frac{5\tau_0}{2}}\cap\big(\Omega_{\frac{5\tau_0}{2}}-h\sigma\big)$ such that

$$\sup_{x+h\sigma,x\in\Omega_{\tau_0}}\frac{w^\tau(x)-u_\tau(x)}{h}=\frac{w^\tau(x_0)-u_\tau(x_0)}{h}=M\geq C_0. \tag{5.11}$$

Take $s>0$ and $r<\tau_0/2$ small enough so that $B_r(x_0)\subset\Omega_{2\tau_0}\cap\big(\Omega_{2\tau_0}-h\sigma\big)$ and define

$$v(x):=u_\tau(x)-w^\tau(x)+Mh+s|x-x_0|^2-sr^2.$$

Then v has a strict minimum at x_0, moreover

$$v(x)\geq 0\quad\text{on }\partial B_r(x_0)\quad\text{and}\quad v(x_0)=-sr^2<0. \tag{5.12}$$

Let us denote by Γ_v the convex envelope of $-v^-$ in $B_{2r}(x_0)$, where we have extended $v\equiv 0$ outside $B_r(x_0)$. Here we use standard techniques from the theory of viscosity solutions, see [2]. Since we do not know if w^τ and u_τ are twice differentiable at x_0, we introduce the convex envelope in order to find a point x_1 of twice differentiability for both w^τ and u_τ such that $w^\tau(x_1)>u_\tau(x_1)$ and $D^2w^\tau(x_1)\leq D^2u_\tau(x_1)$+small corrections. We have that $\Gamma_v\leq 0$ in $B_{2r}(x_0)$. By (b) of Claim 2, for any $x\in\overline{B_r(x_0)}\cap\{v=\Gamma_v\}$ there exists a convex paraboloid with opening independent of x that touches Γ_v by above. By Lemma 3.5 in [2], $\Gamma_v\in C^{1,1}(B_r(x_0))$ and

$$\int_{B_r(x_0)\cap\{v=\Gamma_v\}}\det D^2\Gamma_v\,dx>0.$$

In particular $|B_r(x_0)\cap\{v=\Gamma_v\}|>0$. Since w^τ and u_τ are second order differentiable almost everywhere in $B_r(x_0)$, there exists $x_1\in B_r(x_0)\cap\{v=\Gamma_v\}$ such that w^τ and u_τ are second order differentiable at x_1 and by (d) of

Claim 2,

$$G_\epsilon\left(w^\tau(x_1)-\tau+\frac{1}{\tau}|x_1-x_1^\tau|^2-\kappa|x_1^\tau|^2, D^2w^\tau(x_1)\right)\geq 2\kappa\lambda n \tag{5.13}$$

and

$$G_\epsilon\left(u_\tau(x_1)+\tau-\frac{1}{\tau}|x_1-(x_1)_\tau|^2, D^2u_\tau(x_1)\right)\leq 0. \tag{5.14}$$

Since Γ_v is convex, $\Gamma_v\leq v$ and $\Gamma_v(x_1)=v(x_1)$, we have that $D^2v(x_1)\geq 0$, i.e.,

$$D^2w^\tau(x_1)\leq D^2u_\tau(x_1)+2sI_n. \tag{5.15}$$

Moreover, since Γ_v is negative in $B_r(x_0)$, we have that

$$w^\tau(x_1)>u_\tau(x_1)+s|x_1-x_0|^2-sr^2+Mh.$$

In particular, for s and r small enough

$$w^\tau(x_1)>u_\tau(x_1)+\frac{Mh}{2}. \tag{5.16}$$

Let us denote $\varphi^\tau(x_1):=-\tau+\frac{1}{\tau}|x_1-x_1^\tau|^2$ and $\varphi_\tau(x_1):=\tau-\frac{1}{\tau}|x_1-(x_1)_\tau|^2$. Then, by subtracting the inequalities (5.13) and (5.14), we get

$$\begin{aligned}
2\kappa\lambda n &\leq G_\epsilon\left(w^\tau(x_1)+\varphi^\tau(x_1)-\kappa|x_1^\tau|^2, D^2w^\tau(x_1)\right)-G_\epsilon\left(u_\tau(x_1)+\varphi_\tau(x_1), D^2u_\tau(x_1)\right)\\
&= H_\epsilon\left(w^\tau(x_1)+\varphi^\tau(x_1)-\kappa|x_1^\tau|^2\right)\mathcal{F}^-(w^\tau)(x_1)\\
&+\left(1-H_\epsilon\left(w^\tau(x_1)+\varphi^\tau(x_1)-\kappa|x_1^\tau|^2\right)\right)\mathcal{F}^+(w^\tau)(x_1)\\
&-H_\epsilon\left(u_\tau(x_1)+\varphi_\tau(x_1)\right)\mathcal{F}^-(u_\tau)(x_1)-(1-H_\epsilon\left(u_\tau(x_1)+\varphi_\tau(x_1)\right))\mathcal{F}^+(u_\tau)(x_1).
\end{aligned}$$

Adding and subtracting

$$H_\epsilon\left(u_\tau(x_1)+\varphi_\tau(x_1)\right)\mathcal{F}^-(w^\tau)(x_1)$$

and

$$[1-H_\epsilon\left(u_\tau(x_1)+\varphi_\tau(x_1)\right)]\mathcal{F}^+(w^\tau)(x_1),$$

in the right hand-side of the inequality above, we obtain

$$
\begin{aligned}
2\kappa\lambda n \leq\ & [H_\epsilon\left(w^\tau(x_1)+\varphi^\tau(x_1)-\kappa|x_1^\tau|^2\right)-H_\epsilon\left(u_\tau(x_1)+\varphi_\tau(x_1)\right)]\mathcal{F}^-(w^\tau)(x_1)\\
&+H_\epsilon\left(u_\tau(x_1)+\varphi_\tau(x_1)\right)\mathcal{F}^-(w^\tau)(x_1)\\
&-[H_\epsilon\left(w^\tau(x_1)+\varphi^\tau(x_1)-\kappa|x_1^\tau|^2\right)-H_\epsilon\left(u_\tau(x_1)+\varphi_\tau(x_1)\right)]\mathcal{F}^+(w^\tau)(x_1)\\
&+[1-H_\epsilon\left(u_\tau(x_1)+\varphi_\tau(x_1)\right)]\mathcal{F}^+(w^\tau)(x_1)\\
&-H_\epsilon\left(u_\tau(x_1)+\varphi_\tau(x_1)\right)\mathcal{F}^-(u_\tau)(x_1)-(1-H_\epsilon\left(u_\tau(x_1)+\varphi_\tau(x_1)\right))\mathcal{F}^+(u_\tau)(x_1)\\
&=[H_\epsilon\left(w^\tau(x_1)+\varphi^\tau(x_1)-\kappa|x_1^\tau|^2\right)-H_\epsilon\left(u_\tau(x_1)+\varphi_\tau(x_1)\right)]\mathcal{F}^-(w^\tau)(x_1)\\
&-[H_\epsilon\left(w^\tau(x_1)+\varphi^\tau(x_1)-\kappa|x_1^\tau|^2\right)-H_\epsilon\left(u_\tau(x_1)+\varphi_\tau(x_1)\right)]\mathcal{F}^+(w^\tau)(x_1)\\
&+H_\epsilon\left(u_\tau(x_1)+\varphi_\tau(x_1)\right)[\mathcal{F}^-(w^\tau)(x_1)-\mathcal{F}^-(u_\tau)(x_1)]\\
&+[1-H_\epsilon\left(u_\tau(x_1)+\varphi_\tau(x_1)\right)][\mathcal{F}^+(w^\tau)(x_1)-\mathcal{F}^+(u_\tau)(x_1)]\\
&\leq[H_\epsilon\left(w^\tau(x_1)+\varphi^\tau(x_1)-\kappa|x_1^\tau|^2\right)-H_\epsilon\left(u_\tau(x_1)+\varphi_\tau(x_1)\right)]\\
&\quad[\mathcal{F}^-(w^\tau)(x_1)-\mathcal{F}^+(w^\tau)(x_1)]+\mathcal{M}^+(w^\tau-u_\tau)(x_1).
\end{aligned}
$$

We have obtained

$$
\begin{aligned}
2\kappa\lambda n \leq\ & [H_\epsilon\left(w^\tau(x_1)+\varphi^\tau(x_1)-\kappa|x_1^\tau|^2\right)-H_\epsilon\left(u_\tau(x_1)+\varphi_\tau(x_1)\right)]\\
&[\mathcal{F}^-(w^\tau)(x_1)-\mathcal{F}^+(w^\tau)(x_1)]+\mathcal{M}^+(w^\tau-u_\tau)(x_1).
\end{aligned}
\tag{5.17}
$$

Now, by (5.9) and (5.10) we have that $\varphi^\tau(x_1),\varphi_\tau(x_1)\to 0$ as $\tau\to 0$. This, combined with (5.16), yields

$$w^\tau(x_1)+\varphi^\tau(x_1)-\kappa|x_1^\tau|^2>u_\tau(x_1)+\varphi_\tau(x_1)$$

for s,κ,τ small enough. Since H_ϵ is non-decreasing and $\mathcal{F}^-(w^\tau)(x_1)-\mathcal{F}^+(w^\tau)(x_1)\leq 0$, we infer that

$$[H_\epsilon\left(w^\tau(x_1)+\varphi^\tau(x_1)-\kappa|x_1^\tau|^2\right)-H_\epsilon\left(u_\tau(x_1)+\varphi_\tau(x_1)\right)][\mathcal{F}^-(w^\tau)(x_1)-\mathcal{F}^+(w^\tau)(x_1)]\leq 0.$$

Next, from (5.15),

$$\mathcal{M}^+(w^\tau-u_\tau)(x_1)\leq 2sn\Lambda.$$

Plugging the last two inequalities into (5.17), we obtain

$$2\kappa\lambda n\leq 2sn\Lambda,$$

which is a contradiction for s small enough ($s<\kappa\lambda/\Lambda$).

We have proven that for any $\delta>0$,

$$\sup_{x,x+h\sigma\in\Omega}\frac{u^\epsilon(x+h\sigma)-u^\epsilon(x)}{h}\leq C_0+\delta.$$

Letting δ go to 0, we get (5.4).

Note that comparing u^ϵ with the sub and supersolution introduced in Claim 1, we infer that there exists $C_1 > 0$ independent of ϵ such that

$$\|u^\epsilon\|_{L^\infty(\Omega)} \leq C_1.$$

This bound combined with (5.4) yields a uniform in ϵ estimate of the Lipschitz norm of the solution u^ϵ of (5.3). Thus the theorem is proven. ■

Theorem 5.3. *Under the assumptions of Theorem 5.2, there exists a continuous viscosity solution u^ϵ of the ϵ-problem* (5.3). *Moreover,*

$$\mathcal{F}^-(u^\epsilon) \leq 0 \leq \mathcal{F}^+(u^\epsilon) \tag{5.18}$$

in the viscosity sense in Ω.

Proof: We fix $\epsilon > 0$. For $\alpha \in (1/2, 1)$, let $\Theta := C^{0,\alpha}(\overline{\Omega})$ be the Banach space of α-Hölder continuous functions defined on $\overline{\Omega}$. Let T be the operator defined in the following way, for $u \in \Theta$,

$$T(u) = v$$

if v is the viscosity solution of

$$\begin{cases} H_\epsilon(u)\,\mathcal{F}^-(v) + (1 - H_\epsilon(u))\,\mathcal{F}^+(v) = 0 & \text{in } \Omega \\ v = f & \text{on } \partial\Omega. \end{cases} \tag{5.19}$$

Note that T is well defined. Indeed by Proposition 2.1 the operator $G_\epsilon(x, v) := H_\epsilon(u)\,\mathcal{F}^-(v) + (1 - H_\epsilon(u))\,\mathcal{F}^+(v)$ is uniformly elliptic. Moreover, since $u \in C^{0,\alpha}(\overline{\Omega})$ with $\alpha > 1/2$, $G_\epsilon(x, v)$ satisfies the comparison principle, see [14, Theorem III.1]. Let ψ and φ be the solutions of

$$\begin{cases} \mathcal{M}^-(\psi) = 0 & \text{in } \Omega \\ \psi = f & \text{on } \partial\Omega \end{cases} \quad \text{and} \quad \begin{cases} \mathcal{M}^+(\varphi) = 0 & \text{in } \Omega \\ \varphi = f & \text{on } \partial\Omega. \end{cases}$$

Then, ψ and φ are respectively sub and supersolution of (5.19). Thus, by the Perron's method, there exists a unique viscosity solution of (5.19).

Observe that if T has a fixed point u^ϵ, that is $T(u^\epsilon) = u^\epsilon$, then u^ϵ is solution to (5.3). Moreover, by Remark 5.1, (5.18) also follows. We will prove that we can apply the Leray-Schauder fixed point theorem [12, Theorem 11.3] and conclude that T has a fixed point, which concludes the proof. We have:

(1) $T(\Theta) \subset \Theta$: Let $v = T(u)$, then regularity theory implies that $v \in C^{0,\beta}(\overline{\Omega})$, for any $\beta \in (0,1)$, see [14, Theorem VII.1]. In particular $v \in \Theta$.
(2) *T is continuous:* Let $\{u_n\} \subset \Theta$ be such that $u_n \to \bar{u}$ in Θ. We need to prove that $v_n := T(u_n) \to \bar{v} := T(\bar{u})$ in Θ. By the Hölder estimates for the solutions v_n, [14, Theorem VII.1], we have that $\|v_n\|_{C^{0,\beta}(\overline{\Omega})} \leq C$ for $\beta > \alpha$. Since the subset of Θ of β-Hölder continuous functions on $\overline{\Omega}$ is precompact in Θ, we can extract from $\{v_n\}$ a convergent subsequence. Let $\{v_{n_k}\}$ be any convergent subsequence, $v_{n_k} \to w$ as $k \to +\infty$ in Θ, then by the stability of viscosity solutions under uniform convergence, it follows that w solves (5.19) with $u = \bar{u}$, that is $w = T(\bar{u}) = \bar{v}$. Since every convergent subsequence converges to the same limit function $\bar{v}$, we have that the full sequence $\{v_n\}$ converges to $\bar{v}$ in Θ.
(3) *T is compact:* By the Hölder estimates, T maps bounded set of Θ into bounded sets of $C^{0,\beta}(\overline{\Omega})$, $\beta > \alpha$ which are precompact in Θ.
(4) *There exists $M > 0$ such that $\|u\|_\Theta < M$ for all $u \in \Theta$ and $\sigma \in [0,1]$ satisfying $u = \sigma T(u)$: the equation $u = \sigma T(u)$ is equivalent to the Dirichlet problem*

$$\begin{cases} H_\epsilon(u)\,\mathcal{F}^-(u) + (1 - H_\epsilon(u))\,\mathcal{F}^+(u) = 0 & \textit{in}\,\Omega \\ u = \sigma f & \textit{on}\,\partial\Omega. \end{cases}$$

The estimate $\|u\|_\Theta < M$, for some $M > 0$, then follows from Theorem 5.2.

This concludes the proof of the existence of a fixed point u^ϵ and thus of a solution of (5.3) satisfying (5.18). ■

5.1. Proof of Theorem 3.1. By Theorem 5.3, for any $\epsilon > 0$ there exists u^ϵ viscosity solution of (5.3), satisfying also (5.18). By Theorem 5.2 the sequence $\{u^\epsilon\}$ is uniformly Lipschitz continuous, thus by the Ascoli-Arzelà Theorem there exists a subsequence of $\{u^\epsilon\}$ uniformly convergent to a Lipschitz function u solution to (1.4).

If f^+, $f^- \not\equiv 0$, then by the Lipschitz regularity of u up to the boundary of Ω, we have $u_1 = u^+ \not\equiv 0$ and $u_2 = u^- \not\equiv 0$.

6. Non-degeneracy at regular points

In this section we introduce the definition of regular points for u solution of (1.4). These are points where at least one among u_1 and u_2 has linear growth away from the free boundary, where here and throughout this section we will use the notation introduced in (2.7).

Definition 6.1. Let u be a solution of problem (1.4). Consider x_0 a point on the free boundary Γ. We say that x_0 is a *regular point* if there exist positive constants $\tilde{r} = \tilde{r}(x_0)$ and $M = M(x_0)$ such that

$$\sup_{B_r(x_0)} U \geq Mr, \tag{6.1}$$

for every $0 < r < \tilde{r}$, where

$$U(x) := \max\{u_1(x), u_2(x)\} = |u(x)|.$$

Otherwise, we say that x_0 is a *singular point.*

Lemma 6.2. *Let u be a solution of problem* (1.4)*. If Γ has a ball from one side at $x_0 \in \Gamma$, that is there exists a ball $B_{r_0}(y)$ contained inside the support of either u_1 or u_2, such that $x_0 \in \partial B_{r_0}(y)$, then x_0 is a regular point.*

Proof: Indeed, suppose, without loss of generality, that $B_{r_0}(y) \subset \Omega(u_1)$. By (1.4) u_1 is solution of $\mathcal{F}^-(u_1) = 0$ in $B_{r_0}(y)$. Then by the Harnack inequality we have that $u_1(x) \geq M_1$, for any $x \in \overline{B}_{\frac{r_0}{2}}(y)$ with $M_1 = Cu_1(y)$, where C is a universal constant. Let ϕ be the solution of

$$\begin{cases} \mathcal{M}^-(\phi) = 0 & \text{in } B_{r_0}(y) \setminus B_{\frac{r_0}{2}}(y) \\ \phi = M_1 & \text{on } \partial B_{\frac{r_0}{2}}(y) \\ \phi = 0 & \text{in } \partial B_{r_0}(y), \end{cases}$$

that is, $\phi(x) = M_1 \frac{1}{2^\gamma - 1}\left(\frac{r_0^\gamma}{|x-y|^\gamma} - 1\right)$, where $\gamma = \Lambda(n-1)/\lambda - 1$ and λ and Λ are the elliptic constants of the Pucci's operator $\mathcal{F}^-$ (see Lemma 9.1 in Appendix). Then, since in $B_{r_0}(y) \setminus B_{\frac{r_0}{2}}(y)$

$$\mathcal{F}^-(\phi) \geq \mathcal{M}^-(\phi) = 0 = \mathcal{F}^-(u_1)$$

and $u_1 \geq \phi$ on $\partial B_{r_0}(y) \cup \partial B_{r_0/2}(y)$, the comparison principle and (iii) of Lemma 9.1 imply that for any $x \in B_{r_0}(y) \setminus B_{\frac{r_0}{2}}(y)$,

$$u_1(x) \geq \phi(x) \geq \frac{M_1\gamma}{r_0(2^\gamma - 1)} d(x, \partial B_{r_0}(y)).$$

Hence, for any $r < \frac{r_0}{2}$,

$$\sup_{B_r(x_0)} u_1 \geq \frac{M_1\gamma}{r_0(2^\gamma - 1)} r =: Mr.$$

Therefore, (6.1) holds with $\tilde{r}(x_0) = \frac{r_0}{2}$ and

$$M = \frac{u_1(y)C\gamma}{r_0(2^\gamma - 1)}$$

depending only on x_0, n, λ and Λ. ■

Remark 6.3. *The set of regular points is dense in* Γ. *Indeed, since* $\Omega(u_1)$ *is an open set, the set of points in* Γ *with an interior tangent ball is dense in* Γ. *To see it, let* x *be any point in* Γ. *Let us consider a sequence of points* $\{x_k\}$ *contained in* $\Omega(u_1)$ *and converging to* x *as* $k \to \infty$. *Let* d_k *be the distance of* x_k *from* Γ. *Then the balls* $B_{d_k}(x_k)$ *are contained in* $\Omega(u_1)$ *and there exist points* $y_k \in \Gamma \cap \partial B_{d_k}(x_k)$ *where the* x_k*'s realize the distance from* Γ. *The sequence* $\{y_k\} \subset \Gamma$ *is a sequence of points that have a tangent ball from the inside and converges to* x.

The following lemma states that at regular points both functions u_1 and u_2 have linear growth away from the free boundary.

Lemma 6.4. *Let* u *be a Lipschitz solution of problem* (1.4) *and let* $z \in \Gamma$ *be a regular point. Then, there exist* $c = c(z)$ *and* C *positive constants such that, for any* $0 < r < \tilde{r}(z)$,

$$c\,r \le \sup_{B_r(z)} u_i \le C\,r \qquad i = 1, 2.$$

Proof: The inequality $\sup_{B_r(z)} u_i \le C\,r$ for $i = 1, 2$ follows from the Lipschitz regularity of u. We prove that if (6.1) holds true for $x_0 = z$, then

$$\sup_{B_r(z)} u_i(x) \ge c\,r \quad \text{for any } 0 < r < \tilde{r}(z) < d(z, \partial\Omega), \quad i = 1, 2, \tag{6.2}$$

for some $c = c(M)$. Assume by contradiction that for $\epsilon < \frac{M}{4}$ there exists $0 < \rho < \tilde{r}$ such that, w.l.o.g.

$$\sup_{B_\rho(z)} u_2 < \epsilon\,\rho. \tag{6.3}$$

Set $r_\rho := \frac{\rho}{4} < \tilde{r}$. From (6.3) we have that $\sup_{B_{r_\rho}(z)} u_2 \le \sup_{B_\rho(z)} u_2 < \epsilon\rho$ and hence

$$\sup_{B_{r_\rho}(z)} u_2 < (4\epsilon)\frac{\rho}{4} < M\frac{\rho}{4} = M r_\rho. \tag{6.4}$$

Therefore from (6.4) and the fact that $\sup_{B_{r_\rho}(z)} U \geq Mr_\rho$, where $U = \max\{u_1, u_2\}$, we must have

$$\sup_{B_{r_\rho}(z)} u_1 \geq Mr_\rho;$$

that is, there exists $y \in \overline{B}_{r_\rho}(z)$ such that

$$u_1(y) \geq Mr_\rho.$$

Consider the ball centered at y with radius h, where $h = |y - x_0|$, being $x_0 \in \Gamma$ the closest point to y in Γ. Observe that $h \leq \frac{\rho}{4}$. Next, the ball $B_h(y)$ is contained in $\Omega(u_1)$, therefore by (1.4) $\mathcal{F}^-(u_1) = 0$ in $B_h(y)$. The Harnack inequality then implies $u_1 \geq CMr_\rho$ on $\bar{B}_{\frac{h}{2}}(y)$, where C is a universal constant. Let ϕ be the function defined as follows:

$$\phi(x) = CM\frac{r_\rho}{2^\gamma - 1}\left(\frac{h^\gamma}{|x-y|^\gamma} - 1\right), \tag{6.5}$$

with $\gamma = \frac{\Lambda(n-1)-\lambda}{\lambda}$. Then, ϕ satisfies

$$\begin{cases} \mathcal{M}^-(\phi) = 0 & \text{in } \mathbb{R}^n \setminus \{y\} \\ \phi = CM\, r_\rho & \text{on } \partial B_{\frac{h}{2}}(y) \\ \phi = 0 & \text{on } \partial B_h(y), \end{cases}$$

see Lemma 9.1. In particular, since $u_1 \geq \phi$ on $\partial B_h(y) \cup \partial B_{\frac{h}{2}}(y)$, by the comparison principle,

$$u_1(x) \geq \phi(x), \qquad x \in B_h(y) \setminus B_{\frac{h}{2}}(y).$$

The previous inequality still holds in the complement of $B_h(y)$ in $\Omega(u_1)$, being ϕ negative in that set. Therefore, we have that

$$u(x) = u_1(x) \geq \phi(x) \quad \text{if } x \in \Omega(u_1) \setminus B_{\frac{h}{2}}(y). \tag{6.6}$$

To continue the proof, we will prove that $\phi \leq -u_2$ in a neighborhood of x_0 in $\Omega(u_2)$. If $x \in B_{2h}(y)$ then $d(x,z) \leq d(x,y) + d(y,z) \leq 2h + r_\rho \leq \frac{3}{4}\rho$, therefore $B_{2h}(y) \subset B_\rho(z)$. In particular, by (6.3),

$$\sup_{B_{2h}(y)} u_2 < \epsilon\rho = \epsilon 4 r_\rho.$$

On the other hand,

$$\phi = -\frac{CMr_\rho}{2^\gamma} \text{ on } \partial B_{2h}(y) \quad \text{and} \quad \phi \leq 0 \text{ on } \Gamma.$$

Let ϵ be so small that $4\epsilon \leq \frac{CM}{2^\gamma}$, then

$$\phi \leq -u_2 \quad \text{on } \partial(\Omega(u_2) \cap B_{2h}(y)) \quad \text{ and } \quad \phi < -u_2 \quad \text{on } \Gamma \cap B_{2h}(y).$$

Since in addition, in the set $\Omega(u_2) \cap B_{2h}(y)$ we have

$$\mathcal{F}^-(\phi) \geq \mathcal{M}^-(\phi) = 0 \quad \text{and} \quad \mathcal{F}^-(-u_2) = \mathcal{F}^-(u) \leq 0,$$

the strong maximum principle implies

$$\phi(x) < -u_2(x) \quad \text{for any } x \in \Omega(u_2) \cap B_{2h}(y).$$

Putting all together, by (6.6) and the previous inequality, we conclude that for all $x \in B_{\frac{h}{2}}(x_0)$ the function $u = u_1 - u_2$ satisfies

$$u(x) \geq \phi(x),\ u \not\equiv \phi \text{ and } u(x_0) = \phi(x_0). \tag{6.7}$$

This is in contradiction with the strong maximum principle, since we know that $\mathcal{F}^-(u) \leq 0 \leq \mathcal{F}^-(\phi)$ in $B_{\frac{h}{2}}(x_0)$. The contradiction has followed by assuming that there exists $0 < \rho < \tilde{r}$ such that (6.3) holds true, with ϵ satisfying $\epsilon < \frac{M}{4}$ and $4\epsilon \leq \frac{CM}{2^\gamma}$. Therefore, if we choose for example

$$c = \frac{1}{2} \min\left\{\frac{M}{4}, \frac{CM}{2^{\gamma+2}}\right\},$$

inequalities (6.2) hold true. ■

Lemma 6.5. *Let $z \in \Gamma$ be a regular point and let u be a Lipschitz solution of problem* (1.4)*. Then there exists a constant $C = C(z) > 0$ such that for every $0 < r < \tilde{r}$,*

$$J_r(u_i, z) \geq C, \quad i = 1, 2,$$

where $J_r(u_i, z)$ is defined by (2.8)*.*

Proof: Without loss of generality, we prove the lemma for $i = 1$. By Lemma 6.4 there exists $c = c(z) > 0$ such that for any radius $r < \tilde{r} < d(z, \partial\Omega)$, there exists $y \in \overline{B}_{\frac{r}{4}}(z)$ such that

$$u_1(y) \geq c\frac{r}{4}. \tag{6.8}$$

Let $x_0 \in \Gamma$ be the closest point in the free boundary to y, $h = |y - x_0|$ and consider $B_h(y)$. Note that $h \leq |y - z| \leq r/4$,

$$B_h(y) \subset \Omega(u_1) \cap B_r(z), \tag{6.9}$$

and $B_h(x_0) \subset B_r(z)$. Moreover, since u_1 is Lipschitz continuous in Ω, there exists L such that $|u(x)-u(y)| \le L|x-y|$ for any $x, y \in B_r(z)$. In particular, we have that

$$c\frac{r}{4} \le u_1(y) - u_1(x_0) = u_1(y) \le L|y - x_0| = Lh,$$

which implies

$$\frac{h}{r} \ge \frac{c}{4L}. \tag{6.10}$$

Next, since (6.9) holds, Lemma 6.2 implies that x_0 is also a regular point and for any $x \in B_h(y) \setminus B_{\frac{h}{2}}(y)$,

$$u_1(x) \ge Md(x, \partial B_h(y)) = M(h - |x - y|), \tag{6.11}$$

where $M = \dfrac{u_1(y)C\gamma}{h\,(2^\gamma - 1)}$, (see proof for Lemma 6.2). In particular, for any $s < \frac{h}{2}$,

$$\sup_{B_s(x_0)} u_1 \ge Ms.$$

We now note that, from (6.8) and the inequality $h \le r/4$,

$$M \ge \frac{c\,\gamma\,C}{2^\gamma - 1} =: \tilde{M}, \tag{6.12}$$

where $\tilde{M}$ depends on z but it is independent of x_0, h and r. Since now we have, for any $s < h/2$

$$\sup_{B_s(x_0)} U \ge \tilde{M}\,s,$$

where $U = \max\{u_1, u_2\}$, by Lemma 6.4 there exists $\tilde{c} = \tilde{c}(\tilde{M})$ such that, for any $s < h/2$,

$$\sup_{B_s(x_0)} u_2 \ge \tilde{c}s. \tag{6.13}$$

We are now in conditions to apply a Poincaré-Sobolev type inequality to u_1 (see e.g. [13, Chapter 4, Lemma 2.8] and [15, Theorem 3]). Indeed, we claim that the zero set of u_1 has positive density.

Claim: There exists $\epsilon > 0$ independent of h such that

$$|\{u_1 = 0\} \cap B_{\frac{h}{2}}(x_0)| \ge |\{u_2 > 0\} \cap B_{\frac{h}{2}}(x_0)| \ge \epsilon |B_{\frac{h}{2}}(x_0)|. \tag{6.14}$$

Proof of the claim: Suppose by contradiction that for any $\epsilon > 0$ one has

$\left|\{u_2 > 0\} \cap B_{\frac{h}{2}}(x_0)\right| < \epsilon \left|B_{\frac{h}{2}}(x_0)\right|$. Since u_2 is Lipschitz continuous in Ω, there exists $L > 0$ such that for all $x \in B_{\frac{h}{2}}(x_0)$,

$$u_2(x) \leq L|x - x_0| \leq L\frac{h}{2}. \tag{6.15}$$

Since u_2 is subharmonic, see (4.2), the mean-value Theorem implies that for any $x \in B_{\frac{h}{4}}(x_0)$,

$$u_2(x) \leq \fint_{B_{\frac{h}{4}}(x)} u_2(t)dt \leq \frac{1}{\left|B_{\frac{h}{4}}(x)\right|} \int_{\{u_2>0\}\cap B_{\frac{h}{2}}(x_0)} u_2(t)dt \leq \frac{\epsilon \left|B_{\frac{h}{2}}(x_0)\right|}{\left|B_{\frac{h}{4}}(x)\right|} L\frac{h}{2} = \epsilon 2^{n-1} Lh,$$

which is in contradiction with (6.13) with $s = \frac{h}{4}$ for $\epsilon < \frac{\tilde{c}}{L2^{n+1}}$. This proves (6.14).

Next, to conclude the proof of the lemma, since

$$\frac{1}{r^2}\int_{B_r(z)} \frac{|\nabla u_1(x)|^2}{|x-z|^{n-2}}dx \geq \frac{1}{r^n}\int_{B_r(z)} |\nabla u_1(x)|^2 dx \geq \frac{1}{r^n}\int_{B_{\frac{h}{2}}(x_0)} |\nabla u_1(x)|^2 dx \tag{6.16}$$

we just need to bound from below the last integral.

Since (6.14) holds true, we can apply the Poincaré-Sobolev type inequality to obtain

$$\frac{1}{r^n}\int_{B_{\frac{h}{2}}(x_0)} |\nabla u_1(x)|^2 dx \geq \frac{1}{r^n}\frac{1}{C(n,\epsilon)h^2}\int_{B_{\frac{h}{2}}(x_0)} u_1^2(x)dx. \tag{6.17}$$

Finally, by using (6.11) and (6.12), we get

$$\begin{aligned}
\frac{1}{r^n}\frac{1}{C(n,\epsilon)h^2}\int_{B_{\frac{h}{2}}(x_0)} u_1^2(x)dx &\geq \frac{1}{r^n}\frac{1}{C(n,\epsilon)h^2}\int_{B_{\frac{h}{2}}(x_0)\cap B_h(y)} \tilde{M}^2(h-|x-y|)^2 dx \\
&\geq \frac{1}{r^n}\frac{1}{C(n,\epsilon)h^2}\int_{B_{\frac{h}{2}}(x_0)\cap B_{\frac{7}{8}h}(y)} \tilde{M}^2(h-|x-y|)^2 dx \\
&\geq \frac{\tilde{M}^2 h^2 h^n}{r^n C(n,\epsilon)h^2},
\end{aligned} \tag{6.18}$$

where in the last inequality we have used that $|B_{\frac{h}{2}}(x_0)\cap B_{\frac{7}{8}h}(y)|\geq \bar{c}\,h^n$ and $\tilde{M}^2(h-|x-y|)^2\geq \frac{\tilde{M}^2}{64}h^2$ for any $x\in B_{\frac{7}{8}h}(y)$.

Putting all together, from (6.10), (6.16), (6.17) and (6.18) we infer that there exists $C=C(z)>0$ such that

$$\frac{1}{r^2}\int_{B_r(z)}\frac{|\nabla u_1(x)|^2}{|x-z|^{n-2}}dx\geq C,$$

and this concludes the proof of the lemma. ∎

The following is an immediate corollary of Lemma 6.5.

Corollary 6.6. *Let u be a Lipschitz solution of problem (1.4) and let $z\in\Gamma$ be a regular point. Then there exists a constant $C=C(z)>0$ such that, for any $0<r<\tilde{r}$,*

$$J_r(u,z)\geq C, \tag{6.19}$$

where $J_r(u,z)$ is defined by (2.9).

7. Proof of Theorem 3.2

We start with the analysis of the blow up of the solution at regular points. As in Section 6, throughout this section we will use the notation introduced in (2.7) for u solution of (1.4).

Lemma 7.1. *Let u be a Lipschitz solution of problem (1.4). Let $0\in\Gamma$ be a regular point. Let u_r denote the blow-up sequence*

$$u_r(x):=\frac{1}{r}u(rx),\quad x\in B_2(0),$$

with $r<d(0,\partial\Omega)/2$. Then, u_r admits a uniformly converging subsequence in $B_1(0)$ and for any converging subsequence $u_{r_j}(x)=u(r_jx)/r_j$, $j\in\mathbb{N}$, there exist α, $\beta>0$ and a unit vector n, such that,

$$J_0(u)=c_n\alpha^2\beta^2, \tag{7.1}$$

where $J_0(u)$ is defined as in (4.3), and as $j\to+\infty$,

$$u_{r_j}(x)\to\alpha<x,n>^+-\beta<x,n>^-, \tag{7.2}$$

uniformly in $B_1(0)$.

Proof: Since u is Lipschitz continuous in Ω, the sequence $\{u_r\}$ is uniformly bounded in $C^{0,1}(B_2(0))$. Therefore, by Ascoli-Arzela, there exists a subsequence $\{u_{r_j}\}$ and a Lipschitz function $\bar{u}$, such that, as $j \to +\infty$, $u_{r_j} \to \bar{u}$ uniformly in $B_1(0)$ and weakly in $H^1(B_1(0))$. In particular,

$$\int_{B_1(0)} |\nabla \bar{u}|^2 \, dx \leq \liminf_{j \to +\infty} \int_{B_1(0)} |\nabla u_{r_j}|^2 \, dx. \tag{7.3}$$

We will prove that for any $s \in (0,1)$,

$$J_s(\bar{u}) = J_0(u) > 0, \tag{7.4}$$

where $J_r(u)$ is defined as in (2.9). For that, we truncate $(u_1)_{r_j} = u_1(r_j x)/r_j$ at level ϵ and h, for $0 < \epsilon < h$, by considering $w_{\epsilon,h} := \min\{\max\{(u_1)_{r_j}, \epsilon\}, h\}$. Since each $(u_1)_{r_j}$ is subharmonic (i.e., $\Delta(u_1)_{r_j} = \mu_{r_j} \geq 0$, in the sense of distributions, and μ_j is a Radon measure) and Lipschitz continuous, then we have $w_{\epsilon,h}\nabla(u_1)_{r_j} \in L^\infty(B_1(0))$ and the product rule $\operatorname{div}(w_{\epsilon,h}\nabla(u_1)_{r_j}) = w_{\epsilon,h}\mu_{r_j} + \nabla w_{\epsilon,h} \cdot \nabla(u_1)_{r_j}$ holds in the sense of distributions. Moreover, we can integrate by parts (see [7, 8]) in $B_1(0)$:

$$\int_{B_1(0)} \operatorname{div}(\, w_{\epsilon,h}\nabla(u_1)_{r_j}) = \int_{\partial B_1(0)} (w_{\epsilon,h}\nabla(u_1)_{r_j} \cdot \nu)_{tr} d\mathcal{H}^{n-1}, \tag{7.5}$$

where $(w_{\epsilon,h}\nabla(u_1)_{r_j} \cdot \nu)_{tr} \in L^\infty(\partial B_1(0))$ is the normal trace of the vector field $w_{\epsilon,h}\nabla(u_1)_{r_j}$ and which satisfies

$$(w_{\epsilon,h}\nabla(u_1)_{r_j} \cdot \nu)_{tr} \leq \left\|w_{\epsilon,h}\nabla(u_1)_{r_j}\right\|_{L^\infty(B_1(0))} \leq h \left\|\nabla(u_1)_{r_j}\right\|_{L^\infty(B_1(0))} \leq hL. \tag{7.6}$$

From (7.5), and since $\nabla w_{\epsilon,h} = 0$ a.e. in $B_1(0) \cap \{(u_1)_{r_j} \geq h\}$ and in $B_1(0) \cap \{(u_1)_{r_j} \leq \epsilon\}$, we obtain

$$\begin{aligned} 0 \leq \int_{B_1(0)} w_{\epsilon,h} d\mu_{r_j} &= -\int_{\epsilon \leq (u_1)_{r_j} \leq h\} \cap B_1(0)} |\nabla(u_1)_{r_j}|^2 \, dx + \int_{\partial B_1(0)} (w_{\epsilon,h}\nabla(u_1)_{r_j} \cdot \nu)_{tr} d\mathcal{H}^{n-1} \\ &\leq -\int_{\{\epsilon \leq (u_1)_{r_j} \leq h\} \cap B_1(0)} |\nabla(u_1)_{r_j}|^2 \, dx + Ch, \end{aligned} \tag{7.7}$$

with $C = L\mathcal{H}^{n-1}(\partial B_1(0))$. Using the Lebesgue Dominated Convergence Theorem we let ϵ go to 0, obtaining:

$$\int_{\{0\leq (u_1)_{r_j}\leq h\}\cap B_1(0)} |\nabla (u_1)_{r_j}|^2\,dx \leq Ch. \tag{7.8}$$

Similarly, one gets

$$\int_{\{0\leq (u_2)_{r_j}\leq h\}\cap B_1(0)} |\nabla (u_2)_{r_j}|^2\,dx \leq Ch. \tag{7.9}$$

From (7.8) and (7.9), we obtain

$$\int_{\{|u_{r_j}|\leq h\}\cap B_1(0)} |\nabla u_{r_j}|^2\,dx \leq Ch. \tag{7.10}$$

Next, for j large enough, the set $\{|u_{r_j}|>0\}$ contains $\{|\bar u|>h\}$. Moreover, since u_{r_j} is a Lipschitz viscosity solution of

$$\begin{cases} \mathcal{F}^-(u_{r_j}) = 0 & \text{in } \{u_{r_j}>0\}\cap B_2(0) \\ \mathcal{F}^+(u_{r_j}) = 0 & \text{in } \{u_{r_j}<0\}\cap B_2(0), \end{cases} \tag{7.11}$$

by the interior $C^{2,\alpha}$ estimates for the operators $\mathcal{F}^{\pm}$ (see Remark 2.3), up to a subsequence, $\nabla u_{r_j}\to\nabla\bar u$ uniformly in $\{|\bar u|>h\}\cap B_1(0)$ as $j\to+\infty$, and thus,

$$\lim_{j\to+\infty}\int_{\{|u_{r_j}|>h\}\cap B_1(0)} |\nabla u_{r_j}|^2\,dx = \int_{\{|\bar u|>h\}\cap B_1(0)} |\nabla \bar u|^2\,dx. \tag{7.12}$$

By (7.10) and (7.12) we infer that, for any $h>0$,

$$\limsup_{j\to+\infty}\int_{B_1(0)} |\nabla u_{r_j}|^2\,dx \leq \int_{B_1(0)} |\nabla \bar u|^2\,dx + Ch,$$

which combined with (7.3) yields, letting $h\to 0$,

$$\lim_{j\to+\infty}\int_{B_1(0)} |\nabla u_{r_j}|^2\,dx = \int_{B_1(0)} |\nabla \bar u|^2\,dx. \tag{7.13}$$

By (7.13), $|\nabla u_{r_j}|^2\to|\nabla\bar u|^2$ a.e. in $B_1(0)$. Since in addition $|\nabla u_{r_j}|^2/|x|^{n-2}\leq L^2/|x|^{n-2}\in L^1(B_1(0))$, by the Dominated Convergence Theorem we infer that, for any $s\in(0,1)$,

$$\lim_{j\to+\infty} J_s(u_{r_j}) = J_s(\bar u). \tag{7.14}$$

Next, by Corollary 6.6 and Remark 4.2

$$\lim_{r\to 0^+} J_r(u) = J_0(u) > 0. \tag{7.15}$$

Let $s \in (0,1)$. A change of variables yields:

$$J_s(u_{r_j}) = J_{sr_j}(u). \tag{7.16}$$

Therefore, by (7.14)-(7.16), for any $s \in (0,1)$,

$$J_s(\bar{u}) = \lim_{j\to+\infty} J_s(u_{r_j}) = \lim_{j\to+\infty} J_{sr_j}(u) = J_0(u) > 0,$$

which gives (7.4). The conclusion of the lemma follows from Theorem 4.3. ■

Corollary 7.2. *Under the assumptions of Lemma 7.1, for any $\epsilon > 0$ there exists $J \in \mathbb{N}$ such that for any $j \geq J$, all the level sets of u_{r_j} in $B_1(0)$ are ϵ-flat, in the sense that for any $\lambda \in \mathbb{R}$, we have*

$$\{u_{r_j} = \lambda\} \cap B_1(0) \subset \{x \in \mathbb{R}^n \,|\, d(x, \Pi_\lambda) < c\epsilon\}, \tag{7.17}$$

for some $c > 0$ independent of λ, r_j and ϵ, where

$$\Pi_\lambda = \begin{cases} \{< x, n >= \lambda/\alpha\} & \text{if } \lambda > \epsilon, \\ \{< x, n >= 0\} & \text{if } \lambda \in [-\epsilon, \epsilon], \\ \{< x, n >= \lambda/\beta\} & \text{if } \lambda < -\epsilon. \end{cases}$$

and n, α and β are as in (7.2).

Proof: By Lemma 7.1, for any $\epsilon > 0$, there exists $J \in \mathbb{N}$ such that for any $j \geq J$,

$$\left| u_{r_j}(x) - \alpha < x, n >^+ + \beta < x, n >^- \right| < \frac{\epsilon}{2}, \tag{7.18}$$

for all $x \in B_1(0)$. Let $\lambda > \epsilon$, then by (7.18), if $< x, n >\geq \frac{\lambda+\epsilon}{\alpha}$ then

$$u_{r_j}(x) \geq \lambda + \epsilon - \frac{\epsilon}{2} = \lambda + \frac{\epsilon}{2} > \lambda,$$

and if $< x, n >\leq \frac{\lambda-\epsilon}{\alpha}$, then

$$u_{r_j}(x) \leq \lambda - \frac{\epsilon}{2} < \lambda.$$

We infer that

$$\{u_{r_j} = \lambda\} \cap B_1(0) \subset \left\{ x \in \mathbb{R}^n \,|\, \frac{-\epsilon}{\alpha} \leq < x, n > -\frac{\lambda}{\alpha} \leq \frac{\epsilon}{\alpha} \right\}. \tag{7.19}$$

Similarly, if $\lambda \in [-\epsilon, \epsilon]$,

$$\{u_{r_j} = \lambda\} \cap B_1(0) \subset \left\{ x \in \mathbb{R}^n \mid \frac{-2\epsilon}{\beta} \leq < x, n > \leq \frac{2\epsilon}{\alpha} \right\}, \tag{7.20}$$

and if $\lambda < -\epsilon$,

$$\{u_{r_j} = \lambda\} \cap B_1(0) \subset \left\{ x \in \mathbb{R}^n \mid \frac{-\epsilon}{\beta} \leq < x, n > -\frac{\lambda}{\beta} \leq \frac{\epsilon}{\beta} \right\}. \tag{7.21}$$

Inclusions (7.19), (7.20) and (7.21) give (7.17) with $c = 2\max\{\alpha^{-1}, \beta^{-1}\}$. ■

By Lemma 7.1 we know that if $0 \in \Gamma$ is a regular point, then the blow up sequence $u(rx)/r$ admits a subsequence converging to a two-plane solution of the form (7.2). We next show that if there is a tangent ball from one side to Γ at 0, then the full sequence $u(rx)/r$ converges to a two-plane solution and therefore u has an asymptotic linear behavior at 0.

Lemma 7.3. *Let u be a Lipschitz solution of problem* (1.4). *Let $0 \in \Gamma$. Assume that there exists a tangent ball B from one side to Γ at 0. Then, there exist α, $\beta > 0$ such that*

$$u(x) = \alpha < x, \nu >^+ - \beta < x, \nu >^- + o(|x|), \tag{7.22}$$

where ν is the normal vector of B at 0 pointing inward $\{u > 0\}$.

Proof: By Lemma 6.2, 0 is a regular point. Consider the sequence $u_r(x) = \frac{1}{r}u(rx)$, for r small enough. By Lemma 7.1, there exist a subsequence $\{u_{r_j}\}$, a unit vector n and α, β positive constants satisfying (7.1), such that as $j \to +\infty$,

$$u_{r_j}(x) \to \overline{u}(x) := \alpha < x, n >^+ - \beta < x, n >^-,$$

uniformly in $B_1(0)$.

Assume, without loss of generality, that there exists a ball $B_{r_0}(y) \subset \Omega(u_1)$ such that $0 \in \partial B_{r_0}(y)$. Let ν be the normal vector of $B_{r_0}(y)$ at 0 pointing inward $\Omega(u_1)$.

Claim 1 : We claim that $\nu = n$.

Proof of Claim 1 : Indeed, suppose by contradiction that $\nu \neq n$. Then, there exists $x_0 \in B_1(0)$ such that for any j, $< r_j\, x_0, \nu >> 0$ and $< r_j\, x_0, n >< 0$. Fix $J \in \mathbb{N}$ such that the sequence of points $\{r_j x_0\}$ satisfies $r_j x_0 \in B_{r_0}(y) \subset \Omega(u_1)$ for all $j \geq J$, then

$$\frac{u(r_j x_0)}{r_j} > 0 \quad \text{for all } j \geq J.$$

Passing to the limit as $j \to +\infty$, we get

$$\overline{u}(x_0) \geq 0.$$

On the other hand,

$$\overline{u}(x_0) = -\beta < x_0, n >^- < 0.$$

This is a contradiction. Therefore we must have $\nu = n$.

We now proceed to show that the full sequence u_r converges to $\overline{u}$. Let $\overline{u}$ and $\overline{v}$ be the limits of two converging subsequences of the sequence $\{u_r\}$, then we must have

$$\overline{u} = \alpha_1 < x, \nu >^+ - \beta_1 < x, \nu >^-$$

and

$$\overline{v} = \alpha_2 < x, \nu >^+ - \beta_2 < x, \nu >^- .$$

Claim 2 : We claim that in addition that

$$\alpha_1 = \alpha_2 \quad \text{and} \quad \beta_1 = \beta_2. \tag{7.23}$$

Proof of Claim 2 : To prove this claim, we will construct a barrier to bound u_1 from below. Let ϕ be the solution of

$$\begin{cases} \mathcal{F}^-(\phi) = 0 & \text{in } B_{r_0}(y) \setminus B_{\frac{r_0}{2}}(y) \\ \phi = 1 & \text{on } \partial B_{\frac{r_0}{2}}(y) \\ \phi = 0 & \text{on } \partial B_{r_0}(y). \end{cases}$$

By the comparison principle, for any $x \in B_{r_0}(y) \setminus B_{\frac{r_0}{2}}(y)$

$$u_1(x) \geq c_0 \phi(x), \quad \text{with } c_0 = \min_{\partial B_{\frac{r_0}{2}}(y)} u_1. \tag{7.24}$$

For $k \geq 0$ such that $2^{-k} < r_0/2$, let

$$\tilde{m}_k := \sup\{m \,|\, u_1(x) \geq m\phi(x) \text{ in } B_{2^{-k}}(0) \cap B_{r_0}(y)\}.$$

Notice that the sequence $\tilde{m}_k$ is increasing. Moreover, by (7.24), $\tilde{m}_k \geq c_0$ for any k. Let

$$\tilde{m}_\infty := \sup_k \tilde{m}_k = \lim_{k \to +\infty} \tilde{m}_k.$$

Since u_1 is Lipschitz continuous, $\tilde{m}_\infty < +\infty$.

By Lemma 9.2, there exists $\sigma > 0$ independent of r_0 such that, for any $x \in B_{r_0}(y) \setminus B_{\frac{r_0}{2}}(y)$,

$$\phi(x) = \frac{\sigma}{r_0} < x, \nu > + o(|x|). \tag{7.25}$$

Set

$$m_k := \frac{\sigma}{r_0}\tilde{m}_k \quad \text{and} \quad m_\infty := \frac{\sigma}{r_0}\tilde{m}_\infty,$$

By the definition of m_k and (7.25), for $x \in B_{2^{-k}}(0) \cap B_{r_0}(y)$ we have

$$u_1(x) \geq \tilde{m}_k\phi(x) = m_k < x, \nu > +o(|x|).$$

This implies that $\alpha_1 \geq m_\infty$. Assume by contradiction that $\alpha_1 > m_\infty$. We will show that in this case, there exists $\epsilon > 0$ and a sequence k_j such that, for j large enough,

$$u_1 - \left(\tilde{m}_{k_j} + \epsilon\right)\phi(x) \geq 0, \text{ for all } x \in B_{2^{-k_j}}(0) \cap B_{r_0}(y), \tag{7.26}$$

which is in contradiction with the definition of $\tilde{m}_{k_j}$. For that, if $\alpha_1 > m_\infty$, there exists a sequence $r_j \to 0$ as $j \to +\infty$ such that if $y_j = r_j\nu$, then, for some $\delta_0 > 0$,

$$u_1(y_j) - \tilde{m}_\infty\phi(y_j) \geq \delta_0 r_j.$$

Let $\tilde{m}_{k_j}$ be a subsequence converging to $\tilde{m}_\infty$ as $j \to +\infty$ such that, up to eventually consider a subsequence of $\{r_j\}$, one has that $2r_j \leq 2^{-k_j}$. Then $B_{r_j}(y_j) \subset B_{2^{-k_j}}(0) \cap B_{r_0}(y)$ and since $\tilde{m}_{k_j} \leq \tilde{m}_\infty$,

$$u_1(y_j) - \tilde{m}_{k_j}\phi(y_j) \geq \delta_0 r_j.$$

By the definition of $\tilde{m}_{k_j}$ the function $u_1 - \tilde{m}_{k_j}\phi$ is positive in $B_{2^{-k_j}}(0) \cap B_{r_0}(y)$ and by Proposition 2.1 it satisfies

$$\mathcal{M}^-(u_1 - \tilde{m}_{k_j}\phi) \leq \mathcal{F}^-(u_1 - \tilde{m}_{k_j}\phi) \leq \mathcal{F}^-(u_1) - \mathcal{F}^-(\tilde{m}_{k_j}\phi) = 0,$$

and

$$\mathcal{M}^+(u_1 - \tilde{m}_{k_j}\phi) \geq \mathcal{F}^-(u_1) - \mathcal{F}^-(\tilde{m}_{k_j}\phi) = 0.$$

Therefore, since $B_{r_j}(y_j) \subset B_{2^{-k_j}}(0) \cap B_{r_0}(y)$, by the Harnack inequality,

$$u_1(x) - \tilde{m}_{k_j}\phi(x) \geq c\delta_0 r_j \quad \text{for } x \in B_{\frac{r_j}{2}}(y_j), \tag{7.27}$$

where $c > 0$ is a universal constant. By a barrier argument we see that there exist δ_j and $\tilde{c} > 0$ (independent of j) such that:

$$u_1(x) - \tilde{m}_{k_j}\phi(x) \geq \tilde{c}d(x, \partial B_{r_0}(y)) \quad \text{for } x \in B_{\delta_j}(0) \cap B_{r_0}(y). \tag{7.28}$$

Indeed, let $z \in B_{\frac{r_j}{4}}(y_j) \cap \partial B_{r_0 - r_j}(y)$ and let w be the closest point to z in $\partial B_{r_0}(y)$, that is $w \in \partial B_{r_0}(y) \cap \partial B_{r_j}(z)$. By (7.27), $u_1(x) - \tilde{m}_{k_j}\phi(x) \geq c\delta_0 r_j$

for $x \in B_{\frac{r_j}{4}}(z)$. Let $\psi(x)$ be the solution to

$$\begin{cases} \mathcal{M}^-(\psi) = 0 & \text{in } B_{r_j}(z) \setminus B_{\frac{r_j}{4}}(z) \\ \psi = c\delta_0 r_j & \text{on } \partial B_{\frac{r_j}{4}}(z) \\ \psi = 0 & \text{on } \partial B_{r_j}(z). \end{cases}$$

By Lemma 9.1, $\psi(x) = \frac{c\delta_0 r_j}{4^\gamma - 1}\left(\frac{r_j^\gamma}{|x-z|^\gamma} - 1\right)$, $\gamma = \Lambda(n-1)/\lambda - 1$ and $\psi(x) \geq \frac{c\delta_0\gamma}{4^\gamma - 1}(r_j - |x-z|)$. In particular, for all points x in $B_{r_j}(z) \setminus B_{\frac{r_j}{4}}(z)$ belonging to the segment from z to w we have that $\psi(x) \geq \tilde{c}(r_j - |x-z|) = \tilde{c}d(x, \partial B_{r_0}(y))$, with $\tilde{c} := \frac{c\delta_0\gamma}{4^\gamma - 1}$. Letting z vary in $B_{\frac{r_j}{4}}(y_j) \cap \partial B_{r_0 - r_j}(y)$, we get (7.28).

For every $x \in B_{\delta_j}(0) \cap B_{r_0}(y)$, we have

$$\begin{aligned} u_1 - \tilde{m}_{k_j}\phi(x) &\geq \tilde{c}d(x, \partial B_{r_0}(y)) \\ &\geq 2\epsilon\phi(x) \end{aligned}$$

for some $\epsilon > 0$, hence:

$$u_1 - (\tilde{m}_{k_j} + 2\epsilon)\phi(x) \geq 0, \text{for all } x \in B_{\delta_j}(0) \cap B_{r_0}(y), \text{ for every } j \tag{7.29}$$

Let j_0 be such that $0 < \tilde{m}_\infty - \tilde{m}_{k_j} < \epsilon$ for all $j \geq j_0$. Given j_0, there exists an integer $j_1 \geq j_0$ such that if $j \geq j_1$ then $B_{2^{-k_j}}(0) \cap B_{r_0}(y) \subset B_{\delta_{j_0}} \cap B_{r_0}(y)$. From (7.29),

$$u_1 - (\tilde{m}_{k_{j_0}} + 2\epsilon)\phi(x) \geq 0, \text{ for all } x \in B_{2^{-k_j}}(0) \cap B_{r_0}(y), \quad j \geq j_1.$$

Thus, for $j \geq j_1$ we have

$$\begin{aligned} u_1 - \left(\tilde{m}_{k_j} + \epsilon\right)\phi(x) &= u_1 - \left(\tilde{m}_{k_{j_0}} + \epsilon\right)\phi(x) - (\tilde{m}_{k_j} - \tilde{m}_{k_{j_0}})\phi(x) \\ &\geq u_1 - \left(\tilde{m}_{k_{j_0}} + \epsilon\right)\phi(x) - \epsilon\phi(x) \\ &\geq u_1 - (\tilde{m}_{k_{j_0}} + 2\epsilon)\phi(x) \\ &\geq 0, \text{ for all } x \in B_{2^{-k_j}}(0) \cap B_{r_0}(y), \end{aligned}$$

which contradicts the definition of $\tilde{m}_{k_j}$.

By (7.23) we infer that $\overline{u} = \overline{v}$. Since any subsequence of $\{u_r\}$ converges to the same function, we deduce that the whole sequence $\{u_r\}$ converge, as $r \to 0$, uniformly in $B_1(0)$ to the limit function

$$\overline{u} = \alpha < x, \nu >^+ - \beta < x, \nu >^-$$

for some $\alpha, \beta > 0$. This means that for any $\epsilon > 0$, there exists $\rho > 0$ such that for any $0 < r < \rho$ and any $x \in \overline{B_1(0)}$, then

$$\left| u_r(x) - \alpha < x, \nu >^+ + \beta < x, \nu >^- \right| < \epsilon.$$

Now, fix $\epsilon > 0$ and let $\rho > 0$ be defined as above. Fix $z \in B_\rho(0)$ and let $r = |z|$. Since $\frac{z}{r} \in \overline{B_1(0)}$ we have

$$\left| u_r\left(\frac{z}{r}\right) - \alpha < \frac{z}{r}, \nu >^+ + \beta < \frac{z}{r}, \nu >^- \right| < \epsilon,$$

that is,

$$\left| u(z) - \alpha < z, \nu >^+ + \beta < z, \nu >^- \right| < \epsilon r = \epsilon |z|, \tag{7.30}$$

which proves (7.22). ■

Notice that in Lemma 7.3 we did not use that u is a viscosity solution of $\mathcal{F}^-(u) \leq 0 \leq \mathcal{F}^+(u)$ in Ω. In the next theorem we show that if u satisfies in the viscosity sense these two differential inequalities and has the asymptotic linear behavior (7.22), then we must have $\alpha = \beta$.

Theorem 7.4. *Let u be a Lipschitz solution of problem* (1.4). *Let $0 \in \Gamma$. Assume that exist α, $\beta > 0$ and a unit vector ν such that*

$$u(x) = \alpha < x, \nu >^+ - \beta < x, \nu >^- + o(|x|). \tag{7.31}$$

Then $\alpha = \beta$.

Proof: We first prove that $\beta \geq \alpha$. We argue by contradiction, assuming that $\beta < \alpha$. Fix $h > 0$ and let $y := h\nu$. Notice that $|y| = h$ and ν is the interior normal unit vector to $B_h(y)$ at 0. Consider the function

$$\phi(x) = c\left(\frac{h^\gamma}{|x-y|^\gamma} - 1\right), \quad x \neq y,$$

with $\gamma = \frac{\Lambda(n-1)-\lambda}{\lambda}$ and $c > 0$. Then, by Lemma 9.1,

$$\begin{cases} \phi(x) > 0 & \text{if } |x-y| < h \\ \phi(x) < 0 & \text{if } |x-y| > h \\ \frac{\partial}{\partial \nu}\phi(x) = \frac{c\gamma}{h} & \text{if } |x-y| = h \\ \mathcal{M}^-(\phi)(x) = 0 & \text{if } x \neq y. \end{cases}$$

Since $\beta < \alpha$, there exists $\epsilon > 0$ such that $\beta + \epsilon < \alpha - \epsilon$. Then, we choose $c > 0$ such that

$$\beta + \epsilon < \frac{\partial}{\partial \nu}\phi_{|\partial B_h(y)} = \frac{c\gamma}{h} < \alpha - \epsilon. \tag{7.32}$$

We want to prove that with this choice of c, $\phi \leq u$ in a neighborhood of 0, for h small enough. In order to prove it, we first show that

$$\phi \leq u \quad \text{on } \partial B_{(1-s)h}(y),$$

for h and s small enough.

Observe that the point $w_1 = sh\nu$ belongs to $\partial B_{(1-s)h}(y)$. Moreover, for any $x \in \partial B_{(1-s)h}(y)$, we have that

$$< x, \nu > \geq < w_1, \nu >= sh,$$

from which we get

$$u(x) = \alpha < x, \nu >^+ -\beta < x, \nu >^- +o(|x|) = \alpha < x, \nu >^+ +o(|x|) \geq \alpha sh + o(h). \tag{7.33}$$

Now, let us compute ϕ on $\partial B_{(1-s)h}(y)$. If $|x - y| = (1 - s)h$ and c satisfies (7.32), then

$$\begin{aligned}\phi(x) = c\left(\frac{1}{(1-s)^\gamma} - 1\right) &= \frac{c}{(1-s)^\gamma}\left(1 - (1-s)^\gamma\right) \\ &= \frac{c}{(1-s)^\gamma}\left(\gamma s + o(s)\right) \\ &\leq \frac{(\alpha - \epsilon)sh}{(1-s)^\gamma} + o(s)h.\end{aligned} \tag{7.34}$$

Let $s > 0$ be so small that

$$\frac{(\alpha - \epsilon)s}{(1-s)^\gamma} + o(s) < \left(\alpha - \frac{\epsilon}{2}\right) s.$$

For such s, let h be so small that

$$\alpha sh + o(h) > \left(\alpha - \frac{\epsilon}{2}\right) sh.$$

Then, comparing (7.33) with (7.34), we see that $\phi(x) < u(x)$, for any $x \in \partial B_{(1-s)h}(y)$.

Next, let us prove that $\phi > u$ on $\partial B_{h(1+s)}(y)$, for suitable small h and s. Let $w_2 = -sh\nu$, then w_2 belongs to $\partial B_{h(1+s)}(y)$. Moreover, if $x \in \partial B_{h(1+s)}(y)$, then

$$< x, \nu > \geq < w_2, \nu >= -sh.$$

Therefore, if $x \in \partial B_{h(1+s)}(y)$, and $< x, \nu > \geq 0$, then

$$u(x) = \alpha < x, \nu >^+ +o(|x|) \geq o(h), \tag{7.35}$$

and if $< x, \nu > \leq 0$, then

$$u(x) = -\beta < x, \nu >^- + o(|x|) \geq -\beta sh + o(h). \tag{7.36}$$

Let us now compute the value of ϕ on $\partial B_{h(1+s)}(y)$. If $|x-y| = h(1+s)$, and c satisfies (7.32), then

$$\phi(x) = -\frac{c}{(1+s)^\gamma}\left((1+s)^\gamma - 1\right) = -\frac{c}{(1+s)^\gamma}\left(\gamma s + o(s)\right) \leq -\frac{(\beta+\epsilon)sh}{(1+s)^\gamma} + o(s)h. \tag{7.37}$$

Let s be so small that

$$-\frac{(\beta+\epsilon)s}{(1+s)^\gamma} + o(s) \leq -\left(\beta + \frac{\epsilon}{2}\right) s.$$

For such s, let h be so small that

$$-\beta sh + o(h) \geq -\left(\beta + \frac{\epsilon}{2}\right) sh.$$

Then, by (7.35), (7.36) and (7.37), $\phi < u$ on $\partial B_{h(1+s)}(y)$. Putting all together, we have proven that there exist $s, h > 0$ such that

$$\phi < u \quad \text{on } \partial(B_{h(1+s)}(y) \setminus B_{h(1-s)}(y)).$$

Since, in addition

$$\mathcal{M}^-(\phi) = 0 \geq \mathcal{F}^-(u) \geq \mathcal{M}^-(u) \quad \text{on } B_{h(1+s)}(y) \setminus B_{h(1-s)}(y),$$

the comparison principle combined with the strong maximum principle implies

$$\phi < u \quad \text{in } B_{h(1+s)}(y) \setminus B_{h(1-s)}(y),$$

which gives a contradiction at $x = 0$.

We conclude that we must have $\alpha \leq \beta$. Arguing similarly as before and using that $\mathcal{F}^+(u) \geq 0$ in Ω, one can prove that $\alpha \geq \beta$ and this concludes the proof of the theorem. ■

7.1. Proof of Theorem 3.2. Theorem 3.2 is a corollary of Lemma 7.3 and Theorem 7.4.

8. Proof of Theorem 3.3

Consider the following two phase free boundary problem:

$$\begin{cases} \mathcal{F}^-(u) = 0 & \text{in } \Omega(u^+) \\ \mathcal{F}^+(u) = 0 & \text{in } \Omega(u^-) \\ \frac{\partial u^+}{\partial \nu_+} = \frac{\partial u^-}{\partial \nu_-} & \text{on } \partial\Omega(u^+), \end{cases} \tag{8.1}$$

where $\nu_\pm$ is the inner normal vector to $\Omega(u^\pm) = \{u^\pm > 0\}$.
By Theorem 3.2 we know that any Lipschitz solution to (1.4) satisfies in the viscosity sense (8.1) in Ω. Let us recall the definition of viscosity solution of the problem (8.1) in a given domain $D \subset \mathbb{R}^n$, see [3] for more details.

Definition 8.1. Let u be a continuous function in D. We say that u is a viscosity solution of the problem (8.1) in D, if the following holds.

i) u satisfies in the viscosity sense

$$\begin{cases} \mathcal{F}^-(u) = 0 & \text{in } \{u > 0\} \cap D \\ \mathcal{F}^+(u) = 0 & \text{in } \{u < 0\} \cap D. \end{cases}$$

ii) If there exists a tangent ball at $x_0 \in \partial\{u > 0\} \cap D$, B, such that either $B \subset \{u > 0\} \cap D$ or $B \subset \{u < 0\} \cap D$, then

$$u(x) = \alpha < x - x_0, \nu_+ > +o(|x - x_0|)$$

with $\alpha > 0$ and ν_+ the normal vector to ∂B at x_0 pointing inward to $\{u > 0\} \cap D$.

In this section we prove that for any viscosity solution to the free boundary problem (8.1) the following holds: if the free boundary is flat around 0, meaning that it can be trapped in a small neighborhood of the graph of a Lipschitz function, then in a neighborhood of 0, it is a $C^{1,\alpha}$ surface. Theorem 3.3 will follow as a corollary of this result. To prove that flatness implies $C^{1,\alpha}$, we follow the classical sup-convolution method developed by Caffarelli in the seminal papers [5, 6] for the Laplace operator and extended by Wang [20, 21] to fully-nonlinear elliptic operators. Problem (8.1) differs from the one studied in [20, 21] since u satisfies two different equations in $\Omega(u^+)$ and $\Omega(u^-)$. However the regularity theory developed in those papers can be extended to our problem and some simplifications arise due to the specific free boundary condition here considered: $\frac{\partial u^+}{\partial \nu_+} = \frac{\partial u^-}{\partial \nu_-}$.

Following the classical theory, we first prove that Lipschitz free boundaries are $C^{1,\alpha}$ and then we prove that flat free boundaries are Lipschitz.

8.1. Lipschitz free boundaries are $C^{1,\alpha}$. For $r > 0$, let $\mathcal{C}_r$ be the cylinder defined as $\mathcal{C}_r := B'_r(0) \times (-r, r)$, where $B'_r(0)$ is the ball centered at 0 of radius r of $\mathbb{R}^{n-1}$.

Proposition 8.2. *Let u be a viscosity solution of the problem* (8.1) *in* $\mathcal{C}_1 = B'_1(0) \times (-1, 1)$. *Assume that* $0 \in \partial\Omega(u^+)$ *and that*

$$\mathcal{C}_1 \cap \Omega(u^+) = \{(x', x_n) \,|\, x_n > g(x')\}$$

where g is a Lipschitz continuous function. Then in $B'_{\frac{1}{2}}(0)$, g is a $C^{1,\alpha}$-function, for some $0 < \alpha \leq 1$.

Proof: The proof of the proposition follows by [5] (see also [3]) and [20]. As already pointed out, even though we have different operators on each side of the free boundary, the classical regularity theory still applies. For completion of this paper, we will sketch the main parts of the method highlighting the parts that are simplified in our problem due to the free boundary condition in (8.1).

Step 1: Existence of a cone of monotonicity.

By [20, Lemma 2.5] applied to u^+ and the operator $\mathcal{F}^-$, there exists $\delta > 0$ such that $\partial_{x_n} u^+ \geq 0$ in the set $\mathcal{C}_\delta \cap \{x_n > g(x')\}$. Also, applying the same Lemma to u^- and the operator $F(u) = -\mathcal{F}^+(-u)$, we have that $\partial_{-x_n} u^- = -\partial_{x_n} u^- \geq 0$ on the set $\mathcal{C}_\delta \cap \{x_n < g(x')\}$. Thus, since $u = u^+ - u^-$, we conclude that u is monotone increasing in the direction of $e_n = (0, \ldots, 0, 1)$ in $\mathcal{C}_\delta$. The same is true for any direction τ in the cone determined by L, the Lipschitz constant of g; that is, let $\Gamma(\theta, e_n)$ be the cone with axis e_n and semi-opening θ given by $\cot\theta = L$, then u is monotone increasing in the direction of $\tau \in \Gamma(\theta, e_n)$, in $\mathcal{C}_\delta$. $\Gamma(\theta, e_n)$ is called the monotonicity cone of u.

Step 2: Improvement of the Lipschitz regularity away from the free boundary.

We may suppose that the monotonicity cone exists for all points $x \in \mathcal{C}_1$, by using, if necessary, the invariance by elliptic dilation of the problem. The monotonicity of u along the directions of $\Gamma(\theta, e_n)$ implies that for every small $\tau \in \Gamma\left(\frac{\theta}{2}, e_n\right)$,

$$\sup_{z \in B_\epsilon(x)} u(z - \tau) \leq u(x), \tag{8.2}$$

for every $x \in \mathcal{C}_{1-\epsilon}$, where $\epsilon = |\tau| \sin\left(\frac{\theta}{2}\right)$. Let $x_0 := \frac{3}{4}e_n \in \mathcal{C}_1$. The proof of Lemma 4.6 of [3] which uses Harnack inequality and Schauder estimates, can be adapted to our case to improve the opening of the monotonicity cone in a neighborhood of x_0. The result goes as follows: there exist positive constants b and c such that for every small $\tau \in \Gamma\left(\frac{\theta}{2}, e_n\right)$ and every $x \in B_{\frac{1}{8}}(x_0)$

$$\sup_{z \in B_{(1+b)\epsilon}(x)} u(z-\tau) \leq u(x) - c\epsilon u(x_0), \tag{8.3}$$

with $\epsilon = |\tau| \sin\left(\frac{\theta}{2}\right)$.

Step 3: Construction of a family of subsolutions of variable radii. Here the main technique is the sup-convolution method to construct a family of subsolutions of the form

$$w_\varphi(x) = \sup_{z \in B_{\varphi(x)}(x)} u(z-\tau),$$

for small $\tau \in \Gamma\left(\frac{\theta}{2}, e_n\right)$, to compare with the solution u of (8.1). In order to apply the comparison principle, it is necessary to study the properties of the sup-convolution function and since problem (8.1) is invariant by translations, it is enough to do it before translations, that is with $u(\cdot - \tau)$ replaced by u.

For $0 < r \leq \frac{1}{8}$, $0 < h < 1$, there exists a family of functions φ_t, $0 \leq t \leq 1$, with $\varphi_t \in C^2\left(\overline{B_1(0)} \setminus B_{\frac{r}{2}}(x_0)\right)$, $x_0 = \frac{3}{4}e_n$, with the following properties:

(a) $1 \leq \varphi_t \leq 1 + th$,
(b) $\varphi_t \equiv 1$ outside $B_{\frac{7}{8}}(0)$,
(c) $\varphi_t \geq 1 + \lambda th$, in $B_{\frac{1}{2}}(0)$, for some $\lambda = \lambda(r)$,
(d) $|\nabla \varphi_t| \leq Cth$.

Moreover, if we define

$$v_{\varphi_t}(x) := \sup_{z \in B_{\varphi_t(x)}(x)} u(z), \tag{8.4}$$

(e) then

$$\mathcal{F}^-(v_{\varphi_t}) \geq 0 \quad \text{in } \Omega(v_{\varphi_t}^+),$$

$$\mathcal{F}^+(v_{\varphi_t}) \geq 0 \quad \text{in } \Omega(v_{\varphi_t}^-),$$

and if $|\nabla \varphi_t| < 1$ then

(f) for every point of $\partial\Omega(v_{\varphi_t}^+)$ there is a tangent ball contained in $\Omega(v_{\varphi_t}^+)$,

(g) for every point $x_1 \in \partial\Omega(v^+_{\varphi_t})$, there exists $\overline{\alpha}$ such that

$$v_{\varphi_t}(x) \geq \bar{\alpha} < x - x_1, \bar{\nu} > +o(|x - x_1|), \tag{8.5}$$

where $\bar{\nu}$ is the normal vector of $\partial\Omega(v^+_{\varphi_t})$ pointing inward $\Omega(v^+_{\varphi_t})$.

Properties (a)-(e) are proven in [20, Lemmas 3.4, 3.5]. Since in [20] only concave operators (like $\mathcal{F}^-$) are considered, for the second inequality in (e) we refer to [11, Proposition 1.1] where more general operators, not necessary concave, are taken into account. Property (f) is proven in [3, Lemma 4.9]. Let us prove (g). Note that $u \leq v_{\varphi_t}$, therefore $\Omega(u^+) \subset \Omega(v^+_{\varphi_t})$. Now, let $x_1 \in \partial\Omega(v^+_{\varphi_t})$, then there exists $y_1 \in \partial\Omega(u^+)$ such that $v_{\varphi_t}(x_1) = u(y_1) = 0$. Note that we must have $y_1 \in \partial B_{\varphi_t(x_1)}(x_1)$. Thus, $B_{\varphi_t(x_1)}(x_1)$ is tangent to $\partial\Omega(u^+)$ at y_1 contained in $\Omega(u^-)$ and according to Definition 8.1 we have that

$$u(y) = \alpha < y - y_1, \nu > +o(|y - y_1|), \tag{8.6}$$

where ν is the unit normal vector to $\partial\Omega(u^+)$ at y_1 pointing inward $\Omega(u^+)$. If $y = x + \varphi_t(x)\nu$, since $y_1 = x_1 + \varphi_t(x_1)\nu$, we obtain the asymptotic behavior of v_{φ_t} in a neighborhood of x_1:

$$\begin{aligned} v_{\varphi_t}(x) &\geq u(y) \\ &= \alpha < x + \varphi_t(x)\nu - x_1 - \varphi_t(x_1)\nu, \nu > +o(|x - x_1|) \\ &= \alpha < x - x_1 + (\varphi_t(x) - \varphi_t(x_1))\nu, \nu > +o(|x - x_1|). \end{aligned}$$

We replace $\varphi_t(x) - \varphi_t(x_1)$ by $< x - x_1, \nabla\varphi_t(x_1) > +o(|x - x_1|)$ in the previous inequality and simplify to obtain

$$v_{\varphi_t}(x) \geq \alpha < x - x_1, \nu + \nabla\varphi_t(x_1) > +o(|x - x_1|).$$

Thus, if we let

$$\bar{\alpha} := \alpha|\nu + \nabla\varphi_t(x_1)|, \quad \bar{\nu} := \frac{\nu + \nabla\varphi_t(x_1)}{|\nu + \nabla\varphi_t(x_1)|},$$

we obtain (8.5). By Lemma 4.9 in [3], $\bar{\nu}$ is the unit normal vector to $\partial\Omega^+(v_{\varphi_t})$ at x_1 pointing inward $\Omega^+(v_{\varphi_t})$. We note that in our problem we do not need the correctors used in the sup-convolution method to obtain the correct asymptotic behavior of v_{φ_t} on points on the free boundary (see [3, Lemma 4.12]).

Step 4: Comparison with subsolutions.

In what follows, we will have to compare the solution u of (8.1) with the functions

$$w_t(x) := \sup_{z\in B_{\epsilon\varphi_{bt}(x)}(x)} u(z-\tau), \quad x \in D, \tag{8.7}$$

for small $\tau \in \Gamma\left(\frac{\theta}{2}, e_n\right)$, where $D := B_{\frac{9}{10}}(0) \setminus B_{\frac{1}{8}}(x_0)$, b is defined in (8.3), $\epsilon = |\tau|\sin\left(\frac{\theta}{2}\right)$ and φ_t is the family of functions defined in Step 3. By (d) in Step 3 we can choose h small so that $\epsilon|\nabla\varphi_{bt}| < 1$, therefore by (f), we have that

$$\text{for every point of } \partial\Omega(w^+_{\varphi_t}) \text{ there is a tangent ball contained in } \Omega(w^+_{\varphi_t}). \tag{8.8}$$

Now, having on hands (8.8) and the asymptotic development (8.5) we can show the following comparison result between u and w_t: suppose that

$$u \geq w_t \text{ in } D, \quad u > w_t \quad \text{in } \Omega(w_t^+), \text{ then } \partial\Omega(w_t^+) \text{ and } \partial\Omega(w_t^+) \text{ cannot touch.} \tag{8.9}$$

The proof is given in [3, Lemma 2.1]. We perform it here for reader's convenience. By (8.9), we know that $\Omega(w_t^+) \subset \Omega(u^+)$. Suppose by contradiction that there exists $x_1 \in \partial\Omega(w_t^+)\cap\partial\Omega(u^+)$, then, by (8.8), there exists a tangent ball to $\partial\Omega(u^+)$ at x_1 contained in $\Omega(u^+)$. Thus, according to Definition 8.1, we have

$$u(x) = \alpha < x - x_1, \nu > +o(|x-x_1|). \tag{8.10}$$

and by (8.5), there exists $\eta > 0$ such that

$$w_t(x) \geq \eta < x - x_1, \nu > +o(|x-x_1|). \tag{8.11}$$

Note that here $\bar{\nu} = \nu$. Since $w_t \leq u$ and $w_t(x_1) = u(x_1) = 0$, by (8.10) and (8.11), it follows that

$$\alpha = \eta. \tag{8.12}$$

We have that $u - w_t$ is a supersolution for $\mathcal{F}^-$ in $\Omega(w_t^+)$, since by (c) in Proposition 2.1, (8.1) and (e), in $\Omega(w_t^+) \subset \Omega(u^+)$ we have

$$0 = \mathcal{F}^-(u) \geq \mathcal{F}^-(u-w_t) + \mathcal{F}^-(w_t) \geq \mathcal{F}^-(u-w_t).$$

Since $u > w_t$ in $\Omega(w_t)$, by the Hopf principle there exists $\delta > 0$ such that

$$(u-w_t)(x_1 + h\nu) \geq \delta h,$$

for all small $h > 0$. This is a contradiction, since by (8.10), (8.11) and (8.12), we have that

$$(u-w_t)(x_1 + h\nu) \leq o(h).$$

Thus, we conclude that $\partial\Omega(w_t^+)$ and $\partial\Omega(u^+)$ cannot touch.

Step 5: Carrying the improvement of Step 2 to the free boundary. The improvement obtained in Step 2 needs to be carried to the free boundary, in $B_{1/2}(0)$, giving up a little bit of the interior improvement.

In order to do this, we consider the family of functions w_t defined in (8.7). Let $D := B_{\frac{9}{10}}(0) \setminus B_{\frac{1}{8}}(x_0)$, let us check that the following conditions are satisfied:

i) $w_0 \leq u$ in D,

ii) $w_t \leq u$ on ∂D and $w_t < u$ in $\overline{\Omega(w_t^+)} \cap \partial D$,

iii) the family $\Omega(w_t^+)$ is uniformly continuous, that is, for every $\epsilon > 0$,

$$\Omega(w_{t_1}^+) \subset \mathcal{N}_\epsilon(\Omega(w_{t_2}^+))$$

whenever $|t_1 - t_2| < \delta(\epsilon)$, where $\mathcal{N}_\epsilon(\Omega(w_{t_2}^+))$ is a ϵ-neighborhood of $\Omega(w_{t_2}^+)$.

By (a) in Step 3, $\varphi_0 \equiv 1$ and thus by (8.2), if $x \in D$, we have

$$w_0(x) = \sup_{z\in B_\epsilon(x)} u(z-\tau) \leq u(x), \tag{8.13}$$

which is (i).

By (b) in Step 3, and (8.2) if $x \in \partial B_{\frac{9}{10}}(0)$, then

$$w_t(x) = \sup_{z\in B_\epsilon(x)} u(z-\tau) \leq u(x), \tag{8.14}$$

and the inequality is strict in $\overline{\Omega(w_t^+)}$, by taking any $\epsilon' < \epsilon$ if necessary. If $x \in \partial B_{\frac{1}{8}}(x_0)$ by (a) of Step 3 and (8.3), we have that (since t, $h \leq 1$),

$$w_t(x) \leq \sup_{z\in B_{(1+tbh)\epsilon}(x)} u(z-\tau) \leq \sup_{z\in B_{(1+b)\epsilon}(x)} u(z-\tau) < u(x). \tag{8.15}$$

Combining (8.14) and (8.15) yields (ii).

Finally, (iii) follows from the definition of the functions w_t, (8.7).

Now, from (i)-(iii) and by using (8.9), we can conclude that

$$w_t \leq u \text{ in } D \text{ for every } t \in [0,1]. \tag{8.16}$$

The proof of (8.16) is given in [3, Theorem 2.2] in the case of the Laplace operator and we present it here for the sake of completeness. For that, let $E := \{t \in [0,1] \,|\, v_t \leq u \text{ in } \overline{D}\}$. By (i) $0 \in E$. E is obviously closed. Let us show that it is open. If $t_0 \in E$, that is $v_{t_0} \leq u$ in D, from (ii) and the strong maximum principle it follows that $v_{t_0} < u$ in $\Omega(v_{t_0}^+) \cap D$. By (ii) and

(8.9) we have that $\overline{\Omega(v_{t_0})} \cap D$ is compactly supported in $\Omega(u^+) \cap D$ up to the boundary of D. From (iii), there exists $\delta > 0$ such that $\overline{\Omega(v_t)} \cap D$ is compactly supported in $\Omega(u^+) \cap D$ for all t such that $|t - t_0| < \delta$. Thus, for such values of t, by (ii) and (e) of Claim 1 we have

$$\mathcal{F}^- (v_{\varphi_t}) \geq 0 = \mathcal{F}^-(u) \quad \text{in } \Omega(v^+_{\varphi_t}) \cap D,$$

$$v_{\varphi_t} \leq u \quad \text{on } \partial(\Omega(v^+_{\varphi_t}) \cap D)$$

and by the comparison principle, $v_{\varphi_t} \leq u$ in $\Omega(v^+_{\varphi_t}) \cap D$. Similarly, since

$$\mathcal{F}^+ (v_{\varphi_t}) \geq 0 = \mathcal{F}^+(u) \quad \text{in } \Omega(u^-) \cap D,$$

and

$$v_{\varphi_t} \leq u \quad \text{on } \partial(\Omega(u^-) \cap D),$$

we have that $v_{\varphi_t} \leq u$ in $\Omega(u^-) \cap D$. Clearly $v_{\varphi_t} \leq 0 \leq u$ in $\overline{\Omega(u^+)} \cap \overline{\Omega(v^-_{\varphi_t})} \cap D$. We conclude that $v_{\varphi_t} \leq u$ in D and the openness of E follows. Since E is both an open and closed nonempty subset of $[0,1]$, we must have $E = [0,1]$. This proves (8.16).

Inequality (8.16) holds in particular for $t = 1$ and hence using (c) in Step 3 we obtain that, on $B_{1/2}(0)$,

$$\begin{aligned} u &\geq w_1 \\ &= \sup_{z \in B_{\epsilon\varphi_b(x)}(x)} u(z - \tau) \\ &\geq \sup_{z \in B_{\epsilon(1+(\lambda h)b)}(x)} u(z - \tau), \end{aligned}$$

which implies the desired improvement of the cone of monotonicity across the free boundary. The original radius ϵ in (8.13) was first improved to $\epsilon + \epsilon b$ far from the free boundary (see (8.3)), and at the free boundary the radius became $\epsilon + (\lambda h)\epsilon b$. Since $\lambda h < 1$, a little bit of opening in the cone has to be given up in order to bring the improvement across the free boundary (see Theorem 4.2 and Lemma 4.4 in [3] for details).

Step 6: Basic iteration.

Rescaling and repeating Steps 2-5 we obtain that the free boundary is $C^{1,\alpha}$ in $\mathcal{C}_{\frac{1}{2}}$, see the proof of Theorem 4.1 in [3] for details. ■

8.2. Flat free boundaries are Lipschitz. In this subsection we prove that if u is a solution of the free boundary problem (8.1) and the free boundary can be trapped in a narrow neighborhood in between two Lipschitz graphs, then the free boundary is actually Lipschitz. Let us recall the definition of ϵ-monotone function.

Definition 8.3. We say that u is ϵ-monotone in the cylinder $\mathcal{C}_1$ along a direction τ, with $|\tau| = 1$, if for all $x \in \mathcal{C}_1$,

$$u(x + l\tau) \geq u(x),$$

for all $l \geq \epsilon$ such that $x + l\tau \in \mathcal{C}_1$.

The ϵ-monotonicity can be reformulated equivalently as follows, see [3].

Definition 8.4. We say that u is ϵ_0-monotone in the cylinder $\mathcal{C}_1$ along the directions of the cone $\Gamma(\theta, e)$ if for all $x \in \mathcal{C}_1$,

$$\sup_{y \in B_{\epsilon \sin\theta}(x)} u(y - \epsilon e) \leq u(x),$$

for any $\epsilon \geq \epsilon_0$ such that $B_{\epsilon \sin\theta}(x - \epsilon e) \subset \mathcal{C}_1$.

As in Subsection 8, in the definition above $\Gamma(\theta, e)$ denotes the cone of semi-opening θ and axis e.

Remark 8.5. *If u is ϵ-monotone in $\mathcal{C}_1$ according to Definition 8.4, then the level surfaces of u in $\mathcal{C}_1$, $\partial\{u > t\}$, are contained in a $(1 - \sin\theta)\epsilon$ size of the graph of a Lipschitz function g with Lipschitz constant $L = \mathrm{cotan}\,\theta < 1$, see [3].*

Proposition 8.6. *Let $\frac{\pi}{4} < \theta < \frac{\pi}{2}$ and let u be a viscosity solution of the problem (8.1) in $\mathcal{C}_1 = B_1' \times (-1, 1)$. Assume that $0 \in \partial\Omega(u^+)$. Then there exists $\epsilon = \epsilon(\theta)$ such that if u is ϵ-monotone in $\mathcal{C}_{1-\epsilon} = B_{1-\epsilon}' \times (-1 + \epsilon, 1 - \epsilon)$ along any direction τ in the cone $\Gamma(\theta, e)$, then u is fully monotone in $\mathcal{C}_{\frac{1}{2}} = B_{\frac{1}{2}}' \times \left(-\frac{1}{2}, \frac{1}{2}\right)$ along any direction $\tau \in \Gamma(\theta_1, e)$ with $\theta_1 = \theta_1(\theta, \epsilon)$.*

Proof: The proof of this result follows from [5] (see also [3]) and [20]. We will sketch the proof below.

Step 1: Full monotonicity of u outside a strip of size $M\epsilon$ of the free boundary.

By Lemma 1 in [21] there exists $M > 1$ such that in $\mathcal{C}_1 \setminus \mathcal{N}_{M\epsilon}$, where

$$\mathcal{N}_{M\epsilon} := \{x \in \mathcal{C}_1 \,|\, d(x, \partial\Omega(u^+)) < M\epsilon\}$$

u is actually fully monotone along any direction of $\tau \in \Gamma(\theta, e)$.

Step 2: Construction of a family of subsolutions of variable radii. Following the method developed in [6], we need to construct a family of subsolutions of the form

$$w(x) = \sup_{z \in B_{\varphi(x)}(x)} u(z - \lambda \epsilon e),$$

for some $\lambda \in (0,1)$, to compare with the solution u of (8.1). Up to a change of coordinates, we can assume that

$$e = e_n.$$

Since u is ϵ-monotone, by Remark 8.5 there exists $g : \mathbb{R}^{n-1} \to \mathbb{R}$ with $g(0) = 0$ and Lipschitz constant $L = \operatorname{cotan} \theta < 1$, such that if

$$A := \{(x', x_n) \in \mathbb{R}^n \,|\, x_n = g(x')\}, \tag{8.17}$$

then

$$\partial\Omega(u^+) \subset \mathcal{N}_\epsilon(A), \tag{8.18}$$

where

$$\mathcal{N}_\epsilon(A) := \{x \in \mathcal{C}_1 \,|\, d(x, A) < \epsilon\}.$$

By Lemmas 2 and 3 in [21] and Proposition 1.1 in [11], for any given $\delta > 0$, there exists a family of C^2-functions, φ_t, $0 \le t \le 1$, defined on $\mathcal{C} := \overline{B_1'(0)} \times [-2L, 2L]$, with the following properties:

a) $1 \le \varphi_t \le 1 + t$,

b) $\varphi_t \equiv 1$ on $A_\delta := \{x \in \mathcal{C} \,|\, d(x, A \cap \partial\mathcal{C}) < \delta\}$,

c) in the set $\{x \in \mathcal{C} \,|\, d(x, \partial\mathcal{C}) > \delta\}$,

$$\varphi_t \ge 1 + t\left(1 - \frac{C\delta}{d(x, \partial\mathcal{C})^2}\right),$$

d) $|\nabla \varphi_t| \le \frac{Ct}{\delta}$.

Moreover,

e) if we define

$$v_{\varphi_t}(x) := \sup_{z \in B_{\varphi_t(x)}(x)} u(z),$$

then v_{φ_t} satisfies

$$\mathcal{F}^-(v_{\varphi_t}) \ge 0 \quad \text{in } \Omega(v_{\varphi_t}^+),$$

$$\mathcal{F}^+(v_{\varphi_t}) \ge 0 \quad \text{in } \Omega(v_{\varphi_t}^-),$$

and if $|\nabla\varphi_t| < 1$ then

f) for every point of $\partial\Omega^+(v_{\varphi_t})$ there is a tangent ball contained in $\Omega^+(v_{\varphi_t})$,

g) if

$$0 < \sin\overline{\theta} \le \frac{1}{1+|\nabla\varphi_t|}\left(\sin\theta - \frac{\epsilon}{2\varphi_t}\cos^2\theta - |\nabla\varphi_t|\right),$$

then v_{φ_t} is monotone in the cone $\Gamma(\overline{\theta}, e_n)$; in particular its level surfaces are Lipschitz graphs, in the direction of e_n, with Lipschitz constant $\overline{L} \le \mathrm{cotan}\,\overline{\theta}$.

Finally, as in the proof of Proposition 8.2, if $|\nabla\varphi_t| < 1$, the function v_{φ_t} has the following behavior at points of $\partial\Omega(v^+_{\varphi_t})$

h) for every point $x_1 \in \partial\Omega(v^+_{\varphi_t})$ there exists $\bar{\alpha} > 0$ such that

$$v_{\varphi_t}(x) \ge \bar{\alpha} < x - x_1, \bar{\nu} > +o(|x - x_1|),$$

where $\bar{\nu}$ is the normal vector of $\partial\Omega(v^+_{\varphi_t})$ pointing inward $\Omega(v^+_{\varphi_t})$.

Step 3: Comparison with subsolutions. In what follows, we will have to compare the solution u of (8.1) with the functions

$$w_t(x) := \sup_{z\in B_{\sigma\varphi_t(x)}(x)} u(z - \lambda\epsilon e_n), \tag{8.19}$$

for σ, $\lambda \in (0, 1)$ to be determined, where φ_t is the family of functions defined in Step 2. We first notice that from the ϵ-monotonicity of u (Definition 8.4), for $1 - \lambda < \sqrt{2}/2$, we have

$$\sup_{z\in B_{\epsilon(\sin\theta-(1-\lambda))}(x)} u(z - \lambda\epsilon e_n) \le \sup_{z\in B_{\epsilon\sin\theta}(x)} u(z - \epsilon e_n) \le u(x), \tag{8.20}$$

since $B_{\epsilon(\sin\theta-(1-\lambda))}(x - \lambda\epsilon e_n) \subset B_{\epsilon\sin\theta}(x - \epsilon e_n)$.

For any $\eta > 0$ and A defined as in (8.17), let us denote by $\mathcal{N}_\eta(A)$ the η-neighborhood of A, defined by

$$\mathcal{N}_\eta(A) := \{x \in \mathcal{C} \,|\, d(x, A) < \eta\}.$$

By Step 1 and (8.18), u is fully monotone in the directions of $\Gamma(\theta, e_n)$, outside the set $\mathcal{N}_{2M\epsilon}(A)$. Therefore,

$$\sup_{z\in B_{\lambda\epsilon\sin\theta}(x)} u(z - \lambda\epsilon e_n) \le u(x) \quad \text{for } x \notin \mathcal{N}_{2M\epsilon}(A). \tag{8.21}$$

We now choose

$$\sigma := \epsilon(\sin\theta - (1-\lambda)), \quad \lambda \geq \frac{3}{2} - \frac{\sqrt{2}}{2}, \quad \delta := \epsilon^{\frac{1}{2}}. \tag{8.22}$$

Then the family of functions w_t in (8.19) is well defined in $\mathcal{C}_{1-\epsilon} \cap \mathcal{N}_{2M\epsilon}(A)$. Moreover, (e)-(h) of Step 2 hold true for ϵ (and thus σ) small enough. Since σ defined as in (8.22) satisfies $\sigma < \lambda\epsilon\sin\theta$, by (a) of Step 2 we can choose $\bar{t} > 0$ so small that

$$\sigma\varphi_t \leq \lambda\epsilon\sin\theta, \quad \text{for } 0 \leq t \leq \bar{t}. \tag{8.23}$$

By (e)-(h) of Step 2, the functions w_t, $0 \leq t \leq 1$, satisfy

$$\mathcal{F}^-(w_t) \geq 0 \quad \text{in } \Omega(w_t^+), \tag{8.24}$$

$$\text{for any point of } \partial\Omega(w_t^+) \text{ there is a tangent ball contained in } \Omega(w_t^+) \tag{8.25}$$

$$\begin{aligned} &\text{For every point } x_1 \in \partial\Omega(w_t^+), \ \text{there exists } \bar{\alpha} > 0 \text{ such that} \\ &w_t(x) \geq \bar{\alpha} < x - x_1, \bar{\nu} > + o(|x - x_1|). \end{aligned} \tag{8.26}$$

Let us show that for all $0 \leq t \leq \bar{t}$,

$$w_t(x) \leq u(x) \quad \text{for } x \in \partial(\mathcal{N}_{2M\epsilon}(A) \cap \mathcal{C}_{1-4\epsilon}). \tag{8.27}$$

If $x \in \partial(\mathcal{N}_{2M\epsilon}(A)) \cap \mathcal{C}_{1-4\epsilon}$, then by (8.23) and (8.21), we have that

$$w_t(x) \leq \sup_{z \in B_{\lambda\epsilon\sin\theta}(x)} u(z - \lambda\epsilon e_n) \leq u(x). \tag{8.28}$$

If $x \in \mathcal{N}_{2M\epsilon}(A)) \cap \partial(\mathcal{C}_{1-4\epsilon})$, then, since for ϵ small enough $\delta = \epsilon^{1/2} > 4\epsilon$, by (b) of Step 2, $\varphi_t(x) = 1$. Thus, by the definition of σ in (8.22) and (8.20), for $x \in \mathcal{N}_{2M\epsilon}(A)) \cap \partial(\mathcal{C}_{1-4\epsilon})$,

$$w_t(x) = \sup_{z \in B_{\epsilon(\sin\theta - (1-\lambda))}(x)} u(z - \lambda\epsilon e_n) \leq u(x).$$

This concludes the proof of (8.27).

Finally, by (8.27) and using that the functions w_t satisfy (8.24)-(8.26), arguing as in Step 5 of the proof of Proposition 8.2, we infer that, for $0 \leq t \leq \bar{t}$,

$$w_t(x) \leq u(x) \quad \text{for all } x \in \mathcal{N}_{2M\epsilon}(A) \cap \mathcal{C}_{1-4\epsilon}. \tag{8.29}$$

Step 4: From the ϵ-monotonicity to the $\lambda\epsilon$-monotonicity.

Arguing as in [6] (see also Lemma 5.7 in [3]), by (8.29) and (c) of Step 2, we have that there exists $c_0 > 0$ such that in $\mathcal{N}_{2M\epsilon}(A) \cap \mathcal{C}_{1-4\epsilon^{1/8}}$

$$\sup_{\lambda\epsilon \sin(\theta - c_0\epsilon^{1/4})} u(z - \lambda\epsilon e_n) \leq u(x),$$

that is u is $\lambda\epsilon$-monotone in any direction of the cone of directions $\Gamma(\theta - c_0\epsilon^{1/4}, e_n)$.

Step 5: Basic iteration.

Rescaling and repeating Steps 1-4, we obtain that the free boundary is Lipschitz in $\mathcal{C}_{\frac{1}{2}}$, see the proof of Theorem 5.1 in [3] for details. ■

8.3. Proof of Theorem 3.3. Let u be a solution of (1.4). Then, by Theorem 3.2, u is a solution of the free boundary problem (8.1) in the sense of Definition 8.1. Let $z \in \Gamma$ be a regular point. Assume without loss of generality that $z = 0$. By Corollary 7.2, there exists $r_j \to 0$ as $j \to +\infty$ with the following property: for any $\epsilon > 0$ there exists $J \in \mathbb{N}$ such that for any $j \geq J$, all the level sets of $u_{r_j}(x) = u(r_j x)/r_j$ in $B_2(0)$ are ϵ-flat. Also, by scaling invariance u_{r_j} is solution of (8.1) in the cylinder $\mathcal{C}_1 = B'_1(0) \times (-1, 1)$. We can now apply Propositions 8.2 and 8.6 to conclude that there is $J \in \mathbb{N}$ such that for any $j \geq J$ the set $\partial\Omega((u_1)_{r_j}) \cap B_{\frac{1}{4}}(0)$ is of class $C^{1,\alpha}$ for some $0 < \alpha \leq 1$. Therefore, the same is true for $\Gamma \cap B_{\frac{r_j}{4}}(0)$, as $\Gamma \cap B_{\frac{r_j}{4}}(0) = r_j \partial\Omega((u_1)_{r_j})) \cap B_{\frac{1}{4}}(0))$. Let us prove that the set of regular points is open in Γ.

By the elliptic regularity theory, see Corollary 1.8 in [17], $u_1 \in C^{1,\alpha}(\overline{\Omega(u_1)} \cap B_{\frac{r_j}{8}}(0))$ and $u_2 \in C^{1,\alpha}(\overline{\Omega(u_2)} \cap B_{\frac{r_j}{8}}(0))$, thus

$$u(x) = \frac{\partial u_1}{\partial \nu_1}(0) < x, \nu >^+ - \frac{\partial u_2}{\partial \nu_2}(0) < x, \nu >^- + o(|x|), \tag{8.30}$$

and by Theorem 7.4

$$\frac{\partial u_1}{\partial \nu_1}(0) = \frac{\partial u_2}{\partial \nu_2}(0) > 0,$$

where ν_i is the interior unit normal vector to $\Omega(u_i)$. In particular, u has the asymptotic behavior (3.1) at 0. By the $C^{1,\alpha}$ local regularity of u_1 and u_2 up to the free boundary , there exists $s < r_j/8$, such that:

$$\frac{\partial u_1}{\partial \nu_1}(x_0) > 0, \quad \frac{\partial u_2}{\partial \nu_2}(x_0) > 0, \text{ for any } x_0 \in \Gamma \cap B_s(0), \tag{8.31}$$

and

$$u(x) = \frac{\partial u_1}{\partial \nu_1}(x_0) < x - x_0, \nu >^+ - \frac{\partial u_2}{\partial \nu_2}(x_0) < x - x_0, \nu >^- + o(|x - x_0|).$$

Hence each $x_0 \in \Gamma \cap B_s(0)$ is a regular point of u. Actually, again from Theorem 7.4, we have that $\frac{\partial u_1}{\partial \nu_1}(x_0) = \frac{\partial u_2}{\partial \nu_2}(x_0)$. We have proven that the set of regular points is an open set of Γ, locally of class $C^{1,\alpha}$ and this concludes the proof of the theorem.

9. Appendix

Lemma 9.1. *Assume $r, \gamma, c > 0$, and let*

$$\psi(x) = c\left(\frac{r^\gamma}{|x|^\gamma} - 1\right), \quad x \neq 0.$$

Then, the following holds.

i) $\psi(x) > 0$ *if* $|x| < r$, $\psi(x) = 0$ *if* $|x| = r$, $\psi(x) < 0$ *if* $|x| > r$.

ii) *If ν is the interior normal unit vector of $B_r(0)$, then*

$$\nabla\psi(x) = \frac{c\gamma}{r}\nu \quad \text{for any } x \in \partial B_r(0).$$

iii) *For any $x \in B_r(0)$,*

$$\psi(x) \geq \frac{c\gamma}{r}(r - |x|).$$

iv) *If $\gamma = \frac{\Lambda(n-1)-\lambda}{\lambda}$, then $\mathcal{M}^-(\psi)(x) = 0$ for all $x \neq 0$.*

Proof: Property (i) is immediate.

To prove (ii)-(iv), let us compute the gradient and the Hessian matrix of ψ. We get, for $x \neq 0$,

$$\nabla\psi(x) = -c\gamma r^\gamma \frac{x}{|x|^{\gamma+2}},$$

and

$$D^2\psi(x) = \frac{c\gamma r^\gamma}{|x|^{\gamma+2}}\left((\gamma+2)\frac{x \otimes x}{|x|^2} - I_n\right),$$

where I_n is the $n \times n$ identity matrix.

In particular, if $|x| = r$ and $\nu = -\frac{x}{r}$ is the interior normal unit vector of $B_r(0)$ at x, then we see that

$$\nabla\psi(x) = -\frac{c\gamma}{r}\frac{x}{r} = \frac{c\gamma}{r}\nu,$$

which proves (ii).

To prove (iii), let us denote $\rho = |x|$ and let $\psi(\rho) = c\left(\frac{r^\gamma}{\rho^\gamma} - 1\right)$. Then using that $\psi'(r) = -\frac{c\gamma}{r}$ and that $\psi''(\rho) \geq 0$, we get

$$\psi(\rho) \geq \frac{c\gamma}{r}(r-\rho),$$

which gives (iii).

Next, it is easy to see that, given any $n \times n$-matrix A with eigenvalues $\lambda_1, \ldots, \lambda_n$, then the eigenvalues of $A - I_n$ are $\lambda_1 - 1, \ldots, \lambda_n - 1$. Therefore, since the eigenvalues of $\frac{x\otimes x}{|x|^2}$ are $\lambda_1 = \ldots = \lambda_{n-1} = 0$ and $\lambda_n = 1$, we infer that $(\gamma+2)\frac{x\otimes x}{|x|^2} - I_n$ has $(n-1)$ negative eigenvalues equal to -1 and one positive eigenvalue equal to $(\gamma+1)$. In particular

$$\mathcal{M}^-(\psi) = \frac{c\gamma r^\gamma}{|x|^{\gamma+2}}\left[\lambda(\gamma+1) - \Lambda(n-1)\right].$$

Property (iv) then follows. ∎

Lemma 9.2. *Let ϕ be the solution of*

$$\begin{cases} \mathcal{F}^-(\phi) = 0 & \text{in } B_r(0) \setminus B_{\frac{r}{2}}(0) \\ \phi = 1 & \text{on } \partial B_{\frac{r}{2}}(0) \\ \phi = 0 & \text{on } \partial B_r(0). \end{cases} \tag{9.1}$$

Then, $\phi = \phi(|x|)$ is a radial function and there exists a constant $\sigma > 0$ independent of r such that for $x \in B_r(0) \setminus B_{\frac{r}{2}}(0)$ and $y_0 \in \partial B_r(0)$,

$$\phi(x) = \frac{\sigma}{r} < x - y_0, \nu > + o(|x-y_0|),$$

where ν is the interior normal unit vector of $B_r(0)$ at y_0.

Proof: Let φ be the solution of (9.1) with $r = 1$. Then, since $\mathcal{F}^-$ is a concave operator, we have that $\varphi \in C^{2,\alpha}(\overline{B_1(0)} \setminus B_{\frac{r}{2}}(0))$, see [2]. Let O be any orthogonal matrix and let $v(x) := \varphi(Ox)$. By Proposition 2.2, $\mathcal{F}^-$ is invariant under rotations, thus v is solution of (9.1) and by uniqueness, $\varphi(Ox) = \varphi(x)$. Since the latter equality holds true for any orthogonal matrix O, we infer that φ is a radial function, $\varphi = \varphi(|x|)$.

Let $\psi_1(x) := 1/(2^\gamma - 1)\left(\frac{1}{|x|^\gamma} - 1\right)$ where $\gamma = \frac{\Lambda(n-1)-\lambda}{\lambda}$, and let ψ_2 be the harmonic function solution of

$$\begin{cases} \Delta\psi_2 = 0 & \text{in } B_1(0) \setminus B_{\frac{1}{2}}(0) \\ \psi_2 = 1 & \text{on } \partial B_{\frac{1}{2}}(0) \\ \psi_2 = 0 & \text{on } \partial B_1(0), \end{cases}$$

i.e., for $n > 2$, $\psi_2(x) = 1/(2^{n-2} - 1)\left(\frac{1}{|x|^{n-2}} - 1\right)$. Then by Lemma 9.1 and the comparison principle, for $x \in B_1(0) \setminus B_{\frac{1}{2}}(0)$,

$$\psi_1(x) \leq \varphi(x) \leq \psi_2(x)$$

and thus there exists σ, $\gamma/(2^\gamma - 1) \leq \sigma \leq (n-2)/(2^{n-2} - 1)$, such that if $y_0 \in \partial B_1(0)$,

$$\varphi(x) = \sigma < x - y_0, \nu > +o(|x - y_0|).$$

The lemma is proven by noticing that $\phi(x) = \varphi(x/r)$ is the solution of (9.1). ∎

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