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SOME ASPECTS OF (NON)FUNCTORIALITY OF NATURAL DISCRETE COVERS OF LOCALES

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ABSTRACT: The frame $\mathsf{S_c}(L)$ generated by closed sublocales of a locale L is known to be a natural Boolean ("discrete") extension of a subfit L; also it is known to be its maximal essential extension. In this paper we first show that it is an essential extension of any L and that the maximal essential extensions of L and $\mathsf{S_c}(L)$ are isomorphic. The construction $\mathsf{S_c}$ is not functorial; this leads to the question of individual liftings of homomorphisms $L \to M$ to homomorphisms $\mathsf{S_c}(L) \to \mathsf{S_c}(M)$. This is trivial for Boolean L and easy for a wide class of spatial L, M. Then, we show that one can lift all $h : L \to \mathbf{2}$ for weakly Hausdorff L (and hence the spectra of L and $\mathsf{S_c}(L)$ are naturally isomorphic), and finally present liftings of $h : L \to M$ for regular L and arbitrary Boolean M.

KEYWORDS: frame, locale, sublocale, sublocale lattice, essential extension, subfit, Booleanization.
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Introduction

The trivial discretization $(X, \mathfrak{P}(X)) \to (X, \tau)$ of a topological space allows dealing, inside the category of spaces, with general maps $X \to Y$ (or maps with a weaker continuity) in parallel with continuous maps $(X, \tau) \to (Y, \theta)$. This has in the point-free context a not quite so trivial counterpart in the extension $L \to \mathsf{S_c}(L)$ of a subfit frame to a certain Boolean algebra.

Let us explain in some detail what $\mathsf{S_c}(L)$ is about. It is the lattice of sublocales of L join-generated by the closed ones, which appeared, first, in connection with comparison of fitness and subfitness and in the study of scattered frames ([4]), and later turned out to play a basic role as a discretization tool ([14, 15]). This lattice $\mathsf{S_c}(L)$ is, for any L, a *frame* (although naturally included in the *coframe* $\mathsf{S}(L)$ of all sublocales). Moreover, for subfit frames (a

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very large class of frames, a counterpart of a class of spaces more general than T_1), $\mathsf{S}_{\mathfrak{c}}(L)$ is Boolean (see [15]). Being Boolean is in the point-free context the natural counterpart of discreteness in spaces, and $\mathsf{S}_{\mathfrak{c}}(L)$ indeed turned out to be a satisfactory tool for dealing, e.g., with rings of real functions (continuous, semicontinuous, general) in the point-free context; for instance, unlike other representations it is conservative in the sense that it precisely extends the theory of classical spaces ([14]). Furthermore, for subfit frames the natural extension $L \to \mathsf{S}_{\mathfrak{c}}(L)$ is characterized by being the *maximal essential extension* (see [3]), as is for T_D-spaces the extension of τ to $\mathfrak{P}(X)$ (see 3.2 below).

Unlike the discretization of spaces, the construction $\mathsf{S}_{\mathfrak{c}}(L)$ is not functorial. In fact, it is not to be expected, regarding the non-functoriality of the standard Booleanization ([5]), to which it is, in a way, related. Therefore we have to put up with individual extensions of frame homomorphisms $h\colon L \to M$ to commutative diagrams

$$\begin{tikzcd} \mathsf{S}_{\mathfrak{c}}(L) \arrow[r, "\widetilde{h}"] & \mathsf{S}_{\mathfrak{c}}(M) \\ L \arrow[u] \arrow[r, "h"] & M \arrow[u] \end{tikzcd} \cdot$$

We call them liftings; the liftings for some classes of homomorphisms constitute the main topic of this article.

After Preliminaries introducing the necessary concepts and notation, we discuss in Section 2 the behavior of the $\mathsf{S}_{\mathfrak{c}}(L)$ for general L extending what is known for the subfit case. It is always an essential extension, although it is maximal such (and hence Boolean) only for subfit L. Also, we show that the maximal essential extensions of L and $\mathsf{S}_{\mathfrak{c}}(L)$ are always isomorphic. Then, in Section 3, we compare the classical spaces and the point-free ones from a new perspective and explain the concept of lifting. In Section 4 we show that, in the broad context of weakly Hausdorff (and also other) frames, the homomorphisms $L \to \mathbf{2}$ always lift, which results in the isomorphism of the spectra of L and $\mathsf{S}_{\mathfrak{c}}(L)$. Finally, in Section 5, we prove the liftability of homomorphisms for most (but not quite all) spatial frames and, on the other hand, for Boolean sources resp. targets. The last then results in a canonical connection between $\mathsf{S}_{\mathfrak{c}}(L)$ and the standard Booleanization of L.

1. Preliminaries

1.1. Notation. For a subset A of a poset $(X, \leq)$ we write

$$\uparrow\! A = \{x \in X \mid x \geq a \text{ for some } a \in A\}$$

and abbreviate $\uparrow\!\{x\}$ to $\uparrow\! x$. A join (supremum) of a subset $A \subseteq (X, \leq)$, if it exists, will be denoted by $\bigvee A$, and we write $a \vee b$ for $\bigvee\{a, b\}$; similarly we write $\bigwedge A$ and $a \wedge b$ for meets (infima). The smallest element of a poset (the supremum $\bigvee \emptyset$), if it exists, will be denoted by 0, and the largest element (the infimum $\bigwedge \emptyset$) will be denoted by 1. The dual of a poset X, that is, the poset with the order on X reversed, will be denoted by X^{op}.

1.1.1. Adjoint maps. If X, Y are posets we say that monotone maps $f\colon X \to Y$ and $g\colon Y \to X$ are *adjoint, f to the left and g to the right*, if

$$f(x) \leq y \iff x \leq g(y).$$

Recall that this is characterized by the pair of inequalities $fg(y) \leq y$ and $x \leq gf(x)$, and that f resp. g preserves all the existing suprema resp. infima. Furthermore, if X and Y are complete lattices then a monotone map $f\colon X \to Y$ preserves all suprema iff it is a left adjoint, and a monotone map $g\colon Y \to X$ preserves all infima iff it is a right adjoint.

1.2. Pseudocomplements, complements, Boolean algebras. The *pseudocomplement* of an element a in a lattice L, that is, the largest element b such that $b \wedge a = 0$ (if it exists) will be denoted by a^* (thus, $x \wedge a = 0$ iff $x \leq a^*$). A *complement* of a is a b such that $b \wedge a = 0$ and $b \vee a = 1$. If it exists we say that a is *complemented*. In distributive lattices (which will be always the case in this article), every complement is a pseudocomplement. Hence, we will use the notation a^* also for complements.

A Boolean algebra is a distributive lattice in which every element is complemented. For any distributive lattice with pseudocomplements we have the *Booleanization*

$$\beta_L = (a \mapsto a^{**})\colon L \to \mathfrak{B}L = \{a^{**} \mid a \in L\},$$

a homomorphism the target of which is a Boolean algebra.

1.3. Frames and coframes. A *frame*, resp. *coframe*, is a complete lattice L satisfying the distributivity law

$$(\bigvee A) \wedge b = \bigvee\{a \wedge b \mid a \in A\}, \tag{frm}$$

$$\text{resp.} \quad (\bigwedge A) \vee b = \bigwedge\{a \vee b \mid a \in A\}, \tag{cofrm}$$

for all $A \subseteq L$ and $b \in L$; a *frame homomorphism* preserves all joins and all finite meets. The lattice $\Omega(X)$ of all open subsets of a topological space X provides an example of a frame, and an example of a frame homomorphism $\Omega(f)\colon \Omega(Y) \to \Omega(X)$ is obtained from a continuous $f\colon X \to Y$ by setting $\Omega(f)(U) = f^{-1}[U]$. Thus we obtain a functor $\Omega\colon \mathbf{Top} \to \mathbf{Frm}^{\mathrm{op}}$.

1.3.1. Spectrum. Recall that the functor Ω has a right adjoint $\Sigma\colon \mathbf{Frm}^{\mathrm{op}} \to \mathbf{Top}$, the *spectrum* defined by

$$\Sigma(L) = (\{p \mid p\colon L \to \mathbf{2}\}, \{\Sigma_a \mid a \in L\}), \quad \Sigma h(p) = p \cdot h,$$

where $\mathbf{2} = \{0, 1\}$ is the two-element Boolean frame and $\Sigma_a = \{p \mid p(a) = 1\}$. The $p\colon L \to \mathbf{2}$ are often referred to as *points* of L, and they are in a one-to-one correspondence with the *primes* of L (elements a such that $x \wedge y \le a$ only if either $x \le a$ or $y \le a$) given by the formula $p_a(x) = 0$ iff $x \le a$.

1.3.2. In a frame, (cofrm) generally does not hold, similarly (frm) does not hold in a coframe. But

if b is complemented, then both (frm) *and* (cofrm) *hold for b in any frame and any coframe*

(see, e.g., [7, 11]).

1.4. The Heyting structure. The equality (frm) states that the maps $(x \mapsto x \wedge b)\colon L \to L$ preserve all joins. Hence by 1.1.1 every frame is a Heyting algebra with the Heyting operation $\to$ satisfying

$$a \wedge b \le c \quad \text{iff} \quad a \le b \to c.$$

The pseudocomplements in a frame are obviously given by $a^* = a \to 0$.

1.5. The coframe of sublocales. A *sublocale* of a frame L is a subset $S \subseteq L$ such that

- for every $M \subseteq S$ the meet $\bigwedge M$ is in S, and
- for every $s \in S$ and every $x \in L$, $x \to s$ is in S.

The category $\mathbf{Loc} = \mathbf{Frm}^{\mathrm{op}}$ can be thought of as a category of generalized spaces. It is of advantage to treat it as a concrete category with arrows opposite to frame homomorphisms $h\colon L \to M$ represented as their right Galois adjoints $f\colon M \to L$; these will be referred to as *localic maps*. In emphasizing this point of view, we often speak of frames as *locales*. From this perspective, the sublocales are indeed the sub-locales in the sense that they are precisely the subsets for which the embeddings are extremal monomorphisms in the category **Loc**.

The system $\mathsf{S}(L)$ of all sublocales of L, ordered by inclusion, is a complete lattice with

$$\bigwedge S_i = \bigcap S_i \quad \text{and} \quad \bigvee S_i = \{\bigwedge M \mid M \subseteq \bigcup S_i\}.$$

The least sublocale $\bigvee \emptyset = \{1\}$ is denoted by O and referred to as the *void sublocale*[1]. It is a fundamental fact that the lattice $\mathsf{S}(L)$ is a coframe ([8],[11]).

1.5.1. Open and closed sublocales. With each element $a \in L$ there is associated an *open sublocale* and a *closed sublocale*, respectively

$$\mathfrak{o}(a) = \{x \mid x = a \to x\} = \{a \to x \mid x \in L\} \quad \text{and} \quad \mathfrak{c}(a) = {\uparrow}a.$$

Open and closed sublocales correspond precisely to the *open and closed parts*, respectively, in Isbell's pioneering article [7]. Furthermore, in the spatial case $L = \Omega(X)$ they correspond to open and closed subspaces of X. They are complements of each other in $\mathsf{S}(L)$. We have

$$\mathfrak{o}(0) = \mathsf{O},\ \mathfrak{o}(1) = L,\ \mathfrak{o}(\bigvee a_i) = \bigvee \mathfrak{o}(a_i),\ \mathfrak{o}(a \wedge b) = \mathfrak{o}(a) \wedge \mathfrak{o}(b), \quad \text{and}$$
$$\mathfrak{c}(0) = L,\ \mathfrak{c}(1) = \mathsf{O},\ \mathfrak{c}(\bigvee a_i) = \bigwedge \mathfrak{c}(a_i),\ \mathfrak{c}(a \wedge b) = \mathfrak{c}(a) \vee \mathfrak{c}(b).$$

1.6. Subfitness, fitness, regularity. A frame is *subfit* if for all a and b,

$$a \nleq b \implies \exists c\ (a \vee c = 1 \neq b \vee c).$$

For spaces this is a separation axiom slightly weaker than T_1. The original (and equivalent) definition in [7] characterizes subfitness by the property that

$$\textit{each open sublocale is a join of closed sublocales.} \tag{1.6.1}$$

A frame is *fit* if for all a and b,

$$a \nleq b \implies \exists c\ (a \vee c = 1,\ c \to b \nleq b),$$

A frame is regular if for every a, $a = \bigvee\{x \mid x^* \vee a = 1\}$; note that this corresponds to the classical regularity: a space X is regular iff $\Omega(X)$ is regular in the just defined sense.

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[1]This may sound odd but it makes good sense; if L happens to have points, they are sublocales of the form $\{a, 1\}$ with prime $a \neq 1$.

1.7. The frame $\mathsf{S}_{\mathfrak{c}}(L)$. Set

$$\mathsf{S}_{\mathfrak{c}}(L) \equiv \{S \in \mathsf{S}(L) \mid S \text{ is a join of closed sublocales of } L\}.$$

Somewhat surprisingly, this set of sublocales is always a *frame* (see [15, 4]). For subfit L (and only for them), $\mathsf{S}_{\mathfrak{c}}(L)$ is a Boolean algebra (see [15]).

By the formulas in 1.5.1, $\mathfrak{o}$ constitutes an embedding $L \to \mathsf{S}(L)$. Furthermore, by (1.6.1), for subfit frames we have an embedding

$$\mathfrak{o}\colon L \to \mathsf{S}_{\mathfrak{c}}(L) \qquad (1.7.1)$$

and although the embedding $\mathsf{S}_{\mathfrak{c}}(L) \subseteq \mathsf{S}(L)$ in general does not preserve finite meets, since the intersection $\mathfrak{o}(a) \cap \mathfrak{o}(b) = \mathfrak{o}(a \wedge b)$ in $\mathsf{S}(L)$ is open, it is in $\mathsf{S}_{\mathfrak{c}}(L)$ and hence also the meet in this frame. Hence, (1.7.1) is a frame embedding.

1.7.1. From [15] recall that

- for a subfit L, $\mathsf{S}_{\mathfrak{c}}(L)$ is the (co)Booleanization of $\mathsf{S}(L)$, and
- for a T_1-space X, $\mathsf{S}_{\mathfrak{c}}(\Omega(X))$ picks out precisely the subspaces of X among all the sublocales of $\Omega(X)$.

1.8. The axiom T_D. This is a separation axiom between T_0 and T_1 (defined first in [1]). It requires that

$$\forall x \in X\ \exists \text{ open } U \ni x \text{ such that } U \smallsetminus \{x\} \text{ is open.} \qquad (T_D)$$

For more about frames and locales see, e.g., [8, 11].

2. $\mathsf{S}_{\mathfrak{c}}(L)$ as an essential extension

2.1. A monomorphism $m\colon A \to B$ in a general category is called an *essential extension* if every morphism $f\colon B \to C$ such that $f \cdot m\colon A \to C$ is a monomorphism is itself a monomorphism. If $m\colon A \to B$ is an essential extension we say that A *is essential* in B. We are interested in essential extensions in the category of frames, and for comparison also in the category of distributive lattices **DLat**.

A *maximal essential extension* of L is an essential extension $m\colon L \to M$ such that for every essential extension $n\colon L \to N$ there is precisely one morphism $g\colon N \to M$ such that $g \cdot n = m$.

From [3] let us quote the following facts (also based on the results from [17, 2]).

(2.1.1) *In* **DLat** *and* **Frm**, *$m\colon L \to M$ is essential iff for each pair $x < y$ in M there is a pair $a < b$ in L such that $x \wedge m(b) \leq m(a)$ and $y \vee m(a) \geq m(b)$.*
(2.1.2) *In* **DLat** *and* **Frm**, *every L has a maximal essential extension, and an essential extension $m\colon L \to M$ is maximal iff M is Boolean.*
(2.1.3) *The maximal essential extension of a frame in* **Frm** *coincides with its maximal essential extension in* **DLat**.
(2.1.4) *If L is subfit then $\mathfrak{o}\colon L \to \mathsf{S}_\mathfrak{c}(L)$ is a maximal essential extension.*

2.2. Computing in $\mathsf{S}_\mathfrak{c}(L)$. (1) The embedding of $\mathsf{S}_\mathfrak{c}(L)$ into $\mathsf{S}(L)$ preserves joins but not meets. However, if an element of $\mathsf{S}_\mathfrak{c}(L)$ is complemented in $\mathsf{S}(L)$ and if the complement sits in $\mathsf{S}_\mathfrak{c}(L)$ then it is complemented in $\mathsf{S}_\mathfrak{c}(L)$. In particular, this holds for the closed sublocales.
(2) By (1) we have $(\bigvee {\uparrow} b_i) \cap {\uparrow} a = \bigvee({\uparrow} b_i \cap {\uparrow} a) = \bigvee {\uparrow}(b_i \vee a) \in \mathsf{S}_\mathfrak{c}(L)$ so that for an $S \in \mathsf{S}_\mathfrak{c}(L)$ the meet $S \cap {\uparrow} a$ from $\mathsf{S}(L)$ is also the meet in $\mathsf{S}_\mathfrak{c}(L)$.
(3) We have the join ${\uparrow} a \vee {\uparrow} b = {\uparrow}(a \wedge b)$ both in $\mathsf{S}(L)$ and $\mathsf{S}_\mathfrak{c}(L)$. Indeed: ${\uparrow} a \vee {\uparrow} b = \{u \wedge v \mid u \geq a, v \geq b\}$ and for $x \geq a \wedge b$ we have $x = (x \vee a) \wedge (x \vee b)$.

2.3. Lemma. *Let L be an arbitrary frame. Then in the context of distributive lattices,*

$$(a \mapsto {\uparrow} a)\colon L \to \mathsf{S}_\mathfrak{c}(L)^{op} = M$$

is an essential extension preserving, moreover, all joins.

Proof: (Use 2.1 and 2.2.) It is a lattice embedding preserving all joins:

$${\uparrow} \bigvee a_i = \bigcap {\uparrow} a_i = \bigwedge_{\mathsf{S}_\mathfrak{c}(L)} {\uparrow} a_i = \bigvee_M {\uparrow} a_i,$$

$${\uparrow}(a \wedge b) = {\uparrow} a \vee^{\mathsf{S}_\mathfrak{c}(L)} {\uparrow} b = {\uparrow} a \wedge_M {\uparrow} b.$$

Now let $S <^M T$, that is, $T \subsetneq S$. Hence, there is an x such that ${\uparrow} x \subseteq S$ and ${\uparrow} x \nsubseteq T$ so that there exists an $a > x$ such that $a \notin T$ and as ${\uparrow} a \subseteq {\uparrow} x$, ${\uparrow} a \subseteq S$. Set $b = \bigwedge\{y \in T \mid y \geq a\}$. Then $b \in T$, and as $a \notin T$, $a < b$. Now we have, by the definition,

$${\uparrow} a \cap T \supseteq {\uparrow} b, \quad \text{that is,} \quad {\uparrow} a \vee^M T \geq^M {\uparrow} b,$$

and since ${\uparrow} a \subseteq S$ we also have that

$$S \vee {\uparrow} b \supseteq {\uparrow} a, \quad \text{that is,} \quad {\uparrow} b \wedge_M S \leq^M {\uparrow} a,$$

and we can use (2.1.1). ∎

2.4. Theorem. *For an arbitrary frame L, L and $\mathsf{S}_\mathfrak{c}(L)$ have the same maximal essential extension, up to isomorphism. Hence in particular, for any L, the maximal essential extension of $\mathsf{S}_\mathfrak{c}(L)$ is isomorphic to $\mathfrak{B}(\mathsf{S}(L))^{\mathrm{op}}$.*

Proof: Recall 2.1.

Consider a maximal essential extension B of $\mathsf{S}_\mathfrak{c}(L)$, $m\colon \mathsf{S}_\mathfrak{c}(L) \to B$, and the composed embedding

$$L \xrightarrow{(a\mapsto\uparrow a)} \mathsf{S}_\mathfrak{c}(L)^{\mathrm{op}} \xrightarrow{\ m\ } B^{\mathrm{op}}.$$

By 2.3 it is essential and since B^{op} is Boolean, it is a maximal essential extension. As a Boolean algebra, B is isomorphic to B^{op}. ■

2.5. More precisely. Let us describe the situation in more detail. Consider the following diagram in which the dashed arrows make sense only in the subfit case.

$$\begin{tikzcd}
 & \mathsf{S}(L)^{\mathrm{op}} \arrow[r, "S\mapsto S^{**}"] & \mathfrak{B}(\mathsf{S}(L))^{\mathrm{op}} \\
L \arrow[ur, "(a\mapsto\uparrow a)"] \arrow[r, "(a\mapsto\uparrow a)"] \arrow[dr, dashed, "\mathfrak{o}"'] & \mathsf{S}_\mathfrak{c}(L)^{\mathrm{op}} \arrow[u, "\subseteq"'] \arrow[r, "j"] \arrow[d, "S\mapsto S^{*}"] & B^{\mathrm{op}} \arrow[u, "\cong"'] \arrow[dl, dashed, "\cong"] \\
 & \mathsf{S}_\mathfrak{c}(L) &
\end{tikzcd}$$

We see that the maximal essential extensions of L and of $\mathsf{S}_\mathfrak{c}(L)$ are isomorphic only because a Boolean algebra is isomorphic, by complementation, with its opposite. The embeddings of L into $\mathfrak{B}(\mathsf{S}(L))^{\mathrm{op}}$ and of $\mathsf{S}_\mathfrak{c}(L)$ into B are in fact connected by a *contravariant* isomorphism.

If L is subfit (and only in that case), j is the identity.

3. Essential extension of spaces; subfit frames as generalized spaces, lifting

3.1. By [15, Theorem 4.6], for a T_1-space (X,τ), $\mathsf{S}_\mathfrak{c}(\tau) \cong \mathfrak{P}(X)$ and hence $j\colon \tau \subseteq \mathfrak{P}(X)$ is a maximal essential extension.

3.2. We can prove it directly, and for more general topologies.

Proposition. *If τ is a T_D topology on X then $\mathfrak{P}(X)$ is a maximal essential extension of τ.*

In fact, $\mathfrak{P}(X)$ is a maximal essential extension of τ if and only if τ is T_D.

Proof: Since $\mathfrak{P}(X)$ is Boolean, the point is in the essential extension, the maximality comes for free.

The essential extension of τ to $\mathfrak{P}(X)$ is equivalent to the condition (EE) (recall 2.1) that for any two subsets $A, B \subseteq X$ such that $A \subsetneq B$ there are $U, V \in \tau$ such that $U \subsetneq V$, $A \cap V \subseteq U$ and $B \cup U \supseteq V$.

(EE) $\Rightarrow$ T_D: For an arbitrary $x \in X$ set $A = \emptyset$ and $B = \{x\}$. Take the U, V from (EE). Then we have, in particular, $\{x\} \cup U \supseteq V \supsetneq U$ which yields $U = V \smallsetminus \{x\}$ and $x \in V$.

$T_D \Rightarrow$ (EE): Let $A \subsetneq B$. Choose an $x \in B \smallsetminus A$ and an open $V \ni x$ such that $U = V \smallsetminus \{x\}$ is open. Then $A \cap V \subseteq U$ and $B \cup U \supseteq V$. ∎

3.3. A topological space (X, τ) can be viewed as being carried by the Boolean algebra $\mathfrak{P}(X)$ instead of the set X, that is, as a pair $(\mathfrak{P}(X), \tau)$ with τ a subframe of $\mathfrak{P}(X)$; since maps $f \colon X \to Y$ are in a natural one-one correspondence with Boolean homomorphisms $g \colon \mathfrak{P}(Y) \to \mathfrak{P}(X)$ (setting $g(M) = f^{-1}[M]$) and since the $\mathfrak{P}(X)$'s are, up to isomorphism, precisely the complete atomic Boolean algebras, the category of classical T_D spaces can be viewed as follows.

The objects are

- pairs (B, L) with B *atomic* complete Boolean algebras and $L \subseteq B$ frames that are essential parts

and the morphisms $(B_1, L_1) \to (B_2, L_2)$ are

- Boolean homomorphisms $g \colon B_1 \to B_2$ such that $g[L_1] \subseteq L_2$ (or, the pairs (g, h) where h is the restriction).

3.4. Now a subfit frame L can be viewed, by [3], as a

- pair (B, L) with B an *arbitrary* complete Boolean algebra and $L \subseteq B$ a subframe that is an essential part.

In this sense (replacing "atomic" by "arbitrary") we see (subfit) frames as immediate generalizations of classical topological spaces and one can contemplate the category immediately extending the representation from 3.3 above.

However, the morphisms are not simply the frame homomorphisms $h\colon L_1 \to L_2$. They are pairs (g, h) with commuting squares

$$\begin{tikzcd} B_1 \arrow[r, "g"] & B_2 \\ L_1 \arrow[u, "\subseteq"] \arrow[r, "h"] & L_2 \arrow[u, "\subseteq"'] \end{tikzcd}$$

In such a commuting square, the homomorphism g is uniquely determined by h (see also 5.1 below; of course also h is uniquely determined by g). This yields a formal condition (which can be non-trivial) on h, namely being the restriction of an "L_1L_2-continuous" $g\colon B_1 \to B_2$. When returning to the natural representation of B_i as $\mathsf{S}_{\mathfrak{c}}(L_i)$ this condition means that h can be extended to a commutative diagram

$$\begin{tikzcd} \mathsf{S}_{\mathfrak{c}}(L) \arrow[r, "\widetilde{h}"] & \mathsf{S}_{\mathfrak{c}}(M)\,\cdot \\ L \arrow[u, "\mathfrak{o}_L"] \arrow[r, "h"] & M \arrow[u, "\mathfrak{o}_M"'] \end{tikzcd} \tag{lift}$$

We will speak of a *lifting* of h.

4. The spectrum of $\mathsf{S}_{\mathfrak{c}}(L)$

In this section we will show that for a large class of frames L every homomorphism $L \to \mathbf{2}$, i.e., every point of L, lifts. In other words, we will show that for such frames the spectrum of $\mathsf{S}_{\mathfrak{c}}(L)$ is naturally isomorphic with that of L.

4.1. Proposition. *Let $j\colon L \subseteq M$ be a subframe embedding. Let all the primes in L be maximal and let $j[\max(L)] \subseteq \max(M)$. Then the prime elements of M are all maximal, and coincide with the maximal elements of L.*

Proof: For the right adjoint j_* of j we have $jj_*(x) \le x$. Let $p \in M$ be prime. Since right adjoints preserve primeness, $j_*(p)$ is prime and hence maximal in L, hence $jj_*(p)$ is maximal in M and since $jj_*(p) \le p \ne 1$, $p = jj_*(p)$ and it is maximal. ∎

4.2. Proposition. *Let $L \subseteq M$ be a subframe such that*

(1) *each $s \in L$ is complemented in M, and*

(2) *for every $x \in M$, $x = \bigvee\{s^* \mid s \in L, s^* \leq x\}$.*

Then each maximal element of L is maximal in M.

Proof: Let p be maximal in L and let $p < x < 1$ in M. Then there is an $s \in L$ such that

$$s^* \leq x \quad \text{and} \quad s^* \nleq p.$$

Suppose $p \vee s \neq 1$. This happens in L and hence $p = p \vee s$, that is, $s \leq p$. Then, however, $1 = s \vee s^* \leq p \vee x = x$, a contradiction.

Hence $p \vee s = 1$, and $s^* = s^* \wedge (p \vee s) = s^* \wedge p$, hence $s^* \leq p$, a contradiction again. ∎

4.3. Some Hausdorff type axioms. In the following theorem we will consider frames in which every prime element is maximal.

Recall the numerous Hausdorff type axioms that are variants of the Dowker-Strauss ([6])

(S_2') if $a \vee b = 1$ and $a, b \neq 1$ then there are u, v such that $u \nleq a$, $v \nleq b$ and $u \wedge v = 0$

(Johnstone and Sun Shu-Hao [9], Paseka and Šmarda [10], weakly Hausdorff in [11]). In particular, Johnstone and Sun Shu-Hao require that

(T_2') if $1 \neq a \nleq b$ then there are u, v such that $u \nleq a$, $v \nleq b$ and $u \wedge v = 0$.

Under subfitness they are all equivalent (see [12, Proposition 4]). Let us call the resulting conjunction *naturally Hausdorff* (note that this conjunction is conservative – see [6] –, that is, a space (X, τ) is Hausdorff iff τ is naturally Hausdorff).

4.4. An element $p \neq 1$ is *semiprime* if $u \wedge v = 0$ implies that either $u \leq p$ or $v \leq p$ ([16]).

Lemma. (1) *In a naturally Hausdorff frame each semiprime element is maximal.*

(2) *In a fit frame, every prime element is maximal.*

Proof: (1) Use (T_2'). Let p be semiprime and let $p < x \neq 1$. Then there are u, v such that $u \nleq x$, $v \nleq p$ and $u \wedge v = 0$. By the weak primeness, $u \leq p < x$, a contradiction.

(2) Let L be fit, p prime and $p < x$. Then there is a c such that $c \vee x = 1$ and $c \to p \nleq p$. Since $c \wedge (c \to p) \leq p$ we have $c \leq p < x$ and $x = c \vee x = 1$. ∎

Note. We have seen that the fact that semiprime elements are maximal follows from (T_2') alone, without using subfitness. This is another Hausdorff type axiom, introduced by Rosický and Šmarda in [16].

4.5. Theorem. *Let L be naturally Hausdorff or fit. Then the embedding $\mathfrak{o}\colon L \to \mathsf{S}_\mathfrak{c}(L)$ induces an isomorphism*

$$\Sigma(L) \cong \Sigma(\mathsf{S}_\mathfrak{c}(L)).$$

Proof: The embedding $\mathfrak{o}$ satisfies the condition of 4.2, hence it sends maximal elements to maximal ones. Use 4.1. ■

4.5.1. Note. In fact we have proved more, namely that, moreover, the spectrum $\Sigma(\mathsf{S}_\mathfrak{c}(L))$ is T_1.

5. Some special liftings; in particular the Boolean case

5.1. Let L_i, $i = 1, 2$, be *sober* subfit topologies on sets X_i. Then every frame homomorphism $h\colon L_1 \to L_2$ is $\Omega(f)$ for a continuous map. By the observation in 3.3, hence, every $h\colon L_1 \to L_2$ lifts to a Boolean $\widetilde{h}$.

By Isbell's spatiality theorem in [7] (see also[13]) this also yields the following

Observation. *For compact sober subfit frames L_1 and L_2, every frame homomorphism $h\colon L_1 \to L_2$ lifts.*

5.1.1. Note. This shows that the problem of functoriality of $\mathsf{S}_\mathfrak{c}$ cannot be approached by extra conditions on h that could be non-trivial for continuous maps (openness, closedness, perfectness, etc.). In the (sober subfit) spatial case everything lifts.

5.2. There is of course the trivial case with L_1 Boolean. Then h lifts to $\mathfrak{o}_{L_2} \cdot h \cdot \mathfrak{o}_{L_2}^{-1}$.

5.3. A less trivial case we have encountered in Section 4: under very weak conditions on L every (point) $h\colon L \to \mathbf{2}$ lifts. This will be seen from another perspective shortly.

5.4. To get further let us start with an explicit formula for the lift mapping.

Proposition. *The unique candidate for the $\widetilde{h}$ in the diagram* (lift) *in 3.4 is given by the formula*

$$\widetilde{h}(S) = \bigvee\{\uparrow h(a) \mid \uparrow a \subseteq S\}. \tag{5.4.1}$$

Proof: If the diagram commutes and $\widetilde{h}$ is a homomorphism we have

$$\widetilde{h}(\uparrow a) = \widetilde{h}(\mathfrak{o}(a)^*) = (\widetilde{h}(\mathfrak{o}(a))^* = \mathfrak{o}(h(a))^* = \uparrow h(a)$$

and hence

$$\widetilde{h}(S) = \widetilde{h}(\bigvee\{\uparrow a \mid \uparrow a \subseteq S\}) = \bigvee\{\uparrow h(a) \mid \uparrow a \subseteq S\}. \quad \blacksquare$$

5.4.1. Proposition. *The mapping* $\widetilde{h}$ *from* (5.4.1) *preserves meets.*

Proof: Since $\widetilde{h}$ is obviously monotone, we have $\widetilde{h}(S) \wedge \widetilde{h}(T) \geq \widetilde{h}(S \wedge T)$.

Now, although the meet in $\mathsf{S}_{\mathfrak{c}}(L)$ does not generally coincide with the meet in $\mathsf{S}(L)$ (which is the intersection), we have in $\mathsf{S}_{\mathfrak{c}}(L)$, $\uparrow a \wedge \uparrow b = \uparrow a \cap \uparrow b = \uparrow(a \vee b)$ simply because the intersection is in $\mathsf{S}_{\mathfrak{c}}(L)$. Hence we have, since $\mathsf{S}_{\mathfrak{c}}(L)$ is a frame with the same joins as in $\mathsf{S}(L)$,

$$\begin{aligned}
\widetilde{h}(S) \wedge \widetilde{h}(T) &= \bigvee\{\uparrow h(a) \mid \uparrow a \subseteq S\} \wedge \bigvee\{\uparrow h(b) \mid \uparrow b \subseteq T\} \\
&= \bigvee\{\uparrow h(a) \wedge \uparrow h(b) \mid \uparrow a \subseteq S, \uparrow b \subseteq T\} \\
&= \bigvee\{\uparrow (h(a) \vee h(b)) \mid \uparrow a \subseteq S, \uparrow b \subseteq T\} \\
&= \bigvee\{\uparrow h(a \vee b) \mid \uparrow a \subseteq S, \uparrow b \subseteq T\} \\
&\leq \bigvee\{\uparrow h(c) \mid \uparrow c \subseteq S \wedge T\} \leq \widetilde{h}(S \wedge T). \quad \blacksquare
\end{aligned}$$

5.4.2. Thus, one has to deal with two questions. For a particular h,

- does the $\widetilde{h}$ from (5.4.1) commute in the diagram (lift), and
- does it preserve arbitrary joins?

5.4.3. Observation. *The diagram* (lift) *commutes iff*

$$\uparrow h(a)^* = \bigvee\{\uparrow h(x) \mid x \vee a = 1\}. \tag{5.4.3}$$

(Indeed, we have

$$\widetilde{h}(\mathfrak{o}(a)) = \bigvee\{\uparrow h(x) \mid \uparrow x \subseteq \mathfrak{o}(a)\} = \bigvee\{\uparrow h(x) \mid x \vee a = 1\}.)$$

5.5. The Boolean target. If M is Boolean consider, to simplify the computing, the g in the diagram

$$\begin{tikzcd}
\mathsf{S}_{\mathfrak{c}}(L) \arrow[r, "\widetilde{h}"] \arrow[dr, "g"] & \mathsf{S}_{\mathfrak{c}}(B) \arrow[d, "\mathfrak{o}_M^{-1}"] \\
L \arrow[u, "\mathfrak{o}_L"] \arrow[r, "h"] & M
\end{tikzcd}$$

We have $\mathfrak{o}(a^*) = \mathfrak{o}(a)^{-1} = {\uparrow}a$ and hence $\mathfrak{o}^{-1}({\uparrow}a) = a^*$. Thus,

$$\begin{aligned} g({\uparrow}a) &= \mathfrak{o}_M^{-1}(\widetilde{h}({\uparrow}a)) = \mathfrak{o}_M^{-1}({\uparrow}h(a)) = h(a)^*, \\ g(S) &= \bigvee\{h(x)^* \mid {\uparrow}x \subseteq S\}, \end{aligned} \tag{5.5.1}$$

and hence, in particular, $g(\mathfrak{o}(a)) = \bigvee\{h(x)^* \mid x \vee a = 1\}$ and the condition (5.4.3) transforms to

$$h(a) = \bigvee\{h(x)^* \mid x \vee a = 1\}. \tag{5.5.lift}$$

5.5.1. Proposition. *If M is Boolean then $\widetilde{h}$ is a frame homomorphism.*

Proof: $\widetilde{h}$ preserves finite meets generally.

Now let us prove that g preserves joins. Obviously it suffices to prove that it preserves joins of closed sublocales, that is, that

$$g(\bigvee {\uparrow}a_i) = \bigvee\{h(x)^* \mid {\uparrow}x \subseteq \bigvee {\uparrow}a_i\} = \bigvee g({\uparrow}a_i) = \bigvee h(a_i)^* = (\bigwedge h(a_i))^*.$$

Since $g(\bigvee {\uparrow}a_i) \geq \bigvee g({\uparrow}a_i)$ is trivial, it suffices to prove that

$$\forall x \text{ such that } {\uparrow}x \subseteq \bigvee {\uparrow}a_i, \quad h(x)^* \leq (\bigwedge h(a_i))^*.$$

Hence let ${\uparrow}x \subseteq \bigvee {\uparrow}a_i$. Then $x = \bigwedge_j y_j$ for some $y_j \geq a_j$, hence $h(x) \geq h(a_j)$ for all j, hence $h(x)^* \leq h(a_j)^*$ for all j, and finally $h(x)^* \leq h(a_j)^* \leq (\bigwedge h(a_i))^*$. ■

5.5.2. Theorem. *Let L be regular and let M be Boolean. Then every $h\colon L \to M$ lifts.*

Proof: In view of 5.5.1 it suffices to prove the equation (5.5.lift).

If $x \vee a = 1$ then $h(x)^* = h(x)^* \wedge (h(x) \vee h(a)) = h(x)^* \wedge h(a)$, hence $h(x)^* \leq h(a)$, and we have $h(a) \geq \bigvee\{h(x)^* \mid x \vee a = 1\}$. On the other hand, if L is regular we have

$$\begin{aligned} h(a) = h(\bigvee\{y \mid y \prec a\}) &= \bigvee\{h(y) \mid y^* \vee a = 1\} \\ &\leq \bigvee\{h(y^{**}) \mid y^* \vee a = 1\} \leq \bigvee\{h(x^*) \mid x \vee a = 1\} \\ &\leq \bigvee\{h(x)^* \mid x \vee a = 1\}. \quad ■ \end{aligned}$$

5.5.3. Notes. (1) In particular we have a homomorphism g in

$$\begin{tikzcd} \mathsf{S}_{\mathfrak{c}}(L) \arrow[dr, "g"] & \\ L \arrow[u, "\mathfrak{o}_L"] \arrow[r, "(x \mapsto x^{**})"'] & \mathfrak{B}(L) \end{tikzcd}$$

It is given by the formula
$$g(S) = (\bigwedge S)^*.$$
(Indeed, by (5.5.1),
$$g(S) = \bigvee\{x^* \mid {\uparrow}x \subseteq S\} = (\bigwedge\{x \mid {\uparrow}x \subseteq S\})^* = (\bigwedge S)^*.)$$
(2) For the two-element Boolean algebra $\mathbf{2} = \{0, 1\}$ we have already proved the lifting of any $h\colon L \to \mathbf{2}$ in Section 4, for L with weaker properties than regularity. Let us look at it from the now discussed perspective.

We have a maximal p in a subfit L, and the mapping h is given by $h(x) = 1$ iff $x \leq p$. If $h(a) = 0$ then trivially $\bigvee\{h(x)^* \mid x \vee a = 1\} = 0$. Now if $h(a) = 1$, we have $a \neq p$ and hence there is a c such that $c \vee a = 1$ and $c \vee p \neq 1$. By maximality, $c \vee p = p$, hence $c \leq p$ so that $h(c) = 0$ and $h(c)^* = 0$.

5.6. Example. Let L be the cofinal topology on an infinite set X (consisting of $\emptyset$ and all complements of finite subsets). Here, $p = \emptyset$ is prime. Consider the homomorphism $h\colon L \to \mathbf{2}$ defined by
$$h(x) = 1 \text{ iff } x \nleq p \text{ (that is, } x \neq 0).$$
Consider any element $\xi \in X$ and $a = X \smallsetminus \{\xi\}$. Then $h(a) = 1$ and if $x \vee a = 1$ then $x \neq 0$, hence $h(x) = 1$ and $h(x)^* = 0$. Thus, (5.5.lift) is not satisfied.

This trivial example shows several facts.

(1) There are plenty of homomorphisms that do not lift.[2] Moreover, our example is spatial; thus in 5.1 above the assumption of sobriety was essential.
(2) Also, since L is T_1 and hence subfit, we see that the assumption of regularity in 5.5.2 cannot be relaxed to subfitness.
(3) On the other hand, by 5.5.1 $\widetilde{h}$ is a frame homomorphism; thus, the commutativity question in 5.4.2 is not included in the frame homomorphism one.
(4) In Section 4 we obtained the spectrum result under various conditions of Hausdorff type, or fitness. Again, we see that we cannot go down to subfitness (or even to T_1 in the spatial case – the example reminds us of the fact that T_1 does not necessarily make the primes maximal).

[2]Generally this will occur with homomorphisms $h\colon \Omega(Y) \to \Omega(X)$ that, because of lack of sobriety, cannot be represented as $\Omega(f)$. Further, since a sublocale (resp. closed sublocale) $S \subseteq L_1 \times L_2$ is $S_1 \times S_2$ with S_i (closed) sublocales of L_i, we can easily construct examples of the form $h \times g : L \times K \to M \times K$ from unliftable $h : L \to M$. It would be, however, of interest to find unliftable $h : L \to M$ with trivial spectra of L, M.

Also, it shows that the step from subfitness to fitness jumps far over T_1.

References

[1] C. E. Aull and W. J. Thron, *Separation axioms between T_0 and T_1,* Indag. Math. **24** (1963), 26–37.
[2] R. Balbes, *Projective and inductive distributive lattices,* Pacific J. Math. **21** (1967), 405–420.
[3] R. N. Ball and A. Pultr, *Maximal essential extensions in the context of frames,* to appear in Algebra Universalis.
[4] R. N. Ball, J. Picado and A. Pultr, *On an aspect of scatteredness in the point-free setting,* Portug. Math. **73** (2016), 139–152.
[5] B. Banaschewski and A. Pultr, *Booleanization*, Cahiers Topologie Géom. Différentielle Catég. **37** (1996), 41–60.
[6] C. H. Dowker and D. Strauss, *T_1- and T_2-axioms for frames*, in: Aspects of topology, London Math. Soc. Lecture Note Ser. 93, pp. 325-335, Cambridge Univ. Press, Cambridge, 1985.
[7] J. R. Isbell, Atomless parts of spaces, *Math. Scand.* **31** (1972), 5–32.
[8] P. T. Johnstone, *Stone spaces*, Cambridge Univ. Press, Cambridge, 1982.
[9] P. T. Johnstone and S.-H. Sun, *Weak products and Hausdorff locales*, in: Categorical algebra and its applications (Proc. Louvain-La-Neuve, 1987), Lecture Notes in Math. 1348, Springer, Berlin, 1988, 173–193.
[10] J. Paseka and B. Šmarda, *T_2-frames and almost compact frames,* Czechoslovak Math. J. **42** (1992), 385–402.
[11] J. Picado and A. Pultr, *Frames and locales: Topology without points*, Frontiers in Mathematics, vol. 28, Springer, Basel (2012).
[12] J. Picado and A. Pultr, *More on subfitness and fitness,* Appl. Categ. Struct. **23** (2015), 323–335.
[13] J. Picado and A. Pultr, *New aspects of subfitness in frames and spaces,* Appl. Categ. Struct. **24** (2016), 703–714.
[14] J. Picado and A. Pultr, *A Boolean extension of a frame and a representation of discontinuity,* Quaest. Math. **40** (2017), 1111–1125.
[15] J. Picado, A. Pultr and A. Tozzi, *Joins of closed sublocales,* to appear in Houston J. Math.
[16] J. Rosický and B. Šmarda, *T_1-locales,* Math. Proc. Cambridge Phil. Soc. **98** (1985), 81–86.
[17] R. Sikorski, *A theorem on extensions of homomorphisms,* Ann. Soc. Pol. Math. **21** (1948), 332–335.

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