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PROFINITE TOPOLOGIES

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Abstract: Profinite semigroups are a generalization of finite semigroups that come about naturally when one is interested in considering free structures with respect to classes of finite semigroups. They also appear naturally through dualization of Boolean algebras of regular languages. The additional structure is given by a compact zero-dimensional topology. Profinite topologies may also be considered on arbitrary abstract semigroups by taking the initial topology for homomorphisms into finite semigroups. This text is the proposed chapter of the Handdbook of Automata Theory dedicated to these topics. The general theory is formulated in the setting of universal algebra because it is mostly independent of specific properties of semigroups and more general algebras naturally appear in this context. In the case of semigroups, particular attention is devoted to solvability of systems of equations with respect to a pseudovariety, which is relevant for solving membership problems for pseudovarieties. Focus is also given to relatively free profinite semigroups per se, specially "large" ones, stressing connections with symbolic dynamics that bring light to their structure.

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1. Introduction

Profinite semigroups and profinite topologies in semigroups have become an important tool in the theory of finite automata. There are many reasons for this fact.

First, since finite automata describe transition finite semigroups (of transformations or relations) by giving the action of their generators on a finite set of states, the separation power of words by a class $\mathcal{C}$ of finite automata translates in algebraic terms to the separation power of homomorphisms from

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free semigroups into the corresponding transition semigroups. More generally, the homomorphisms from a semigroup S into such semigroups determine an initial topology on S, namely the corresponding profinite topology. The topological separation axiom of Hausdorff is the familiar algebraic property of being residually in $\mathcal{C}$. But, actually, homomorphisms into finite semigroups give a finer structure, namely a uniform structure, or even a metric structure in case the semigroup S is finitely generated. Thus, there is a natural completion associated with our separation scheme, which is called the pro-$\mathcal{C}$ completion of S. The topological semigroups thus obtained are so-called *pro-$\mathcal{C}$ semigroups*. For the class $\mathcal{C}$ of all finite semigroups, the attribute "pro-$\mathcal{C}$" becomes simply "profinite".

Another explanation for the importance of profinite topologies comes from duality. Applying the above recipe to the free semigroup A^+, with $\mathcal{C}$ a pseudovariety V of finite semigroups, the resulting pro-V semigroup is known as the free pro-V semigroup, for indeed it has the expected universal property. It turns out that the topological structure of this free pro-V semigroup is precisely the Stone dual of the Boolean algebra of regular languages over the alphabet A that can be recognized by members of V. The further dualization of residual operations determines the multiplication [58, 59].

Another fundamental reason why free profinite semigroups are important is that their elements, sometimes called *pseudowords*, play the role of terms in classical universal algebra. Indeed, pseudovarieties can be defined by formal equalities between pseudowords.

To be able to apply these connections with the profinite world, some knowledge of the structure of free pro-V semigroups is usually necessary for suitable pseudovarieties V. The thus motivated structural investigation of these semigroups is in general quite hard and has only been carried out in a very limited number of cases.

Another major difficulty lies in the fact that in most interesting cases, free pro-V semigroups are uncountable. Thus, there are delicate questions when trying to obtain decidability results using pseudowords. An important idea in this context is to replace arbitrary pseudowords by those of a special kind, namely the elements of the subalgebra with respect to a suitably enriched language. This leads to the notions of reducibility and tameness which are involved in some of the deepest results using profinite methods.

The aim of this chapter is to efficiently introduce these topics, illustrating with examples and results the wide range of application of profinite methods. We introduce profinite topologies in the context of general algebraic structures. Although they were originally considered in this context by Birkhoff [44], so far they have not been much studied outside the realm of group and semigroup theories. In the context of ring theory, there is an analog topology, which may or may not be profinite, and which is known as the *Krull topology*. It is determined on a ring by a filtration by ideals. For instance, for the ring of p-adic integers, the filtration consists of the ideals generated by the powers of the prime p and the topology is "profinite" in the sense that the quotient rings $\mathbb{Z}/p^n\mathbb{Z}$ are finite.

Since most of the theory is independent of the concrete algebraic structures in which one may be interested, and a lot of attention has been given to general algebraic structures as recognizing devices for tree languages (see Chapter 22), it seems worthwhile to formulate the theory in the more general context. Moreover, the reducibility and tameness properties involve themselves general algebraic structures, even when semigroups are the aim of the investigations. In Section 2, results are formulated in the context of general algebras. Section 3 deals with applications in the special case of semigroups. Section 4 introduces recent results concerning the structure of free profinite semigroups over large pseudovarieties, where connections with symbolic dynamics play an important role.

2. Profinite topologies for general algebras

This section introduces profinite topologies for general (topological abstract) algebras. The treatment presented here is meant to be a quick guide to the main general results in this area. For most proofs, the reader is referred to the bibliography. Occasionally, simple generalizations of the previously published results are presented here for we believe this contributes to understanding the theory, and may be helpful in applications.

2.1. General algebraic structures. This subsection introduces the basics of Universal Algebra. The reader is referred to [45] for further details.

By an *algebraic signature* we mean a set σ, of *operation symbols*, together with an *arity* function $\nu : \sigma \to \mathbb{N}$ into the set of non-negative integers. We denote $\nu^{-1}(n)$ by σ_n. A σ-*algebra* consists of a nonempty set S together with an interpretation function assigning to each operation symbol $f \in \sigma$ a $\nu(f)$-ary operation $f^S : S^{\nu(f)} \to S$. The operations on S of this form are called the *basic operations*. Usually, the interpretation function is understood and we talk about the *algebra* S. Moreover, unless explicit mention of the signature σ is required, which is usually understood from the context, we will omit reference to it. An algebra S is *trivial* if S is a singleton.

From hereon, whenever we talk about algebras and their classes, unless otherwise stated, we always assume that the same signature is involved.

A *homomorphism* is a mapping $\varphi : S \to T$ between two algebras such that, for every arity n and $f \in \sigma_n$, and for all $s_1, \ldots, s_n \in S$, the equality $\varphi\big(f^S(s_1, \ldots, s_n)\big) = f^T\big(\varphi(s_1), \ldots, \varphi(s_n)\big)$ holds. For an algebra T, a nonempty subset S closed under the interpretation in T of the operation symbols is an algebra under the induced operations; we then say that S is a *subalgebra* of T. For a family $(S_i)_{i\in I}$ of algebras, their *direct product* $\prod_{i\in I} S_i$ is the Cartesian product with operation symbols interpreted component-wise. Note that, if $I = \emptyset$, then $\prod_{i\in I} S_i$ is a trivial algebra.

A *congruence* on an algebra S is an equivalence relation θ on S such that θ is a subalgebra of $S \times S$. For a congruence θ on S, we may interpret each operation symbol $f \in \sigma_n$, on the quotient set S/θ by putting $f^{S/\theta}(s_1/\theta, \ldots, s_n/\theta) = f^S(s_1, \ldots, s_n)/\theta$, whenever $s_1, \ldots, s_n \in S$, where s/θ denotes the θ-class of s; this is called the *quotient algebra* of S by θ. The chosen structure of S/θ is the unique way of defining the quotient algebra so that the natural mapping $S \to S/\theta$, which sends s to s/θ, is a homomorphism.

Given an algebra S and a nonempty family $(\theta_i)_{i\in I}$ of congruences on S, there is a natural injective homomorphism $S/(\bigcap_{i\in I} \theta_i) \to \prod_{i\in I} S/\theta_i$. For a class of algebras $\mathcal{C}$ containing trivial algebras, and an algebra S, we denote by $\theta_{\mathcal{C}}$ the intersection of the family of all congruences θ on S such that $S/\theta \in \mathcal{C}$. We say that S is *residually in* $\mathcal{C}$ if $\theta_{\mathcal{C}}$ is the equality relation Δ_S on the set S.

A *variety* is a class $\mathcal{V}$ of algebras which is closed under taking homomorphic images, subalgebras and arbitrary direct products. Since the intersection of a nonempty family of varieties is again a variety, we may consider the *variety generated by* any class $\mathcal{C}$ of algebras, denoted $\mathcal{V}(\mathcal{C})$. By a well-known theorem of Birkhoff [43], for every nonempty set A and every variety $\mathcal{V}$, there is a *$\mathcal{V}$-free algebra on A*, that is an algebra $F_A\mathcal{V}$ together with a mapping $\iota : A \to F_A\mathcal{V}$ such that, for every mapping $\varphi : A \to S$ into an algebra S from $\mathcal{V}$, there is a unique homomorphism $\hat{\varphi} : F_A\mathcal{V} \to S$ such that $\hat{\varphi} \circ \iota = \varphi$. By the usual 'abstract nonsense', such an algebra is unique up to isomorphism and depends only on the variety $\mathcal{V}$ and the cardinality of the set A. In case A is a finite set of cardinality n, we may write $F_n\mathcal{V}$ instead of $F_A\mathcal{V}$. A similar convention applies for other notations for free algebras that are used in this chapter.

In particular, the class of all σ-algebras is a variety. The corresponding free algebra on A is the algebra $T_A^{(\sigma)}$ of formal *(σ-)terms*, constructed recursively from the elements of A by formally applying the operation symbols, which also defines their interpretation:

- for each $a \in A$, we have $a \in T_A$;
- if each $t_1, \ldots, t_n$ is in T_A and $f \in \sigma_n$, then $f(t_1, \ldots, t_n)$ is also in T_A;
- all elements of T_A are obtained by applying the preceding rules.

In fact the algebra $F_A\mathcal{V}$ is naturally constructed as the quotient algebra $T_A/\theta_{\mathcal{V}}$.

For a variety $\mathcal{V}$, each element w of $F_A\mathcal{V}$ determines a function $w_S : S^A \to S$ on each algebra S from $\mathcal{V}$ by letting $w_S(\varphi) = \hat{\varphi}(w)$ for a function $\varphi : A \to S$. In case $A = \{a_1, \ldots, a_n\}$, one may prefer to view w_S as a function from S^n to S, by putting $w_S(s_1, \ldots, s_n) = \hat{\varphi}(w)$, where $\varphi : A \to S$ maps a_i to s_i $(i = 1, \ldots, n)$.

An *identity* is a formal equality $u = v$ with $u, v \in T_A$ for some set A. We say that an algebra S satisfies the identity $u = v$ if $u_S = v_S$. For a set Σ of identities, the class $[\Sigma]$ consisting of all algebras that satisfy all the identities from Σ is easily seen to be a variety. Birkhoff's variety theorem [43] states that every variety is of this form.

A *pseudovariety* is a nonempty class V of finite algebras that is closed under taking homomorphic images, subalgebras and finite direct products. The *pseudovariety generated by* a class $\mathcal{C}$ of finite algebras, denoted $\mathsf{V}(\mathcal{C})$, is the intersection of all pseudovarieties that contain $\mathcal{C}$. A class of finite algebras closed under taking isomorphic algebras, subalgebras, and finite direct products is called a *pseudoquasivariety*.

Example 2.1. (1) For the signature consisting of a single binary operation, the class $\mathcal{S}$ of all semigroups is a variety, defined by the identity $x(yz) = (xy)z$. Its free algebra $F_A\mathcal{S}$ is the semigroup of words A^+. The class S of all finite semigroups is a pseudovariety. The classes $\mathcal{M}$, of all monoids, and $\mathcal{G}$, of all groups, are not varieties: they are not closed under taking subalgebras. The class G of all finite groups is a pseudovariety, but the class M of all finite monoids is not a pseudovariety.

(2) For the signature consisting of a binary and a nullary operation (or *constant*), the class $\mathcal{M}$ is a variety and the class M is a pseudovariety.

(3) For the signature consisting of a binary operation, a unary operation and a nullary operation, the class $\mathcal{G}$ of all groups is a variety.

(4) For the signature of (1), consider the class of all finite semigroups such that, if an element s generates a subsemigroup whose subgroups are trivial, then $s^2 = s$. This is a pseudoquasivariety but not a pseudovariety (for instance, the 3-element semigroup with presentation $\langle a : a^4 = a^2 \rangle$ belongs to the class but its quotient $\langle a : a^3 = a^2 \rangle$ does not).

Given an algebra S and a subset L of S, the *syntactic congruence* of L on S is the largest congruence $\sim_L$ such that L is a union of $\sim_L$-classes. It is characterized by the following property: for $s, s' \in S$, the relation $s \sim_L s'$ holds if and only if, for all $n \geq 1$, $t \in T_n$, and $s_2, \ldots, s_n \in S$, we have $t_S(s, s_2, \ldots, s_n) \in L$ if and only if $t_S(s', s_2, \ldots, s_n) \in L$. For some varieties, such as of semigroups, monoids, groups, or rings, and for any finitely generated variety of lattices, it turns out that, rather than considering all terms in the preceding equivalence, it suffices to consider a finite number of them. For instance, for the variety of monoids, it suffices to consider the single term $t = (xy)z$, as in the usual definition of the syntactic congruence for monoids. See Clark *et al.* [46] for alternative characterizations of varieties with such a finiteness property.

For an algebra S, we say that a subset L of S is *recognized* by a homomorphism $\varphi : S \to T$ if $L = \varphi^{-1}\varphi L$. In other words, L is a union of classes of the *kernel* congruence $\ker\varphi = (\varphi \times \varphi)^{-1}\Delta_S$ or, equivalently, $\ker\varphi$ is contained in $\sim_L$. For a class $\mathcal{C}$ of algebras, we say that a subset L of S is *$\mathcal{C}$-recognizable* if L is recognized by a homomorphism $\varphi : S \to T$ into some algebra T from $\mathcal{C}$. In particular L is recognizable by some finite algebra if and only if $\sim_L$ has finite index, in which case we also say simply that L is *recognizable*.

2.2. Pseudometric and uniform spaces. A *pseudometric* on a set X is a function d from $X \times X$ to the non-negative reals such that the following conditions hold:

i. $d(x, x) = 0$ for every $x \in X$;
ii. $d(x, y) = d(y, x)$ for all $x, y \in X$;
iii. (triangle inequality) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

In case, additionally, $d(x, y) = 0$ implies $x = y$, then we say that d is a *metric* on X. If, instead of the triangle inequality, we impose the stronger

iv. (ultrametric inequality) $d(x, z) \leq \max\{d(x, y), d(y, z)\}$ for all $x, y, z \in X$,

then we refer respectively to a *pseudo-ultrametric* and an *ultrametric*. For each of these types of "something" metrics, a *"something" metric space* is a set endowed with a "same thing" metric.

The remainder of this section is dedicated to recalling the notion of a uniform space. We build up here on the approach of [32]. The reader may prefer to consult a book on general topology such as [109].

Definition 2.1. A *uniformity* on a set X is a set $\mathscr{U}$ of reflexive binary relations on X such that the following conditions hold:

(1) if $R_1 \in \mathscr{U}$ and $R_1 \subseteq R_2$, then $R_2 \in \mathscr{U}$;
(2) if $R_1, R_2 \in \mathscr{U}$, then there exists $R_3 \in \mathscr{U}$ such that $R_3 \subseteq R_1 \cap R_2$;
(3) if $R \in \mathscr{U}$, then there exists $R' \in \mathscr{U}$ such that $R' \circ R' \subseteq R$;
(4) if $R \in \mathscr{U}$, then $R^{-1} \in \mathscr{U}$.

An element of a uniformity is called an *entourage*. A *uniform space* is a set endowed with a uniformity, which is usually understood and not mentioned explicitly.

A *uniformity basis* on a set X is a set $\mathscr{U}$ of reflexive binary relations on X satisfying the above conditions (2)–(4). The *uniformity generated* by $\mathscr{U}$ consists of all binary relations on X that contain some member of $\mathscr{U}$.

A uniformity $\mathscr{U}$ is *transitive* if it admits a basis consisting of transitive relations.

The notion of a uniform space generalizes that of a pseudometric space. In this respect, the following notation is suggestive of the intuition behind the generalization. For an entourage R and elements $x, y \in X$, we write $d(x, y) < R$ to indicate that $(x, y) \in R$. Indeed, given a metric d on X, if we let R_ϵ denote the set of pairs $(x, y) \in X \times X$ such that $d(x, y) < \epsilon$, then the set $\mathscr{U}_d$ of all R_ϵ, with $\epsilon > 0$, is a uniformity basis on X such that $d(x, y) < R_\epsilon$ if and only if $d(x, y) < \epsilon$. The uniformity $\mathscr{U}_d$ is said to be *defined* by d.

The *topology of* a uniform space X (or *induced by* its uniformity) has neighborhood basis for each $x \in X$ consisting of all sets of the form $B_R(x) = \{y \in X : d(x, y) < R\}$. Not every topology is induced by a uniformity [109, Theorem 38.2].

Note that the topology induced by a uniformity $\mathscr{U}$ on X is Hausdorff if and only if the intersection $\bigcap \mathscr{U}$ is the diagonal (equality) relation Δ_X. In general, it follows from the definition of uniformity that $\bigcap \mathscr{U}$ is an equivalence relation on X. The quotient set $X/\bigcap\mathscr{U}$ is then naturally endowed with the *quotient uniformity*, whose entourages are the relations $R/\bigcap\mathscr{U}$, with $R\in\mathscr{U}$. Of course, the *quotient space* $X/\bigcap\mathscr{U}$ is Hausdorff and we call it the *Hausdorffization* of X while the natural mapping $X\to X/\bigcap\mathscr{U}$ is called the *natural Hausdorffization mapping*. Given a uniformity $\mathscr{U}$ on a set X and a subset Y, the *relative uniformity* on Y consists of the entourages of the form $R\cap(Y\times Y)$ with $R\in\mathscr{U}$. Endowed with this uniformity, Y is said to be a *uniform subspace* of X.

Recall that a *net* in a set X is a function $f: I\to X$, where I is a directed set, meaning a set endowed with a partial order $\leq$ such that, for all $i,j\in I$, there is some $k\in I$ with $i\leq k$ and $j\leq k$. A subnet of such a net is a net $g: J\to X$ for which there is an order-preserving function $\lambda: J\to I$ such that $g=f\circ\lambda$ and, for every $i\in I$, there is some $j\in J$ with $i\leq\lambda(j)$, that is, λ has cofinal image in I. Usually, the net f is represented by $(x_i)_{i\in I}$, where $x_i=f(i)$. The subnet g is then represented by $(x_{i_j})_{j\in J}$, where $i_j=\lambda(j)$. In case X is a topological space, we say that the net $(x_i)_{i\in I}$ *converges to* $x\in X$ if, for every neighborhood N of x, there is some $i\in I$ such that $x_j\in N$ whenever $j\geq i$.

A net $(x_i)_{i\in I}$ in a uniform space X is said to be a *Cauchy net* if, for every entourage R, there is some $i\in I$ such that $d(x_j,x_k)<R$ whenever $j,k\geq i$. A uniform space is said to be *complete* if every Cauchy net converges.

A Hausdorff topological space X is said to be *compact* if every open covering of X contains a finite covering. Equivalently, every net in X has a convergent subnet. A topological space is said to be *zero-dimensional* if it admits a basis consisting of *clopen* sets, that is sets that are both closed and open. It is well known that a compact space is zero-dimensional if and only if it is *totally disconnected*, meaning that all its connected components are singleton sets. One can also show that a compact space has a unique uniformity that induces its topology [109, Theorem 36.19].

A uniform space X is *totally bounded* if, for every entourage R, there is a finite cover $X=U_1\cup\cdots\cup U_n$ such that $\bigcup_{k=1}^n U_k\times U_k\subseteq R$. It is well known that a Hausdorff uniform space is compact if and only if it is complete and totally bounded [109, Theorem 39.9].

A function $\varphi: X\to Y$ between two uniform spaces is *uniformly continuous* if, for every entourage R of Y, there is some entourage R' of X such that $d(x_1,x_2)<R'$ implies $d(\varphi(x_1),\varphi(x_2))<R$. Equivalently, φ maps Cauchy nets to Cauchy nets. We say that φ is a *uniform isomorphism* if it

is a uniformly continuous bijection whose inverse is also uniformly continuous. The function φ is a *uniform embedding* if φ is a uniform isomorphism of X with a subspace of Y. Note that, if $\varphi : X \to Y$ is a uniformly continuous function, then φ induces a unique uniformly continuous function $\psi : X/\bigcap \mathscr{U}_X \to Y/\bigcap \mathscr{U}_Y$ between the corresponding Hausdorffizations such that $\psi \circ \pi_X = \pi_Y \circ \varphi$, where π_X and π_Y are the natural Hausdorffization mappings. We call ψ the *Hausdorffization* of φ.

One can show [109, Theorem 38.3] that a uniformity is defined by some pseudometric (respectively by a pseudo-ultrametric) if and only if it has a countable basis (and, respectively, it is transitive). In the Hausdorff case, one can remove the prefix "pseudo". Moreover, every uniform space can be uniformly embedded in a product of pseudometric spaces [109, Theorem 39.11].

For every uniform space X there is a complete uniform space $\hat{X}$ such that X embeds uniformly in $\hat{X}$ as a dense subspace. This can be done by first uniformly embedding X in a product of pseudometric spaces and then completing each factor by diagonally embedding it in the space of equivalence classes of Cauchy sequences under the relation $(x_n)_n \approx (y_n)_n$ if $\lim d(x_n, y_n) = 0$ (cf. [109, Theorems 39.12 and 24.4]).

Such a space $\hat{X}$ is unique in the sense that, given any other complete uniform space Y in which X embeds uniformly as a dense subspace, there is a unique uniform isomorphism $\hat{X} \to Y$ leaving X pointwise fixed. The uniform space $\hat{X}$ is called the *completion* of X. It is easy to verify that the Hausdorffization of the completion of X is the completion of the Hausdorffization of X; it is known as the *Hausdorff completion* of X. Moreover, the Hausdorff completion of X is compact if and only if X is totally bounded. The following is a key property of completions.

Proposition 2.1. *Let X and Y be uniform spaces and let $\varphi : X \to Y$ be a uniformly continuous function. Then there is a unique extension of φ to a uniformly continuous function $\hat{\varphi} : \hat{X} \to \hat{Y}$.*

Let I be a nonempty set. If $\mathscr{U}_i$ is a uniformity on a set X_i for each $i \in I$, then the Cartesian product $\prod_{i\in I} X_i$ may be endowed with the *product uniformity*, with basis consisting of all sets of the form $p_{i_1}^{-1}(R_1) \cap \cdots \cap p_{i_n}^{-1}(R_n)$, where each $R_j \in \mathscr{U}_{i_j}$ and each $p_i : X \times X \to X_i \times X_i$ is the natural projection on each component. From the fact that a nonempty product of complete uniform spaces is complete [109, Theorem 39.6], it follows that completion and product commute. One can also easily show that Hausdorffization and product commute.

2.3. Profinite uniformities and metrics. By a *topological algebra* we mean an algebra endowed with a topology with respect to which each basic operation is continuous. A *compact algebra* is a topological algebra whose topology is compact. We view finite algebras as topological algebras with respect to the discrete topology. When we write that two topological algebras are isomorphic we mean that there is an algebraic isomorphism between them which is also a homeomorphism. A subset X of a topological algebra S is said to *generate* S if it generates a dense subalgebra of S.

Similarly, a *uniform algebra* is an algebra endowed with a uniformity such that the basic operations are uniformly continuous. Note that a uniform algebra is also a topological algebra for the topology induced by the uniformity and that, in case the topology is compact, the basic operations are continuous if and only if they are uniformly continuous (for the unique uniformity inducing the topology). Consistently with the choice of the discrete topology for finite algebras, we endow them with the *discrete uniformity*, in which every reflexive relation is an entourage.

Let $\mathcal{F}$ be a class of finite algebras. A subset L of a topological (respectively uniform) algebra S is said to be *$\mathcal{F}$-recognizable* if there is a continuous (resp. uniformly continuous) homomorphism $\varphi : S \to P$ into some $P \in \mathcal{F}$ such that $L = \varphi^{-1}\varphi L$. In case $\mathcal{F}$ consists of all finite algebras, we say simply that L is *recognizable* to mean that it is $\mathcal{F}$-recognizable.

Let $\mathcal{T}$ be a class of topological algebras. A topological algebra S is said to be *residually in $\mathcal{T}$* if, for every pair of distinct points $s, t \in S$, there exists a continuous homomorphism $\varphi : S \to P$, into some $P \in \mathcal{T}$, such that $\varphi(s) \neq \varphi(t)$.

Suppose that S is a topological algebra and Q is a pseudoquasivariety. The case that will interest us the most is when Q is a pseudovariety and S is a discrete algebra. The *pro-Q uniformity* on S, denoted $\mathscr{U}_{\mathsf{Q}}$, is generated by the basis consisting of all congruences θ such that $S/\theta \in \mathsf{Q}$ and the natural mapping $S \to S/\theta$ is continuous. Note that $\mathscr{U}_{\mathsf{Q}}$ is indeed a uniformity on S, which is transitive. In case Q consists of all finite algebras, we also call the pro-Q uniformity the *profinite uniformity*. The pro-Q uniformity on S is Hausdorff if and only if S is residually in Q as a topological algebra. More precisely, the Hausdorffization of S is given by the pro-Q uniform structure of S/θ_{Q}, under the quotient topology. The topology induced by the pro-Q uniformity of the algebra S is also called its *pro-Q topology*. Sets that are open in this topology are also said to be Q-*open* and a similar terminology is adopted for closed and clopen sets. Similar notions can be defined if we start with a uniform algebra instead of a topological algebra, replacing continuity by uniform continuity, but we will have no use for them here.

Note that the pro-Q uniformity $\mathscr{U}_{\mathsf{Q}}$ is totally bounded for a pseudoquasivariety Q. Given a subset L of an algebra S, we denote by E_L the equivalence relation whose classes are L and its complement $S \setminus L$. Note that, for a congruence θ on S, we have $\theta = \bigcap_L E_L$, where the intersection runs over all θ-classes. The following is now immediate.

Proposition 2.2. *Suppose that Q is a pseudoquasivariety and S is a topological algebra.*

(1) The Hausdorff completion of S under $\mathscr{U}_{\mathsf{Q}}$ is compact.
(2) A subset L of S is Q-recognizable if and only if E_L belongs to $\mathscr{U}_{\mathsf{Q}}$. In case Q is a pseudovariety, a further equivalent condition is that the syntactic congruence $\sim_L$ belong to $\mathscr{U}_{\mathsf{Q}}$.
(3) The Q-recognizable subsets of S are Q-clopen and constitute a basis of the pro-Q topology of S. In particular, the pro-Q topology of S is zero-dimensional and a subset L of S is Q-open if and only if L is a union of Q-recognizable sets.

In contrast, not every Q-clopen subset of an algebra S needs to be Q-recognizable. For instance, for the pseudovariety N, of all finite nilpotent semigroups, one may easily show that the pro-N topology on the (discrete) free semigroup A^+ over a finite alphabet A is discrete, and so every subset is clopen, while it is well-known that the N-recognizable subsets of A^+ are the finite and cofinite languages.

For a pseudoquasivariety Q and a topological algebra S, we define two functions on $S \times S$ as follows. For $s,t \in S$, $r_{\mathsf{Q}}(s,t)$ is the minimum of the cardinalities of algebras P from Q for which there is some continuous homomorphism $\varphi : S \to P$ such that $\varphi(s) \neq \varphi(t)$, where we set $\min \emptyset = \infty$. We then put $d_{\mathsf{Q}}(s,t) = 2^{-r_{\mathsf{Q}}(s,t)}$ with the convention that $2^{-\infty} = 0$. One can easily check that d_{Q} is a pseudo-ultrametric on S, which is called the *pro-Q pseudo-ultrametric* on S.

The following result is an immediate generalization of [86, Section 3], where the hypothesis that the signature is finite serves to guarantee that there are at most countably many isomorphism classes of finite σ-algebras.

Proposition 2.3. *Suppose that σ is a finite signature. For a pseudoquasivariety Q and a topological algebra S, the following conditions are equivalent:*

(1) the pro-Q uniformity on S is defined by the pro-Q pseudo-ultrametric on S;
(2) the pro-Q uniformity on S is defined by some pseudo-ultrametric on S;
(3) there are at most countably many Q-recognizable subsets of S;
(4) for every $P \in \mathsf{Q}$, there are at most countably many homomorphisms $S \to P$.

In particular, all these conditions hold in case S is finitely generated. Moreover, if Q contains nontrivial algebras then, for the discrete free algebra $F_A\mathcal{Q}$ over the variety generated by Q, the pro-Q uniformity is defined by the pro-Q pseudo-ultrametric if and only if A is finite.

The next result gives a different way of looking into pro-Q topologies and uniformities.

Proposition 2.4. *Let S be a topological algebra and Q a pseudoquasivariety.*

(1) The pro-Q uniformity of S is the smallest uniformity $\mathscr{U}$ on S for which all continuous homomorphisms from S into members of Q are uniformly continuous.
(2) The pro-Q topology of S is the smallest topology $\mathscr{T}$ on S for which all continuous homomorphisms from S into members of Q remain continuous.
(3) The algebra S is a uniform algebra with respect to its pro-Q uniformity. In particular, it is a topological algebra for its pro-Q topology.

Following [86], we say that a function $\varphi : S \to T$ between two topological algebras is (Q,R)-*uniformly continuous* if it is uniformly continuous with respect to the uniformities $\mathscr{U}_\mathsf{Q}$, on S, and $\mathscr{U}_\mathsf{R}$, on T. Similarly, we say that φ is (Q,R)-*continuous* if it is continuous with respect to the Q-topology of S and the R-topology of T.

It is now easy to deduce the following result, which is a straightforward generalization of [86, Theorem 4.1].

Proposition 2.5. *Let Q and R be two pseudoquasivarieties, S and T be two topological algebras, and $\varphi : S \to T$ an arbitrary function.*

(1) *The function* φ *is* (Q, R)*-uniformly continuous if and only if, for every* R*-recognizable subset* L *of* T*,* $\varphi^{-1}L$ *is a* Q*-recognizable subset of* S*.*
(2) *The function* φ *is* (Q, R)*-continuous if and only if, for every* R*-recognizable subset* L *of* T*,* $\varphi^{-1}L$ *is a union of* Q*-recognizable subsets of* S*.*

Proposition 2.5 was motivated by the work of Pin and Silva [87] on non-commutative versions of Mahler's theorem in p-adic Number Theory, which states that a function $\mathbb{N} \to \mathbb{Z}$ is uniformly continuous with respect to the p-adic metric if and only if it can be uniformly approximated by polynomial functions.

2.4. Profinite algebras. This subsection is mostly based on [10], where the reader may find further details.

For a class $\mathcal{T}$ of topological algebras, a *pro-*$\mathcal{T}$ *algebra* is a compact algebra that is residually in $\mathcal{T}$. A *profinite algebra* is a pro-$\mathcal{T}$ algebra where $\mathcal{T}$ is the class of all finite algebras.

An *inverse system* $\mathcal{I} = (I, S_i, \varphi_{ij})$ of topological algebras consists of a family $(S_i)_{i\in I}$ of such algebras, indexed by a directed set I, together with a family $(\varphi_{ij})_{i,j\in I; i\geq j}$ of functions, the *connecting homomorphisms*, such that the following conditions hold:

i. each φ_{ij} is a continuous homomorphism $S_i \to S_j$;
ii. each φ_{ii} is the identity function on S_i;
iii. for all $i, j, k \in I$ such that $i \geq j \geq k$, the equality $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$ holds.

The *inverse limit* of an inverse system $\mathcal{I} = (I, S_i, \varphi_{ij})$ is the subspace $\varprojlim \mathcal{I}$ of $\prod_{i\in I} S_i$ consisting of the families $(s_i)_{i\in I}$ such that $\varphi_{ij}(s_i) = s_j$ whenever $i \geq j$. Note that, in case $\varprojlim \mathcal{I}$ is nonempty, it is a subalgebra of $\prod_{i\in I} S_i$ and, therefore, a topological algebra. The inverse limit may be empty. For instance, the inverse limit of the inverse system $(\mathbb{N}, [n, +\infty[, \varphi_{nm})$ is empty, where the intervals are viewed as semilattices under the usual ordering and with the inclusion mappings as connecting homomorphisms φ_{nm}. In contrast, if all the S_i are compact algebras, then so is $\varprojlim \mathcal{I}$ [109, Exercise 29C].

The following is a key property of pro-V algebras for a pseudovariety V.

Proposition 2.6. *Let* V *be a pseudovariety,* S *a pro-*V *algebra, and* $\varphi : S \to T$ *a continuous homomorphism onto a finite algebra. Then* T *belongs to* V*.*

More generally, for a pseudoquasivariety Q, the following alternative characterizations of pro-Q algebras are straighforward extensions of the pseudovariety case for semigroups, which can be found, for instance, in [10, Proposition 4.3].

Proposition 2.7. *Let Q be a pseudoquasivariety. Then the class $\bar{\mathsf{Q}}$ of all pro-Q algebras consists of all inverse limits of algebras from Q and it is the smallest class of topological algebras containing Q that is closed under taking isomorphic algebras, closed subalgebras, and arbitrary direct products. The classes $\bar{\mathsf{Q}}$ and Q have the same finite members. In case Q is a pseudovariety, the class $\bar{\mathsf{Q}}$ is additionally closed under taking profinite continuous homomorphic images.*

Since every compact metric space is a continuous image of the Cantor set [109, Theorem 30.7], the profiniteness assumption in the second part of Proposition 2.7 cannot be dropped.

The nontrivial parts of the next theorem were first observed in [36] to follow from the arguments in [4], which in turn extend the case of semigroups, due to Numakura [81], through the approach of Hunter [67]. The key ingredient is the following lemma, first stated explicitly and proved by Hunter [67, Lemma 4] for semigroups although, in this case, it can also be extracted from [81].

Lemma 2.8. *Let S be a compact zero-dimensional algebra and let L be a subset of S for which the syntactic congruence is determined by finitely many terms. Then L is recognizable if and only if L is clopen.*

The reader may wish to compare Lemma 2.8 with Proposition 2.2(3) and the subsequent comments.

Theorem 2.9. *Let S be a compact algebra and consider the following conditions:*

(1) S is profinite;
(2) S is an inverse limit of an inverse system of finite algebras;
(3) S is isomorphic to a closed subalgebra of a direct product of finite algebras;
(4) S is a compact zero-dimensional algebra.

Then the implications $(1) \Leftrightarrow (2) \Leftrightarrow (3) \Rightarrow (4)$ *always hold, while* $(4) \Rightarrow (3)$ *also holds in case the syntactic congruence of S is determined by a finite number of terms.*

One can find in [46] explicit proofs of Lemma 2.8 and Theorem 2.9. As mentioned in Section 2.1, the same paper provides characterizations of the finiteness assumption in Theorem 2.9. In particular, compact zero-dimensional semigroups, monoids, groups, rings, and lattices in finitely generated varieties of lattices are profinite.

The finitely generated case of the following variant of Lemma 2.8 can be found in [4]. The essential step for the proof of the general case can be found in [10, Lemma 4.1].

Proposition 2.10. *Let* Q *be a pseudoquasivariety and let* S *be a pro-*Q *algebra. Then a subset* L *of* S *is clopen if and only if it is* Q*-recognizable, if and only if it is recognizable. In particular, the topology of* S *is the smallest topology for which all continuous homomorphisms from* S *into algebras from* Q *(or, alternatively, into finite algebras) are continuous with respect to it. Hence, a topological algebra is a pro-*Q *algebra if and only if it is compact and its topology coincides with its pro-*Q *topology.*

A way of constructing profinite algebras is via the Hausdorff completion of an arbitrary topological algebra S with respect to its pro-Q uniformity. We denote this completion by $C_{\mathsf{Q}}(S)$. The next result can be easily deduced from Propositions 2.1, 2.2, and 2.4.

Proposition 2.11. *Let* S *be a topological algebra and* Q *a pseudoquasivariety. Then* $C_{\mathsf{Q}}(S)$ *is a pro-*Q *algebra. Moreover, if* S *is residually in* Q*, then the topology of* S *coincides with the induced topology as a subspace of* $C_{\mathsf{Q}}(S)$*.*

It is important to keep in mind that the topology of a pro-Q algebra S may not be its pro-Q topology when S is viewed as a discrete algebra. To give an example, we introduce a pseudovariety which is central in the theory of finite semigroups: the class A of all finite *aperiodic* semigroups whose subgroups are trivial.

Example 2.2. Let $\mathbb{N}$ be the discrete additive semigroup of natural numbers and consider its pro-A completion $C_{\mathsf{A}}(\mathbb{N})$, which is obtained by adding one point, denote it ∞, which is such that $n + \infty = \infty + n = \infty$ and $\lim n = \infty$. Then the mapping that sends natural numbers to 1 and ∞ to 0 is a homomorphism into the semilattice $\{0, 1\}$ which is not continuous for the topology of $C_{\mathsf{A}}(\mathbb{N})$ but which is continuous for the pro-A topology.

In contrast, it is a deep and difficult result that, for every finitely generated profinite group, its topology coincides with its profinite topology as a discrete group [80]. The proof of this result depends on the classification of finite simple groups.

The Q-recognizable subsets of an algebra S constitute a subalgebra $\mathcal{P}_{\mathsf{Q}}(S)$ of the Boolean algebra $\mathcal{P}(S)$ of all its subsets. On the other hand, a compact zero-dimensional space is also known as a *Boolean space*. The two types of Boolean structures are linked through *Stone duality* (cf. [45, Section IV.4]), whose easily described direction associates with a Boolean space its Boolean algebra of clopen subsets; every Boolean algebra is obtained in this way. The following result shows that the Boolean space $C_{\mathsf{Q}}(S)$ and the Boolean algebra $\mathcal{P}_{\mathsf{Q}}(S)$ are Stone duals. In it, we adopt a convenient abuse of notation: for the natural mapping $\iota : S \to C_{\mathsf{Q}}(S)$ and a subset K of $C_{\mathsf{Q}}(S)$, we write $K \cap S$ for $\iota^{-1}K$, while, for a subset L of S, we write $\overline{L}$ for the closure of ιL in $C_{\mathsf{Q}}(S)$.

Theorem 2.12. *Let Q be a pseudoquasivariety and let S be an arbitrary topological algebra. Then the following are equivalent for a subset L of S:*

(1) the set L is Q-recognizable;
(2) the set L is of the form $K \cap S$ for some clopen subset K of $C_{\mathsf{Q}}(S)$;
(3) the set $\overline{L}$ is open and $\overline{L} \cap S = L$.

When the pro-Q topology of S is discrete, a further equivalent condition is that $\overline{L}$ is open. Moreover, the clopen sets of the form $\overline{L}$ with L a Q-recognizable subset of S form a basis of the pro-Q topology of S.

Since $C_{\mathsf{Q}}(S)$ has further structure involved besides its topology, which is the sole to intervene in Stone duality, one may ask what further structure is reflected in the Boolean algebra. This question has been investigated in [58, 59], in the context of the theory of semigroups and its connections with regular languages.

For a topological algebra S, we denote by $\mathrm{End}(S)$ the monoid of continuous endomorphisms of S. It can be viewed as a subspace of the product space S^S, that is with the *pointwise convergence topology*. A classical alternative is the *compact-open topology*, for which a basis consists of all sets of the form (K, U), which in turn consist of all self maps φ of S such that $\varphi(K) \subseteq U$, where K is compact and U is open. These two topologies on a space of self maps of S in general do not coincide. However, for finitely generated profinite algebras they coincide on $\mathrm{End}(S)$. This was first proved by Hunter [66, Proposition 1] and rediscovered by the first author [12, Theorem 4.14] in the context of profinite semigroups. Steinberg [105] showed how this is related with the classical theorem of Ascoli on function spaces. The proofs extend easily to an arbitrary algebraic setting.

Theorem 2.13. *For a finitely generated profinite algebra S, the pointwise convergence and compact-open topologies coincide on* $\mathrm{End}(S)$ *and turn it into a profinite monoid such that the evaluation mapping* $\mathrm{End}(S) \times S \to S$*, sending (φ, s) to $\varphi(s)$, is continuous.*

A further result from [105] that extends to the general algebraic setting is that finitely generated profinite algebras are *Hopfian* in the sense that all continuous onto endomorphisms are automorphisms.

Denote by $\mathrm{Aut}(S)$ the group of units of $\mathrm{End}(S)$, consisting of all continuous automorphisms of S whose inverse is also continuous, the latter restriction being superfluous in case S is compact. From Theorem 2.13, it follows that, for a finitely generated profinite algebra S, $\mathrm{Aut}(S)$ is a profinite group. In case S is a profinite group, this result as well as the Hopfian property of S are well known in group theory [99].

2.5. Relatively free profinite algebras. Let Q be a pseudoquasivariety. We say that a pro-Q algebra S is *free pro-*Q over a set A if there is a mapping $\iota : A \to S$ satisfying the following universal property: for every function $\varphi : A \to T$ into a pro-Q algebra, there is a unique continuous homomorphism $\hat{\varphi} : S \to T$ such that $\hat{\varphi} \circ \iota = \varphi$. The mapping ι is usually not unique and it is said to be a *choice of free generators*. The following result is well known [10].

Proposition 2.14. *For every pseudoquasivariety* Q *and every set A, there exists a free pro-*Q *algebra over A, namely the inverse limit of all A-generated algebras from* Q*, with connecting homomorphisms respecting the choice of generators. Up to isomorphism respecting the choice of free generators, it is unique.*

We denote the free pro-Q algebra over a set A by $\overline{\Omega}_A\mathsf{Q}$. The notation is justified below.

An alternative way of constructing free pro-Q algebras is through the pro-Q Hausdorff completion of free algebras.

Proposition 2.15. *Let* Q *be a pseudoquasivariety and let A be a set. Let $\mathcal{V}$ be the variety generated by* Q*. Then the pro-*Q *Hausdorff completion of the free algebra $F_A\mathcal{V}$ is a free pro-*Q *algebra over A.*

Note that, by Proposition 2.3, if A is finite, then $\overline{\Omega}_A\mathsf{Q}$ is metrizable. In contrast, the argument presented in [32, end of Section 3] for pseudovarieties of monoids may be extended to every nontrivial pseudoquasivariety Q to show that, if A is infinite, then $\overline{\Omega}_A\mathsf{Q}$ is not metrizable.

A topological algebra S is *self-free* with *basis* A if A is a generating subset of S such that every mapping $A \to S$ extends uniquely to a continuous endomorphism of S.

Theorem 2.16. *The following conditions are equivalent for a profinite algebra S:*

(1) the topological algebra S is self-free with basis A;
(2) there is a pseudoquasivariety Q such that S is isomorphic with $\overline{\Omega}_A\mathsf{Q}$;
(3) there is a pseudovariety V such that S is isomorphic with $\overline{\Omega}_A\mathsf{V}$.

Proof: The implications (3) $\Rightarrow$ (2) $\Rightarrow$ (1) are obvious, so it remains to prove that (1) $\Rightarrow$ (3). Suppose that (1) holds and let V be the pseudovariety generated by all finite algebras that are continuous homomorphic images of S. We claim that S is isomorphic with $\overline{\Omega}_A\mathsf{V}$.

We first observe that, since S is a profinite algebra, it is an inverse limit of finite algebras, which may be chosen to be continuous homomorphic images of S. Hence S is a pro-V algebra and, therefore, there is a unique continuous homomorphism $\varphi : \overline{\Omega}_A\mathsf{V} \to S$ such that, for a choice of free generators $\iota : A \to \overline{\Omega}_A\mathsf{V}$, the composite $\varphi \circ \iota$ is the inclusion mapping $A \hookrightarrow S$. Since S is generated by A as a topological algebra, the function φ is surjective. It suffices to show that it is injective.

Let u, v be distinct points of $\overline{\Omega}_A\mathsf{V}$. Since $\overline{\Omega}_A\mathsf{V}$ is residually in V, there is some continuous homomorphism $\psi : \overline{\Omega}_A\mathsf{V} \to T$, onto some $T \in \mathsf{V}$, such that $\psi(u) \neq \psi(v)$. By the definition of V, there are continuous homomorphisms $\xi_i : S \to V_i$ $(i = 1, \ldots, n)$ onto finite algebras, a subalgebra U of $\prod_{i=1}^n V_i$, and a surjective homomorphism $\rho : U \to T$. Since ρ is surjective, there is a mapping $\eta : A \to U$ such that $\rho \circ \eta = \psi \circ \iota$. Let $\pi_i : \prod_{j=1}^n V_j \to V_i$ be the ith component projection. Since ξ_i is surjective, there is a function $\mu_i : A \to S$ such that $\xi_i \circ \mu_i = \pi_i \circ \eta$. By self-freeness of S, with basis A, it follows that there is a continuous endomorphism $\hat{\mu}_i$ of S such that $\hat{\mu}_i|_A = \mu_i$. Let $\zeta : S \to \prod_{i=1}^n V_i$ be the unique continuous homomorphism such that $\pi_i \circ \zeta = \xi_i \circ \hat{\mu}_i$ for $i = 1, \ldots, n$. The following diagram depicts the relationships between these mappings.

$$\begin{tikzcd}
 & A \arrow[r, hook] \arrow[dl, "\iota"'] \arrow[dr, "\eta"] \arrow[rr, bend left, "\mu_i"] & S \arrow[r] \arrow[d, dashed, "\zeta"] \arrow[dr, "\hat{\mu}_i"] \arrow[dr, "\zeta"'] & S \arrow[dr, "\xi_i"] & \\
\overline{\Omega}_A\mathsf{V} \arrow[r, "\psi"'] \arrow[urr, "\varphi"] & T & U \arrow[l, "\rho"] \arrow[r, hook] & \prod_{j=1}^n V_j \arrow[r, "\pi_i"'] & V_i
\end{tikzcd}$$

Note that $\pi_i \circ \zeta|_A = \xi_i \circ \mu_i = \pi_i \circ \eta$ for $i = 1, \ldots, n$, which shows that $\zeta|_A = \eta$ and so the image of ζ is contained in U and the chain of equalities $\rho \circ \zeta \circ \varphi \circ \iota = \rho \circ \zeta|_A = \rho \circ \eta = \psi \circ \iota$ holds, which yields $\rho \circ \zeta \circ \varphi = \psi$. Since $\psi(u) \neq \psi(v)$, we deduce that $\varphi(u) \neq \varphi(v)$, which establishes the claim that φ is injective. ∎

Theorem 2.16 not only gives a characterization of relatively free profinite algebras in terms of properties that only involve the algebras themselves, but also shows that, when talking about such algebras, we may as well deal only with pseudovarieties.

Yet another description of relatively free profinite algebras is given by algebras of implicit operations, which further provide a useful viewpoint. For a class $\mathcal{C}$ of profinite algebras and a set A, an *A-ary implicit operation w* on $\mathcal{C}$ is a correspondence associating with each $S \in \mathcal{C}$ a continuous operation $w_S : S^A \to S$ such that, for every continuous homomorphism $\varphi : S \to T$ between members of $\mathcal{C}$, the equality $w_T(\varphi \circ f) = \varphi(w_S(f))$ holds for every $f \in S^A$. We call w_S the *interpretation* of w in S.

Proposition 2.17. *Let C be a class of finite algebras, let V be the pseudovariety it generates, and let A be a set. For $w \in \overline{\Omega}_A\mathsf{V}$ and a pro-V algebra S, let $\bar{w}_S : S^A \to S$ be defined by $\bar{w}_S(\varphi) = \hat{\varphi}(w)$, where $\hat{\varphi}$ is the unique continuous homomorphism $\overline{\Omega}_A\mathsf{V} \to S$ such that $\hat{\varphi} \circ \iota = \varphi$. Then $\bar{w}$ is an A-ary implicit operation on the class of all pro-V algebras and every such operation is of this form. Moreover, the correspondence associating to w the restriction of $\bar{w}$ to C is injective and, therefore, so is the correpondence $w \mapsto \bar{w}$.*

Thus, we may as well identify each $w \in \overline{\Omega}_A\mathsf{V}$ with the implicit operation $\bar{w}$ that it determines. In terms of implicit operations, the interpretation of the basic operations is quite transparent: for an n-ary operation symbol f, implicit operations $w_1, \ldots, w_n \in \overline{\Omega}_A\mathsf{V}$, a pro-$\mathsf{V}$ algebra S, and a function

$\varphi \in S^A$, we have

$$\big(f^{\overline{\Omega}_A\mathsf{V}}(w_1,\ldots,w_n)\big)_S(\varphi) = f^S\big((w_1)_S(\varphi),\ldots,(w_n)_S(\varphi)\big).$$

In other words, the basic operations are interpreted pointwise.

Among the implicit operations on the class of all profinite algebras, we have the projections x_a. More precisely, for a set A and $a \in A$, the A-ary projection on the a-component is interpreted in a profinite algebra S by $(x_a)_S(\varphi) = \varphi(a)$ for each $\varphi \in S^A$. By restriction to pro-V algebras, we also obtain corresponding implicit operations, which we still denote x_a. The subalgebra of $\overline{\Omega}_A\mathsf{V}$ generated by the x_a with $a \in A$ is denoted $\Omega_A\mathsf{V}$. Its elements are also known as *A-ary explicit operations* on pro-V algebras. From the universal property of $\overline{\Omega}_A\mathsf{V}$, it follows immediately that $\Omega_A\mathsf{V}$ is the free algebra $F_A\mathcal{V}$, where $\mathcal{V}$ is the variety generated by V. The following result explains the notation.

Proposition 2.18. *Let V be a pseudovariety. Then the algebra $\Omega_A\mathsf{V}$ is dense in $\overline{\Omega}_A\mathsf{V}$.*

The operational point of view has the advantage that pro-V algebras are automatically endowed with a stucture of profinite algebras over any enriched signature obtained by adding implicit operations on V. This idea is essential for Subsection 2.6.

A formal equality $u = v$ between members of some $\overline{\Omega}_A\mathsf{V}$ is said to be a *pseudoidentity for* V; the elements of A are called the *variables* of the pseudoidentity. It is said to *hold* in a pro-V algebra S if $u_S = v_S$. In case V is the pseudovariety of all finite algebras, we omit reference to V. For a set Σ of pseudoidentities for V, the class of all algebras from V that satisfy all pseudoidentities from Σ is denoted $[\![\Sigma]\!]$; this class is said to be *defined* by Σ and Σ to be a *basis of pseudoidentities* for it.

Theorem 2.19 (Reiterman [93])**.** *A subclass of a pseudovariety V is a pseudovariety if and only if it is defined by some set of pseudoidentities for V.*

There are many alternative proofs of Reiterman's theorem, as well as extensions to various generalizations of the algebras considered in this chapter. The most relevant in the context of this handbook seems to be the one obtained by Molchanov [78] for "pseudovarieties" of algebras with predicates, also proved independently by Pin and Weil [89].

The interest in Reiterman's theorem stems from the fact that it provides a language to obtain elegant descriptions of pseudovarieties. Moreover, namely through the techniques described in the next subsection, they sometimes lead to decidability results, even if in a somewhat indirect way.

2.6. Decidability and tameness. In the theory of regular word or tree languages, pseudovarieties serve the purpose of providing an algebraic classification tool for certain combinatorial properties. The properties that are amenable to this approach have been identified, first by Eilenberg [56] for word languages, and later by the first author [5, 6] and Steinby [106] for tree languages. By considering additional relational structure on the algebras, further combinatorial properties may be captured (see [84, 92]).

Basically, in such an algebraic approach, one seeks to decide whether a language has a certain combinatorial property by testing whether its syntactic algebra has the corresponding algebraic property, that is, if this algebra belongs to a certain pseudovariety. Thus, a property of major interest that pseudovarieties may have is decidability of the membership problem: given a finite algebra, decide whether or not it belongs to the pseudovariety. We then simply say that the pseudovariety is *decidable*.

One way to establish that a pseudovariety is decidable is to prove that it has a finite basis of pseudoidentities which are equalities between implicit operations that can be effectively computed, so that the pseudoidentities in the basis can be effectively checked. In fact, for most commonly encountered implicit operations, the computation can be done in polynomial time, in terms of the size of the algebra, and so the verification of the basic pseudoidentities can then be done in polynomial time.

However, many pseudovarieties of interest are not finitely based. For instance, it is easy to see that, if a pseudovariety is generated by a single algebra, then it is decidable, but it may not be finitely based, an important example being the pseudovariety generated by the syntactic monoid B_2^1 of the language $(ab)^*$ over the 2-letter alphabet [82, 100]. Moreover, contrary to a conjecture proposed by the first author [6], a pseudovariety for which the membership problem is solvable in polynomial time may not admit a finite basis of pseudoidentities [107]. Sapir has even shown that there is a finite semigroup that generates such a pseudovariety [70, Theorem 3.53]. It has recently been announced by M. Jackson that the membership problem for $\mathsf{V}(B_2^1)$ is NP-hard and so, provided P $\neq$ NP, that problem cannot be decided in polynomial time, which would solve [70, Problem 3.11].

Pseudovarieties are often described by (infinite) generating sets of algebras. This comes about by applying some natural operator on other pseudovarieties, like the join in the lattice of pseudovarieties. In general, for any construction $C(S_1, \ldots, S_n)$ of an algebra from given algebras S_i, perhaps under suitable restrictions or additional data (like in the definition of semidirect product, where an action of one of the factors on the other is required), one may consider the pseudovariety $C(\mathsf{V}_1, \ldots, \mathsf{V}_n)$ generated by all algebras of the form $C(S_1, \ldots, S_n)$ with each S_i in a given pseudovariety V_i. The join is obtained in this way by considering the usual direct product. Another type of operator of interest is the following: for two pseudovarieties V and W, their *Mal'cev product* $\mathsf{V} \textcircled{m} \mathsf{W}$ is the pseudovariety generated by all algebras S for which there is a congruence θ such that S/θ belongs to W, and each class which is a subalgebra belongs to V.

Since most such natural operators in the case of semigroups do not preserve decidability [1, 42], it is of interest to develop methods that, under suitable additional assumptions on the given pseudovarieties, guarantee that the operator produces a decidable pseudovariety. The starting point in the profinite approach is to obtain a basis of pseudoidentities for the resulting pseudovariety. In the context of semigroups and monoids, bases theorems of this kind have been established for Mal'cev products [88] and various types of semidirect products [38]. Unfortunately, there is a gap in the proof of the latter, so that the results are only known to hold under certain additional finiteness hypotheses.* The bases provided by such theorems for a binary operator $C(\mathsf{V}, \mathsf{W})$ consist of pseudoidentities which are built from pseudoidentities determined by V by substituting the variables by certain implicit operations. The implicit operations that should be considered to test membership in $C(\mathsf{V}, \mathsf{W})$ of a given finite A-generated algebra S are the solutions of certain systems of equations in $\overline{\Omega}_A\mathsf{W}$, determined by the operator C, subject to regular constraints determined by each specific evaluation of the variables in S which is to be tested. This approach was first introduced in [8, 7], improved in [30], and later extended in [10] and, independently and in a much more systematic way, also in [96]. The reader is referred to [7, 30, 10] for the proofs of the results presented in this section.

We proceed to formalize the above ideas. Consider a set Σ of pseudoidentities, which we view as a *system of equations*. The sides of the equations

*See [96] for a discussion and a general basis theorem, which in turn has not led to decidability results.

$u = v$ in Σ are implicit operations $u, v \in \overline{\Omega}_X\mathsf{U}$ on a suitable ambient pseudovariety U over a fixed alphabet X, whose letters are called the *variables* of the system. We may say that Σ consists of U-*equations* to emphasize this condition. Additionally, we impose for each variable x a clopen constraint $K_x \subseteq \overline{\Omega}_A\mathsf{U}$ over another fixed alphabet. The constraints are thus recognizable subsets of $\overline{\Omega}_A\mathsf{U}$. We say that the constrained system has a *solution* γ in an A-generated pro-U algebra T if $\gamma : X \to \overline{\Omega}_A\mathsf{U}$ is a function such that the following two conditions hold, where $\hat{\gamma} : \overline{\Omega}_X\mathsf{U} \to \overline{\Omega}_A\mathsf{U}$ and $\pi : \overline{\Omega}_A\mathsf{U} \to T$ are the unique continuous homomorphisms respectively extending γ and respecting the choice of generators of T:

(1) for each variable $x \in X$, the constraint $\gamma(x) \in K_x$ is satisfied;
(2) for each equation $u = v$ in Σ, the equality $\pi(\hat{\gamma}(u)) = \pi(\hat{\gamma}(v))$ holds.

The following is a simple compactness result which can be found for instance in [10].

Theorem 2.20. *A system of* U-*equations over a set of variables* X *with clopen constraints* $K_x \subseteq \overline{\Omega}_A\mathsf{U}$ $(x \in X)$ *has a solution in every* A-*generated algebra from a given subpseudovariety* V *of* U *if and only if it has a solution in* $\overline{\Omega}_A\mathsf{V}$.

If the set of variables X is finite, which we assume from hereon, then there is a continuous homomorphism $\varphi : \overline{\Omega}_A\mathsf{U} \to S$ into a finite algebra S which recognizes all the given constraints $K_x \subseteq \overline{\Omega}_A\mathsf{U}$ $(x \in X)$. Then the existence of a solution for the system in an A-generated algebra $T \in \mathsf{U}$ is equivalent to the existence of a solution in T for the same system for at least one of a certain set of constraints of the form $K'_x = \varphi^{-1}(s)$ with $s \in S$. Thus, one may prefer to give the constraints in the form of a function $X \to S$ into an A-generated finite algebra S.

Another formulation of the above ideas is in terms of relational morphisms, which is the perspective initially taken in [7] and which prevails in [96]. A *relational morphism* between two topological algebras S and T is a closed subalgebra μ of the direct product $S \times T$ whose projection in the first component is onto. Note that, if S and T are pro-U algebras then so is μ and if μ is A-generated, then the induced continuous homomorphisms $\varphi : \overline{\Omega}_A\mathsf{U} \to S$ and $\psi : \overline{\Omega}_A\mathsf{U} \to T$ are such that μ is obtained by composing the relations $\varphi^{-1} \subseteq S \times \overline{\Omega}_A\mathsf{U}$ and $\psi \subseteq \overline{\Omega}_A\mathsf{U} \times T$. This is called a *canonical factorization* of μ.

An example of such a relational morphism is obtained as follows. Let $\varphi : A \to S$ be a generating mapping for a pro-U algebra S and let V be a subpseudovariety of U. Consider the unique continuous homomorphisms $\hat{\varphi} : \overline{\Omega}_A\mathsf{U} \to S$ and $\psi : \overline{\Omega}_A\mathsf{U} \to \overline{\Omega}_A\mathsf{V}$ respecting the choice of generators. Then $\mu_{\mathsf{V},A} = \hat{\varphi}^{-1}\psi$ is a relational morphism from S to $\overline{\Omega}_A\mathsf{V}$.

We say that the system of U-equations Σ with constraints given by a function $\xi : X \to S$ into a finite algebra S is *inevitable* with respect to a relational morphism $\mu \subseteq S \times T$, where T is a profinite algebra, if there is a continuous homomorphism $\delta : \overline{\Omega}_X\mathsf{U} \to T$ such that the following conditions hold:

(1) for each variable $x \in X$, the constraint $(\xi(x), \delta(x)) \in \mu$ is satisfied;
(2) for each equation $u = v$ in Σ, the equality $\delta(u) = \delta(v)$ holds.

One can easily check that this property is equivalent to the existence of a solution of the system subject to the constraints $K_x = \hat{\varphi}^{-1}(\xi(x)) \subseteq \overline{\Omega}_A\mathsf{U}$, where $\mu = \hat{\varphi}^{-1}\psi$ is the canonical factorization associated with a finite generating set A for μ. Theorem 2.20 then yields the following similar compactness theorem for inevitability.

Theorem 2.21. *For a system of* U*-equations over a finite set* X*, of variables, with constraints given by a mapping* $X \to S$ *into a finite algebra* S *and a subpseudovariety* V *of* U*, the following conditions are equivalent:*

(1) the constrained system is inevitable with respect to every relational morphism μ *from* S *into an arbitrary algebra from* V*;*
(2) the constrained system is inevitable with respect to every relational morphism μ *from* S *into an arbitrary pro-*V *algebra;*
(3) for some *finite generating set* A *of* S*, the constrained system is inevitable with respect to the relational morphism* $\mu_{\mathsf{V},A}$*;*
(4) for every *finite generating set* A *of* S*, the constrained system is inevitable with respect to the relational morphism* $\mu_{\mathsf{V},A}$*.*

Let V be a subpseudovariety of U. We say that a constrained system is V-*inevitable* if it satisfies the equivalent conditions of Theorem 2.21. The pseudovariety V is said to be *hyperdecidable* with respect to a class $\mathcal{S}$ of systems of U-equations with constraints in algebras from U if there is an algorithm that decides, for each constrained system in $\mathcal{S}$, whether it is V-inevitable.

An approach to prove hyperdecidability which was devised by Steinberg and the first author [30, 31], inspired by seminal work of Ash [41], was to draw this property from other either more familiar or more conceptual properties. Assume that the class $\mathcal{S}$ consists of finite systems, that it is recursively enumerable, and that the implicit operations that appear on the sides of the equations of the systems are computable. Moreover, suppose that V is recursively enumerable. One can then effectively check whether a constrained system in $\mathcal{S}$ is inevitable with respect to a relational morphism from the constraining algebra into an algebra from V, which gives a semi-algorithm to enumerate the constrained systems which are not V-inevitable. To decide whether a constrained system from $\mathcal{S}$ has a solution in $\overline{\Omega}_A\mathsf{V}$ it thus suffices to add hypotheses to guarantee that there is also a semi-algorithm to enumerate the systems that are V-inevitable. To do so, the idea is to prove that if the system is V-inevitable, then there is a solution of a special kind, so that the candidates for such special solutions can be effectively enumerated and whether such a candidate is indeed a solution can be effectively checked.

To formalize this idea, consider a recursively enumerable set τ of computable implicit operations on U, including the basic operations. We call such a set τ a *computable implicit signature* over U. Note that every pro-U algebra has automatically the structure of a τ-algebra (see Proposition 2.17). For a subpseudovariety V of U, we denote by $\Omega_A^\tau\mathsf{V}$ the τ-subalgebra of $\overline{\Omega}_A\mathsf{V}$ generated by A. It follows from the definition of free pro-V algebra that $\Omega_A^\tau\mathsf{V}$ is freely generated by A in the variety of τ-algebras generated by V. The *word problem* for $\Omega_A^\tau\mathsf{V}$ consists in, given two τ-terms over the alphabet A, deciding whether they represent the same element of $\Omega_A^\tau\mathsf{V}$. We may now state the following key definition.

Definition 2.2. Let V be a recursively enumerable subpseudovariety of U and let $\mathcal{S}$ be a class of constrained systems of U-equations. We say that V is *τ-reducible* with respect to $\mathcal{S}$ if, whenever a constrained system in $\mathcal{S}$ has a solution $\gamma : X \to \overline{\Omega}_A\mathsf{U}$ in $\overline{\Omega}_A\mathsf{V}$, it has a solution $\gamma' : X \to \Omega_A^\tau\mathsf{U}$ in $\overline{\Omega}_A\mathsf{V}$.[†] If, moreover, the word problem for $\Omega_A^\tau\mathsf{V}$ is decidable, then we say that V is *τ-tame* with respect to $\mathcal{S}$. We say that V is *completely τ-tame* if it is τ-tame with respect to the class of all finite constrained systems of equations of τ-terms.

The following result summarizes the above discussion.

[†]A topological formulation of the notion of τ-reducibility was recently found in [26]. It simply states that, for each system from $\mathcal{S}$, forgetting the constraints, the solutions in $\overline{\Omega}_A\mathsf{V}$ from $\mathcal{S}$ taking values in $\Omega_A^\tau\mathsf{U}$ are dense in the set of all solutions in $\overline{\Omega}_A\mathsf{V}$.

Theorem 2.22. *Let U be a recursively enumerable pseudovariety and let τ be a computable implicit signature over U. Let $\mathcal{S}$ be a recursively enumerable class of constrained systems of equations between τ-terms. Finally, let V be a subpseudovariety of U. If V is τ-tame with respect to $\mathcal{S}$, then V is hyperdecidable with respect to $\mathcal{S}$.*

Several important examples of tame pseudovarieties are discussed in Subsection 3.2. Here, we only present tameness results which hold in the general algebraic context to which this section is dedicated. Before doing so, we introduce a weaker version of tameness which is also of interest.

Let S be an A-generated algebra from U and let τ be a computable implicit signature. The relational morphism $\bar{\mu}^{\tau}_{\mathsf{V},A} \subseteq S \times \Omega^{\tau}_{A}\mathsf{V}$ is obtained by taking the intersection of $\mu_{\mathsf{V},A}$ with $S \times \Omega^{\tau}_{A}\mathsf{V}$. We say that V is *weakly τ-reducible* for a class $\mathcal{S}$ of constrained systems of U-equations if, for every V-inevitable constrained system in $\mathcal{S}$, say with constraints in the A-generated algebra $S \in \mathsf{U}$, the system is inevitable with respect to the relational morphism $\bar{\mu}^{\tau}_{\mathsf{V},A}$. Replacing τ-reducibility by weak τ-reducibility in the definition of τ-tameness we speak of *weak τ-tameness*.

Viewing $\Omega^{\tau}_{A}\mathsf{V}$ as a discrete algebra, there is another natural relational morphism $\mu^{\tau}_{\mathsf{V},A} \subseteq S \times \Omega^{\tau}_{A}\mathsf{V}$, namely the τ-subalgebra generated by the pairs of the form (a, a) with $a \in A$. The notation is justified since, as it is easily proved, the relation $\bar{\mu}^{\tau}_{\mathsf{V},A}$ is the closure of $\mu^{\tau}_{\mathsf{V},A}$ in $S \times \Omega^{\tau}_{A}\mathsf{V}$ with respect to the discrete topology in the first component and the pro-V topology in the second component. We say that V is *τ-full* if the two relational morphisms coincide for every A-generated algebra S from U. Note that a weakly τ-reducible τ-full pseudovariety is τ-reducible. Conversely, the terminology is justified by the fact that, if V is τ-reducible with respect to a constrained system Σ of U-equations, then it is also weakly τ-reducible with respect to Σ.

We say that the pseudovariety V has *computable τ-closures* if there is an algorithm such that, given a finite alphabet A, a regular subset L of $\Omega_{A}\mathsf{V}$ and an element $v \in \Omega^{\tau}_{A}\mathsf{V}$, determines whether or not v belongs to the closure of L in the pro-V topology of $\Omega^{\tau}_{A}\mathsf{V}$. The following combines a couple of results from [30].

Theorem 2.23. *Let V be a recursively enumerable subpseudovariety of a recursively enumerable pseudovariety U, let τ be a computable implicit signature, and suppose that the word problem for each $\Omega^{\tau}_{A}\mathsf{V}$ is decidable.*

(1) If V is τ-full then V has computable τ-closures.

(2) *If* V *is weakly* τ*-reducible for a class* $\mathcal{S}$ *of constrained systems of* U*-equations and* V *has computable* τ*-closures, then* V *is hyperdecidable with respect to* $\mathcal{S}$*.*

We say that a class of algebras is *locally finite* if all finitely generated algebras in the variety it generates are finite. This is the case, for instance, for a pseudovariety generated by a single algebra but not every locally finite pseudovariety is of this kind. A well-known example in the realm of semigroups is provided by the pseudovariety of all finite bands (in which every element is idempotent).

A decidable locally finite pseudovariety V is said to be *order computable* if the function that associates with each positive integer n the cardinality of the algebra $\overline{\Omega}_n\mathsf{V}$ is computable. It seems to be an open problem whether every locally finite pseudovariety is order computable. The following result is an immediate extension of [30, Theorem 4.18], which is based on the "slice theorem" of Steinberg [102].

Theorem 2.24. *Let* V *be a* τ*-tame pseudovariety with respect to a class* $\mathcal{S}$ *of systems of equations and let* W *be an order-computable pseudovariety. Then the join* $\mathsf{V} \vee \mathsf{W}$ *is also* τ*-tame with respect to* $\mathcal{S}$*.*

One of the ingredients behind the proof of Theorem 2.24 is that $\overline{\Omega}_A\mathsf{W} = \Omega_A^\tau\mathsf{W}$ for every locally finite pseudovariety W and every implicit signature τ. Under this weaker property for a computable implicit signature τ, tameness becomes much simpler. The following result is a simple corollary of some of the above results. We do not know whether the τ-fullness hypothesis can be dropped.

Proposition 2.25. *Let* τ *be a computable implicit signature and let* V *be a recursively enumerable pseudovariety such that the equality* $\overline{\Omega}_A\mathsf{V} = \Omega_A^\tau\mathsf{V}$ *holds for every finite set* A *and* V *is* τ*-full. Then* V *is completely* τ*-tame if and only if the word problem for each* $\Omega_A^\tau\mathsf{V}$ *is decidable.*

3. The case of semigroups

The motivation to study profinite topologies in finite semigroup theory comes from automata and language theory: Eilenberg's correspondence theorem [56] shows the relevance of investigating pseudovarieties of semigroups and monoids.

The results mentioned in this section by no means cover entirely the literature in the area that is presently available. In particular, we stick to the more classical case of semigroups, while for instance the cases of ordered semigroups or stamps have come to play a significant role, as can be seen in Chapter 16. It turns out that in all these cases the same relatively free profinite semigroups intervene and the results are also often quite similar, although sometimes their proofs involve additional technical difficulties.

3.1. Computing profinite closures. There are several reasons why profinite topologies are relevant for automata theory and Sections 1 and 2 provide many of them. We start this subsection by formulating a simple problem which has a direct translation in terms of profinite topologies.

Let A be a finite alphabet and let $L \subseteq A^+$ be a regular language. Membership in L of a word $w \in A^+$ can be effectively tested by checking whether the action of w on the initial state of the minimal automaton of L leads to a final state, or whether the syntactic image of w belongs to that of L. But, one may be interested in a weaker test such as whether w may be separated from L by a regular language K of a particular type, for instance a group language (i.e., a language whose syntactic semigroup is a group): does there exist a group language $K \subseteq A^+$ such that $w \in K$ and $K \cap L = \emptyset$?

Let G denote the pseudovariety of all finite groups. In view of Proposition 2.2(3), the above separation property is equivalent to being able to separate w from L by some open set in the pro-G topology of A^+. Thus, in terms of the pro-G topology, the above question translates into testing membership of w in the closure of L in the pro-G topology of A^+. More generally, we have the following result, where we denote by $\mathrm{cl}_{\mathsf{V}}(L)$ the closure of L in the pro-V topology of A^+.

Proposition 3.1. *Let A be a finite alphabet and let V be a pseudovariety of semigroups. For a regular language $L \subseteq A^+$, a word $w \in A^+$ can be separated from L by a V-recognizable language if and only if $w \notin \mathrm{cl}_{\mathsf{V}}(L)$.*

Thus, for a pseudovariety V to have computable τ-closures for the implicit signature reduced to multiplication (see Subsection 2.6) is a property that has immediate automata-theoretic relevance.

The special case of separation by group languages has particular historical importance. It was first considered by Pin and Reutenauer [85], who proposed the following recursive procedure to compute $\mathrm{cl}_{\mathsf{G}}(L)$, where $FG(A)$ denotes the group freely generated by A.

Theorem 3.2 ([85, Theorem 2.4])**.** *Given a regular expression for a language $L \subseteq A^+$, replace the operation $K \mapsto K^+$ by that of taking the subgroup of $FG(A)$ generated by the argument K. The resulting expression describes a subset of $FG(A)$ and $\mathrm{cl}_{\mathsf{G}}(L)$ is its intersection with A^+.*

The correctness of the algorithm described in Theorem 3.2 was reduced to the proposed conjecture that the product of finitely many finitely generated subgroups of a free group is closed in its profinite topology, thus generalizing M. Hall's result that finitely generated subgroups of the free group are closed in the profinite topology [61]. This conjecture was established by Ribes and Zalesskiĭ [97] using profinite group theory. The original motivation for computing $\mathrm{cl}_{\mathsf{G}}(L)$ comes from the fact that Pin and Reutenauer also showed that the correctness of their procedure implies that the "type II conjecture" holds. This other conjecture gives a constructive description of the *group kernel* $K_{\mathsf{G}}(M)$ of a finite monoid M. More generally, for a pseudovariety H of groups, the H-*kernel* $K_{\mathsf{H}}(M)$ consists of all $m \in M$ such that, for every relational morphism μ from M to a group in H, $(m, 1)$ belongs to μ. The type II conjecture states that $K_{\mathsf{G}}(M)$ is the smallest submonoid of M that contains the idempotents and is closed under the operation that sends m to amb if $aba = a$ or $bab = b$ (*weak conjugation*). An independent proof of the type II conjecture was obtained by Ash [41] and is discussed in Subsection 3.2.

The pro-V closure of regular languages in A^+ has also been considered for other pseudovarieties V. For a pseudovariety H of groups, the motivation comes from the membership problem in the pseudovariety $\mathsf{W} \textcircled{m} \mathsf{H}$ for a pseudovariety of monoids W. Indeed, it is easy to show that a finite monoid M belongs to this Mal'cev product if and only if $K_{\mathsf{H}}(M)$ belongs to W. On the other hand, Pin [83] observed that, if $\varphi : A^* \to M$ is an onto homomorphism and $m \in M$, then $m \in K_{\mathsf{H}}(M)$ if and only if the empty word 1 belongs to $\mathrm{cl}_{\mathsf{H}}(\varphi^{-1}(m))$. Thus, we have the following result.

Proposition 3.3. *Let W be a decidable pseudovariety of monoids and H a pseudovariety of groups such that one can decide whether, given a regular language $L \subseteq A^*$, the empty word belongs to $\mathrm{cl}_{\mathsf{H}}(L)$. Then $\mathsf{W} \textcircled{m} \mathsf{H}$ is decidable.*

The problem of computing the pro-H closure in A^* of a regular language $L \subseteq A^*$ has been considered for other pseudovarieties of groups such as Ab (Abelian groups) [53], G_p (p-groups), $\mathsf{G}_{\mathrm{nil}}$ (nilpotent groups), and $\mathsf{G}_{\mathrm{sol}}$ (solvable groups) [98, 76].

Suppose that H is a pseudovariety of groups such that the free group $FG(A)$ is residually in H. Then we have natural embeddings $A^* \hookrightarrow FG(A) \hookrightarrow \overline{\Omega}_A\mathsf{H}$. The pro-$\mathsf{H}$ topology of a subalgebra of $\overline{\Omega}_A\mathsf{H}$ is its subspace topology by Propositions 2.11 and 2.15. Thus, an equivalent problem to computing the H-kernel of a finite monoid is to decide whether 1 belongs to the closure in $FG(A)$ of a regular language $L \subseteq A^*$. In case H is closed under extensions (or, equivalently, under semidirect product), Ribes and Zalesskiĭ [98, Theorem 5.1] have shown that, for the pro-H topology, the product of finitely many finitely generated closed subgroups of $FG(A)$ is also closed. Using the Pin-Reutenauer techniques, they reduced the computation of the pro-H closure of a regular language $L \subseteq A^*$ to the computation of the pro-H closure in $FG(A)$ of a given finitely generated subgroup. That these results also apply to $\mathsf{G}_{\mathrm{nil}}$ has been recently shown in [27]. Algorithms for the computation of the pro-G_p and pro-$\mathsf{G}_{\mathrm{nil}}$ closures of finitely generated subgroups of a free group can be found in [98, 76]. The case of $\mathsf{G}_{\mathrm{sol}}$ remains open.

For an element s of a profinite semigroup S, s^ω denotes the unique idempotent in the closed subsemigroup T generated by s, and $s^{\omega-1}$ the inverse of $s^{\omega+1} = ss^\omega$ in the maximal subgroup of T. Consider the implicit signature κ consisting of multiplication together with the unary operation $x \mapsto x^{\omega-1}$. The free group $FG(A)$ may be then identified with the free algebra $\Omega_A^\kappa\mathsf{G}$. This suggests generalizations of the Pin-Reutenauer procedure for computing the pro-G closure of a regular language $L \subseteq A^*$ to other pseudovarieties. The analog of the procedure is shown in [22] to hold for the pseudovariety A (and the signature κ).

Another important application of the separation problem has been given by Place and Zeitoun [90] in the study of the decidability problem for the Straubing-Thérien hierarchy of star-free languages.

3.2. Tameness. There is a natural way of associating a system of semigroup equations to a finite digraph that is relevant for the computation of semidirect products of pseudovarieties of semigroups [7]. Namely, the variables of the system are the vertices and the arrows, and each arrow $u \xrightarrow{e} v$ gives rise to an equation $ue = v$ which relates the act of following the arrow with multiplication as in a Cayley graph. The term *tameness* was introduced in [31] to refer to tameness with respect to such systems of equations in the sense of Subsection 2.6. To avoid confusion, we prefer to call it here *graph* *tameness*. We adopt a similar convention for other properties parametrized by systems of equations, such as hyperdecidability and reducibility.

For example, for the one-vertex one-loop digraph, the corresponding equation is $xy = x$. It is easy to verify that, with constraints given by a function ξ into a finite monoid M, this equation is G-inevitable if and only if $\xi(y) \in K_{\mathsf{G}}(M)$. For the two-vertex digraph with n arrows from one vertex to the other one, the associated system of equations has the form $xy_i = z$ $(i = 1, \ldots, n)$. If ξ is a constraining function into a finite monoid M such that $\xi(x) = 1$, then the system is G-inevitable if and only if, for every relational morphism μ from M into an arbitrary $G \in \mathsf{G}$, there is some $g \in G$ such that $\xi(\{y_1, \ldots, y_n, z\}) \times \{g\} \subseteq \mu$. Replacing G by an arbitrary pseudovariety V of monoids, the latter condition is expressed by saying that the subset $\xi(\{y_1, \ldots, y_n, z\})$ of M is V-*pointlike*.

The first and best known example of a graph tame pseudovariety is that of the pseudovariety G. This result has been discovered in different disguises, first by Ash [41], as a means of establishing the type II (see Subsection 3.1 and [69]) and *pointlike* [63] conjectures. In Ash's formulation, the arrows of finite digraphs are labeled with elements of a finite monoid M and the result is said to be *inevitable* if, for every relational morphism μ from M into an arbitrary finite group G, each label can be replaced by a μ-related label in G such that, for every (not necessarily directed) cycle, the product of the labels of the arrows, or their inverses for backward arrows, is equal to 1 in G. In the notation of Subsection 2.6 and also taking into account [30, Lemma 4.8], Ash's theorem states that such a labeled digraph is inevitable if and only if the preceding property holds for the relational morphism $\mu^{\kappa}_{\mathsf{G},A}$ associated with any choice of generating set A for M. It then follows easily that Ash's theorem translates to the statement that G is graph κ-tame.

Fix a finite relational language. A class $\mathcal{R}$ of relational structures is said to satisfy the *finite extension property for partial automorphisms* (FEPPA) if, for every finite structure R in $\mathcal{R}$ and every set P of isomorphisms between substructures of R, if there exists in $\mathcal{R}$ an extension S of R in which all $f \in P$ extend to automorphisms of S, then there is such an extension S which is finite. A *homomorphism* of relational structures is a function that preserves the relations in the forward direction. The *exclusion* of a class $\mathcal{R}$ of relational structures is the class of relational structures S such that there is no homomorphism $R \to S$ with $R \in \mathcal{R}$. Herwig and Lascar [65] showed that, for a finite class $\mathcal{R}$ of finite relational structures, its exclusion class satisfies

FEPPA. They also gave an equivalent formulation of this result in terms of a property of free groups, which Delgado and the first author [23, 24] proved to be equivalent to the graph κ-tameness of G.

On the other hand, it follows from results of Coulbois and Khélif [52] that the pseudovariety G is not completely κ-tame. It would be of interest to find a signature τ such that G is completely τ-tame, if any such signature exists.

Theorem 3.4. *The pseudovariety* G *is graph* κ*-tame but not completely* κ*-tame.*

Tameness has also been investigated for other pseudovarieties of groups. The pseudovariety Ab is completely κ-tame [25]. On the other hand, a pseudovariety of Abelian groups is completely hyperdecidable if and only if it is decidable while it is completely κ-tame if and only if it is locally finite or Ab [54]. For the pseudovariety G_p, the situation is more complicated. Steinberg [101, Theorem 11.12] showed that, for every nontrivial extension-closed pseudovariety of groups H such that the pro-H closure of a finitely generated subgroup of a free group is again finitely generated, H is graph weakly κ-reducible. On the other hand, a graph κ-reducible pseudovariety must admit a basis of pseudoidentities consisting of κ-identities [30, Proposition 4.2] which, since free groups are residually in G_p, entails that G_p is not graph κ-tame. Using symbolic dynamics techniques to generate a suitable infinite implicit signature, the first author has established the following result [9].

Theorem 3.5. *There is a signature* τ *such that* G_p *is graph* τ*-tame.*

Building on the approach of [9], Alibabaei has constructed for each decidable pseudovariety H of Abelian groups an implicit signature with respect to which H is completely tame [2] and also an implicit signature with respect to which $\mathsf{G}_{\mathrm{nil}}$ is graph tame [3].

A semigroup is said to be *completely regular* if every element lies in some subgroup. The pseudovariety CR consists of all completely regular finite monoids and OCR is the subpseudovariety consisting of those in which the idempotents constitute a submonoid. Both these pseudovarieties have been shown to be graph κ-tame [34, 33],[‡] results which depend heavily on Theorem 3.4 together with structure theorems for the corresponding relatively free profinite monoids.

Several aperiodic pseudovarieties have also been investigated. An interesting example is that of the pseudovariety J of all finite $\mathcal{J}$-trivial semigroups, corresponding to the variety of piecewise testable languages (see [56]). The first author has shown that, for a finite alphabet A, $\overline{\Omega}_A\mathsf{J} = \Omega_A^\kappa\mathsf{J}$ and also solved the word problem for $\Omega_A^\kappa\mathsf{J}$ (see [6, Section 8.2]). Since it is an easy exercise to deduce that J is κ-full, it follows from Proposition 2.25 that J is completely κ-tame, and therefore graph hyperdecidable. The construction of a "real algorithm" to decide inevitability turns out to be much more involved [39].

For the pseudovariety R, consisting of all finite $\mathcal{R}$-trivial semigroups, constructing a concrete algorithm to show that R is graph hyperdecidable is technically complicated, even when only strongly connected digraphs are considered [29]. Building on seminal ideas of Makanin [75] and taking into account the structure of free pro-R semigroups [37], Costa, Zeitoun and the first author [21] have established the following result.

Theorem 3.6. *The pseudovariety* R *is completely* κ*-tame.*

This result has been extended in [13] to pseudovarieties of the form DRH, consisting of all finite semigroups in which every regular $\mathcal{R}$-class is a group from the pseudovariety H of groups.

Consider next the pseudovariety LSl of all finite local semilattices, which corresponds to the variety of locally testable languages (see [56]). The proof of the following result involves very delicate combinatorics on words [51, 50].

Theorem 3.7. *The pseudovariety* LSl *is completely* κ*-tame.*

J. Rhodes announced in a conference held in 1998 in Lincoln, Nebraska, that A is graph κ-tame. The only part of the program to establish such a result

[‡]The conjecture to which the graph tameness of CR is reduced in [34] has been observed by K. Auinger (private communication) to hold using the methods of [23, 24]. There is also another difficulty which comes from the fact that free profinite semigroupoids over profinite graphs are considered. As has been shown in [14], there are some rather delicate aspects in the description of such structures when the graph has infinitely many vertices, namely the free subsemigroupoid generated by a dense subgraph of the profinite graph may not be dense, and one needs in general to transfinitely iterate algebraic and topological closures. However, one can check that for the free profinite semigroupoid in question, the iteration stops in one step, from which it follows that the required properties of the suitable free profinite semigroupoid are guaranteed [16].

that has been published is McCammond's solution of the word problem for $\Omega_A^\kappa\mathsf{A}$ [77]. Another, earlier ingredient in Rhodes' ideas comes from Henckell's computation of the A-pointlike subsets of a given finite semigroup [62]. See [91] for a different proof. In [64] there is also an alternative proof and the following generalization.

Theorem 3.8. *If π is a recursive set of prime integers, then there is an algorithm to compute pointlike sets of finite semigroups with respect to the pseudovariety $\overline{\mathsf{G}_\pi}$, consisting of all finite semigroups whose subgroups are π-groups.*

Once it was discovered that there was a gap in the proof of the *basis theorem* (see the discussion in Subsection 2.6), which invalidated the reduction of the decidability of the Krohn-Rhodes complexity to proving that A is tame announced in [30], Rhodes withdrew several manuscripts that he claimed would prove that A is tame. The Krohn-Rhodes complexity pseudovarieties are defined recursively by $\mathsf{C}_0 = \mathsf{A}$ and $\mathsf{C}_{n+1} = \mathsf{C}_n * \mathsf{G} * \mathsf{A}$ [71], which determines a complete and strict hierarchy for the pseudovariety of all finite semigroups. Here, $*$ denotes the *semidirect product* of pseudovarieties of semigroups as defined in [6, Section 10.1], which is an associative operation.

It has also been investigated whether tameness is preserved under the operators of join and semidirect product. Since tameness is apparently much stronger than decidability, if tameness is preserved by semidirect product then the decidability of the Krohn-Rhodes complexity is indeed reduced to proving that A is tame. But, so far, only very special cases have been treated. An example is the following result [19], which improves on [28].

Theorem 3.9. *Let V be a graph κ-tame pseudovariety and let W be an order computable pseudovariety. Then $\mathsf{V} * \mathsf{W}$ is graph κ-tame.*

It is also unknown whether tameness is preserved under join. Yet, several positive results have been obtained. The following theorem combines results from [30, 20].

Theorem 3.10. *Let $\mathcal{C}$ be a class of constrained systems of equations.*

(1) Let V be an order-computable pseudovariety. If a pseudovariety W is τ-tame with respect to $\mathcal{C}$, then so is $\mathsf{V} \vee \mathsf{W}$.
(2) Let V be a recursively enumerable κ-full subpseudovariety of J such that the word problem for $\Omega_A^\kappa\mathsf{V}$ is decidable. If a pseudovariety W is τ-tame with respect to $\mathcal{C}$, then so is $\mathsf{V} \vee \mathsf{W}$.

(3) Let W *be a pseudovariety satisfying some pseudoidentity of the form*

$$x_1 \cdots x_n y^{\omega+1} z t^{\omega} = x_1 \cdots x_n y z t^{\omega}.$$

If W *is* τ*-tame with respect to* $\mathcal{C}$*, then so is* $\mathsf{R} \vee \mathsf{W}$*.*

Theorem 3.10 yields for instance that the join $\mathsf{J} \vee \mathsf{G}$ is graph κ-tame, a result which was also proved by Steinberg [102].

4. Relatively free profinite semigroups

Several representation theorems and structural results about relatively free profinite semigroups have been obtained for various pseudovarieties, such as J ($\mathcal{J}$-trivial) [6, Section 8.2], R ($\mathcal{R}$-trivial) [37, 40], DA (regular elements are idempotent) [79], and LSl (local semilattices) [49]. Much remains unknown, particularly in the case of pseudovarieties containing LSl. However, progress has been made in this case too. For instance, in [60, 18] faithful representations of finitely generated free profinite semigroups over A were obtained. There is a common trend in these faithful representations of free profinite semigroups over A, R, or DA, and also in the partial faithful representations obtained in [68, 72] for many other pseudovarieties: it is the fact they consist in viewing pseudowords as linearly ordered sets whose elements are labeled with letters, generalizing the fact that words are nothing else than such sets with a finite cardinal.

In the most general case, that of the pseudovariety S of all finite semigroups, no meaningful faithful representation is known (albeit we can always get partial information on the elements of $\overline{\Omega}_A\mathsf{S}$ by looking at their projection on $\overline{\Omega}_A\mathsf{V}$, for some semigroup pseudovariety V, when a suitable representation for $\overline{\Omega}_A\mathsf{V}$ is available). This adds motivation for studying the structure of free profinite semigroups over S and other "large" pseudovarieties. In this section we review some results on this subject, mainly about Green's relations, with an emphasis on maximal subgroups. A substantial part of the results originated in connections with symbolic dynamics, most introduced by the first author, sometimes in co-authorship. We highlight some of the progress in this front. Other approaches, for the most part developed by Rhodes and Steinberg, based on expansions of finite semigroups or on wreath product techniques, also led to results about structural properties of free profinite semigroups over many pseudovarieties containing LSl, as is the case in [94, 104, 103, 48]. We mention in Subsection 4.2 two results where these other approaches played a key role, namely Theorems 4.9 and 4.10.

4.1. Connections with symbolic dynamics. For a good reference book on symbolic dynamics, see [73]. Even though an introduction to symbolic dynamics appears in Chapter 27, for the convenience of the presentation we include our own brief introduction. Let A be a finite alphabet. Since A is compact, the product space $A^{\mathbb{Z}}$ is compact. The *shift* on $A^{\mathbb{Z}}$ is the homeomorphism $\sigma : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ sending $(x_i)_{i\in\mathbb{Z}}$ to $(x_{i+1})_{i\in\mathbb{Z}}$. A *symbolic dynamical system*, also called *shift space* or *subshift*, is a nonempty[§] closed subspace $\mathcal{X}$ of $A^{\mathbb{Z}}$ such that $\sigma(\mathcal{X}) = \mathcal{X}$, for some finite alphabet A. A shift space $\mathcal{X}$ is *minimal*, if $\mathcal{X}$ does not contain subshifts other than $\mathcal{X}$. A *block* of $(x_i)_{i\in\mathbb{Z}}$ is a word $x_i x_{i+1} \cdots x_{i+n}$, with $i \in \mathbb{Z}$ and $n \geq 0$. Denote by $\mathcal{B}(\mathcal{X})$ the set of all blocks of elements of $\mathcal{X}$. One has $\mathcal{X} \subseteq \mathcal{Y}$ if and only if $\mathcal{B}(\mathcal{X}) \subseteq \mathcal{B}(\mathcal{Y})$.

Often, one may define a subshift by an effectively computable amount of data. This happens for example if $\mathcal{B}(\mathcal{X})$ is a rational language, in which case we say that $\mathcal{X}$ is *sofic*. Sofic subshifts are considered in Chapter 27. Another class of examples, extensively studied, comes from subshifts defined by *primitive substitutions* [57]. Here, by a *substitution* over a finite alphabet A we mean an endomorphism φ of A^+. A substitution φ over A is *primitive* if there is $n \geq 1$ such that all letters of A are factors of $\varphi^n(a)$, for every $a \in A$. For such a primitive substitution, there is a unique minimal subshift $\mathcal{X}_\varphi$ such that $\mathcal{B}(\mathcal{X}_\varphi)$ is the set of all factors of words of the form $\varphi^n(a)$, where $n \geq 1$ and $a \in A$.

A subset L of a semigroup S is *irreducible* if $u, v \in L$ implies $uwv \in L$ for some $w \in S$. A subshift $\mathcal{X}$ of $A^{\mathbb{Z}}$ is *irreducible* if $\mathcal{B}(\mathcal{X})$ is an irreducible language of A^+. Minimal subshifts are irreducible. A subshift $\mathcal{X}$ is *periodic* if $\mathcal{X}$ is a finite set of the form $\{\sigma^n(x) : n \in \mathbb{Z}\}$ for some $x \in \mathcal{X}$. An irreducible subshift is either periodic or infinite.

For the remainder of this subsection, V is a pseudovariety of semigroups containing all finite nilpotent semigroups. Then A^+ is isomorphic with $\Omega_A\mathsf{V}$ and embeds in $\overline{\Omega}_A\mathsf{V}$. The elements of A^+ are isolated in $\overline{\Omega}_A\mathsf{V}$. Hence $\overline{L} \cap A^+ = L$ holds for every language L of A^+. Therefore $\overline{\mathcal{B}(\mathcal{X})}$ captures all information about $\mathcal{X}$. Clearly, $\mathcal{B}(\mathcal{X})$ is closed under taking factors; when $\mathsf{V} = \mathsf{A} \circledcirc \mathsf{V}$, the topological closure of $\mathcal{B}(\mathcal{X})$ in $\overline{\Omega}_A\mathsf{V}$ is also closed under taking factors, a fact that follows from the multiplication being open in $\overline{\Omega}_A\mathsf{V}$ when $\mathsf{V} = \mathsf{A} \circledcirc \mathsf{V}$, cf. [14, Lemma 2.3 and Proposition 2.4].

[§]The empty set is frequently considered a subshift in the literature (*e.g.*, in Chapter 27).

Using a compactness argument, in case $\mathcal{X}$ is irreducible one shows the existence of a unique $\leq_{\mathcal{J}}$-minimal $\mathcal{J}$-class $\mathfrak{J}_{\mathsf{V}}(\mathcal{X})$ consisting of factors of $\overline{\mathcal{B}(\mathcal{X})}$. If V contains LSl then $\overline{\mathcal{B}(\mathcal{X})}$ consists of elements of $\overline{\Omega}_A\mathsf{V}$ whose finite factors belong to $\mathcal{B}(\mathcal{X})$ [47]. From this one gets the following proposition, which a particular case of [48, Proposition 3.6].

Proposition 4.1. *Let* V *be a pseudovariety of semigroups containing* LSl*. Let* $\mathcal{X}$ *and* $\mathcal{Y}$ *be irreducible subshifts. Then* $\mathcal{X} \subseteq \mathcal{Y}$ *if and only if* $\mathfrak{J}_{\mathsf{V}}(\mathcal{Y}) \leq_{\mathcal{J}} \mathfrak{J}_{\mathsf{V}}(\mathcal{X})$*.*

The following result is taken from [11].

Theorem 4.2. *If* V *contains* LSl*, then the mapping* $\mathcal{X} \mapsto \mathfrak{J}_{\mathsf{V}}(\mathcal{X})$ *is a bijection from the set of minimal subshifts of* $A^{\mathbb{Z}}$ *onto the set of* $\leq_{\mathcal{J}}$*-maximal regular* $\mathcal{J}$*-classes of* $\overline{\Omega}_A\mathsf{V}$*.*

If $|A| \geq 2$, then there are $2^{\aleph_0}$ minimal subshifts of $A^{\mathbb{Z}}$ (cf. [74, Chapter 2]), and a chain with $2^{\aleph_0}$ irreducible subshifts of $A^{\mathbb{Z}}$ [108, Section 7.3]. Hence, from Theorem 4.2 and Proposition 4.1 we obtain the following result (a weaker version appears in [49]).

Theorem 4.3. *Let* V *be a pseudovariety containing* LSl *and let* A *be an alphabet with at least two letters. For the relation* $<_{\mathcal{J}}$ *in* $\overline{\Omega}_A\mathsf{V}$*, there are both chains and anti-chains with* $2^{\aleph_0}$ *regular elements.*

For a subshift $\mathcal{X}$ of $A^{\mathbb{Z}}$, the real number $h(\mathcal{X}) = \lim \frac{1}{n} \log_2 |\mathcal{B}(\mathcal{X}) \cap A^n|$ is its *entropy*. This fundamental concept is also considered in Chapter 27. For V containing LSl, and for $w \in \overline{\Omega}_A\mathsf{V}$, let $q_w(n)$ be the number of finite factors of w of length n. The real number $h(w) = \lim \frac{1}{n} \log_2 q_w(n)$ exists if $w \in \overline{\Omega}_A\mathsf{V} \setminus A^+$; we define $h(w) = 0$ for $w \in A^+$. The number $h(w)$ is also called the *entropy of* w and was used in [35] to get structural information about $\overline{\Omega}_A\mathsf{V}$.

Note that $h(\mathcal{X}) \leq \log_2 |A|$. One can also show that $h(\mathcal{X}) = \log_2 |A|$ if and only if $\mathcal{X} = A^{\mathbb{Z}}$, for every subshift $\mathcal{X}$ of $A^{\mathbb{Z}}$. The following is a similar result.

Proposition 4.4 ([35])**.** *Let* V *be a pseudovariety containing* LSl*. Suppose* $|A| \geq 2$*. Let* $w \in \overline{\Omega}_A\mathsf{V}$*. Then* $h(w) \leq \log_2 |A|$*, and equality holds if and only if* w *belongs to the minimum ideal of* $\overline{\Omega}_A\mathsf{V}$*.*

For each k such that $0 < k \leq \log_2 |A|$, consider the set E_k of all $w \in \overline{\Omega}_A\mathsf{V}$ with $h(w) < k$. In particular, thanks to Proposition 4.4, $E_{\log_2 |A|}$ is the complement of the minimum ideal of $\overline{\Omega}_A\mathsf{V}$. The following summarizes the most important results from [35].

Theorem 4.5. *Let* V *be a pseudovariety containing* LSI *and suppose* $0 < k \leq \log_2 |A|$.

(1) For all $u, v \in \overline{\Omega}_A\mathsf{V}$, *we have* $h(uv) = \max\{h(u), h(v)\}$, *and so* E_k *is a subsemigroup of* $\overline{\Omega}_A\mathsf{V}$. *In particular, the minimum ideal is prime.*
(2) The set E_k *is stable under the application of every* n*-ary implicit operation* w *such that* $h(w) < k \cdot \log_2 |A| \cdot \log_n |A|$.
(3) The set E_k *is also stable under the iterated application of a continuous endomorphism* φ *of* $\overline{\Omega}_A\mathsf{V}$ *such that* $\varphi(A) \subseteq E_k$, *in the following sense: if* ψ *belongs to the closed subsemigroup of* $\mathrm{End}(\overline{\Omega}_A\mathsf{V})$ *generated by* φ, *then* $\psi(E_k) \subseteq E_k$.

If $\mathcal{Y}$ is a proper subshift of an irreducible sofic subshift $\mathcal{X}$, then $h(\mathcal{Y}) < h(\mathcal{X})$, see [73, Corollary 4.4.9]. By a reduction to this result, the following theorem generalizing some of the above mentioned properties of the minimum ideal of $\overline{\Omega}_A\mathsf{V}$ is proved in [48].

Theorem 4.6. *Let* V *be a pseudovariety of semigroups containing* LSI *and let* $\mathcal{X}$ *be a sofic subshift of* $A^{\mathbb{Z}}$. *Suppose that* $\mathsf{V} = \mathsf{A} \circledcirc \mathsf{V}$ *or* $\mathcal{B}(\mathcal{X})$ *is* V*-recognizable. Then* $h(w) < h(\mathcal{X})$ *whenever* $w \in \overline{\mathcal{B}(\mathcal{X})} \setminus \mathcal{J}_{\mathsf{V}}(\mathcal{X})$. *Moreover,* $\overline{\Omega}_A\mathsf{V} \setminus \mathcal{J}_{\mathsf{V}}(\mathcal{X})$ *is a subsemigroup of* $\overline{\Omega}_A\mathsf{V}$.

4.2. Closed subgroups of relatively free profinite semigroups. Note that maximal subgroups of profinite semigroups are closed. If a closed subsemigroup of a profinite semigroup is a group then, for the induced topology, it is a profinite group. This subsection presents results on the structure of closed subgroups of relatively free profinite semigroups, with an emphasis on maximal subgroups, using symbolic dynamics.

We shall see examples of maximal subgroups that are (relatively) free profinite groups. When A is a finite set and H is a nontrivial pseudovariety of groups, it is customary to refer to the cardinal of A as being the *rank* of $\overline{\Omega}_A\mathsf{H}$.

A *retract* of $\overline{\Omega}_A\mathsf{S}$ is the image of a continuous idempotent endomorphism of $\overline{\Omega}_A\mathsf{S}$. The free profinite subgroups of $\overline{\Omega}_A\mathsf{S}$ of rank $|A|$ that are retracts of $\overline{\Omega}_A\mathsf{S}$ are characterized in [35, Theorem 4.4]. Combining that characterization with results from [11], leads to the following theorem.

Theorem 4.7. *For every finite alphabet A, there are maximal subgroups H of $\overline{\Omega}_A\mathsf{S}$ such that H is a retract of $\overline{\Omega}_A\mathsf{S}$ and a free profinite group of rank $|A|$.*

The maximal subgroups in a $\mathcal{J}$-class of a profinite semigroup are isomorphic profinite groups (cf. [96, Theorem A.3.9]). When $\mathcal{X}$ is an irreducible subshift, we may consider the (isomorphism class of the) maximal profinite subgroup of $\mathfrak{J}_{\mathsf{V}}(\mathcal{X})$, denoted $G_{\mathsf{V}}(\mathcal{X})$. It is invariant under isomorphisms of subshifts, as long as $\mathsf{V} = \mathsf{V} * \mathsf{D}$ and V contains all finite semilattices [47], where D denotes the pseudovariety of all finite semigroups whose idempotents are right zeros.

Let $\varphi : A^+ \to A^+$ be a primitive substitution. The substitution φ is called *periodic* if the associated minimal subshift $\mathcal{X}_\varphi$ is periodic. If there are $b, c \in A$ such that $\varphi(a)$ starts with b and ends with c for every $a \in A$, then the substitution is said to be *proper*. Denote respectively by φ_{S} and by φ_{G} the unique extension of φ to a continuous endomorphism of $\overline{\Omega}_A\mathsf{S}$ and to a (continuous) endomorphism of $\overline{\Omega}_A\mathsf{G}$. The following is a result from [15].

Theorem 4.8. *If φ is a proper non-periodic primitive substitution over A, then the retract $\varphi_{\mathsf{S}}^{\omega}(\overline{\Omega}_A\mathsf{S})$ is a maximal subgroup of $\mathfrak{J}_{\mathsf{S}}(\mathcal{X}_\varphi)$, which is presented as a profinite group by the set of generators A subject to the relations of the form $\varphi_{\mathsf{G}}^{\omega}(a) = a$ ($a \in A$).*

For the general case where ψ is a primitive (not necessarily proper) non-periodic substitution, one finds in [55] an algorithm to build a proper primitive substitution φ such that $\mathcal{X}_\varphi$ is isomorphic to $\mathcal{X}_\psi$, and so the general case can be reduced to the proper case via the invariance of the maximal subgroup under isomorphism of subshifts. An alternative finite presentation for $G_{\mathsf{S}}(\mathcal{X}_\psi)$ as a profinite group is given in [15]. These results yield that it is decidable whether a given finite group is a (continuous) homomorphic image of $G_{\mathsf{S}}(\mathcal{X}_\psi)$.

Note that, if in Theorem 4.8 the extension of φ to the free group over A is invertible, then we immediately get that $G_{\mathsf{S}}(\mathcal{X}_\varphi)$ is a free profinite group of rank $|A|$, which is a particular case of [11, Corollary 5.7]. On the other hand, it was proved in [15] that if τ is the *Prouhet-Thue-Morse substitution*, that is, the substitution given by $\tau(a) = ab$ and $\tau(b) = ba$, then $G_{\mathsf{S}}(\mathcal{X}_\tau)$ is not a relatively free profinite group.

In [17], further knowledge on $G_{\mathsf{S}}(\mathcal{X})$ was obtained, when $\mathcal{X}$ is minimal, without requiring that $\mathcal{X}$ is defined by a substitution. Namely, it was shown that $G_{\mathsf{S}}(\mathcal{X})$ is an inverse limit of profinite completions of fundamental groups of a special family of finite graphs that is naturally associated to $\mathcal{X}$.

For the sofic case, concerning groups of the form $G_{\mathsf{V}}(\mathcal{X})$, we have first to introduce a definition which is similar with the definition, given in Section 2, of a free profinite group over a pseudovariety H of groups (cf. [99, Chapter 3]). A subset X of a profinite group G is said to *converge to the identity* if every neighborhood of the identity element of G contains all but finitely many elements of X. A profinite group F is a *free pro-*H *group with a basis* X *converging to the identity* if X is a subset of F for which every mapping $\varphi : X \to G$, with G a pro-H group such that $\varphi(X)$ converges to the identity, has a unique extension to a continuous group homomorphism $\hat{\varphi} : F \to G$. All bases of F converging to the identity have the same cardinality, which is called the *rank* of F, and if F and F' have bases converging to the identity with the same cardinal, then F and F' are isomorphic as profinite groups. Note that, if $|X|$ is finite, then F is isomorphic with $\overline{\Omega}_X\mathsf{H}$, and so this definition of rank extends the one given for finitely generated relatively free profinite groups. A free pro-H group in the former sense is also free pro-H with some basis converging to one, but the converse is not true; indeed, as follows from Theorem 4.9, for a nontrivial pseudovariety of groups H, the free pro-H group of countable rank is metrizable, but $\overline{\Omega}_A\mathsf{H}$ is not metrizable when A is infinite (see note following Proposition 2.15).

For a pseudovariety H of groups, denote by $\overline{\mathsf{H}}$ the pseudovariety of all finite semigroups whose subgroups belong to H. Note that $\mathsf{S} = \overline{\mathsf{G}}$. We are now able to cite the result from [48] about maximal subgroups of the from $G_{\overline{\mathsf{H}}}(\mathcal{X})$. Note that the minimal sofic subshifts are periodic subshifts.

Theorem 4.9. *Let* H *be a nontrivial pseudovariety of groups and* $\mathcal{X}$ *an irreducible sofic subshift. If* $\mathcal{X}$ *is periodic, then* $G_{\overline{\mathsf{H}}}(\mathcal{X})$ *is a free pro-*H *group of rank* 1*. If* $\mathcal{X}$ *is non-periodic and* $\mathcal{B}(\mathcal{X})$ *is* $\overline{\mathsf{H}}$*-recognizable then* $G_{\overline{\mathsf{H}}}(\mathcal{X})$ *is a free pro-*H *group of rank* $\aleph_0$*, provided* H *is extension-closed and contains nontrivial* p*-groups for infinitely many primes* p*.*

Note that Theorem 4.9 applies to $\mathcal{X} = A^{\mathbb{Z}}$, in which case $\mathcal{J}_{\overline{\mathsf{H}}}(\mathcal{X})$ is the minimum ideal of $\overline{\Omega}_A\overline{\mathsf{H}}$. This case was previously shown in [104]. For further results on the structure of the minimum ideal of $\overline{\Omega}_A\mathsf{V}$, where V may be among pseudovarieties other than those in Theorem 4.9, see [94, 104]. In contrast with Theorems 4.8 and 4.9, the $\mathcal{H}$-class of a non-regular element of $\overline{\Omega}_A\mathsf{S}$ is a singleton [94, Corollary 13.2].

While not all closed subgroups of $\overline{\Omega}_A\mathsf{S}$ are free profinite groups, they do have a property resembling freeness. A profinite group G is *projective* if, for all continuous onto homomorphisms of profinite groups $\varphi : G \to K$ and $\alpha : H \to K$, there is a continuous homomorphism $\hat{\varphi} : G \to H$ such that $\alpha \circ \hat{\varphi} = \varphi$. It is easy to see that all (finitely generated) projective profinite groups embed into (finitely generated) free profinite groups [99]. The following converse is much more difficult to prove.

Theorem 4.10 ([95]). *Let* V *be a pseudovariety of semigroups such that* $(\mathsf{V} \cap \mathsf{Ab}) * \mathsf{V} = \mathsf{V}$. *Then every closed subgroup of a free pro-*V *semigroup is a projective profinite group.*

The definition of projective profinite group can be considered for other algebras. The *projective profinite semigroups* embed into free profinite semigroups but, in contrast with Theorem 4.10, there are finite subsemigroups of $\overline{\Omega}_A\mathsf{S}$ that are not projective. For further details about finite subsemigroups of $\overline{\Omega}_A\mathsf{S}$ (and $\overline{\Omega}_A\mathsf{V}$ for other V), and their interplay with projective profinite semigroups, see [96, Remark 4.1.34].

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