Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 18–19

ENTOURAGES, DENSITY, CAUCHY STRUCTURE, AND COMPLETION

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ABSTRACT: We study uniformities and quasi-uniformities (uniformities without the symmetry axiom) in the common language of entourages. The techniques developed allow for a general theory in which uniformities are the symmetric part. In particular we have a natural Cauchy structure independent of symmetry and a very simple general completion procedure (perhaps more transparent and simpler than the usual symmetric one).

KEYWORDS: Frame, locale, sublocale, uniform frame, quasi-uniform frame, entourage, uniform map, uniform dense embedding, Cauchy map, complete, completion.

AMS SUBJECT CLASSIFICATION (2010): 06D22, 18B30, 54E15, 54D35.

Introduction

For the description of (symmetric) uniformity in the point-free context one typically uses the geometrically very natural Tukey type approach of a specific system of covers. For the non-symmetric case (although also here an approach using a specific system of pairs of covers is possible), it is natural to adopt an alternative definition using *entourages* (modelling the system of neighborhoods of the diagonal in the product $X \times X$ from the classical Weyl definition). It is not immediately clear that the two approaches are equivalent: the point-free (localic) products are not conservative, that is, the square $L \times L$ of locales does not quite correspond to the square $X \times X$ of topological spaces (see e.g. [15, 24]). But they are ([20, 21]). This fact is somewhat surprising; it is deeper than the classical equivalence, and can be used to advantage (see e.g. the solution of an open problem in [23]).

Received May 28, 2018.
Support from the projects P202/12/G061 (Grant Agency of the Czech Republic) and MTM2015-63608-P (Ministry of Economy and Competitiveness of Spain) and from the Centre for Mathematics of the University of Coimbra (UID/MAT/00324/2013 funded by the Portuguese Government through FCT/MCTES and co-funded by the European Regional Development Fund through the Partnership Agreement PT2020) is gratefully acknowledged.

As the title suggests, one of the topics of our paper is the concept of entourages. It is an ideal background for treating uniformities in (not necessary symmetric) generality (with the symmetric uniformity as a – simpler – special case). It provides an expedient common language enabling us, in particular, to present a natural theory of completeness independent on symmetry. We discuss this technique in more detail than necessary just for this purpose, believing that some of its aspects are interesting in themselves.

Density of uniform sublocales is crucial for understanding completeness. It also has some features that have not been discussed in literature (in particular, in the non-symmetric case, with entourages). Therefore we give it a somewhat more extensive treatment.

As for the completeness questions, the reader is probably aware of a rich literature concerning this topic. To name just a few: there is the pioneering Isbell's paper [14], then Kříž [17], Banaschewski [2, 3], Banaschewski and Pultr [6, 7, 8], Pultr and Tozzi [25], Banaschewski, Hong and Pultr [4], Hong and Kim [13]. Extending the topic to the non-symmetric case opened new vistas and brought more understanding to some of the aspects (Frith and Hunsaker [11], Frith, Hunsaker and Walters-Wayland [12], Kim [16]). The perspectives differ: there is the Isbell's theory of feet, another one using the idea of Samuel compactification, yet another focusing in the behavior of uniform dense embeddings; in the non-symmetric case one uses the technique of biframes. Our approach is closest to that of the uniform dense embeddings, but the perspective is different. The central point is in the Cauchy structure. The natural construction of a maximal frame in which a uniform frame is still densely embedded has a feature that has not been so far properly exploited: it yields a universal Cauchy map. Focusing on this we obtain a simple and more transparent completion procedure. Moreover, this universality is independent on the symmetry so that this approach to completion automatically yields the desired general result (with the symmetric one as a special case).

The paper is organized as follows. After Preliminaries we concentrate in Section 2 on the technique of entourages. This is then used throughout the rest of the paper. In particular it is consequently applied in Section 3 for introducing uniformities and quasi-uniformities and their basic properties. Next, in Section 4 we discuss density in uniform and quasi-uniform contexts; as in the whole of the paper we use, whenever it is of advantage, the localic technique. Section 5 is devoted to the pivot of our approach to completeness, namely to Cauchy maps (it should be noted that it was B. Banaschewski who proposed to view Cauchy maps as *generalized Cauchy points*). It turns out that the construction used so far to produce a minimal dense superlocale in fact produces a universal Cauchy map (the proof of this is very easy, easier than the original minimal density one, probably because the Cauchy context is somewhat more natural). Then, in Section 6, we obtain what we want to have about completeness and completion as simple applications. In Section 7 we add a few remarks on the non-symmetric case and its relation with the symmetric one.

1. Preliminaries

1.1. Frames and locales. Recall that a *frame* is a complete lattice L satisfying the distributive law

$$a \wedge (\bigvee B) = \bigvee\{a \wedge b \mid b \in B\} \tag{dist}$$

for all $a \in L$ and all subsets $B \subseteq L$. A frame homomorphism $h\colon L \to M$ preserves all joins (including the void one, the bottom 0) and all *finite* meets (including the top 1). The resulting category will be denoted by **Frm**. A typical frame is the lattice $\Omega(X)$ of all open sets of a topological space X; if $f\colon X \to Y$ is a continuous map then $\Omega(f) = (U \mapsto f^{-1}[U])\colon \Omega(Y) \to \Omega(X)$ is a frame homomorphism. Thus one has a contravariant functor $\Omega\colon$ **Top** $\to$ **Frm** (where **Top** is the category of topological spaces). Setting **Loc** $=$ **Frm**$^{\mathrm{op}}$ one obtains the *category of locales*. Then Ω becomes a contravariant functor **Top** $\to$ **Loc**; furthermore, restricted to the subcategory of sober spaces it is a full embedding. Thus one can think of locales as a generalization of (sober) topological spaces.

1.1.1. It is of advantage to view **Loc** as a concrete category with the arrow opposite to a frame homomorphism $h\colon L \to M$ represented by its right Galois adjoint $f = h_*\colon M \to L$ (uniquely determined by the fact that h preserves all joins). Such f's will be referred to as *localic maps*.

1.1.2. By (dist), if L is a frame and $a \in L$, the map $a \wedge (\cdot)\colon L \to L$ preserves arbitrary joins and thus has a right adjoint $a \to (\cdot)\colon L \to L$ determined by $c \leq a \to b$ iff $a \wedge c \leq b$. Hence, every frame is a Heyting algebra (note, however, that a frame homomorphism is not necessarily a Heyting one).

The *pseudocomplement* of $a \in L$ is $a^* = a \to 0$. We will use the standard facts like $a \leq b \Rightarrow b^* \leq a^*$, $a \leq a^{**}$ or $a^{***} = a^*$.

1.1.3. Localic maps are just the infima preserving maps $f\colon L \to M$ (with left adjoint f^*) such that $f[L\smallsetminus\{1\}] \subseteq M\smallsetminus\{1\}$ and $f(f^*(a) \to b) = a \to f(b)$ for every $a \in M$ and $b \in L$ ([22]).

1.1.4. We set $a \prec b$ for $a^* \vee b = 1$. One speaks of a *regular* frame (or locale) if $a = \bigvee\{x \mid x \prec a\}$ for every a.

Further, let $\prec\!\!\prec$ be the largest *interpolative* relation (i.e. such that $a \prec\!\!\prec b \;\Rightarrow\; \exists c \in L\colon a \prec\!\!\prec c \prec\!\!\prec b$) contained in $\prec$. One speaks of a *completely regular* frame (or locale) if $a = \bigvee\{x \mid x \prec\!\!\prec a\}$ for every a.

1.2. Down-set frame. The *down-set frame* $\mathfrak{D}L$ of a frame L is the frame $\{U \subseteq L \mid \emptyset \neq U = {\downarrow}U\}$ (where ${\downarrow}U = \{x \mid x \leq a \in U\}$, as usual) with meets and joins given respectively by intersections and unions. The mapping

$$\lambda' = (a \mapsto {\downarrow}a)\colon L \to \mathfrak{D}L$$

is a localic map (it is the right adjoint to $v = (U \mapsto \bigvee U)\colon \mathfrak{D}L \to L$). Note that for any meet homomorphism $\phi\colon L \to M$ there is a unique frame homomorphism $g\colon \mathfrak{D}L \to M$ such that $g\lambda' = \phi$, given by $g(U) = \bigvee\{\phi(u) \mid u \in U\}$.

1.3. Taking quotients of a frame (saturation). For a relation $R \subseteq L\times L$ on a frame L, call an $s \in L$ *saturated* (more precisely, *R-saturated*) if

$$aRb \quad\Rightarrow\quad \forall c,\; a \wedge c \leq s \;\text{ iff }\; b \wedge c \leq s.$$

A meet of saturated elements is saturated, we have a monotone mapping $\kappa = (x \mapsto \kappa(x)) = \bigwedge\{s \mid x \leq s,\; s \text{ saturated}\}$ satisfying $x \leq \kappa(x)$, $\kappa\kappa(x) = \kappa(x)$ and, moreover, $\kappa(x \wedge y) = \kappa(x) \wedge \kappa(y)$ (the *nucleus* of R), and if we set

$$L/R = \{x \mid x = \kappa(x)\}$$

we obtain a frame homomorphism $\kappa' = (x \mapsto \kappa(x))\colon L \to L/R$ such that

(1.3.1) $xRy \;\Rightarrow\; \kappa'(x) = \kappa'(y)$, and

(1.3.2) if for a frame homomorphism $h\colon L \to K$ one has $xRy \Rightarrow h(x) = h(y)$ then there is precisely one frame homomorphism $\overline{h}\colon L/R \to K$ such that $\overline{h} \cdot \kappa' = h$; moreover, for $x \in L/R$, $\overline{h}(x) = h(x)$

(see e.g. [22]).

1.4. Sublocales. Subspaces of locales (viewed as generalized spaces) are represented by *sublocales*. A subset S of a frame L is a *sublocale* of L if, for any $A \subseteq S$, $x \in L$ and $a \in S$, we have $\bigwedge A \in S$ (in particular, $1 \in S$) and $x \to a \in S$. Sublocales are precisely such subsets for which the embedding map $j\colon S \to L$ is a (one-to-one) localic map (recall 1.1.1 and 1.1.3) — in fact, extremal monomorphisms in **Loc**.

The set $\mathcal{S}(L)$ of all sublocales of L forms a *coframe* (i.e., a complete lattice satisfying the dual of (dist)) under inclusion, in which arbitrary infima coincide with intersections, $\{1\}$ is the bottom element and L is the top element. When S is a sublocale of L we also say that L is a *superlocale* of S.

For more about frames and locales see e.g. [15, 22].

2. Technique of entourages

2.1. Coproduct ideals. Take the Cartesian product $L \times L$ as a poset and consider the frame $\mathfrak{D}(L \times L)$. Call $U \in \mathfrak{D}(L \times L)$ a *coproduct ideal* (shortly, a *cp-ideal*) if

(2.1.1) for any $A \subseteq L$ and $b \in L$, if $A \times \{b\} \subseteq U$ then $(\bigvee A, b) \in U$, and
(2.1.2) for any $a \in L$ and $B \subseteq L$, if $\{a\} \times B \subseteq U$ then $(a, \bigvee B) \in U$.

Note that cp-ideals are precisely the elements in $\mathfrak{D}(L \times L)$ saturated with respect to the relation

$$R = \{(\bigcup_{x \in A} {\downarrow}(x,b), {\downarrow}(\bigvee A, b)), (\bigcup_{y \in B} {\downarrow}(a,y), {\downarrow}(a, \bigvee B)) \mid A, B \subseteq L, a, b \in L\}.$$

The sets A and B in (2.1.1)-(2.1.2) can be void; hence, in particular, each cp-ideal contains as a subset $\mathsf{n} = \{(0,b),(a,0) \mid a,b \in L\}$. It is easy to check that for each $(a,b) \in L \times L$,

$$a \oplus b = {\downarrow}(a,b) \cup \mathsf{n}$$

is a cp-ideal.

Intersections of cp-ideals are obviously cp-ideals and thus we have a complete lattice, which is indeed a frame; however, unions of cp-ideals are not necessarily cp-ideals and their joins (suprema) in this frame are typically bigger.

2.2. Binary products. Let L be a frame. Recall (see e.g. [15, 22]; it should be noted that the first construction of frame coproducts appeared in [9]) that the coproduct $L \oplus L$ in **Frm** (product in **Loc**) is just the frame

$$L \oplus L = \{U \in \mathfrak{D}(L \times L) \mid U \text{ is a cp-ideal}\}$$

with coproduct injections

$$\iota_1 = (a \mapsto a \oplus 1)\colon L \to L \oplus L, \quad \iota_2 = (b \mapsto 1 \oplus b)\colon L \to L \oplus L.$$

Note that for each $U \in L \oplus L$,

$$U = \bigvee\{a \oplus b \mid (a,b) \in U\} = \bigcup\{a \oplus b \mid (a,b) \in U\}.$$

Notation. Using the symbol $L \times L$ when speaking of $L \oplus L$ as a product in the category **Loc** would probably obscure the matter. Therefore, we will keep the notation $L \oplus L$ also in **Loc**. We only have to keep in mind that then the injections ι become projections $p_1, p_2 \colon L \oplus L \to L$.

2.3. Composition of decreasing binary relations. For any $U \in \mathfrak{D}(L \times L)$ let

$$\kappa_0(U) = \{(\bigvee A, b), (a, \bigvee B) \mid A \times \{b\} \subseteq U, \{a\} \times B \subseteq U\}$$

and let $\langle U \rangle = \bigvee\{a \oplus b \mid (a,b) \in U\}$ be the cp-ideal generated by U. Of course, U is a cp-ideal iff $\langle U \rangle = U$ iff $\kappa_0(U) = U$.

For any $U, V \in \mathfrak{D}(L \times L)$, the composite $U \circ V$ is the cp-ideal

$$U \circ V = \bigvee\{a \oplus b \mid \exists c \neq 0, \quad (a,c) \in U \text{ and } (c,b) \in V\},$$

that is, the cp-ideal generated by the down-set

$$U \cdot V = \{(a,b) \mid \exists c \neq 0, \quad (a,c) \in U \text{ and } (c,b) \in V\}.$$

It is easy to check that

$$\kappa_0(U) \cdot V \subseteq \kappa_0(U \cdot V) \tag{$*$}$$

(and, by symmetry, we also have $U \cdot \kappa_0(V) \subseteq \kappa_0(U \cdot V)$).

The following technical lemma plays an important role in the entourage approach to frame uniformities (cf. [19, Lemma 3.1]).

2.3.1. Lemma. *For any $U, V \in \mathfrak{D}(L \times L)$, $\langle U \rangle \circ \langle V \rangle = U \circ V$.*

Proof: It suffices to show that $\langle U \rangle \cdot \langle V \rangle \subseteq \langle U \cdot V \rangle$. For that, consider the non-empty set

$$\mathbb{D} = \{S \in \mathfrak{D}(L \times L) \mid U \subseteq S \subseteq \langle U \rangle, S \cdot V \subseteq \langle U \cdot V \rangle\}.$$

By $(*)$, $S \in \mathbb{D}$ implies $k_0(S) \in \mathbb{D}$. On the other hand, for any non-void subset of $\mathbb{D}$, its union is also in $\mathbb{D}$ and thus $\mathbb{D}$ has a largest element D. But $k_0(D) \in \mathbb{D}$ so $D = k_0(D)$, i.e., D is a cp-ideal. Hence $\langle U \rangle = D \in \mathbb{D}$, that is, $\langle U \rangle \cdot V \subseteq \langle U \cdot V \rangle$, and by symmetry, $U \cdot \langle V \rangle \subseteq \langle U \cdot V \rangle$. Then the inclusion $\langle U \rangle \cdot \langle V \rangle \subseteq \langle U \cdot V \rangle$ follows since $\langle \cdot \rangle$ is an idempotent operator. ■

2.4. Entourages. An *entourage* in a frame L is an element $E \in L \oplus L$ such that

$$\bigvee\{u \mid (u,u) \in E\} = 1.$$

2.4.1. Lemma. *If E is an entourage then $E \le E \circ E$.*

Proof: Let $(a,b) \in E$ with $b \neq 0$. We have

$$b = \bigvee\{b \wedge u \mid (u,u) \in E\} = \bigvee\{b \wedge u \mid (u,u) \in E,\ u \wedge b \neq 0\}.$$

Now we have, for $b \wedge u \neq 0$, $a \oplus (b \wedge u) \le E$ and $(b \wedge u) \oplus (b \wedge u) \le E$. Hence $a \oplus (b \wedge u) \le E \circ E$ and, finally,

$$a \oplus b = a \oplus \bigvee\{b \wedge u \mid (u,u) \in E\} = \bigvee\{a \oplus (b \wedge u) \mid (u,u) \in E\} \le E \circ E.$$

■

2.5. For each $x \in L$ and $E \in L \oplus L$, let

$$E_1 x = \bigvee\{a \in L \mid (a,b) \in E,\ b \wedge x \neq 0\},$$

$$E_2 x = \bigvee\{b \in L \mid (a,b) \in E,\ a \wedge x \neq 0\}$$

and

$$Ex = \bigvee\{a \in L \mid (a,a) \in E,\ a \wedge x \neq 0\}.$$

The following formulas are easy to check (for details see [19, 21]):

(2.5.1) $F_1(E_1 x) \le (F \circ E)_1 x$ and $F_2(E_2 x) \le (E \circ F)_2 x$.
(2.5.2) If E is an entourage then $x \le Ex \le E_1 x \wedge E_2 x$.
(2.5.3) $\langle E \rangle_i x = E_i x$ $(i = 1,2)$.

2.5.1. Lemma. *For each entourage E, we have:*

(a) $Ex^{**} = Ex$ *and* $E_i x^{**} = E_i x$ $(i = 1,2)$.
(b) $(Ex, Ex) \in E^3$ *for any* $(x,x) \in E$.
(c) $(E_1 x, E_2 y) \in E^3$ *for any* $(x,y) \in E$.

Proof: (a) Follows immediately from the fact that $y \wedge x = 0$ iff $y \wedge x^{**} = 0$.

(b) If $(y,y), (x,x), (z,z) \in E$ with $y \wedge x \neq 0$ and $x \wedge z \neq 0$, then $(y,z) \in E^3$.

(c) If $(a,b), (x,y), (c,d) \in E$ with $b \wedge x \neq 0$ and $y \wedge c \neq 0$, then $(a,d) \in E^3$. ■

2.6. For an entourage E set

$$\widetilde{E} = \bigvee\{x^{**} \oplus y^{**} \mid (x, y) \in E\}.$$

Fact. *This is clearly an entourage of* L *and* $\widetilde{E} \leq E^3$.

Indeed, by (a) and (c) above we have, for any $(x, y) \in E$,

$$(x^{**}, y^{**}) \leq (E_1 x^{**}, E_2 y^{**}) = (E_1 x, E_2 y) \in E^3.$$

3. Uniformities and quasi-uniformities

3.1. Entourage uniformities. For an entourage E (resp. a set of entourages $\mathcal{E}$) write

$$b \lhd_E a \ \text{ if } \ E \circ (b \oplus b) \leq a \oplus a, \quad \text{and} \quad b \lhd_{\mathcal{E}} a \ \text{ if } \ \exists E \in \mathcal{E},\ b \lhd_E a.$$

A system of entourages $\mathcal{E}$ is said to be *admissible* if

$$\forall a \in L, \quad a = \bigvee\{b \mid b \lhd_{\mathcal{E}} a\}. \tag{3.1.1}$$

Further, for an entourage E set

$$E^{-1} = \{(a, b) \mid (b, a) \in E\}$$

(which is obviously an entourage again).

3.1.1. Definition. A (*entourage*) *uniformity* [19] on a frame L is an admissible system of entourages $\mathcal{E}$ such that

(E1) if $E \in \mathcal{E}$ and $E \leq F$ then $F \in \mathcal{E}$,
(E2) if $E, F \in \mathcal{E}$ then $E \cap F \in \mathcal{E}$,
(E3) for every $E \in \mathcal{E}$ there is an $F \in \mathcal{E}$ such that $F \circ F \leq E$, and
(E4) if $E \in \mathcal{E}$ then E^{-1} is in $\mathcal{E}$.

A *uniform frame* (or *locale*) is a pair $(L, \mathcal{E})$ where $\mathcal{E}$ is a uniformity on L. In particular, the admissibility condition (3.1.1) implies that L is always regular (in fact, completely regular).

A *basis of a uniformity* $\mathcal{B}$ is a system satisfying just (E2), (E3) and (E4) and a uniformity is then obtained by taking all the $F \geq E$ with $E \in \mathcal{B}$.

3.2. Non-symmetric uniformities. For a system $\mathcal{E}$ of entourages satisfying (E1),(E2) and (E3) define, first, $\overline{\mathcal{E}}$ as the filter of entourages generated by $\mathcal{E} \cup \{E^{-1} \mid E \in \mathcal{E}\}$ (of course, if $\mathcal{E}$ is a uniformity, $\overline{\mathcal{E}} = \mathcal{E}$). If $\overline{\mathcal{E}}$ is admissible we speak of $\mathcal{E}$ as of a *quasi-uniformity* and of $(L, \mathcal{E})$ as of a *quasi-uniform frame* ([20, 21]).

Because of the lack of symmetry condition (E4), the system $\mathcal{E}$ unmodified induces two distinct order relations:

$$x \triangleleft^1_{\mathcal{E}} y \quad \equiv \quad \exists E \in \mathcal{E},\ E \circ (x \oplus x) \leq y \oplus y,$$
$$x \triangleleft^2_{\mathcal{E}} y \quad \equiv \quad \exists E \in \mathcal{E},\ (x \oplus x) \circ E \leq y \oplus y.$$

Note that $x \triangleleft^i_{\mathcal{E}} y$ iff $E_i x \leq y$ for some $E \in \mathcal{E}$ $(i = 1, 2)$.

3.2.1. Notes. (1) Let $\mathcal{E}$ be a quasi-uniformity on a frame L. Then $\mathcal{E}^{-1} = \{E^{-1} \mid E \in \mathcal{E}\}$ is also a quasi-uniformity (the *conjugate quasi-uniformity* on L) and $\overline{\mathcal{E}}$ is the coarsest uniformity containing $\mathcal{E}$.
(2) The two relations $\triangleleft^1_{\mathcal{E}}$ and $\triangleleft^2_{\mathcal{E}}$ yield two subframes of L, namely

$$L_i(\mathcal{E}) = \{x \in L \mid x = \bigvee \{y \in L \mid y \triangleleft^i_{\mathcal{E}} x\}\} \ \ (i = 1, 2).$$

(3) Recall that a *biframe* is a triple (L, L_1, L_2) of frames where L_1 and L_2 are subframes of L and

$$\forall a \in L,\ a = \bigvee \{a_1 \wedge a_2 \mid a_i \in L_i,\ a_1 \wedge a_2 \leq a\}.$$

Biframes are the pointfree counterpart of bitopological spaces ([5]).

The admissibility condition on a quasi-uniform frame $(L, \mathcal{E})$ means precisely that the triple $(L, L_1(\mathcal{E}), L_2(\mathcal{E}))$ is a (completely regular) biframe (see [20, Remark 3.4]). Thus every quasi-uniformity on a frame L induces a natural biframe structure on L, in the same way as every quasi-uniformity on a set X endows X with a bitopological structure ([10]).

3.3. Proposition. *Let $\mathcal{E}$ be a quasi-uniformity on a frame L. If $b_1 \triangleleft^1_{\mathcal{E}} a_1$ and $b_2 \triangleleft^2_{\mathcal{E}} a_2$ then $(b_1 \wedge b_2) \triangleleft_{\overline{\mathcal{E}}} (a_1 \wedge a_2)$.*

Proof: If $E \circ (b_1 \oplus b_1) \leq a_1 \oplus a_1$ and $(b_2 \oplus b_2) \circ F \leq a_2 \oplus a_2$ then $F^{-1} \circ (b_2 \oplus b_2) \leq a_2 \oplus a_2$. Hence $(E \cap F)^{-1} \circ (b_1 \wedge b_2 \oplus b_1 \wedge b_2) \leq a_1 \wedge a_2 \oplus a_1 \wedge a_2$. ■

3.4. Let $\mathcal{E}$ be a quasi-uniformity on L. For $E \in \mathcal{E}$ set

$$E^{\triangleleft} = \bigvee \{x \oplus y \mid x \triangleleft^1_{\mathcal{E}} a,\ y \triangleleft^2_{\mathcal{E}} b,\ (a, b) \in E\} \leq E.$$

3.4.1. Proposition. *For any F such that $F^3 \leq E$, $F \leq E^{\triangleleft}$. Hence $E^{\triangleleft} \in \mathcal{E}$ and $\{E^{\triangleleft} \mid E \in \mathcal{E}\}$ is a basis for $\mathcal{E}$.*

Proof: Let $(x,y) \in F$. By 2.5(c), $(F_1x, F_2y) \in F^3 \le E$, and, of course, $x \lhd^1_{\mathcal{E}} F_1x$ as well as $y \lhd^2_{\mathcal{E}} F_2y$. ■

3.5. The fact in 2.6 provides an alternative basis for $\mathcal{E}$:

3.5.1. Proposition. *Let $\mathcal{E}$ be a quasi-uniformity on L. Then $\{\widetilde{E} \mid E \in \mathcal{E}\}$ is a basis of $\mathcal{E}$.* ■

3.6. Product maps. By the universal property of products, for each localic map $f\colon L \to M$ there is the (unique) localic map

$$f \times f\colon L \oplus L \to M \oplus M$$

such that $p_i(f \times f) = fp_i$ for $i = 1, 2$. Its left adjoint $(f \times f)^*$ in **Frm** is the unique frame homomorphism

$$h \oplus h\colon M \oplus M \to L \oplus L$$

such that $(h \oplus h)\iota_i = \iota_i h$ (where $h = f^*$). Clearly,

$$(h \oplus h)(E) = \bigvee_{(a,b)\in E} (h(a) \oplus h(b)). \tag{3.6.1}$$

Furthermore, for every $E \in L \oplus L$,

$$(h \oplus h)(E^{-1}) = (h \oplus h)(E)^{-1}. \tag{3.6.2}$$

Indeed, we have $(h \oplus h)(E^{-1}) = \bigvee\{h(a) \oplus h(b) \mid (b,a) \in E\}$ while

$$(h \oplus h)(E)^{-1} = \bigvee\{x \oplus y \mid (y,x) \in (h \oplus h)(E)\}$$

and, clearly, $(h \oplus h)(E^{-1}) \le (h \oplus h)(E)^{-1}$. Regarding the reverse inclusion, it is equivalent to

$$(h \oplus h)(E) \le (h \oplus h)(E^{-1})^{-1}$$

and this follows immediately from the obvious

$$\bigcup_{(a,b)\in E} (h(a) \oplus h(b)) \le (h \oplus h)(E^{-1})^{-1}.$$

3.6.1. Uniform maps. If $(L,\mathcal{E})$ and $(M,\mathcal{F})$ are quasi-uniform locales and $f\colon L \to M$ is a localic map satisfying

$$\forall F \in \mathcal{F}, \quad (f \times f)^*(F) \in \mathcal{E}.$$

then we speak of a *uniform map* $f\colon (L,\mathcal{E}) \to (M,\mathcal{F})$ (and the corresponding left adjoint $h = f^*\colon (M,\mathcal{F}) \to (L,\mathcal{E})$ is called a *uniform frame homomorphism*). We then have the category **QULoc** of quasi-uniform locales and uniform maps and its full subcategory **ULoc** of uniform locales.

3.6.2. Uniform embeddings. Let $j\colon L \to M$ be an embedding in **Loc** with adjoint onto frame homomorphism $h = j^*\colon M \to L$. Since regular monomorphisms are preserved by products, $j \times j$ is also one-to-one.

By a *uniform embedding* $j\colon (L, \mathcal{E}) \to (M, \mathcal{F})$ we mean a one-to-one uniform map such that

$$\mathcal{E} = \{(j \times j)^*(F) \mid F \in \mathcal{F}\}.$$

4. Dense uniform embeddings

4.1. Dense maps and dense sublocales. Recall that a frame homomorphism $h\colon L \to M$ is *dense* if

$$h(x) = 0 \;\Rightarrow\; x = 0,$$

and that a sublocale $S \subseteq L$ is *dense* if

$$\overline{S} = L, \text{ that is, } 0 \in S.$$

Consider the localic map $f = h_*\colon M \to L$ associated with h.

4.1.1. Proposition. *The following are equivalent:*

(i) *h is dense.*
(ii) *$f(0) = 0$.*
(iii) *$f[M]$ is dense in L.*

Proof: (a)⇔(b): $x \le 0$ iff $h(x) \le 0$ iff $x \le f(0)$ makes $f(0) = 0$.
(b)⇔(c) is trivial. ■

4.1.2. Regarding uniform maps $f\colon (L, \mathcal{E}) \to (M, \mathcal{F})$ we will use the fact that f is epic if it is dense, a consequence of the regularity of the frames involved ([22, §V.6.5]).

4.2. Proposition (Banaschewski [2])**.** *Let $S \subseteq L$ be a dense sublocale of a regular locale. Then S join-generates L.*

In particular, if for $a, b \in L$, $\{x \mid x \in S, x \le a\} = \{x \mid x \in S, x \le b\}$, then $a = b$.

Proof: Since $x \prec a$ implies $x^{**} \prec a$ ($1 = x^* \vee a = x^{***} \vee a$), a regular frame is generated by its Booleanization $\mathfrak{B}L = \{x^{**} \mid x \in L\}$, contained in every dense sublocale by Isbell's density theorem ([22, §III.8.3]. ■

4.3. From now on we will consider an embedding $j\colon L \to M$ with adjoint onto frame homomorphism $h = j^*\colon M \to L$. Since j is one-to-one, we infer from $jhj = j$ that $hj = \mathrm{id}$. In case j is dense (by the density we mean, of course, that h is dense, that is, by 4.1.1, that $j(0) = 0$) then $j(x) = 0$ iff $x = 0$.

4.3.1. Proposition. *For each $E \in L \oplus L$,*

$$(j \times j)(E) = \bigvee\{j(a)\oplus j(b) \mid (a,b) \in E\}.$$

Proof: Let $E \in L \oplus L$. We have

$$(j \times j)(E) = \bigvee\{F \in M \oplus M \mid (h \oplus h)(F) \leq E\}.$$

The latter is equal to $\bigvee\{j(a)\oplus j(b) \mid (a,b) \in E\}$. Indeed:

$\geq$:

$$(h \oplus h)(\bigvee_{(a,b)\in E}(j(a)\oplus j(b)) = \bigvee_{(a,b)\in E}(hj(a)\oplus hj(b)) = \bigvee_{(a,b)\in E}(a\oplus b) = E$$

since $hj = \mathrm{id}$.

$\leq$: $(h\oplus h)(F) \leq E$ iff $\bigvee_{(x,y)\in F}(h(x)\oplus h(y)) \leq E$ and the latter clearly implies that $F \leq \bigvee_{(a,b)\in E}(j(a)\oplus j(b))$ since

$$(x,y) \in F \Rightarrow (h(x),h(y)) \in E \Rightarrow (jh(x),jh(y)) \in \bigvee_{(a,b)\in E}(j(a)\oplus j(b))$$

and $(x,y) \leq (jh(x),jh(y))$. ■

Since $hj = \mathrm{id}$, we have immediately:

4.3.2. Corollary. *For each $E \in L \oplus L$, $(h \oplus h)(j \times j)(E) = E$.* ■

4.4. If j is moreover dense, then $jh(x) \leq x^{**}$ (for $h(jh(x) \wedge x^*) = h(x) \wedge h(x^*) = h(0) = 0$ implies $jh(x) \wedge x^* = 0$), and therefore

$$\bigcup_{(a,b)\in E}(jh(a)\oplus jh(b)) \leq \widetilde{E}.$$

Hence:

4.4.1. Corollary. *Let $j\colon L \to M$ be a dense embedding. For each $F \in M \oplus M$, $F \leq (j \times j)(h \oplus h)(F) \leq \widetilde{F}$.* ■

4.5. Proposition. *Let $j\colon (L,\mathcal{E}) \to (M,\mathcal{F})$ be a uniform map with left adjoint $h = j^*$. Then j is a dense uniform embedding if and only if*

$$\{(j\times j)(E) \mid E \in \mathcal{E}\} \text{ is a basis of } \mathcal{F} \tag{1}$$

and

$$(h\oplus h)(j\times j)(E) = E \text{ for every } E \in L\oplus L. \tag{2}$$

Proof: $\Rightarrow$: By definition of a uniform embedding,

$$\mathcal{E} = \{(j\times j)^*(F) = (h\oplus h)(F) \mid F \in \mathcal{F}\}$$

and hence $(j\times j)(E) = (j\times j)(h\oplus h)(F) \geq F$ for some $F\in\mathcal{F}$, which shows that every $(j\times j)(E)$ is an entourage in $\mathcal{F}$. Now for each $F\in\mathcal{F}$ take $G\in\mathcal{F}$ such that $\widetilde{G}\leq F$ (recall Proposition 2.2). By 4.4.1,

$$G \leq (j\times j)(h\oplus h)(G) \leq \widetilde{G} \leq F.$$

This proves condition (1) while (2) was already proved in 4.3.2.
$\Leftarrow$: We need to show that $\mathcal{E} = \{(h\oplus h)(F) \mid F\in\mathcal{F}\}$. If $E\in\mathcal{E}$ then $E = (h\oplus h)(j\times j)(E)$ by (2), and $(j\times j)(E)\in\mathcal{F}$ by (1). Conversely, condition (1) ensures that each $(h\oplus h)(F)$ contains some $(h\oplus h)(j\times j)(E) = E\in\mathcal{E}$. ■

4.6. Proposition. *Let $j\colon (L,\mathcal{E}) \to (M,\mathcal{F})$ be a dense uniform embedding. Then, for every $x,y\in L$, we have:*

(1) $x \lhd^i_{\mathcal{E}} y$ iff $j(x) \lhd^i_{\mathcal{F}} j(y)$ ($i=1,2$).
(2) $x \lhd_{\overline{\mathcal{E}}} y$ iff $j(x) \lhd_{\overline{\mathcal{F}}} j(y)$.

Proof: (1) $\Rightarrow$: If $x\lhd^1_{\mathcal{E}} y$ then there is some $E\in\mathcal{E}$ such that $E_1x\leq y$ (recall 2.5). Take $F=(j\times j)(E)\in\mathcal{F}$ (by the previous proposition). By (2.5.3),

$$F_1 j(x) = \bigvee\{a \mid (a,b)\in \bigcup_{(a',b')\in E} (j(a')\oplus j(b')),\ b\wedge j(x)\neq 0\}$$

and therefore for each such (a,b) we have $(a,b)\leq (j(a'),j(b'))$ for some $(a',b')\in E$. Then $0\neq b\wedge j(x)\leq j(b')\wedge j(x) = j(b'\wedge x)$ implies $b'\wedge x\neq 0$ and thus $a'\leq E_1x\leq y$ and hence $a\leq j(a')\leq j(y)$. In conclusion, $F_1j(x)\leq j(y)$.
$\Leftarrow$: If $j(x)\lhd^1_{\mathcal{F}} j(y)$ apply the preceding proposition to get some $E\in\mathcal{E}$ such that

$$((j\times j)(E))_i j(x) \leq j(y).$$

Then if $(a,b)\in E$ with $b\wedge x\neq 0$ we have $(j(a),j(b))\in (j\times j)(E)$ and $j(b)\wedge j(x)\neq 0$, hence $j(a)\leq j(y)$, and since j is an embedding, $a\leq y$. Hence $x\lhd^1_{\mathcal{E}} y$.

(2) can be proved similarly. ■

4.7. Let $j\colon L\to M$ be a dense embedding. Using the formula in 4.3.1, a proof similar to the one for frame homomorphisms in (3.6.2) gives us the analogous property for $j\times j$:

$$\text{for every } E\in L\oplus L,\ (j\times j)(E^{-1})=(j\times j)(E)^{-1}.$$

It follows that for any dense uniform embedding $j\colon(L,\mathcal{E})\to(M,\mathcal{F})$,

$$(j\times j)(E_1\cap E_2^{-1})=(j\times j)(E_1)\cap(j\times j)(E_2)^{-1},$$

and thus that

$$(j\times j)(E)\in\overline{\mathcal{F}} \text{ for every } E\in\overline{\mathcal{E}}. \tag{4.7.1}$$

4.7.1. Proposition. *If $j\colon(L,\mathcal{E})\to(M,\mathcal{F})$ is a dense uniform embedding, then $j\colon(L,\overline{\mathcal{E}})\to(M,\overline{\mathcal{F}})$ is a dense uniform embedding.*

Proof: We need to show that for $h=j^*$,

$$\{(h\oplus h)(F)\mid F\in\overline{\mathcal{F}}\}=\overline{\mathcal{E}}.$$

$\subseteq$: For each $F\in\overline{\mathcal{F}}$ let $G_1,G_2\in\mathcal{F}$ such that $G_1\cap G_2^{-1}\subseteq F$. Then, using (3.6.2), we get

$$(h\oplus h)(F)\supseteq(h\oplus h)(G_1\cap G_2^{-1})=(h\oplus h)(G_1)\cap(h\oplus h)(G_2)^{-1}\in\overline{\mathcal{E}}.$$

$\supseteq$: Let $E\in\overline{\mathcal{E}}$. By 4.3.2, $E=(h\oplus h)(j\times j)(E)$, where $(j\times j)(E)\in\overline{\mathcal{F}}$ by (4.7.1). ■

5. Cauchy maps, and a universal one

5.1. Cauchy and weakly Cauchy maps. Let $(L,\mathcal{E})$ be a quasi-uniform frame (or, more particularly, a uniform frame) and let M be a frame. A *Cauchy map* $\phi\colon(L,\mathcal{E})\to M$ is a bounded meet homomorphism $\phi\colon L\to M$ such that

(C1) for all $E\in\mathcal{E}$, $\bigvee\{\phi(a)\mid(a,a)\in E\}=1$, and
(C2) for each $a\in L$, $\phi(a)=\bigvee\{\phi(b)\mid b\lhd_{\overline{\mathcal{E}}}a\}$.

If we do not require (C2) we speak of a *weak Cauchy map*.

5.1.1. Notes. (1) Recall the *Cauchy points* from classical completions of uniform spaces $(X,\mathcal{E})$, that is, filters F in $L=\Omega(X)$ such that

$$F\cap\{a\mid(a,a)\in E\}\neq\emptyset^1,$$

[1] Recall 2.4; the sets $\{a\in L\mid(a,a)\in E\}$ are the covers replacing entourages in the covering description of a uniformity, see e.g. [21].

for all $E \in \mathcal{E}$, and for every $a \in F$ there is a $b \in F$ such that $b \lhd_{\mathcal{E}} a$.

Taking their characteristic maps we have Cauchy points represented as bounded meet semilattice homomorphisms $\phi\colon (L, \mathcal{E}) \to \mathbf{2} = \{0, 1\}$ such that

- for all $E \in \mathcal{E}$, $\bigvee\{\phi(a) \mid (a, a) \in E\} = 1$, and
- for every $a \in L$, $\phi(a) = \bigvee\{\phi(b) \mid b \lhd_{\mathcal{E}} a\}$.

Thus, Cauchy maps are Cauchy points with a general frame M instead of $\mathbf{2}$.
(2) In the standard terminology one speaks of *regular Cauchy maps*, and the term "Cauchy map" is preserved for what we call "weak Cauchy map". We have decided to simplify the terminology using the shorter expression for the fundamental concept, and the longer one for the technical modification.

5.2. Observations. (1) If ϕ is Cauchy and h a frame homomorphism then $h\phi$ is Cauchy.
(2) If ϕ is weakly Cauchy and h a uniform frame homomorphism then ϕh is weakly Cauchy.

5.2.1. Note. The statement (2) does not hold for Cauchy maps. An example will be given in 6.5.1.

5.3. $\mathbf{C}(L, \mathcal{E})$ and the mapping $\lambda\colon (L, \mathcal{E}) \to \mathbf{C}(L, \mathcal{E})$. Recall the mapping $\lambda' = (a \mapsto {\downarrow}a)\colon L \to \mathfrak{D}(L)$ and the fact that a meet homomorphism $\phi\colon L \to M$ is uniquely extendable to a frame homomorphism $g\colon \mathfrak{D}(L) \to M$ such that $g\lambda' = \phi$, namely the one given by $g(U) = \bigvee\{\phi(u) \mid u \in U\}$.

Defining on $\mathfrak{D}(L)$ the relation

$$R_{\mathcal{E}} = \{(\bigcup\{{\downarrow}x \mid x \lhd_{\overline{\mathcal{E}}} y\}, {\downarrow}y), (\bigcup\{{\downarrow}a \mid (a, a) \in E\}, L) \mid y \in L, E \in \mathcal{E}\}$$

we obtain a frame

$$\mathbf{C}(L, \mathcal{E}) = \mathfrak{D}(L)/R_{\mathcal{E}}.$$

It is easy to check that the saturated sets constituting this frame (they will be referred to as *Cauchy ideals* in $(L, \mathcal{E})$) are the $U \in \mathfrak{D}(L)$ such that

(R1) if $\{x \mid x \lhd_{\overline{\mathcal{E}}} y\} \subseteq U$ then $y \in U$, and
(R2) if for some $E \in \mathcal{E}$, $E \wedge \{x\} = \{a \wedge x \mid (a, a) \in E\} \subseteq U$, then $x \in U$.

The restriction of λ' to a mapping $L \to \mathbf{C}(L, \mathcal{E})$ will be denoted by $\lambda_{(L,\mathcal{E})}$.

5.3.1. Note. $\mathbf{C}(L, \mathcal{E})$ was used in several texts (e.g., [3, 2, 4, 22]) (typically for the cover uniformities) as an underlying frame of the completion of $(L, \mathcal{E})$.

In fact it is a special case of a general construction of a frame extension of a site (see [1]). A *site* is a meet semilattice (here a frame) together with a *covering*, a subset $\sqsubseteq$ of $L \times \mathfrak{D}(L)$; $(x, U) \in \sqsubseteq$ means that we want to force x to become a supremum of U. In our case the covering is constituted by the couples

$$(\downarrow y, \{\downarrow x \mid x \lhd_{\overline{\mathcal{E}}} y\}),\ y \in L, \quad \text{and} \quad (L, \{\downarrow a \mid (a,a) \in E\}),\ E \in \mathcal{E}.$$

This site covering makes in a canonical way $\lambda\colon (L, \mathcal{E}) \to \mathbf{C}(L, \mathcal{E})$ a Cauchy map, and the following very easy theorem comes as no surprise. In the next section we will then show that a completion procedure, symmetric or not, can be obtained as an easy consequence.

5.4. Theorem. *$\lambda_{(L,\mathcal{E})}$ is a Cauchy map and it is a universal Cauchy map in the sense that for every Cauchy map $\phi\colon (L, \mathcal{E}) \to M$ there is precisely one frame homomorphism $h\colon \mathbf{C}(L, \mathcal{E}) \to M$ such that the diagram*

$$\begin{tikzcd}
(L,\mathcal{E}) \arrow[r, "\lambda_{(L,\mathcal{E})}"] \arrow[dr, "\phi"'] & \mathbf{C}(L,\mathcal{E}) \arrow[d, "h"] \\
& M
\end{tikzcd}$$

commutes.

Proof: $\lambda_{(L,\mathcal{E})}$ is obviously Cauchy by the definition of the relation $R_{\mathcal{E}}$ above.

Now let $\phi\colon (L, \mathcal{E}) \to M$ be Cauchy. It is a bounded meet homomorphism and hence we have the frame homomorphism $g\colon \mathfrak{D}(L) \to M$ as in 5.3. We have

$$g(\bigcup\{\downarrow x \mid x \lhd_{\overline{\mathcal{E}}} y\}) = \bigvee\{\phi(x) \mid x \lhd_{\overline{\mathcal{E}}} y\} = \phi(y) = g(\downarrow y), \quad \text{and}$$
$$\text{for } E \in \mathcal{E}, \quad g(\bigcup\{\downarrow a \mid (a,a) \in E\}) = \bigvee\{\phi(a) \mid (a,a) \in E\} = 1,$$

hence g respects the relation $R_{\mathcal{E}}$ and restricts to a frame homomorphism $h\colon \mathbf{C}(L, \mathcal{E}) \to M$. ■

5.5. Proposition. *Each dense uniform embedding $j\colon (L, \mathcal{E}) \subseteq (M, \mathcal{F})$ is a Cauchy map.*

Proof: (C1) is obvious. For (C2) take an $a \in L$. We have, by 4.6,

$$\begin{aligned}
j(a) &= \bigvee\{b \in M \mid b \lhd_{\overline{\mathcal{F}}} a = j(a)\} \\
&= \bigvee\{j(x) \in M \mid j(x) \lhd_{\overline{\mathcal{F}}} j(a) = a\} \\
&= \bigvee\{j(x) \in M \mid x \lhd_{\overline{\mathcal{E}}} a\}. \quad ■
\end{aligned}$$

6. Applications: Completion

6.1. Completions. A (quasi-)uniform locale $(L, \mathcal{E})$ is said to be *complete* if every dense uniform embedding with domain $(L, \mathcal{E})$ is an isomorphism.

Let $(L, \mathcal{E})$ and $(M, \mathcal{F})$ be (quasi-)uniform locales with $(M, \mathcal{F})$ complete. Then $(M, \mathcal{F})$ is a *completion* of $(L, \mathcal{E})$ provided that there is a dense uniform embedding

$$j\colon (L, \mathcal{E}) \to (M, \mathcal{F}).$$

6.2. (Quasi-)Uniform structure on $\mathbf{C}(L, \mathcal{E})$. Entourages will allow us to construct the completion for uniform frames and quasi-uniform frames at once.

For a Cauchy map $\phi\colon (L, \mathcal{E}) \to M$ define $\phi \times \phi\colon L \oplus L \to M \oplus M$ by setting

$$(\phi \times \phi)(E) = \bigvee \{\phi(a) \oplus \phi(b) \mid (a, b) \in E\}.$$

6.2.1. Proposition. *Let $\phi\colon (L, \mathcal{E}) \to M$ be a Cauchy map. If $\mathcal{E}$ is a quasi-uniformity (resp. uniformity), then*

$$\{(\phi \times \phi)(E) \mid E \in \mathcal{E}\}$$

is a basis for a quasi-uniformity (resp. uniformity) on M.

Proof: The fact that each $(\phi \times \phi)(E)$ is an entourage of M follows from (C1) while the admissibility condition follows from (C2). Condition (E2) follows easily from the fact that ϕ is a meet homomorphism and, in the symmetric case, (E4) is a consequence of the obvious formula $((\phi \times \phi)(E))^{-1} = (\phi \times \phi)(E^{-1})$. Finally, in order to check condition (E3) it suffices to prove the following:

$$F \circ F \leq E \;\Rightarrow\; (\phi \times \phi)(F) \circ (\phi \times \phi)(F) \leq (\phi \times \phi)(E).$$

By 2.3, $(\phi \times \phi)(F) \circ (\phi \times \phi)(F)$ is equal to

$$\bigcup \{\phi(a) \oplus \phi(b) \mid (a, b) \in F\} \circ \bigcup \{\phi(a) \oplus \phi(b) \mid (a, b) \in F\}.$$

So let $(a, b), (c, d) \in F$ with $\phi(b) \wedge \phi(c) \neq 0$. Then $b \wedge c \neq 0$ and therefore $(a, d) \in F \circ F \leq E$. Hence $(\phi(a), \phi(d)) \in (\phi \times \phi)(E)$. ■

The frame $\mathbf{C}(L, \mathcal{E})$ will be endowed with the quasi-uniformity (resp. uniformity) $\mathbf{C}(\mathcal{E})$ induced by $\mathcal{E}$ via $\lambda_{(L,\mathcal{E})}$, using the preceding result.

6.3. Theorem. *Let $(L,\mathcal{E}),(M,\mathcal{F})$ be quasi-uniform (resp. uniform) locales. For every dense uniform embedding $j\colon (L,\mathcal{E}) \to (M,\mathcal{F})$ there is a dense quasi-uniform (resp. uniform) sublocale*

$$d\colon S(M,\mathcal{F}) \subseteq \mathbf{C}(L,\mathcal{E})$$

and an isomorphism $\alpha\colon (M,\mathcal{F}) \cong S(M,\mathcal{F})$ such that $\lambda_{(L,\mathcal{E})} = d\alpha j$.

Proof: By 5.4 and 5.5 we have a frame homomorphism $h\colon \mathbf{C}(L,\mathcal{E}) \to M$ such that $h\lambda = j$. By 4.2, M is join-generated by L. Hence we have, for an arbitrary $y \in M$, $y = \bigvee\{x \in L \mid x \le y\}$ so that $y = h(\bigvee\{\downarrow x \mid x \in L, x \le y\}$, that is, h is onto. Consequently the right adjoint $f = h_*$ is one-to-one, it is a dense embedding and we can set $S(M,\mathcal{F}) = f[M]$ and endow it with the (quasi-)uniformity transferred from M. ∎

6.3.1. Remark: The coframe of dense embeddings. Note that if two dense uniform embeddings $j_i\colon (L,\mathcal{E}) \to (M_i,\mathcal{F}_i)$ are equivalent, that is, if there is an isomorphism $\iota\colon M_1 \to M_2$ such that $\iota j_1 = j_2$, then $S(M_1,\mathcal{F}_1) = S(M_2,\mathcal{F}_2)$. Furthermore, there is the smallest dense uniform embedding of $(L,\mathcal{E})$, namely the identity, and each embedding $\lambda[L] \subseteq S$ with S a sublocale of $\mathbf{C}(L,\mathcal{E})$ is dense. Thus, the construction $S(-)$ provides a one-to-one correspondence between the system of all (equivalence classes of) uniform embeddings $j\colon (L,\mathcal{E}) \to (M,\mathcal{F})$ and the lattice of all sublocales of $\mathbf{C}(L,\mathcal{E})$ containing $\lambda[L]$.

Consequently, the system of all (equivalence classes of) uniform dense embeddings of a fixed uniform frame $(L,\mathcal{E})$ naturally constitutes a sub-coframe of $\mathbf{C}(L,\mathcal{E})$.

To have the statement quite correct, we have to complete the theorem above by the following simple observation that shows that the sublocale $S(M,\mathcal{F})$ is uniquely determined. Consider the following diagram in **Loc**

$$\begin{tikzcd}
& S_1 \arrow[dr, "j_1=\subseteq"] \arrow[dd, shift left, "\alpha"] & \\
S \arrow[ur, "k_1=\subseteq"] \arrow[dr, "k_2=\subseteq"'] & & L \\
& S_2 \arrow[ur, "j_2=\subseteq"'] \arrow[uu, shift left, "\beta"] &
\end{tikzcd}$$

with j_i, k_i dense embeddings, α,β mutually inverse isomorphisms, $k_2\alpha = k_1$, and $k_1\beta = k_2$. Then $j_1\beta k_2 = j_1 k_1 = j_2 k_2$ and since k_2, being dense, is an epimorphism in **Loc**, $j_1\beta = j_2$ is an actual embedding, we have β an actual embedding, and similarly for α; hence α, β are identities, and $S_1 = S_2$.

6.4. Proposition. *Let $(L, \mathcal{E})$ be a quasi-uniform frame, resp. a uniform frame. In each case, $\mathbf{C}(L, \mathcal{E})$ is complete, and hence a completion of $(L, \mathcal{E})$.*

Proof: Let $k\colon \mathbf{C}(L, \mathcal{E}) \to (M, \mathcal{F})$ be a dense uniform embedding. Consider the diagram

$$\begin{tikzcd}
(L,\mathcal{E}) \arrow[r, "\lambda_{(L,\mathcal{E})}"] \arrow[rr, bend left, "j"] & \mathbf{C}(L,\mathcal{E}) \arrow[r, "k"] & (M,\mathcal{F}) \arrow[dl, "\alpha"] \\
 & S(M,\mathcal{F}) \arrow[u, "d"] &
\end{tikzcd}$$

with $j = k\lambda_{(L,\mathcal{E})}$ and α and d given by 6.3. Then $\lambda = \lambda_{(L,\mathcal{E})} = d\alpha j = d\alpha k\lambda$. Since λ and k are epimorphisms, this yields $d\alpha k = \mathrm{id}_{\mathbf{C}(L,\mathcal{E})}$ which makes k an isomorphism. ■

The symmetric case in the following result is due to B. Banaschewski.

6.5. Theorem. *Let $(L, \mathcal{E})$ be a quasi-uniform frame, resp. a uniform frame. Then $(L, \mathcal{E})$ is complete iff each Cauchy map $(L, \mathcal{E}) \to M$ is a frame homomorphism.*

Proof: If $(L, \mathcal{E})$ is complete then λ is an isomorphism, and hence we have $\phi = h\lambda$ a homomorphism. On the other hand, if each Cauchy map $(L, \mathcal{E}) \to M$ is a frame homomorphism, then in particular $\lambda\colon (L, \mathcal{E}) \to \mathbf{C}(L, \mathcal{E})$ is one. For the $v = v_{(L,\mathcal{E})}$ we have $v\lambda = \mathrm{id}$ and since v, being dense, is a monomorphism, we also have $\lambda v = \mathrm{id}$, and λ is an isomorphism. ■

6.5.1. Example. Let $L = (L, \mathcal{E})$ be a complete uniform frame which is not Boolean, and let $B = \mathfrak{B}(L, \mathcal{E})$ be its Booleanization endowed with the uniformity induced by $\mathcal{E}$. Since the right adjoint of the frame homomorphism $h = (a \mapsto a^{**})\colon L \to B$ is a uniform dense embedding and not an isomorphism, B is not complete. Hence there exists a Cauchy map $\phi\colon B \to M$ which is not a homomorphism. Then $\phi \cdot h$ is not Cauchy, because it is not a homomorphism (L is complete) — if it were, ϕ would be one as well, as h is onto.

6.6. Lemma. *Let $(L,\mathcal{E})$ be a quasi-uniform frame. For a weakly Cauchy map $\phi\colon (L,\mathcal{E})\to M$ define*

$$\phi^\circ\colon (L,\mathcal{E})\to M$$

by setting $\phi^\circ(a)=\bigvee\{\phi(b)\mid b\lhd_{\overline{\mathcal{E}}}a\}$. Then ϕ° is a Cauchy map.

Proof: If $E\in\mathcal{E}$ consider the $E^\lhd$ from 3.4. Then

$$\bigvee\{\phi^\circ(a)\mid (a,a)\in E\}=\bigvee\{\phi(b)\mid b\lhd_{\overline{\mathcal{E}}}a,(a,a)\in E\}$$

and, using the property in 3.3,

$$\bigvee\{\phi(b)\mid b\lhd_{\overline{\mathcal{E}}}a,(a,a)\in E\}\geq\bigvee\{\phi(b)\mid (b,b)\in E^\lhd\}=1$$

(indeed, for any $(b,b)\in E^\lhd$ with $b\lhd^1_{\mathcal{E}}x$, $b\lhd^2_{\mathcal{E}}y$ and $(x,y)\in E$, 3.3 yields $b\lhd_{\overline{\mathcal{E}}}x\wedge y$ and, of course, $(x\wedge y,x\wedge y)\in E$).

Next,

$$\bigvee\{\phi^\circ(b)\mid b\lhd_{\overline{\mathcal{E}}}a\}=\bigvee\{\phi(c)\mid c\lhd_{\overline{\mathcal{E}}}b\lhd_{\overline{\mathcal{E}}}a\}=\bigvee\{\phi(c)\mid c\lhd_{\overline{\mathcal{E}}}a\}=\phi^\circ(a)$$

since $\lhd_{\overline{\mathcal{E}}}$ interpolates. ∎

In particular, when $(L,\mathcal{E})$ is a uniform frame, the Cauchy map ϕ° is given by

$$\phi^\circ(a)=\bigvee\{\phi(b)\mid b\lhd_{\mathcal{E}}a\}.$$

6.7. Theorem. *The completion $\mathbf{C}$ is functorial in both symmetric and non-symmetric cases.*

Proof: For a uniform frame homomorphism $h\colon (L,\mathcal{E})\to(M,\mathcal{F})$ consider the weak Cauchy map $\lambda\cdot h\colon (L,\mathcal{E})\to\mathbf{C}(M,\mathcal{F})$ and the Cauchy map

$$(\lambda\cdot h)^\circ\colon (L,\mathcal{E})\to\mathbf{C}(M,\mathcal{F}).$$

Then, we have the frame homomorphism $g\colon\mathbf{C}(L,\mathcal{E})\to\mathbf{C}(M,\mathcal{F})$ defined by

$$g(U)=\bigvee\{(\lambda h)^\circ(u)\mid u\in U\}=\bigvee\{{\downarrow}h(x)\mid x\lhd_{\overline{\mathcal{E}}}u\in U\}.$$

Denoting this g by $\mathbf{C}(h)$ we obtain that

$$\begin{aligned} v_{(M,\mathcal{F})}\cdot\mathbf{C}(h) &= \bigvee\{v({\downarrow}h(x))\mid x\lhd_{\overline{\mathcal{E}}}u\in U\}=\\ &=\bigvee\{h(x)\mid x\lhd_{\overline{\mathcal{E}}}u\in U\}=\\ &=\bigvee\{h(u)\mid u\in U\}=\\ &=h(\bigvee\{u\mid u\in U\})=h\cdot v_{(L,\mathcal{E})}.\end{aligned}$$

Since the homomorphisms $v_{(L,\mathcal{E})}$ are dense and hence monomorphic in our category, it immediately follows that $\mathbf{C}(h_1h_2)=\mathbf{C}(h_1)\mathbf{C}(h_2)$.

Checking that $\mathbf{C}(h) = g$ is uniform is straightforward. ■

7. More on the asymmetric case

7.1. Let $(L, \mathcal{E})$ be a quasi-uniform frame. Consider the associated uniformity $\overline{\mathcal{E}}$ on L and the corresponding completion

$$\lambda = \lambda_{(L,\overline{\mathcal{E}})} \colon (L, \overline{\mathcal{E}}) \to (\mathbf{C}(L, \overline{\mathcal{E}}), \mathbf{C}(\overline{\mathcal{E}})).$$

By 6.2.1,

$$\{(\lambda \times \lambda)(E) \mid E \in \mathcal{E}\}$$

forms a basis for a quasi-uniformity $\mathbf{C}(\mathcal{E})$ on $\mathbf{C}(L, \overline{\mathcal{E}})$ and thus we have a quasi-uniform frame

$$(\mathbf{C}(L, \overline{\mathcal{E}}), \mathbf{C}(\mathcal{E}))$$

also defined in the sublocale $\mathbf{C}(L, \overline{\mathcal{E}})$ of $\mathfrak{D}L$.

7.1.1. Proposition. $\overline{\mathbf{C}(\mathcal{E})} = \mathbf{C}(\overline{\mathcal{E}})$.

Proof: $\overline{\mathbf{C}(\mathcal{E})} \subseteq \mathbf{C}(\overline{\mathcal{E}})$ follows from the fact that $\mathbf{C}(\overline{\mathcal{E}})$ is a uniformity and $\mathbf{C}(\mathcal{E}) \subseteq \mathbf{C}(\overline{\mathcal{E}})$ (since $\{(\lambda \times \lambda)(E) \mid E \in \overline{\mathcal{E}}\}$ is a basis of $\mathbf{C}(\overline{\mathcal{E}})$, by 4.5).

On the other hand, by the characterization in 4.5,

$$\lambda \colon (L, \mathcal{E}) \to ((\mathbf{C}(L, \overline{\mathcal{E}}), \mathbf{C}(\mathcal{E}))$$

is clearly a dense uniform embedding. Hence, by 4.7.1,

$$\lambda \colon (L, \overline{\mathcal{E}}) \to (\mathbf{C}(L, \overline{\mathcal{E}}), \overline{\mathbf{C}(\mathcal{E})})$$

is a dense uniform embedding. Since $(\mathbf{C}(L, \overline{\mathcal{E}}), \mathbf{C}(\overline{\mathcal{E}}))$ is the completion of $(L, \overline{\mathcal{E}})$, this implies immediately that $\mathbf{C}(\overline{\mathcal{E}}) = \overline{\mathbf{C}(\mathcal{E})}$. ■

7.2. Proposition. *Let $(L, \mathcal{E})$ be a quasi-uniform frame. Then $(L, \mathcal{E})$ is complete iff it is bicomplete (i.e. $(L, \overline{\mathcal{E}})$ is complete).*

Proof: $\Leftarrow$: Let $j \colon (L, \mathcal{E}) \to (M, \mathcal{F})$ be a dense uniform embedding. By 4.7.1, $j \colon (L, \overline{\mathcal{E}}) \to (M, \overline{\mathcal{F}})$ is a dense uniform embedding and thus it is an isomorphism.

$\Rightarrow$: Let $j \colon (L, \overline{\mathcal{E}}) \to (M, \overline{\mathcal{F}})$ be a dense uniform embedding. By 6.2.1, $\{(j \times j)(E) \mid E \in \mathcal{E}\}$ is a basis for a quasi-uniformity $\mathcal{G}$ in M, and by 4.5, $j \colon (L, \mathcal{E}) \to (M, \mathcal{G})$ is a dense uniform embedding hence an isomorphism. ■

7.3. Theorem. $(\mathbf{C}(L, \overline{\mathcal{E}}), \mathbf{C}(\mathcal{E}))$ *is the completion of* $(L, \mathcal{E})$.

Proof: $(\mathbf{C}(L,\overline{\mathcal{E}}),\mathbf{C}(\mathcal{E}))$ is complete by the two preceding propositions and

$$\lambda\colon (L,\mathcal{E}) \to ((\mathbf{C}(L,\overline{\mathcal{E}}),\mathbf{C}(\mathcal{E}))$$

is a dense uniform embedding by 4.5. ■

7.4. Corollary. *If the given $(L,\mathcal{E})$ is a uniform frame, then the quasi-uniform completion $(\mathbf{C}(L,\overline{\mathcal{E}}),\mathbf{C}(\mathcal{E}))$ is the uniform completion of $(L,\mathcal{E})$.* ■

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