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ACCURACY OF A COUPLED MIXED AND GALERKIN FINITE ELEMENT APPROXIMATION FOR POROELASTICITY

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ABSTRACT: In this paper, we consider a coupling mixed finite element and continuous Galerkin finite element formulation for a coupled flow and geomechanics model. We use the lowest order Raviart-Thomas space for the spatial approximation of the flow variables and continuous piecewise linear finite elements for the deformation variable while we consider the backward Euler method for the time discretization. This numerical scheme appears to be one common approach applied to existing reservoir engineering simulators. Theoretical convergence error estimates are derived in a discrete-in-time setting. Previous *a priori* error estimates described in the literature *e.g.* [2][19], which are optimal, show first order convergency with respect to the L^2-norm for the pressure and for the average fluid velocity approximation errors and with respect to the H^1-norm for the displacement approximation error. Here we prove one extra order of convergence for the displacement approximation with respect to the L^2-norm. We also demonstrate that, by including a post-processing step in the scheme, the order of convergence for the approximation of pressure can be improved. Even though this result is critical for deriving the L^2-norm error estimates for the approximation of the deformation variable, surprisingly the corresponding gain of one convergence order holds independently of including or not the post-processing step in the method.

KEYWORDS: Maxwell's equations, leap-frog DG method, stability and convergence.

1. Introduction

Poroelasticity theory is used to model the interaction of fluid flow and the mechanical response in fluid-saturated porous media. The deformation of the medium influences the flow of fluid and vice versa. The development of the necessity coupled geomechanics and flow models emerged in the context soil mechanics, in particular with the seminal work of Karl von Terzaghi [24] and the theory proposed by Maurice Anthony Biot [4][5], known as Biot Theory.

Poroelasticity models are widely used in geomechanics and reservoir engineering, and they have relevance in diverse other fields as, for example, biomechanics and environmental engineering. Due to high interest of applications there is an ever-growing demand for reliable models and numerical tools.

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Applications range from reservoir simulation [9][20][22][23], modelling carbon sequestration [12], the study the mechanical behaviour of fluid-saturated living bone tissue [11], among others as highlighted in [18].

As a prototype of the geomechanical coupling between the single-phase flow of pore fluids and the deformation of the solid skeleton, in this paper we consider the linear poroelastic Biot Theory. The flow (pressures and fluxes) and deformations (displacements) in the poroelastic medium are modeled based on the Darcy's law and the momentum and mass conservation principles. The momentum equation is similar to linear elasticity, with a fluid pressure term acting as a force.

We summarize the governing equations below. Let $\Omega \subset \mathbb{R}^d$, $d = 2$ or 3, denote the domain of interest. The coupled balance equations are written as follow: find $(\mathbf{u}, p)$ such that

$$\begin{aligned} -(\lambda+\mu)\nabla(\nabla\cdot\mathbf{u}) - \mu\nabla^2\mathbf{u} + \alpha\nabla p &= \mathbf{f} \text{ in } \Omega\times(0,T] \\ \frac{\partial}{\partial t}(c_o p + \alpha\nabla\cdot\mathbf{u}) - \frac{1}{\mu_f}\nabla\cdot K(\nabla p - \rho_f\mathbf{g}) &= s_f \text{ in } \Omega\times(0,T] \\ p &= p_D \text{ on } \Gamma_p\times(0,T] \\ -\frac{1}{\mu_f}K(\nabla p - \rho_f\mathbf{g})\cdot\eta &= q \text{ on } \Gamma_f\times(0,T] \\ \mathbf{u} &= \mathbf{u}_D \text{ on } \Gamma_0\times(0,T] \\ \tilde{\sigma}\eta &= \mathbf{r}_N \text{ on } \Gamma_N\times(0,T] \\ p(0) &= p^0 \text{ in } \Omega, \end{aligned} \tag{1}$$

where $\partial\Omega = \Gamma_p \cup \Gamma_f$ and $\partial\Omega = \Gamma_0 \cup \Gamma_N$, with $\text{meas}(\Gamma_0) > 0$. The symbol η represents the outward normal vector on $\partial\Omega$. The primary variables are the pressure p and the deformation $\mathbf{u}$. The physical parameters of the model are: λ, μ, the Lamé constants, c_o, the constrained specific storage coefficient, α, the Biot-Willis constant, μ_f, the fluid viscosity, ρ_f, the fluid mass density and $\mathbf{g}$, the body force per unit of mass. The effective stress σ, is the standard stress tensor from elasticity,

$$\sigma(\mathbf{u}) = 2\mu\epsilon(\mathbf{u}) + \lambda\text{tr}(\epsilon(\mathbf{u}))I,$$

where

$$\epsilon(\mathbf{u}) = \frac{1}{2}\Big(\text{grad}\,\mathbf{u} + (\text{grad}\,\mathbf{u})^t\Big),$$

and I is the identity matrix in $\mathbb{R}^d\times\mathbb{R}^d$, and the total stress, $\tilde{\sigma}$, is given by

$$\tilde{\sigma}(\mathbf{u}, p) = \sigma(\mathbf{u}) - \alpha p I.$$

The Biot-Willis constant has the range of values $0 < \alpha \leq 1$. K denotes the symmetric permeability tensor. We require the existence of the inverse of the operator K and we assume that K^{-1} is uniformly bounded and positive definite, that is, there exists a positive constant ζ such that, for all $\mathbf{s} \in (L^2(\Omega))^d$,

$$(K^{-1}(\mathbf{x},t)\mathbf{s},\mathbf{s}) \geq \zeta\|\mathbf{s}\|_{L^2(\Omega)}, \quad \forall \mathbf{x} \in \Omega, t \in [0,T], \tag{2}$$

and we assume the storage coefficient to be strictly positive and uniformly bounded,

$$0 < \gamma_c \leq c_o(\mathbf{x}) \leq L_c, \ \forall \mathbf{x} \in \Omega. \tag{3}$$

In practice, if the initial condition p^0 is unknown, then p^0 can be found by considering $\nabla p(0) = \rho_f \mathbf{g}$ and then use the first equation of (1) to find $\mathbf{u}(0)$.

The complete system (1) can be solved either simultaneously, in a fully coupled approach, or sequentially, in a loosely coupled scheme. The analysis of the fully coupled numerical method, combining a mixed method and a continuous or discontinuous Galerkin method, was considered *e.g.* in [2], [13] and [19]. The iteratively coupled solution methods were considered *e.g.* in [14], [16] and [27].

In this paper we focus on the fully coupled method which combines lowest order Raviart-Thomas mixed finite elements for the Darcy flow and Galerkin piecewise linear finite elements for elasticity. We analyze the effect on convergence of considering a post-processing step in the scheme and we prove second order of convergence in space for the pressure approximation. Moreover we derive L^2-error estimates for the approximation of the deformation and we also obtain second order of convergence in space. Both results, which are here proved considering the fully coupled approach, are also useful to analyse the iteratively coupled schemes which converge to fully coupled schemes [27].

2. The coupled variational formulation

In order to introduce the mixed formulation for the flow [17] [21], we consider the variable for the flux $\mathbf{z} = -\frac{1}{\mu_f}K(\nabla p - \rho_f \mathbf{g})$.

For the mixed variational formulation of the problem (1), the function space for pressure is $L^2(\Omega)$. The space used for the flux variable is

$$\mathbf{H}(\mathrm{div}) := \{\mathbf{s} \in (L^2(\Omega))^d : \nabla \cdot \mathbf{s} \in L^2(\Omega)\}$$

and we define its subset

$$\mathbf{S}_0 := \{\mathbf{s} \in \mathbf{H}(\mathrm{div}) : \mathbf{s} \cdot \eta|_{\Gamma_f} = 0\}.$$

The function space for the deformation is

$$\mathbf{V}_0 := \{\mathbf{v} \in H^1((\Omega))^d : \mathbf{v}|_{\Gamma_0} = 0\}.$$

Associated to this space we define the bilinear form $a_{\mathbf{u}}(.,.)$ by

$$a_{\mathbf{u}}(\mathbf{u},\mathbf{v}) := \int_\Omega \sigma(\mathbf{u}) : \epsilon(\mathbf{v})\, d\mathbf{x},$$

or equivalently

$$a_{\mathbf{u}}(\mathbf{u},\mathbf{v}) = \int_\Omega (2\mu(\epsilon(\mathbf{u}) : \epsilon(\mathbf{v})) + \lambda(\nabla\cdot\mathbf{u})(\nabla\cdot\mathbf{u}))\, d\mathbf{x}.$$

The bilinear form is continuous and coercive in $\mathbf{V}_0\times\mathbf{V}_0$ ([7]); therefore, for some positive real number C_{cont} and C_{coer} holds

$$a_{\mathbf{u}}(\mathbf{u},\mathbf{v}) \le C_{cont}\|\mathbf{u}\|_{H^1}\|\mathbf{v}\|_{H^1},\ \forall \mathbf{u},\mathbf{v}\in\mathbf{V}_0, \tag{4}$$

$$a_{\mathbf{u}}(\mathbf{v},\mathbf{v}) \ge C_{coer}\|\mathbf{v}\|^2_{H^1},\ \forall \mathbf{v}\in\mathbf{V}_0.$$

We define the linear functional

$$\begin{aligned}
\ell_1(\mathbf{v}) &= \int_\Omega \mathbf{f}\cdot\mathbf{v} + \int_{\Gamma_N} \mathbf{r}_N\cdot\mathbf{v}, \quad \mathbf{v}\in\mathbf{V}_0,\\
\ell_2(w) &= \int_\Omega s_f w, \quad w\in L^2(\Omega),\\
\ell_3(\mathbf{s}) &= -\int_{\Gamma_p} p_D\mathbf{s}\cdot\eta + \int_\Omega \rho_f\mathbf{g}\cdot\mathbf{s}, \quad \mathbf{s}\in\mathbf{S}_0.
\end{aligned}$$

Since the boundary conditions are allowed to be inhomogeneous, we need to select, for each $t\in[0,T]$, a function $\mathbf{u}_d(.,t)\in(H^1(\Omega))^d$ such that $\mathbf{u}_d(.,t)|_{\Gamma_0} = \mathbf{u}_D(.,t)$ and a function $\mathbf{z}_d(.,t)\in\mathbf{H}(\mathrm{div})$ such that $\mathbf{z}_d(.,t)|_{\Gamma_f}\cdot\eta = q(.,t)$.

The variational problem becomes: find $\mathbf{u}\in\mathbf{u}_d + H^1([0,T];\mathbf{V}_0)$, $p\in H^1([0,T];\,L^2(\Omega))$ and $\mathbf{z}\in\mathbf{z}_d + L^2([0,T];\mathbf{S}_0)$ such that

$$a_{\mathbf{u}}(\mathbf{u},\mathbf{v}) - \alpha(\nabla\cdot\mathbf{v},p) = \ell_1(\mathbf{v}), \tag{5}$$

$$\Big(c_o\frac{\partial p}{\partial t},w\Big) + \alpha\Big(\frac{\partial}{\partial t}\nabla\cdot\mathbf{u},w\Big) + (\nabla\cdot\mathbf{z},w) = \ell_2(w), \tag{6}$$

$$\mu_f(K^{-1}\mathbf{z},\mathbf{s}) - (p,\nabla\cdot\mathbf{s}) = \ell_3(\mathbf{s}) \tag{7}$$

holds for all $(\mathbf{v},w,\mathbf{s})\in(\mathbf{V}_0,L^2(\Omega),\mathbf{S}_0)$ and $t\in[0,T]$.

We also make the following smoothness assumptions, in order the above variational formulation makes sense:

$$
\begin{aligned}
&\mathbf{f} \in C^1([0,T];(H^{-1}(\Omega))^d),\\
&s_f \in C([0,T];L^2(\Omega)),\\
&p_D \in C([0,T];L^2(\Gamma_p)),\\
&q \in C([0,T];TrS), \quad TrS = \{\mathbf{s}\cdot\eta_{|\Gamma_f} : \mathbf{s} \in \mathbf{H}(\mathrm{div})\},\\
&\mathbf{u}_D \in C^1([0,T];(H^{1/2}(\Gamma_0))^d),\\
&\mathbf{r}_N \in C^1([0,T];(H^{-1/2}(\Gamma_N)^d)),\\
&\mathbf{g} \in C([0,T];(L^2(\Omega))^d),\\
&\mathbf{u}^0 \in (H^1(\Omega))^d,\\
&p^0 \in L^2(\Omega).
\end{aligned}
$$

In order to approximate the variational problem (5)-(7) with a finite element scheme we need to provide some definitions.

Let $\mathcal{E}_h$ and $\mathcal{E}_H$ be two nondegenerate partitions of the polyhedral domain Ω, with maximal element diameter h and H, respectively. The elements of $\mathcal{E}_h$ and $\mathcal{E}_H$ are triangles, if $d = 2$, and tetrahedra, if $d = 3$.

Let $(W_h, \mathbf{S}_h) \subset (L^2(\Omega) \times \mathbf{H}(\mathrm{div}))$ denote a standard mixed finite element space on $\mathcal{E}_h$, called lowest order Raviart-Thomas approximating space (RT0) (e.g. [8], [21]) and

$$\mathbf{S}_{h,0} := \{\mathbf{s} \in \mathbf{S}_h : \mathbf{s}\cdot\eta_{\,|\Gamma_f} = 0\}.$$

We consider the linear operators $\Pi_h : \mathbf{H}(\mathrm{div}) \to \mathbf{S}_h$ and $I_h : L^2 \to W_h$ which satisfy the following properties:

$$
\begin{gathered}
(\nabla\cdot(\mathbf{s}-\Pi_h\mathbf{s}), w) = 0, \quad \forall w \in W_h,\\
\|\mathbf{s}-\Pi_h\mathbf{s}\|_{L^2(\Omega)} \le Ch\|\mathbf{s}\|_{H^1(\Omega)},\\
\nabla\cdot\Pi_h = I_h\nabla\cdot,\\
(\nabla\cdot\mathbf{s}_h, p - I_hp) = 0, \quad \forall \mathbf{s}_h \in \mathbf{S}_h,\\
\|p - I_hp\|_{L^2(\Omega)} \le Ch\|p\|_{H^1(\Omega)}.
\end{gathered}
$$

Let $\mathbf{V}_H$ be the space of continuous piecewise polynomials of degree 1 defined on $\mathcal{E}_H$ and

$$\mathbf{V}_{H,0} := \{\mathbf{v} \in \mathbf{V}_H : \mathbf{v}|_{\Gamma_0} = 0\}.$$

The elliptic projector $E_H : (H^1(\Omega))^d \to \mathbf{V}_H$ is defined by

$$a_{\mathbf{u}}(\mathbf{u} - E_H\mathbf{u}, \mathbf{v}_H) = 0, \ \forall \mathbf{v}_H \in \mathbf{V}_H, \tag{8}$$

and satisfies (see [7])

$$\|\mathbf{u} - E_H\mathbf{u}\|_{a_\mathbf{u}} \le CH\|\mathbf{u}\|_{(H^2(\Omega))^d}. \tag{9}$$

Let $\mathbf{u}_{dH}(\mathbf{x}, t) = E_H\mathbf{u}_d(\mathbf{x}, t)$ and $\mathbf{z}_{dh}(\mathbf{x}, t) = \Pi_h\mathbf{z}_d(\mathbf{x}, t)$. We define $\Delta t = T/N$, where N denotes the number of time steps and $t^n = n\Delta t$ and we will use the following notation $g^n = g(., t^n)$.

The fully discrete method is derived by discretizing the time derivatives. Here we considered the backward Euler method.

The complete numerical formulation becomes: find $\mathbf{u}_H^n \in \mathbf{u}_{dH}^n + \mathbf{V}_{H,0}$, $p_h^n \in W_h$, $\mathbf{z}_h^n \in \mathbf{z}_{dh}^n + \mathbf{S}_{h,0}$ such that

$$a_\mathbf{u}(\mathbf{u}_H^n, \mathbf{v}) - \alpha(p_h^n, \nabla\cdot\mathbf{v}) = \ell_1^n(\mathbf{v}), \tag{10}$$

$$\Big(c_o\frac{p_h^n - p_h^{n-1}}{\Delta t}, w\Big) + \alpha\Big(\nabla\cdot\frac{\mathbf{u}_H^n - \mathbf{u}_H^{n-1}}{\Delta t}, w\Big) + (\nabla\cdot\mathbf{z}_h^n, w) = \ell_2^n(w), \tag{11}$$

$$\mu_f((K^n)^{-1}\mathbf{z}_h^n, \mathbf{s}) - (p_h^n, \nabla\cdot\mathbf{s}) = \ell_3^3(\mathbf{s}), \tag{12}$$

for all $(\mathbf{v}, w, \mathbf{s}) \in (\mathbf{V}_{H,0}, W_h, \mathbf{S}_{h,0})$. Here $(\sigma_m)_H^n$ is defined locally in $\mathcal{E}_H$ by

$$(\sigma_m)_H^n = \frac{1}{d}\mathrm{tr}(\sigma(\mathbf{u}_H^n)).$$

Additionally, we consider the initial conditions $\mathbf{u}_H^0 \in \mathbf{u}_{dH}^0 + \mathbf{V}_{H,0}$, $p_h^0 \in W_h$, such that

$$a_\mathbf{u}(\mathbf{u}_H, \mathbf{v})_{|t=0} = a_\mathbf{u}(\mathbf{u}^0, \mathbf{v}),\ \forall\mathbf{v} \in \mathbf{V}_H,$$

$$(p_h, w)_{|t=0} = (p^0, w),\ \forall w \in W_h.$$

The fully coupled scheme involves calculating $\mathbf{u}_H^n$, p_h^n and $\mathbf{z}_h^n$ simultaneously.

The convergence result in the next theorem can be found in [2], [19].

Theorem 1. *Let $(\mathbf{u}, p, \mathbf{z})$ be the solution of (5)–(7) and $(\mathbf{u}_H, p_h, \mathbf{z}_h)$ be the solution of (10)–(12). Then, for Δt small enough, there exists $C > 0$ such that*

$$\|\mathbf{u} - \mathbf{u}_H\|_{L^\infty(H^1)} + \|p - p_h\|_{L^\infty(L^2)} + \|\mathbf{z} - \mathbf{z}_h\|_{L^2(L^2)} \le C(H + h) + \mathcal{O}(\Delta t), \tag{13}$$

where C depends on the model parameters, and on the true solution but is not dependent on H, h and Δt.

3. Post-processing step for pressure

The objective of this section is to obtain a higher order approximation for pressure. To improve accuracy, a post-processing step can be included in the numerical scheme, following the idea by Arbogast and Wheeler in [1].

We start by defining the space $\tilde{W}_h$ consisting of functions that are discontinuous and piecewise linear over the grid $\mathcal{E}_h$. We locally post-process the pressure by finding $\tilde{p}_h \in \tilde{W}_h$ such that on each element of $R \in \mathcal{E}_h$,

$$(c_o(\tilde{p}_h^n - p_h^n), w)_R = 0 \quad \forall w \in W_h, \tag{14}$$

$$(K^n \nabla \tilde{p}_h^n + \mathbf{z}_h^n, \nabla w)_R = 0 \quad \forall w \in \tilde{W}_h. \tag{15}$$

We will demonstrate that this post-processing technique improves the approximation p_h^n so that the L^2-error between $\tilde{p}_h^n$ and $p(t^n)$ is of second order in space.

In the error analysis we will compare the post-processed finite element solution to an elliptic projection of the solution of (5)-(7). We define the projection $(P_h, \mathbf{Z}_h) \in W_h \times \mathbf{S}_h$ of $(p, \mathbf{z})$ [1] [26], by

$$(c_o(P_h - p), w) + (\nabla \cdot (\mathbf{Z}_h - \mathbf{z}), w) = 0 \quad \forall w \in W_h, \tag{16}$$

$$\mu_f \left(K^{-1}(\mathbf{Z}_h - \mathbf{z}), \mathbf{s}\right) = (P_h - p, \nabla \cdot \mathbf{s}) \quad \forall \mathbf{s} \in \mathbf{S}_h, \tag{17}$$

and on each element $R \in \mathcal{E}_h$ we define $\tilde{P}_h \in \tilde{W}_h$ by

$$\Big(c_o(\tilde{P}_h - P_h), w\Big)_R = 0 \quad \forall w \in W_h, \tag{18}$$

$$\Big(K \nabla \tilde{P}_h + \mathbf{Z}_h, \nabla w\Big)_R = 0 \quad \forall w \in \tilde{W}_h. \tag{19}$$

For convenience, we now introduce some additional notation, in particular for auxiliary and projection errors:

$$\xi^n = p_h^n - P_h^n \in W_h, \quad \tilde{\xi}^n = \tilde{p}_h^n - \tilde{P}_h^n \in \tilde{W}_h, \quad \zeta^n = \mathbf{z}_h^n - \mathbf{Z}_h^n \in \mathbf{S}_h,$$

and

$$\eta^n = P_h^n - p^n, \quad \tilde{\eta}^n = \tilde{P}_h^n - p^n.$$

To simplify the notation in we use $\|.\|_0$, $\|.\|_\infty$ and $\|.\|_1$, respectively, for the L^2, L^∞ and H^1 norms.

Lemma 1. *The following inequalities hold*

$$\|\sqrt{c_o}\xi^n\|_0 \leq \|\sqrt{c_o}\tilde{\xi}^n\|_0, \tag{20}$$

$$\left(c_o(\tilde{\xi}^n - \xi^n), \tilde{\xi}^n\right) \leq Q\|(K^n)^{-1/2}\zeta^n\|_0^2 h^2, \tag{21}$$

where Q depends on the positive upper and lower bounds for c_o and K^n.

Proof: For any element $R \in \mathcal{E}_h$, by (14), (15), (18) and (19) we note that

$$\left(c_o(\tilde{\xi}^n - \xi^n), 1\right)_R = 0, \tag{22}$$

$$(K^n \nabla \xi^n + \zeta^n, \nabla w)_R = 0, \; w \in \tilde{W}_h. \tag{23}$$

Since ξ^n is constant on R, from (22) we have

$$(c_o \xi^n, \xi^n)_R = \left(c_o(\xi^n - \tilde{\xi}^n), \xi^n\right)_R + \left(c_o \tilde{\xi}^n, \xi^n\right)_R,$$

and (20) follows.

For a good choice of the constant C, we get

$$\begin{aligned}\left(c_o(\tilde{\xi}^n - \xi^n), \tilde{\xi}^n - \xi^n\right)_R = \left(c_o(\tilde{\xi}^n - \xi^n), \tilde{\xi}^n\right)_R &= \left(c_o(\tilde{\xi}^n - \xi^n), \tilde{\xi}^n - C\right)_R \\ &\leq Q\|\sqrt{c}_o(\tilde{\xi}^n - \xi^n)\|_{0,R}\|\nabla\tilde{\xi}^n\|_{0,R}\, h\end{aligned}$$

and then

$$\|\sqrt{c}_o(\tilde{\xi}^n - \xi^n)\|_{0,R} \leq Q\|\nabla\tilde{\xi}^n\|_{0,R}\, h.$$

Taking $w = \tilde{\xi}^n$ in (23) results

$$\|(K^n)^{1/2}\nabla\tilde{\xi}^n\|_{0,R} \leq \|(K^n)^{-1/2}\zeta^n\|_{0,R},$$

and we obtain (21). ∎

The detailed arguments that proof of the next lemma can be found in the demonstration of Theorem 2 of [1].

Lemma 2. *Assume sufficient regularity of data and of the true solution of (5)-(7). For each $t \in (0, T]$ and for h sufficiently small, holds*

$$\|\eta\|_0 = \|P_h - p\|_0 \leq C\|\mathbf{z}\|_1 h, \tag{24}$$

$$\|\tilde{\eta}\|_0 = \|\tilde{P}_h - p\|_0 \leq C\left(\|\mathbf{z}\|_1 + \|\nabla \cdot \mathbf{z}\|_1\right) h^2, \tag{25}$$

$$\|(\tilde{\eta})_t\|_0 = \|(\tilde{P}_h - p)_t\|_0 \leq C\left(\|\mathbf{z}\|_1 + \|\nabla \cdot \mathbf{z}\|_1 + \|(\mathbf{z})_t\|_1 + \|\nabla \cdot \mathbf{z}_t\|_1\right) h^2, \tag{26}$$

where C is independent of t, p, h and Δt.

The next result will be central in the convergence analysis.

Lemma 3. *Let E_H be defined by (8). The following estimate holds*

$$\|\nabla \cdot E_H \mathbf{u} - \nabla \cdot \mathbf{u}_H\|_0 \le \frac{\alpha}{\lambda}\|p - p_h\|_0. \tag{27}$$

Proof: For any element $R \in \mathcal{E}_H$ we have

$$\begin{aligned}
\lambda\|\nabla \cdot (E_H\mathbf{u} - \mathbf{u}_H)\|^2_{0,R} &\le a_{\mathbf{u}}(E_H\mathbf{u} - \mathbf{u}_H, E_H\mathbf{u} - \mathbf{u}_H) \\
&= \alpha\,(p - p_h, \nabla \cdot (E_H\mathbf{u} - \mathbf{u}_H))_R \\
&= \alpha\,(p - \tilde{p}_h, \nabla \cdot (E_H\mathbf{u} - \mathbf{u}_H))_R \\
&\le \alpha\|p - \tilde{p}_h\|_{0,R}\|\nabla \cdot (E_H\mathbf{u} - \mathbf{u}_H)\|_{0,R}.
\end{aligned}$$

■

The convergence result for the post-processed numerical solution for pressure is given in the next theorem.

Theorem 2. *Consider that $\mathcal{E}_h$ and $\mathcal{E}_H$ coincide or $\mathcal{E}_h$ to be a refinement of $\mathcal{E}_H$. Assume sufficient regularity of the true solution and that the initialization error satisfy*

$$\|\tilde{p}^0_h - p^0\|_0 \le C_0(p_0)h^2,$$

for some constant C_0 depending on p_0. Let Δt satisfy $\Delta t\frac{\mu_f}{2} > Qh^2$ and $\Delta t = c'H^2$ for some positive constant c'. If $\frac{\alpha^2}{\lambda} < \frac{1}{4}\gamma_c$, then for h and H sufficiently small,

$$\max_n \|p^n - \tilde{p}^n_h\|_0 \le C(p)(h^2 + H^2), \tag{28}$$

where $C(p)$ depends on p but not on h, H or Δt.

Proof: For convenience, we use the notation

$$\ell_4(\mathbf{u}, w) = \alpha\Big(\frac{\partial}{\partial t}\nabla \cdot \mathbf{u}, w\Big),$$

$$\bar{\ell}_4(\mathbf{u}^n_H, w) = \alpha\Big(\nabla \cdot \frac{\mathbf{u}^n_H - \mathbf{u}^{n-1}_H}{\Delta t}, w\Big).$$

Combining (6), (11) and (14), we obtain, for all $w \in W_h$,

$$\begin{aligned}
(c_o(\tilde{p}^n_h - p^n), w) - \big(c_o(\tilde{p}^{n-1}_h - p^{n-1}), w\big) + \Delta t(\nabla \cdot \mathbf{z}^n_h, w) \\
- \int_{t^{n-1}}^{t^n} (\nabla \cdot \mathbf{z}, w)\, dt \\
= \Delta t\,\big(\ell_2(w) - \bar{\ell}_4(\mathbf{u}^n_H, w)\big) - \int_{t^{n-1}}^{t^n} (\ell_2(w) - \ell_4(\mathbf{u}, w))\, dt.
\end{aligned}$$

Then

$$\begin{aligned}\Big(c_o(\tilde{\xi}^n+\tilde{\eta}^n),w\Big)-\Big(c_o(\tilde{\xi}^{n-1}+\tilde{\eta}^{n-1}),w\Big)+\Delta t(\nabla\cdot\zeta^n,w)\\
+\Delta t(\nabla\cdot\mathbf{Z}_h^n,w)-\int_{t^{n-1}}^{t^n}(\nabla\cdot\mathbf{z},w)\,dt\\
=\Delta t\left(\ell_2(w)-\bar{\ell}_4(\mathbf{u}_H^n,w)\right)-\int_{t^{n-1}}^{t^n}\left(\ell_2(w)-\ell_4(\mathbf{u},w)\right)\,dt,\end{aligned}$$

and by (16) we get

$$\begin{aligned}&\Big(c_o(\tilde{\xi}^n+\tilde{\eta}^n),w\Big)-\Big(c_o(\tilde{\xi}^{n-1}+\tilde{\eta}^{n-1}),w\Big)+\Delta t(\nabla\cdot\zeta^n,w)\\
&+\int_{t^{n-1}}^{t^n}\left(\ell_2(w)-\ell_4(\mathbf{u},w)\right)\,dt\\
&=\Delta t\left(\ell_2(w)-\bar{\ell}_4(\mathbf{u}_H^n,w)\right)+\int_{t^{n-1}}^{t^n}(\nabla\cdot\mathbf{z},w)\,dt-\Delta t(\nabla\cdot\mathbf{z}^n,w)\\
&+\Delta t\left(c_o\eta^n,w\right).\end{aligned}\tag{29}$$

Combining (7), (12) and (17), we have

$$\begin{aligned}&\mu_f\left(K^{-1}\zeta^n,\mathbf{s}\right)\\
&=(p_h^n,\nabla\cdot\mathbf{s})+(\ell_3,\mathbf{s})-\mu_f\left(K^{-1}(\mathbf{Z}_h^n-\mathbf{z}^n),\mathbf{s}\right)-\mu_f\left((K^n)^{-1}\mathbf{z}^n,\mathbf{s}\right)\\
&=(p_h^n,\nabla\cdot\mathbf{s})+(\ell_3,\mathbf{s})-(P_h^n,\nabla\cdot\mathbf{s})-\mu_f\left((K^n)^{-1}\mathbf{z}^n,\mathbf{s}\right)+(p^n,\nabla\cdot\mathbf{s})\\
&=(\xi^n,\nabla\cdot\mathbf{s}).\end{aligned}\tag{30}$$

Taking in (29) and (30) $w=\xi^n$ and $\mathbf{s}=\zeta^n$, respectively, we obtain

$$\begin{aligned}\Big(c_o(\tilde{\xi}^n+\tilde{\eta}^n),\xi^n\Big)-\Big(c_o(\tilde{\xi}^{n-1}+\tilde{\eta}^{n-1}),\xi^n\Big)+\Delta t(\nabla\cdot\zeta^n,\xi^n)\\
+\int_{t^{n-1}}^{t^n}\left(\ell_2(\xi^n)-\ell_4(\mathbf{u},\xi^n)\right)\,dt\\
=\Delta t\left(\ell_2(\xi^n)-\bar{\ell}_4(\mathbf{u}_H^n,\xi^n)\right)+\int_{t^{n-1}}^{t^n}(\nabla\cdot\mathbf{z},\xi^n)\,dt-\Delta t(\nabla\cdot\mathbf{z}^n,\xi^n)\\
+\Delta t(c_o\eta^n,\xi^n)\end{aligned}$$

and

$$\mu_f\left(K^{-1}\zeta^n,\zeta^n\right)=(\xi^n,\nabla\cdot\zeta^n).\tag{31}$$

Using (18) we get

$$\begin{aligned}\left(c_o\tilde{\xi}^n,\xi^n\right)-\left(c_o\tilde{\xi}^{n-1},\xi^n\right)+\Delta t\mu_f\left(K^{-1}\zeta^n,\zeta^n\right)\\=\int_{t^{n-1}}^{t^n}(\nabla\cdot\mathbf{z}-\nabla\cdot\mathbf{z}^n,\xi^n)\,dt+\int_{t^{n-1}}^{t^n}\ell_4(\mathbf{u},\xi^n)\,dt-\Delta t\bar{\ell}_4(\mathbf{u}_H^n,\xi^n)\\-(1-\Delta t)(c_o\tilde{\eta}^n,\xi^n)+(c_o\tilde{\eta}^{n-1},\xi^n).\end{aligned}\tag{32}$$

Since

$$\left(c_o\tilde{\xi}^{n-1},\xi^n\right)\leq\frac{1}{2}\left(c_o\tilde{\xi}^{n-1},\tilde{\xi}^{n-1}\right)+\frac{1}{2}\left(c_o\xi^n,\xi^n\right)$$

then

$$\left(c_o\tilde{\xi}^n,\xi^n\right)-\left(c_o\tilde{\xi}^{n-1},\xi^n\right)\geq\left(c_o\tilde{\xi}^n,\xi^n\right)-\frac{1}{2}\left(c_o\tilde{\xi}^{n-1},\tilde{\xi}^{n-1}\right)-\frac{1}{2}\left(c_o\xi^n,\xi^n\right).$$

From (14) and (18) we obtain $\left(c_o\tilde{\xi}^n,\xi^n\right)=\left(c_o\xi^n,\xi^n\right)$ and consequently,

$$\begin{aligned}\left(c_o\tilde{\xi}^n,\xi^n\right)-\left(c_o\tilde{\xi}^{n-1},\xi^n\right)\geq\frac{1}{2}\left(c_o\tilde{\xi}^n,\xi^n\right)-\frac{1}{2}\left(c_o\tilde{\xi}^{n-1},\tilde{\xi}^{n-1}\right)\\=\frac{1}{2}\left(c_o\tilde{\xi}^n,\tilde{\xi}^n\right)-\frac{1}{2}\left(c_o\tilde{\xi}^{n-1},\tilde{\xi}^{n-1}\right)-\frac{1}{2}\left(c_o(\tilde{\xi}^n-\xi^n),\tilde{\xi}^n\right).\end{aligned}\tag{33}$$

We will now analyze the right-hand side of (32). Bramble-Hilbert Lemma (*e.g.* [10]) implies that

$$\|\int_{t^{n-1}}^{t^n}\nabla\cdot\mathbf{z}-\nabla\cdot\mathbf{z}^n\,dt\|_0\leq C(\Delta t)^{3/2}\|\int_{t^{n-1}}^{t^n}(\nabla\cdot\mathbf{z})_t\,dt\|_0,$$

where C is independent of t, $\mathbf{z}$, h and Δt. Hence,

$$\int_{t^{n-1}}^{t^n}(\nabla\cdot\mathbf{z}-\nabla\cdot\mathbf{z}^n,\xi^n)\,dt\leq C\left(\|\xi^n\|_0^2\Delta t+\int_{t^{n-1}}^{t^n}\|(\nabla\cdot\mathbf{z})_t\|_0^2\,dt(\Delta t)^2\right).\tag{34}$$

Summing and subtracting $(E_H\mathbf{u})^n$, where E_H is the elliptic projector defined by (9), we have that

$$\begin{aligned}\int_{t^{n-1}}^{t^n}\ell_4(\mathbf{u},\xi^n)\,dt-\Delta t\bar{\ell}_4(\mathbf{u}_H^n,\xi^n)=\int_{t^{n-1}}^{t^n}\ell_4(\mathbf{u}-E_H\mathbf{u},\xi^n)\,dt\\+\Delta t\bar{\ell}_4((E_H\mathbf{u})^n-\mathbf{u}_H^n,\xi^n).\end{aligned}\tag{35}$$

Let us now consider the first term of the right-hand side of (35). For any $\epsilon > 0$, holds

$$\int_{t^{n-1}}^{t^n} \ell_4(\mathbf{u} - E_H\mathbf{u}, \xi^n)\, dt \leq C\Delta t H \|\mathbf{u}_t\|_{L^\infty(H^2)} \|\xi^n\|_0$$
$$\leq \frac{C}{4\epsilon} \Delta t H^4 \|\mathbf{u}_t\|^2_{L^\infty(H^2)} + \epsilon \Delta t H^{-2} \|\xi^n\|_0^2. \qquad (36)$$

For the other term, we use Lemma 3 and (25) to obtain the estimate

$$\begin{aligned}\Delta t \bar{\ell}_4(E_H\mathbf{u} - \mathbf{u}_H^n, \xi^n) &\leq \frac{\alpha^2}{\lambda}(\|p^n - \tilde{p}_h^n\|_0 + \|p^{n-1} - \tilde{p}_h^{n-1}\|_0)\|\xi^n\|_0 \\ &\leq \frac{\alpha^2}{\lambda}(\|\tilde{\eta}^n\|_0 + \|\tilde{\eta}^{n-1}\|_0 + \|\tilde{\xi}^n\|_0 + \|\tilde{\xi}^{n-1}\|_0)\|\xi^n\|_0 \\ &\leq C\left(\|\mathbf{z}\|_{L^\infty(H^1)} + \|\nabla\cdot\mathbf{z}\|_{L^\infty(H^1)}\right) h^2 \|\tilde{\xi}^n\|_0 \\ &+ \frac{3}{2}\frac{\alpha^2}{\lambda}\|\tilde{\xi}^n\|_0^2 + \frac{1}{2}\frac{\alpha^2}{\lambda}\|\tilde{\xi}^{n-1}\|_0^2. \end{aligned} \qquad (37)$$

It remains to analyze the last term of the right hand-side of (32). Using Lemma 2 we deduce that

$$\begin{aligned}&-(c_o\tilde{\eta}^n, \xi^n) + (c_o\tilde{\eta}^{n-1}, \xi^n) = -\int_{t^{n-1}}^{t^n} ((c_o\tilde{\eta})_t, \xi^n)\, dt \\ &\leq C\left(\int_{t^{n-1}}^{t^n} \|\tilde{\eta}\|_0^2\, dt + \|\xi^n\|_0^2 \Delta t\right) \\ &\leq C\left(h^4 \int_{t^{n-1}}^{t^n} \|\mathbf{z}\|_1^2 + \|\nabla\cdot\mathbf{z}\|_1^2 + \|(\mathbf{z})_t\|_1^2 + \|\nabla\cdot\mathbf{z}_t\|_1^2\, dt + \|\xi^n\|_0^2\Delta t\right), \end{aligned} \qquad (38)$$

and

$$\Delta t(c_o\tilde{\eta}^n, \xi^n) \leq Ch^2\Delta t\left(\|\mathbf{z}^n\|_1^2 + \|\nabla\cdot\mathbf{z}^n\|_1^2\right) + \|\xi^n\|_0^2\Delta t. \qquad (39)$$

Combining (32) with (33) and using (21), (34), (36), (37), (38) and (39) we obtain

$$\begin{aligned}
&\frac{1}{2}\left(\left(c_o\tilde{\xi}^n,\tilde{\xi}^n\right)-\left(c_o\tilde{\xi}^{n-1},\tilde{\xi}^{n-1}\right)\right)+\left((K^n)^{-1}\zeta^n,\zeta^n\right)(\Delta t\mu_f-\frac{1}{2}Qh^2)\\
&\leq C\left(\|\tilde{\xi}^n\|_0^2+\int_{t^{n-1}}^{t^n}\|(\nabla\cdot\mathbf{z})_t\|_0^2dt\Delta t\right)\Delta t\\
&+\frac{C}{4\epsilon}\Delta tH^4\|\mathbf{u}_t\|^2_{L^\infty((t^{n-1},t^n),H^2)}+\epsilon\Delta tH^{-2}\|\xi^n\|_0^2+\frac{3}{2}\frac{\alpha^2}{\lambda}\|\tilde{\xi}^n\|_0^2+\frac{1}{2}\frac{\alpha^2}{\lambda}\|\tilde{\xi}^{n-1}\|_0^2\\
&+C\left(h^4\int_{t^{n-1}}^{t^n}\|\mathbf{z}\|_1^2+\|\nabla\cdot\mathbf{z}\|_1^2+\|(\mathbf{z})_t\|_1^2+\|\nabla\cdot\mathbf{z}_t\|_1^2\,dt\right)\\
&+Ch^4\Delta t\left(\|\mathbf{z}^n\|_1^2+\|\nabla\cdot\mathbf{z}^n\|_1^2\right).
\end{aligned}\tag{40}$$

A summation on n, and an application of Gronwall's inequality yield to

$$\begin{aligned}
\max_n\|\tilde{\xi}^n\|_0^2\leq C\Bigg(&\|\tilde{\xi}^0\|_0^2+h^4\int_0^T\|\mathbf{z}\|_1^2+\|\nabla\cdot\mathbf{z}\|_1^2+\|(\mathbf{z})_t\|_1^2+\|\nabla\cdot\mathbf{z}_t\|_1^2\,dt\\
&+h^4\left(\|\mathbf{z}\|^2_{L^\infty((0,T),H^1)}+\|\nabla\cdot\mathbf{z}\|^2_{L^\infty((0,T),H^1)}\right)+H^4\|\mathbf{u}_t\|^2_{L^\infty((0,T),H^2)}\Bigg).
\end{aligned}\tag{41}$$

■

Remark 1. *It is interesting to observe that equation (10) remains unaltered if we replace $\tilde{p}_h^n$ by p_h^n, under the assumption that $\mathcal{E}_h$ and $\mathcal{E}_H$ coincide or that $\mathcal{E}_h$ is a refinement of $\mathcal{E}_H$. In fact, for any test function $\mathbf{v}\in\mathbf{V}_{H,0}$ we have that $\nabla\cdot\mathbf{v}$ is constant in every element $R\in\mathcal{E}_h$ and consequently $(p_h^n,\nabla\cdot\mathbf{v})_R=(\tilde{p}_h^n,\nabla\cdot\mathbf{v})_R$.*

4. The L^2 estimates for deformation

The objective of this section is to derive the convergence order for the displacement approximation error with respect to the L^2-norm. The estimate we will derive is based on duality techniques.

Let $\mathbf{e}_H^n=\mathbf{u}^n-\mathbf{u}_H^n$. We will restrict our study to the case $\mathbf{e}_H\in\mathbf{V}_0$, which is satisfied for example when the Dirichlet condition for $\mathbf{u}$ in Γ_0 is homogeneous. If the general case of inhomogeneous Dirichlet data for $\mathbf{u}$ in Γ_0, the analysis required is more involving. We refer the paper [3] for some insight in this question, even though therein the study is restricted to the Laplace equation.

Consider the dual problem: find $\phi \in \mathbf{V}_0$, such that

$$a_{\mathbf{u}}(\phi, \mathbf{v}) = (\mathbf{e}_H^n, \mathbf{v}), \ \forall \mathbf{v} \in \mathbf{V}_0. \tag{42}$$

For the derivation of L^2 error estimates we assume that the problem (42) is H^2-regular, that is, $\phi \in H^2(\Omega)$ and

$$\|\phi\|_{H^2} \leq C_{reg} \|\mathbf{e}_H^n\|_0, \tag{43}$$

where C_{reg} is a positive constant which depends on the domain Ω. A sufficient condition for the H^2 regularity estimate (43) to hold is that the domain Ω is a convex polygonal domain in $\mathbb{R}^2$ and that (42) is a pure displacement problem ($\Gamma_0 = \partial\Omega$) [6]. Other conditions which guarantee (43) to be true are discussed for instance in [15] and [25].

In the next theorem we present the L^2-estimates.

Theorem 3. *Under the foregoing assumptions of Section 4 and the same conditions as in Theorem 2, the following estimate holds:*

$$\|\mathbf{u} - \mathbf{u}_H\|_{L^\infty(L^2)} \leq C(H^2 + h^2). \tag{44}$$

Proof: Let $I_H\phi \in \mathbf{V}_{H,0}$ be the nodal interpolation of ϕ. It is well known that

$$\|\phi - I_H\phi\|_1 \leq C_{interp} H \|\phi\|_{H^2}. \tag{45}$$

Since $\mathbf{e}_H \in \mathbf{V}_0$ then,

$$\begin{aligned}
\|\mathbf{e}_H^n\|_0^2 &= a_{\mathbf{u}}(\phi, \mathbf{e}_H^n) = a_{\mathbf{u}}(\mathbf{e}_H^n, \phi - I_H\phi) + a_{\mathbf{u}}(\mathbf{e}_H^n, I_H\phi) \\
&= a_{\mathbf{u}}(\mathbf{e}_H^n, \phi - I_H\phi) + \alpha(\nabla \cdot I_H\phi, p^n - p_h^n) \\
&= a_{\mathbf{u}}(\mathbf{e}_H^n, \phi - I_H\phi) + \alpha(\nabla \cdot (I_H\phi - \phi), p^n - p_h^n) + \alpha(\nabla \cdot \phi, p^n - p_h^n).
\end{aligned}$$

Now, the trick is to sum and subtract the post-processed approximation for pressure. We get

$$\|\mathbf{e}_H^n\|_0^2 = a_{\mathbf{u}}(\mathbf{e}_H^n, \phi - I_H\phi) + \alpha(\nabla \cdot (I_H\phi - \phi), p^n - \tilde{p}_h^n) + \alpha(\nabla \cdot \phi, p^n - \tilde{p}_h^n).$$

Using (4), (43) and (45), we obtain

$$\begin{aligned}
\|\mathbf{e}_H^n\|_0^2 \leq{}& C_{cont} C_{interp} C_{reg} H \|\mathbf{e}_H^n\|_1 \|\mathbf{e}_H^n\|_0 + \alpha C_{cont} C_{interp} C_{reg} H \|\mathbf{e}_H^n\|_0 \|p^n - \tilde{p}_h^n\|_0 \\
&+ \alpha C_{reg} \|\mathbf{e}_H^n\|_0 \|p^n - \tilde{p}_h^n\|_0,
\end{aligned} \tag{46}$$

and consequently,

$$\|\mathbf{e}_H^n\|_0 \leq C \left(H \|\mathbf{e}_H^n\|_1 + H \|p^n - \tilde{p}_h^n\|_0 + \|p^n - \tilde{p}_h^n\|_0 \right). \tag{47}$$

■

5. Conclusion

In this paper we have analyzed the convergence of a fully coupled numerical method for a coupled flow and geomechanics model. The numerical scheme combines lowest order mixed finite elements and Galerkin piecewise linear finite elements. We proposed a post-processing procedure to increase the order of convergence of the numerical approximation of pressure. Moreover, we were able to gain one order of convergence for the numerical approximation of displacement, estimating the error in the L^2-norm when compared to the error in the H^1-norm.

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