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THE SYMMETRIC SEMI-CLASSICAL ORTHOGONAL POLYNOMIALS OF CLASS TWO AND SOME OF THEIR EXTENSIONS

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Abstract: We study symmetric sequences of orthogonal polynomials of class two related to Stieltjes functions satisfying a Riccati type differential equation with polynomial coefficients. We show difference equations for the recurrence coefficients of the orthogonal polynomials as well as for related quantities. Some of such recurrences are identified with discrete Painlevé equations.

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1. Motivation and preliminary results

Symmetric orthogonal polynomials on the real line, $\{P_n(x) = x^n + \dots\}_{n\geq 0}$, are characterized in terms of the three term recurrence relation

$$P_{n+1}(x) = xP_n(x) - \gamma_n P_{n-1}(x)\,, \quad n = 0, 1, \dots\,, \tag{1}$$

with $P_{-1}(x) = 0$, $P_0(x) = 1$. The parameters γ_n, known as recurrence relation coefficients, satisfy $\gamma_n \neq 0$, $n \geq 1$. Integrating with respect to the orthogonality measure, μ, gives us the representation

$$\gamma_n = \frac{1}{h_n}\int_I xP_nP_{n-1}(x)d\mu(x)\,, \quad h_n = \int_I P_{n-1}^2(x)d\mu(x)\,, \ \ n \geq 1\,.$$

Here, I is the support of μ.

A very well-known class of symmetric orthogonal polynomials is the one related to semi-classical weights, characterized through a differential equation

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known as Pearson equation [19],

$$\frac{1}{w(x)}\frac{d}{dx}w(x)=\frac{C(x)}{A(x)}\,,$$

with the property of symmetry in w. A systematic study of symmetric semi-classical weights and the corresponding orthogonal polynomials, began with G. Freud in the 1970's (see [18]). Weights of the form

$$w(x)=\exp(-Q(x))\,,$$

with Q an even, non-negative and continuous real valued function defined on the real line (satisfying certain conditions involving its first and second derivatives), are nowadays commonly known as Freud-type weights. The cases $Q(x)=|x|^m$, $m\in\mathbb{N}$, have been extensively studied, main references and results can be found in the introduction section of [11]. A common topic of research concerns the derivation and study of the systems of non-linear difference equations satisfied by the recurrence relation coefficients of the corresponding orthogonal polynomials. These systems of recurrences are known, at least since the works of A.P. Magnus [22, 23] as the Laguerre-Freud equations (see also [5, 20]).

The Laguerre-Freud equations for orthogonal polynomials related to semi-classical weights are often identified with discrete forms of Painlevé equations [31]. Early examples of such identification concern the weight studied by G. Freud [18],

$$w(x,t)=\exp(-x^4+tx^2)\,,\quad x\in\mathbb{R}\,, \tag{2}$$

where t is a parameter, and the case $t=0$ in (2), studied by J. Shohat in [29]. Here, the Laguerre Freud equations are $4\gamma_n(\gamma_{n-1}+\gamma_n+\gamma_{n+1}-\frac{t}{2})=n$, $n=1,2,\ldots$, which are a form of dPI (see [2, 17]). Many other examples of discrete Painlevé equations for the recurrence relation coefficients of orthogonal polynomials have been studied (see [6, 14, 16, 24]). Applications of Laguerre-Freud equations to the study of asymptotics for the orthogonal polynomials, properties of zeroes, estimates for derivatives, inequalities, etc, can be found in a vast list of references (see, amongst many others, [2, 11, 32] and its lists of references).

In this paper we shall consider extensions of semi-classical orthogonal polynomials. We will take the family of Laguerre-Hahn orthogonal polynomials

[13, 21, 26, 33], that is, the sequences of orthogonal polynomials whose Stieltjes function satisfies a Riccati type differential equation with polynomial coefficients,

$$AS' = BS^2 + CS + D\,, \quad A \neq 0\,, \tag{3}$$

where A, B, C, D are co-prime. On a general setting, S is the formal moment generating function, defined through the asymptotic expansion

$$S(x) = \sum_{n=0}^{+\infty} u_n x^{-n-1}\,, \tag{4}$$

given the moments $(u_n)_{n\geq 0}$ of the orthogonality measure. Here, we take, without loss of generality, the normalized sequence of moments, that is, $u_0 = 1$ [33]. The Laguerre-Hahn families of orthogonal polynomials include, as special cases, the semi-classical orthogonal polynomials as well as their standard modifications [12, 33]. The semi-classical case appears whenever $B \equiv 0$ in (3).

The study of Laguerre-Freud equations for Laguerre-Hahn orthogonal polynomials has been done for several instances of the polynomials A, B, C, D in (3) (see [1, 7, 14, 15, 28]). In this paper we focus on the symmetric class two, that is, we take $s = 2$ in [1, Prop. 3.1], thus, we consider the symmetric case under the bounds

$$\max\{\deg(C) - 1, \max\{\deg(A), \deg(B)\} - 2\} = 2 \tag{5}$$

in equation (3). In the symmetric case, the moments in (4) satisfy $u_{2n-1} = 0,\ n \geq 1$. We shall take sequences of monic orthogonal polynomials, $P_n(x) = x^n+$ lower degree terms, $n \geq 0$, satisfying (1), and we denote them by SMOP. We also consider the sequence of associated polynomials of the first kind, $\{P_n^{(1)}\}_{n\geq 0}$, satisfying a three-term recurrence relation

$$P_n^{(1)}(x) = xP_{n-1}^{(1)}(x) - \gamma_n P_{n-2}^{(1)}(x)\,, \quad n = 1, 2, ... \tag{6}$$

with $P_{-1}^{(1)}(x) = 0,\ P_0^{(1)}(x) = 1$. Combining the recurrence relations (1) and (6) in the matrix form, yields

$$Y_n = \mathcal{A}_n Y_{n-1}\,, \quad Y_n = \begin{bmatrix} P_{n+1} & P_n^{(1)} \\ P_n & P_{n-1}^{(1)} \end{bmatrix}, \quad \mathcal{A}_n = \begin{bmatrix} x & -\gamma_n \\ 1 & 0 \end{bmatrix}, \quad n \geq 1\,,$$

with initial conditions

$$Y_0 = \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}. \tag{7}$$

With the matrices Y_n defined above, the SMOP related to the Riccati equation (3), $AS' = BS^2 + CS + D$, satisfy differential systems that can be put into the matrix form as the matrix Sylvester equation [8],

$$AY_n' = \mathcal{B}_n Y_n - Y_n \mathcal{C} \,, \quad n \geq 0 \,, \tag{8}$$

where $\mathcal{C} = \begin{bmatrix} C/2 & -D \\ B & -C/2 \end{bmatrix}$ and the matrices $\mathcal{B}_n$ are defined in terms of polynomials l_n, Θ_n of uniformly bounded degrees,

$$\mathcal{B}_n = \begin{bmatrix} l_n & \Theta_n \\ -\Theta_{n-1}/\gamma_n & l_{n-1} + x\Theta_{n-1}/\gamma_n \end{bmatrix} \,,\ n \geq 1 \,,\ \mathcal{B}_0 = \begin{bmatrix} l_0 & \Theta_0 \\ -\Theta_{-1} & l_{-1} + x\Theta_{-1} \end{bmatrix} , \tag{9}$$

where, in the account of (7), the following initial conditions hold:

$$\Theta_{-1} = D \,,\ \Theta_0 = A + x(C/2 - l_0) + B \,, \tag{10}$$

$$l_{-1} = C/2 \,,\ l_0 = -C/2 - xD \,. \tag{11}$$

Furthermore, combining the recurrence relation (1) with the differential system (8) yields the Lax pair

$$\begin{cases} Y_n = \mathcal{A}_n Y_{n-1} \,, \\ AY_n' = \mathcal{B}_n Y_n - Y_n \mathcal{C} \,, \end{cases}$$

and, consequently, we get the compatibility conditions for the matrices $\mathcal{A}_n$,

$$A\mathcal{A}_n' = \mathcal{B}_n \mathcal{A}_n - \mathcal{A}_n \mathcal{B}_{n-1} \,, \quad n \geq 1 \,. \tag{12}$$

In turn, equations (12) yield the following identities:

$$\operatorname{tr} \mathcal{B}_n = 0 \,, \quad n \geq 0 \,, \tag{13}$$

$$\det \mathcal{B}_n = \det \mathcal{B}_0 + A \sum_{k=1}^{n} \frac{\Theta_{k-1}}{\gamma_k} \,, \quad n \geq 1 \,, \tag{14}$$

where $\det \mathcal{B}_0 = D(A+B) - (C/2)^2$.

Let us emphasize that the Sylvester equations (8) can be regarded as an extension of the differential systems for semi-classical orthogonal polynomials in [24, Eq. 17]. Equation (14) is the analogue of A.P. Magnus' summation formula [24, Eq. 20].

The differential systems enclosed by (12), together with the identities (13) and (14), will be our main tools to deduce difference equations for the recurrence coefficients γ_n as well as for other relevant coefficients of the orthogonal polynomials related to the Riccati equation (3). Let us also emphasize that, in the semi-classical case ($B \equiv 0$ in (3)), other methods to get the difference equations for the recurrence coefficients are available, for instance, the Ladder Operator technique [4, Section 4], and the Riemann-Hilbert method [6, 32].

The reminder of the paper is organized as follows. In Section 2 we deduce recurrences involving the coefficients γ_n. We stress the results in Theorems 4 and 5, where we deduce discrete Painlevé equations when $\deg(A) = 0$ and $\deg(A) = 2$; it is deduced a d-PI and d-PII, respectively. In section 3 we give applications of the previous results, we show examples related to semi-classical as well as to non semi-classical orthogonal polynomials.

2. The symmetric Laguerre-Hahn class two

2.1. Fundamental quantities. In this section we derive fundamental quantities to be used throughout the paper.

Henceforth we will use the following convention: if $i > j$, then $\sum_i^j \cdot = 0$.

Taking into account the recurrence relations (1) and (6), we obtain the expansions given in the following lemma.

Lemma 1. *Let $\{P_n\}_{n\geq 0}$ be a symmetric SMOP. The following expansions hold, for all $n \geq 1$:*

$$P_{n+1}(x) = x^{n+1} + \mathbf{p}_1(n+1)x^{n-1} + p_{n+1,n-3}x^{n-3} + \dots , \tag{15}$$

$$P_n^{(1)}(x) = x^n + \nu_n x^{n-2} + p_{n,n-4}^{(1)} x^{n-4} + \dots , \tag{16}$$

with

$$\mathbf{p}_1(n+1) = -\sum_{k=1}^{n} \gamma_k , \quad \mathbf{p}_1(1) = 0 , \quad \nu_n = -\sum_{k=2}^{n} \gamma_k , \tag{17}$$

and

$$p_{n+1,n-3} = \gamma_1\gamma_3 + \sum_{k=4}^{n} \gamma_k(\gamma_1 + \cdots + \gamma_{k-2}) , \quad n \geq 3 , \tag{18}$$

$$p_{n,n-4}^{(1)} = \gamma_2\gamma_4 + \sum_{k=5}^{n} \gamma_k(\gamma_2 + \cdots + \gamma_{k-2}) , \quad n \geq 4 . \tag{19}$$

Also, the following relation holds:

$$p_{n+1,n-3} = -\gamma_1(\gamma_1 + \gamma_2) - \gamma_1\mathbf{p}_1(n+1) + p_{n,n-4}^{(1)} . \tag{20}$$

Recall that throughout the paper we are considering the Riccati equation $AS' = BS^2 + CS + D$ in the symmetric setting with the sequence of moments (u_n) normalized, that is, $u_{2n-1} = 0, \ \geq 1, \ u_0 = 1$. Let the bounds (5) hold. According to [1, Prop. 3.1], A, B must be even, and C odd. Let us write

$$A(x) = a_4x^4 + a_2x^2 + a_0 \,, \quad B(x) = b_4x^4 + b_2x^2 + b_0 \,, \tag{21}$$

$$C(x) = c_3x^3 + c_1x \,, \quad D(x) = d_2x^2 + d_0 \,. \tag{22}$$

The polynomial D is defined in terms of A, B, C as follows:

$$d_2 = -a_4 - b_4 - c_3 \,, \quad d_0 = -a_2 - b_2 - c_1 - \gamma_1(3a_4 + 2b4 + c_3) \,. \tag{23}$$

The data from (23) is obtained by equating coefficients of x^{n+3} and x^{n+1}, respectively, from the equation enclosed in position $(1, 2)$ from (8),

$$A\left(P_n^{(1)}\right)' = (l_n + C/2)P_n^{(1)} + \Theta_n P_{n-1}^{(1)} + DP_{n+1} \,. \tag{24}$$

Indeed, d_2 is determined once we use $l_{n,3}$ given by (25) and d_0 is determined once we use $l_{n,1}$ given by (26).

Furthermore, the parameter γ_1 is related to the moment of order two. Indeed, the coefficient of x^0 in (3) gives us, in the account of the asymptotic expansion (4) with $u_0 = 1$, $d_0 = -a_2 - b_2 - c_1 - (3a_4 + 2b_4 + c_3)u_2$. This, combined with d_0 given in (23), yields $\gamma_1 = u_2$.

Lemma 2. *Let S be a Stieltjes function satisfying $AS' = BS^2 + CS + D$ with A, B, C, D given as in (21)–(22), with D given through (23). Let $\{P_n\}_{n\geq 0}$ be the symmetric SMOP associated with S, satisfying the recurrence relation $P_{n+1}(x) = xP_n(x) - \gamma_n P_{n-1}(x) \,, \quad n = 0, 1, 2, \ldots$. The polynomials l_n, Θ_n in (9) are defined by*

$$l_n(x) = l_{n,3}x^3 + l_{n,1}x \,, \quad \Theta_n(x) = \Theta_{n,2}x^2 + \Theta_{n,0} \,.$$

where, for all $n \geq 1$,

$$l_{n,3} = (n+1)a_4 + b_4 + c_3/2 \,, \tag{25}$$

$$l_{n,1} = -2a_4\mathbf{p}_1(n+1) + (n+1)a_2 + \lambda - \Theta_{n,2} \,, \tag{26}$$

$$\frac{\Theta_{n,2}}{\gamma_{n+1}} = -((2n+3)a_4 + \mu) \,, \tag{27}$$

$$\Theta_{n,0} = -\gamma_n\Theta_{n,2} - 2a_2\mathbf{p}_1(n+1) + 2a_4\mathbf{p}_1^2(n+1) - 4a_4p_{n+1,n-3} + (n+1)a_0 + \tau \,, \tag{28}$$

where

$$\lambda=\gamma_1 b_4+b_2+c_1/2\,,\quad \mu=2b_4+c_3\,,\quad \tau=b_4(\gamma_1+\gamma_2)\gamma_1+b_2\gamma_1+b_0\,. \tag{29}$$

Alternatively, $\Theta_{n,0}$ *is given, for all* $n\geq 1$*, by*

$$\frac{\Theta_{n,0}}{\gamma_{n+1}}=4a_4\mathbf{p}_1(n+1)-(2n+3)a_2-2\lambda-((2n+5)a_4+\mu)(\gamma_{n+2}+\gamma_{n+1})\,. \tag{30}$$

Also, we have the initial conditions

$$l_{0,3}=a_4+b_4+\frac{c_3}{2}\,,\quad l_{0,1}=a_2+b_2+\frac{c_1}{2}+(3a_4+2b_4+c_3)\gamma_1\,, \tag{31}$$

$$\frac{\Theta_{0,2}}{\gamma_1}=-3a_4-2b_4-c_3\,, \tag{32}$$

$$\frac{\Theta_{0,0}}{\gamma_1}=-3a_2-2b_2-c_1-(5a_4+c_3+3b_4)\gamma_1-(5a_4+\mu)\gamma_2\,. \tag{33}$$

Proof: Take the condition enclosed by position $(1,1)$ in (8),

$$AP'_{n+1}=(l_n-C/2)P_{n+1}+\Theta_nP_n-BP_n^{(1)}\,. \tag{34}$$

For all $n\geq 1$, the coefficient of x^{n+4} gives us (25). The coefficient of x^{n+3} gives us

$$l_{n,2}=0\,,\quad n\geq 0\,. \tag{35}$$

For all $n\geq 1$, the coefficient of x^{n+2} gives us (26). The coefficient of x^{n+1} gives us

$$l_{n,0}=-\Theta_{n,1}\,,\quad n\geq 0\,. \tag{36}$$

For all $n\geq 1$, the coefficient of x^n gives us (28).

On the other hand, let us take (13) for $n\geq 1$. It reads

$$l_n(x)+l_{n-1}(x)+x\frac{\Theta_{n-1}(x)}{\gamma_n}=0\,,\quad n\geq 1\,. \tag{37}$$

The coefficient of x^2 in (37) gives us, in the account of (35),

$$\frac{\Theta_{n-1,1}}{\gamma_n}=0,\quad n\geq 1\,.$$

Therefore, from (36), we obtain

$$l_{n,0}=0\,,\quad n\geq 0\,.$$

The coefficient of x^3 in (37) yields $l_{n,3}+l_{n-1,3}+\dfrac{\Theta_{n-1,2}}{\gamma_n}=0$, from which we get (27).

The alternative form for $\Theta_{n,0}$ is obtained from the coefficient of x in (37), $l_{n,1}+l_{n-1,1}+\frac{\Theta_{n-1,0}}{\gamma_n}=0$, from which we get (30).

To obtain (31), we use $l_0=-C/2-xD$ (cf. (11)), thus we get

$$l_{0,3}=-\frac{c_3}{2}-d_2\,,\quad l_{0,1}=-\frac{c_1}{2}-d_0\,, \tag{38}$$

which gives the required identities.

To get (32) and (33) we use (37) with $n=1$, hence,

$$\frac{\Theta_{0,2}}{\gamma_1}=-l_{1,3}-l_{0,3}\,,\quad \frac{\Theta_{0,0}}{\gamma_1}=-l_{1,1}-l_{0,1}\,, \tag{39}$$

which gives the required identities. ■

Lemma 3. *Let the previous notations hold. The coefficients γ_2 and γ_3 are defined in terms of γ_1 through the following equations:*

$$\frac{a_0+b_0}{\gamma_1}=-3a_2-2b_2-c_1-(5a_4+c_3+3b_4)\gamma_1-(5a_4+\mu)\gamma_2\,, \tag{40}$$

$$((-4a_4-2b_4)\gamma_1-5a_2-2b_2-c_1-(7a_4+\mu)(\gamma_3+\gamma_2))\gamma_2=a_0+\gamma_1 d_0. \tag{41}$$

Proof: From (10) we get $\Theta_{0,0}=a_0+b_0$, which we combine with (33), thus getting (40).

Taking $n=1$ in equation (24) and equating the independent term, we get $a_0=\Theta_{1,0}-d_0\gamma_1$. Thus, we get (41). ■

Remark . Alternatively, (40) can be obtained as follows: the coefficient of x^n in (34) and the coefficient of x^{n-1} in (24) give us, respectively, after computations where we use (20),

$$\gamma_n\Theta_{n,2}+\Theta_{n,0}=-2a_2\mathbf{p}_1(n+1)+2a_4\mathbf{p}_1^2(n+1)$$
$$-4a_4p_{n+1,n-3}+(n+1)a_0+\tau\,, \tag{42}$$

$$\gamma_n\Theta_{n,2}+\Theta_{n,0}=-2a_2\mathbf{p}_1(n+1)+2a_4\mathbf{p}_1^2(n+1)$$
$$-4a_4p_{n+1,n-3}+na_0+\tilde{\tau}\,, \tag{43}$$

with

$$\tau=b_4(\gamma_1+\gamma_2)\gamma_1+b_2\gamma_1+b_0,$$
$$\tilde{\tau}=-(3a_2+b_2+c_1+(5a_4+c_3+2b_4)\gamma_1+(5a_4+c_3+b_4)\gamma_2)\gamma_1\,.$$

Equating (42) with (43), we get $a_0+\tau=\tilde{\tau}$, thus obtaining (40).

2.2. Difference equations for γ_n and $\mathbf{p}_1(n)$.

Theorem 1. *Let S be a Stieltjes function satisfying $AS' = BS^2+CS+D$ with A, B, C, D given as in (21)–(22), with D given through (23). Let $\{P_n\}_{n\geq 0}$ be the symmetric SMOP associated with S, satisfying the recurrence relation (1), $P_{n+1}(x) = xP_n(x) - \gamma_n P_{n-1}(x)\,,\quad n = 0,1,2,\ldots$. Let the previous notations hold. The recurrence coefficients γ_n satisfy the following equation:*

$$\gamma_{n+1}T_{n+1} = \gamma_n T_{n-1} + a_0\,,\quad n \geq 2\,, \tag{44}$$

with

$$T_n = 4a_4\mathbf{p}_1(n) - (2n+1)a_2 - 2\lambda - ((2n+3)a_4 + \mu)(\gamma_{n+1} + \gamma_n)\,,\quad n \geq 2\,,$$

and the initial condition $T_1 = \frac{\Theta_{0,0}}{\gamma_1}$. The quantities λ, μ are given in (29).

Proof: The independent term of the equation enclosed in position $(1,1)$ of (12), that is,

$$A = x(l_n - l_{n-1}) + \Theta_n - \gamma_n\frac{\Theta_{n-2}}{\gamma_{n-1}}\,, \tag{45}$$

gives us

$$a_0 = \Theta_{n,0} - \gamma_n\frac{\Theta_{n-2,0}}{\gamma_{n-1}}\,. \tag{46}$$

Hence, we have

$$a_0 = \gamma_{n+1}T_{n+1} - \gamma_n T_{n-1}\,,$$

with the identification $T_n = \frac{\Theta_{n-1,0}}{\gamma_n}$, with $\frac{\Theta_{n-1,0}}{\gamma_n}$ given by (30). ∎

Corollary 1. *Take $a_0 = 0$ in Theorem 1. The quantities T_n satisfy*

$$T_{n+1}T_n = \frac{\gamma_2 T_2 T_1}{\gamma_{n+1}}\,,\quad n \geq 2\,. \tag{47}$$

Proof: If $a_0 = 0$, then from (44) we get

$$\gamma_{n+1}T_{n+1}T_n = \gamma_n T_n T_{n-1}\,,\ \ n \geq 2\,.$$

Iteration gives us

$$\gamma_{n+1}T_{n+1}T_n = \gamma_2 T_2 T_1\,.$$

Thus, we get (47). ∎

Corollary 2. *The recurrence coefficients γ_n may be determined recursively through the following equation:*

$$\gamma_{n+2} = \frac{-a_0 + E_n\gamma_{n+1} - F_n\gamma_n}{((2n+5)a_4+\mu)\gamma_{n+1}}\,, \quad n \geq 2\,, \tag{48}$$

$$\gamma_3 = -\gamma_2 - \frac{(a_0+\gamma_1 d_0)/\gamma_2 + (4a_4+2b_4)\gamma_1 + 5a_2 + 2b_2 + c_1}{7a_4+\mu}\,,$$

$$\gamma_2 = -\frac{(a_0+b_0)/\gamma_1 + (5a_4+c_3+3b_4)\gamma_1 + 3a_2 + 2b_2 + c_1}{5a_4+\mu}\,,$$

with

$$E_n = 4a_4\mathbf{p}_1(n+1) - (2n+3)a_2 - 2\lambda - ((2n+5)a_4+\mu)\gamma_{n+1}\,, \tag{49}$$

$$F_n = 4a_4\mathbf{p}_1(n-1) - (2n-1)a_2 - 2\lambda - ((2n+1)a_4+\mu)(\gamma_n+\gamma_{n-1})\,. \tag{50}$$

Proof: Solving (44) for γ_{n+2} gives us (48). The coefficients γ_2 and γ_3 are given from (40) and (41). ■

Corollary 3. *If $a_4 \neq 0$, the coefficient $\mathbf{p}_1(n)$ is determined in terms of $\gamma_{n+2}, \gamma_{n+1}, \gamma_n, \gamma_{n-1}$ through the following equation:*

$$\mathbf{p}_1(n) = \frac{a_0 + G_n\gamma_{n+1} - H_n\gamma_n}{4a_4(\gamma_{n+1}-\gamma_n)}\,, \quad n \geq 2\,, \tag{51}$$

with

$$G_n = 4a_4\gamma_n + (2n+3)a_2 + 2\lambda + ((2n+5)a_4+\mu)(\gamma_{n+2}+\gamma_{n+1})\,, \tag{52}$$

$$H_n = -4a_4\gamma_{n-1} + (2n-1)a_2 + 2\lambda + ((2n+1)a_4+\mu)(\gamma_n+\gamma_{n-1}). \tag{53}$$

Furthermore, we have:

$$\mathbf{p}_1(n-1) = \frac{a_0 + \tilde{G}_n\gamma_{n+1} - \tilde{H}_n\gamma_n}{4a_4(\gamma_{n+1}-\gamma_n)}\,, \quad n \geq 2\,, \tag{54}$$

with

$$\tilde{G}_n = G_n + 4a_4\gamma_{n-1}\,, \quad \tilde{H}_n = H_n + 4a_4\gamma_{n-1}. \tag{55}$$

Proof: Using $\mathbf{p}_1(n+1) = \mathbf{p}_1(n) - \gamma_n$ and $\mathbf{p}_1(n-1) = \mathbf{p}_1(n) + \gamma_{n-1}$ in (44) and solving for $\mathbf{p}_1(n)$ we get (51).

Using $\mathbf{p}_1(n+1) = \mathbf{p}_1(n-1) - \gamma_{n-1} - \gamma_n$ in (44) and solving for $\mathbf{p}_1(n-1)$ we get (54). ■

Doing the shift $n \to n-1$ in (51) and equating to (54) we get a fourth order difference equation for γ_n, as stated in the following corollary.

Corollary 4. *The recurrence coefficients γ_n satisfy the fourth order difference equation*

$$\gamma_{n+2} = \frac{\mathcal{F}(\gamma_{n+1},\gamma_n,\gamma_{n-1},\gamma_{n-2})}{((2n+5)a_4+\mu)\gamma_{n+1}(\gamma_{n-1}-\gamma_n)}\,, \quad n\geq 2\,,$$

with

$$\begin{aligned}\mathcal{F}(\gamma_{n+1},\gamma_n,\gamma_{n-1},\gamma_{n-2}) &= -a_0(\gamma_{n+1}-2\gamma_n+\gamma_{n-1})\\ &+\gamma_{n+1}\gamma_n\left[2a_4\gamma_{n+1}-((2n-1)a_4+\mu)\gamma_n+2a_2\right]\\ +\gamma_{n+1}\gamma_{n-1}&\left[-((2n+5)a_4+\mu)\gamma_{n+1}-4a_4\gamma_n+((2n-5)a_4+\mu)(\gamma_{n-1}+\gamma_{n-2})-6a_2\right]\\ &+\gamma_n^2\left[((2n+3)a_4+\mu)\gamma_{n+1}+2a_4\gamma_n-((2n-3)a_4+\mu)\gamma_{n-1}+2a_2\right]\\ &+\gamma_n\gamma_{n-1}\left[((2n+1)a_4+\mu)\gamma_n+2a_4\gamma_{n-1}-((2n-5)a_4+\mu)\gamma_{n-2}+2a_2\right]\,.\end{aligned}$$

Theorem 2. *Let S be a Stieltjes function satisfying $AS'=BS^2+CS+D$ with A,B,C,D given as in (21)–(22), with D given through (23). Let $\{P_n\}_{n\geq 0}$ be the symmetric SMOP associated with S, satisfying the recurrence relation (1), $P_{n+1}(x)=xP_n(x)-\gamma_nP_{n-1}(x)\,,\quad n=0,1,2,\ldots$. Let the previous notations hold, as well as*

$$\mu_n=(2n+1)a_4+\mu\,,\qquad \lambda_n=(2n+1)a_2+2\lambda\,,$$

where λ and μ are given in (29). The coefficients $\mathbf{p}_1(n)$ given in (17) satisfy the following quadratic equations:

$$4a_4^3\,\mathbf{p}_1^2(n+1)+B_{n+1}\mathbf{p}_1(n+1)+C_{n+1}=0\,, \tag{56}$$
$$16a_4^3\gamma_{n+1}\,\mathbf{p}_1^2(n+1)+\tilde{B}_{n+1}\mathbf{p}_1(n+1)+\tilde{C}_{n+1}=0\,, \tag{57}$$

where

$$B_{n+1}=4a_4\left[a_2\left(\frac{c_3}{2}+b_4\right)-a_4\lambda+a_4\mu_n\gamma_{n+1}\right]\,, \tag{58}$$
$$\begin{aligned}C_{n+1}=-a_4&\left[\mu_n\mu_{n+2}\gamma_{n+2}+\mu_n\mu_{n+2}\gamma_{n+1}+\mu_{n+1}\mu_{n-1}\gamma_n\right.\\ &\left.+\mu_n\lambda_{n+1}\right]\gamma_{n+1}+\tau_n\,,\end{aligned} \tag{59}$$
$$\begin{aligned}\tilde{B}_{n+1}=4a_4&\left[-a_4\mu_{n+2}\gamma_{n+2}\gamma_{n+1}-a_4(\mu_{n+1}+\mu_{n+2})\gamma_{n+1}^2\right.\\ &\left.+a_4(4a_4-\mu_{n+1})\gamma_{n+1}\gamma_n-a_4(\lambda_n+\lambda_{n+1})\gamma_{n+1}-a_0l_{n,3}\right]\,,\end{aligned} \tag{60}$$
$$\begin{aligned}\tilde{C}_{n+1}=\gamma_{n+1}&\left[a_4(\lambda_{n+1}+\mu_{n+2}(\gamma_{n+2}+\gamma_{n+1}))(\lambda_n+\mu_{n+1}\gamma_{n+1}+\mu_{n-1}\gamma_n)\right.\\ &\left.+\mu_{n+1}(2a_0l_{n,3}+a_0\mu_n)\right]+\tilde{\tau}_n\,,\end{aligned} \tag{61}$$

where

$$\tau_n = (a_4\xi_{0,2} - a_2\xi_{0,4}) + (a_0a_4 - a_2^2)\left(\frac{\Theta_{0,2}}{\gamma_1} - (n-1)((n+3)a_4+\mu)\right) - ((n+1)a_2+\lambda)\left(a_4((n+1)a_2-\lambda)+a_2\mu\right),$$

$$\tilde{\tau}_n = 2a_0 l_{n,3}((n+1)a_2+\lambda)+a_0\xi_{0,4}-a_4\xi_{0,0}+a_0a_2\left(\frac{\Theta_{0,2}}{\gamma_1} - (n-1)((n+3)a_4+\mu)\right).$$

Here, $\xi_{0,j}$ denotes the coefficient of x^j in $\det \mathcal{B}_0$.

Proof: Take the coefficients of x^4, x^2 and x^0 in (14), i.e., in equation

$$-l_n^2(x) + \Theta_n(x)\frac{\Theta_{n-1}(x)}{\gamma_n} = \det \mathcal{B}_0 + A\sum_{k=1}^{n}\frac{\Theta_{k-1}}{\gamma_k}, \quad n \geq 1. \tag{62}$$

We get, respectively,

$$-2l_{n,1}l_{n,3} + \Theta_{n,2}\frac{\Theta_{n-1,2}}{\gamma_n} = \xi_{0,4} + a_4\sum_{k=1}^{n}\frac{\Theta_{k-1,0}}{\gamma_k} + a_2\sum_{k=1}^{n}\frac{\Theta_{k-1,2}}{\gamma_k}, \tag{63}$$

$$-l_{n,1}^2 + \Theta_{n,2}\frac{\Theta_{n-1,0}}{\gamma_n} + \Theta_{n,0}\frac{\Theta_{n-1,2}}{\gamma_n} = \xi_{0,2} + a_2\sum_{k=1}^{n}\frac{\Theta_{k-1,0}}{\gamma_k} + a_0\sum_{k=1}^{n}\frac{\Theta_{k-1,2}}{\gamma_k} \tag{64}$$

$$\Theta_{n,0}\frac{\Theta_{n-1,0}}{\gamma_n} = \xi_{0,0} + a_0\sum_{k=1}^{n}\frac{\Theta_{k-1,0}}{\gamma_k}. \tag{65}$$

Eliminating $\sum_{k=1}^{n}\frac{\Theta_{k-1,0}}{\gamma_k}$ between (63) and (64), and using the data from Lemma 2, we get, after simplifications, (56). Eliminating $\sum_{k=1}^{n}\frac{\Theta_{k-1,0}}{\gamma_k}$ between (63) and (65), and using the data from Lemma 2, we get, after simplifications, (57). ■

Theorem 3. *Let the notations and conditions of Theorem 2 hold. The recurrence coefficients γ_n satisfy the following second order difference equation:*

$$\sum_{p=0}^{3}\sum_{q=0}^{6}\sum_{r=0}^{3} c_{p,q,r}\gamma_n^p\gamma_{n+1}^q\gamma_{n+2}^r = 0. \tag{66}$$

Proof: Eliminating the quadratic term between (56) and (57), we get

$$\mathbf{p}_1(n+1) = \frac{\tilde{C}_{n+1} - 4\gamma_{n+1}C_{n+1}}{4\gamma_{n+1}B_{n+1} - \tilde{B}_{n+1}},$$

from which we obtain, by substitution into (56), the equation

$$4a_4^3(\tilde{C}_{n+1}-4\gamma_{n+1}C_{n+1})^2+B_{n+1}\left(\tilde{C}_{n+1}-4\gamma_{n+1}C_{n+1}\right)\left(4\gamma_{n+1}B_{n+1}-\tilde{B}_{n+1}\right)\\ +C_{n+1}(4\gamma_{n+1}B_{n+1}-\tilde{B}_{n+1})^2=0\,.$$

Therefore, we obtain (66) with coefficients $c_{p,q,r}$ defined in terms of λ_n,μ_n,τ_n, and $\tilde{\tau}_n$. ■

2.3. Difference equations of the Painlevé type. In this subsection we derive difference equations of the Painlevé type for symmetric Laguerre-Hahn orthogonal polynomials of class two. The fundamental tools are the identities for the trace and determinant given by (13) and (14). Recall these identities reading as (37) and (62), respectively:

$$l_n(x)+l_{n-1}(x)+x\frac{\Theta_{n-1}(x)}{\gamma_n}=0\,,\quad n\geq 0\;,$$

$$-l_n^2(x)+\Theta_n(x)\frac{\Theta_{n-1}(x)}{\gamma_n}=\det\mathcal{B}_0+A\sum_{k=1}^{n}\frac{\Theta_{k-1}}{\gamma_k}\,,\quad n\geq 1\,.$$

In what follows we consider the cases $\deg(A)=0$ and $\deg(A)=2$ in the Riccati equation $AS'=BS^2+CS+D$. Without loss of generality, the polynomial A will be taken as monic.

Theorem 4. *Let S be a Stieltjes function satisfying $AS'=BS^2+CS+D$ with*

$$A(x)=1\,,\quad B(x)=b_4x^4+b_2x^2+b_0\,,\quad C(x)=c_3x^3+c_1x\,,\quad D(x)=d_2x^2+d_0\,,\tag{67}$$

with d_2,d_0 given in (23). Let $\{P_n\}_{n\geq 0}$ be the symmetric SMOP associated with S, satisfying the recurrence relation $P_{n+1}(x)=xP_n(x)-\gamma_nP_{n-1}(x)$, $n=0,1,2,\ldots$. The coefficients γ_n satisfy the discrete Painlevé I equation

$$\gamma_n\left(\mu(\gamma_{n-1}+\gamma_n+\gamma_{n+1})+2\lambda\right)=-n-\tau\,,\quad n\geq 1\,,\tag{68}$$

with λ,μ,τ given in (29),

$$\lambda=\gamma_1b_4+b_2+c_1/2\,,\quad \mu=2b_4+c_3\,,\quad \tau=b_4(\gamma_1+\gamma_2)\gamma_1+b_2\gamma_1+b_0\,.$$

Proof: From Lemma 2, we have

$$l_{n,1}=\lambda-\Theta_{n,2}\,,\quad \Theta_{n,0}=-\gamma_n\Theta_{n,2}+n+1+\tau\,,\quad \Theta_{n,2}=-\mu\gamma_{n+1}\,,$$

with λ, μ, τ given in (29). Using these equalities into the relation that follows from the linear term in the equation for the trace (37),

$$l_{n,1} + l_{n-1,1} + \frac{\Theta_{n-1,0}}{\gamma_n} = 0\,, \tag{69}$$

we obtain (68). ■

Theorem 5. *Let S be a Stieltjes function satisfying $AS' = BS^2 + CS + D$ with*

$$A(x) = x^2 + a_0\,,\ B(x) = b_4x^4 + b_2x^2 + b_0\,,\ C(x) = c_3x^3 + c_1x\,,\ D(x) = d_2x^2 + d_0\,, \tag{70}$$

with d_2, d_0 given in (23). Let $\{P_n\}_{n\geq 0}$ be the symmetric SMOP associated with S, satisfying the recurrence relation $P_{n+1}(x) = xP_n(x) - \gamma_n P_{n-1}(x)$, $n = 0, 1, 2, \ldots$.

If

$$4(a_0b_2 - b_0)d_0 = a_0^3\mu^2 - 4a_0^2\mu(1+\lambda) - 4a_0(d_2(a_0+b_0) - c_1^2/4)\,, \tag{71}$$

where $\lambda = \gamma_1 b_4 + b_2 + c_1/2$, $\mu = 2b_4 + c_3$, then the expression $x_n = n + 1 + \lambda + \mu\gamma_{n+1} - \frac{\mu a_0}{2}$ satisfies the discrete Painlevé II equation

$$(x_{n-1} + x_n)(x_n + x_{n+1}) = \frac{-4x_n^2}{(\hat{\lambda}x_n + z_n)}\,, \quad n \geq 1\,, \tag{72}$$

with $\hat{\lambda} = \frac{4}{\mu a_0}$, $z_n = 2 - \frac{4(n+1+\lambda)}{\mu a_0}$.

Proof: The independent term in the equation for the determinant (62) gives us

$$\sum_{k=1}^{n} \frac{\Theta_{k-1,0}}{\gamma_k} = \frac{1}{a_0}\left(\gamma_{n+1}\frac{\Theta_{n,0}}{\gamma_{n+1}}\frac{\Theta_{n-1,0}}{\gamma_n} - \xi_{0,0}\right).$$

Using the equation above as well as $\Theta_{n,2} = -\mu\gamma_{n+1}$, $n \geq 0$, into the coefficient of the quadratic term of (62),

$$-l_{n,1}^2 + \Theta_{n,2}\frac{\Theta_{n-1,0}}{\gamma_n} + \Theta_{n,0}\frac{\Theta_{n-1,2}}{\gamma_n} = \xi_{0,2} + a_2\sum_{k=1}^{n}\frac{\Theta_{k-1,0}}{\gamma_k} + a_0\sum_{k=1}^{n}\frac{\Theta_{k-1,2}}{\gamma_k}\,,$$

we obtain, after some computations,

$$-a_0 l_{n,1}^2 - \kappa_n = \mu a_0\gamma_{n+1}\left(\frac{\Theta_{n-1,0}}{\gamma_n} + \frac{\Theta_{n,0}}{\gamma_{n+1}}\right) + \gamma_{n+1}\frac{\Theta_{n,0}}{\gamma_{n+1}}\frac{\Theta_{n-1,0}}{\gamma_n}\,,$$

with $\kappa_n = a_0\xi_{0,2} - \mu a_0^2 n - \xi_{0,0}$. Thus, we have

$$\gamma_{n+1}(\mu a_0)^2 - a_0 l_{n,1}^2 - \kappa_n = \gamma_{n+1}\left(\frac{\Theta_{n-1,0}}{\gamma_n} + \mu a_0\right)\left(\frac{\Theta_{n,0}}{\gamma_{n+1}} + \mu a_0\right). \tag{73}$$

Taking into account that

$$l_{n,1} = n + 1 + \lambda + \mu\gamma_{n+1}\,, \tag{74}$$

the left hand side of (73) is quadratic in γ_{n+1}. Due to condition (71), the left hand side of (73) factorizes as

$$-a_0\mu^2(\gamma_{n+1} + \varepsilon_n)^2\,,$$

with $\varepsilon_n = \dfrac{2(n+1+\lambda) - \mu a_0}{2\mu}$.

Hence, we have

$$-a_0\mu^2(\gamma_{n+1} + \varepsilon_n)^2 = \gamma_{n+1}\left(\frac{\Theta_{n-1,0}}{\gamma_n} + \mu a_0\right)\left(\frac{\Theta_{n,0}}{\gamma_{n+1}} + \mu a_0\right). \tag{75}$$

Now we use the identities

$$\gamma_{n+1} = \frac{l_{n,1} - (n+1+\lambda)}{\mu}\,, \quad -(l_{n,1} + l_{n-1,1}) = \frac{\Theta_{n-1,0}}{\gamma_n}$$

(cf.(74) and (69)) into (75), thus obtaining

$$\begin{aligned}\left(l_{n,1} - \frac{\mu a_0}{2}\right)^2 = \frac{(l_{n,1} - (n+1+\lambda))}{-\mu a_0}&\left(\left(l_{n+1,1} - \frac{\mu a_0}{2}\right) + \left(l_{n,1} - \frac{\mu a_0}{2}\right)\right)\\ &\times\left(\left(l_{n,1} - \frac{\mu a_0}{2}\right) + \left(l_{n-1,1} - \frac{\mu a_0}{2}\right)\right).\end{aligned}$$

With the identification $x_n = l_{n,1} - \dfrac{\mu a_0}{2}$, the previous equation is written as

$$(x_{n+1} + x_n)(x_n + x_{n-1})(\tilde{\lambda}x_n + \tilde{z}_n) = x_n^2\,,$$

with $\tilde{\lambda} = -\frac{1}{\mu a_0}$, $\tilde{z}_n = \frac{n+1+\lambda-(\mu a_0)/2}{\mu a_0}$. Hence, we get the discrete Painlevé II

$$(x_{n+1} + x_n)(x_n + x_{n-1}) = \frac{-4x_n^2}{(\hat{\lambda}x_n + z_n)}\,,$$

with $\hat{\lambda} = \frac{4}{\mu a_0}$, $z_n = 2 - \frac{4(n+1+\lambda)}{\mu a_0}$. ∎

Remark . If no cancellations occur, condition (71) yields conditions on γ_1 (i.e, on the normalized moment of order two) for the Painlevé equation to hold.

3. Examples

A way of generating orthogonal polynomials with respect to a symmetric measure is doing a quadratic transformation from the weights related to the classical orthogonal polynomials (see [9]). Some of such transformations falling into the class two are given in the following examples.

3.1. Example 1. Let us take the modified Freud weight [18, 29],

$$w(x,t) = \exp(-x^4 + tx^2)\,, \quad x \in \mathbb{R}\,. \tag{76}$$

Here, t is a parameter, which, in some contexts, is interpreted as the time variable (see [24]).

w satisfies the Pearson equation $\dfrac{1}{w}\dfrac{d}{dx}w = \dfrac{C}{A}$, where $A(x) = 1$, $C(x) = -4x^3 + 2tx$. Thus, in our previous notations,

$$a_4 = a_2 = 0\,,\ a_0 = 1\,,\ c_3 = -4\,,\ c_1 = 2t\,. \tag{77}$$

We take the Stieltjes function related to w, satisfying $AS' = CS + D$, where $D(x) = d_2x^2 + d_0$, with $d_2 = -a_4 - c_3$, $d_0 = -a_2 - c_1 - (3a_4 + c_3)\gamma_1$. Thus, we have

$$d_2 = 4\,,\ \ d_0 = -2t + 4\gamma_1\,,$$

where γ_1 is the normalized moment of order two,

$$\gamma_1 = \frac{\int_{\mathbb{R}} x^2 \exp(-x^4 + tx^2)dx}{\int_{\mathbb{R}} \exp(-x^4 + tx^2)dx}\,. \tag{78}$$

In the account of the above data we have, from Lemma 2, $\lambda = t$, $\mu = -4$, $\tau = 0$. From Theorem 4, we have the following d-PI (see [17, 24]),

$$4\gamma_n\left(\gamma_{n-1} + \gamma_n + \gamma_{n+1} - \frac{t}{2}\right) = n\,, \quad n \geq 1\,, \tag{79}$$

with initial conditions $\gamma_0 = 0$ and γ_1 given by (78).

Let us note that $n = 1$ in (79) is compatible with (40) provided $\gamma_0 = 0$, and $n = 2$ in (79) agrees with (41).

3.2. Example 2. Let us take the modified Hermite weight [3, 25],

$$w(x,a) = \exp(-x^2)\,, \quad x \in I(a) =]-\infty, -a] \cup [a, +\infty[\,, \tag{80}$$

where a is some positive real number.

w satisfies the Pearson equation $\dfrac{1}{w}\dfrac{d}{dx}w = \dfrac{C}{A}$, where $A(x) = x^2 - a^2$, $C(x) = -2x^3 + 2a^2x$ (see [3]). Thus, in our previous notations,

$$a_4 = 0\,,\ a_2 = 1,\ a_0 = -a^2\,,\ c_3 = -2\,,\ c_1 = 2a^2\,. \tag{81}$$

We take the Stieltjes function related to w, satisfying $AS' = CS + D$, where $D(x) = d_2x^2 + d_0$, with $d_2 = -a_4 - c_3$, $d_0 = -a_2 - c_1 - (3a_4 + c_3)\gamma_1$. Thus, we have

$$d_2 = 2\,, \quad d_0 = -1 - 2a^2 + 2\gamma_1\,,$$

where γ_1 is the normalized moment of order two,

$$\gamma_1 = \frac{\int_{I(a)} x^2 \exp(-x^2)dx}{\int_{I(a)} \exp(-x^2)dx}\,. \tag{82}$$

The constants λ and μ in Theorem 5 are given by $\lambda = a^2$, $\mu = -2$. Note that condition (71) holds. Therefore, the expression $x_n = n + 1 - 2\gamma_{n+1}$ satisfies the following d-PII,

$$(x_{n-1} + x_n)(x_n + x_{n+1}) = \frac{-4x_n^2}{(\hat{\lambda}x_n + z_n)}\,, \quad n \geq 1\,, \tag{83}$$

with $\hat{\lambda} = \frac{2}{a^2}$, $z_n = -\frac{2(n+1)}{a^2}$. Indeed, the formula (83) holds for $n = 0$, under the initial conditions $x_{-1} = 0, x_0 = 1 - 2\gamma_1$, where γ_1 is given by (82).

3.3. Example 3. Let us take the modified Jacobi weight [4],

$$w(x,k) = (1-x^2)^{\alpha}(1-k^2x^2)^{\beta}\,,\ \ x \in [-1,1],\ \alpha > -1\,, \beta \in \mathbb{R}\,,\ \ k^2 \in]0,1[\,. \tag{84}$$

w satisfies the Pearson equation $\dfrac{1}{w}\dfrac{d}{dx}w = \dfrac{C}{A}$, where $A(x) = (x^2-1)(x^2 - 1/k^2)$, $C(x) = (2\alpha+2\beta)x^3 + (-2\alpha/k^2 - 2\beta)x$. Thus, in our previous notations,

$$a_4 = 1\,,\ a_2 = -(1+1/k^2)\,,\ a_0 = 1/k^2\,,\ c_3 = 2\alpha + 2\beta\,,\ c_1 = -2\alpha/k^2 - 2\beta\,.$$

We take the Stieltjes function related to w, satisfying $AS' = CS + D$, where $D(x) = d_2x^2 + d_0$, with $d_2 = -a_4 - c_3$, $d_0 = -a_2 - c_1 - (3a_4 + c_3)\gamma_1$. Thus, we have

$$d_2 = -1 - 2\alpha - 2\beta\,, \quad d_0 = (1 + 1/k^2) + 2\alpha/k^2 + 2\beta - (3 + 2\alpha + 2\beta)\gamma_1\,,$$

where γ_1 is the normalized moment of order two,

$$\gamma_1 = \frac{\int_{-1}^{1} x^2(1-x^2)^\alpha(1-k^2x^2)^\beta dx}{\int_{-1}^{1}(1-x^2)^\alpha(1-k^2x^2)^\beta dx}\,. \tag{85}$$

The coefficients γ_n of the SMOP related to (84) are governed by the difference equations described in Theorems 1 and 3, with $b_4 = b_2 = b_0$ in all formulae.

Remark . The difference equations given in Theorems 1 and 3 with $B \equiv 0$ were also derived in [4], using the ladder operator technique.

3.4. Further examples - non semi-classical orthogonal polynomials. Let us now look at some examples of orthogonal polynomials not semi-classical whose results from Section 2 apply.

Let us take the Stieltjes function related to the previous examples, satisfying the differential equation $AS' = CS + D$. We now consider the Stieltjes function $S^{(1)}$ [33]

$$\gamma_1 S^{(1)}(x) = -\frac{1}{S(x)} + x\,. \tag{86}$$

$S^{(1)}$ is the Stieltjes function related to the associated polynomials of the first kind, $\{P_n^{(1)}\}_{n\geq 0}$. As S satisfies the linear differential equation $AS' = CS+D$, then $S^{(1)}$ satisfies the Riccati equation

$$A_1\left(S^{(1)}\right)' = B_1\left(S^{(1)}\right)^2 + C_1 S^{(1)} + D_1\,, \tag{87}$$

with

$$A_1 = A\,,\ B_1 = \gamma_1 D\,,\ C_1 = -(C+2xD)\,,\ D_1 = (A+xC+x^2D+B)/\gamma_1\,, \tag{88}$$

Note that $\deg(D_1) = 2$. The degrees of the polynomials in (88) satisfy (5), therefore, the sequences of orthogonal polynomials $\{P_n^{(1)}\}_{n\geq 0}$ are Laguerre-Hahn sequences of class two. The recurrence coefficients of $\{P_n^{(1)}\}_{n\geq 0}$, satisfy the difference equations given in Theorems 1 and 3, as well as in Theorems 4 and 5.

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