Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 18–25

ANOTHER PROOF OF BANASCHEWSKI'S SURJECTION THEOREM

DHARMANAND BABOOLAL, JORGE PICADO AND ALEŠ PULTR

Abstract: We present a new proof of Banaschewski's theorem stating that the completion lift of a uniform surjection is a surjection. The new procedure allows to extend the fact (and, similarly, the related theorem on closed uniform sublocales of complete uniform frames) to quasi-uniformities ("not necessarily symmetric uniformities"). Further, we show how a (regular) Cauchy point on a closed uniform sublocale can be extended to a (regular) Cauchy point on the larger (quasi-)uniform frame.

Keywords: Frame (locale), sublocale, closed sublocale, uniform frame, quasi-uniform frame, uniform embedding, uniform dense embedding, complete uniform frame, completion, Cauchy map, Cauchy filter, Cauchy complete.

Math. Subject Classification (2010): 06D22, 54D35, 54E15.

Introduction

The present paper originated in our ignorance. Considering a certain sublocale of a complete uniform frame we came across the question whether a closed sublocale of a complete locale is complete, like in the classical setting; this then naturally led to the question whether the completion image of a uniform surjection is a surjection. After a somewhat superficial unsuccessful search in the literature we found a positive proof and wrote it down. Then it was pointed out to us by Themba Dube (to whom we are very thankful) that it was a result of Banaschewski from [1, Section 6]; on the other hand he kindly indicated that we might consider to publish the proof in some form anyway, and this is what we present.

Our proof differs from the Banaschewski's one by being direct in two aspects: (1) in [1] the closedness theorem was proved first and then used as a tool (in our procedure we work immediately with the image; the closedness

Received July 18, 2018.

The authors gratefully acknowledge support under Grant No. 113840 from the National Research Foundation (South Africa), the Centre for Mathematics of the University of Coimbra (UID/MAT/00324/2013 funded by FCT/MCTES and FEDER through the Partnership Agreement PT2020), and grants P202/12/G061 (Grant Agency of the Czech Republic) and MTM2015-63608-P (Ministry of Economy and Competitiveness of Spain).

theorem comes as a consequence), and (2) the regularity of the Cauchy maps adjoint to the completion homomorphism is the immediate feature of the reasoning.

Furthermore, we present the theorem in the quasi-uniform (not necessarily symmetric) setting of entourages which is not only a technical difference, but makes the results more general in the direction to non-symmetry (while, of course, the Banaschewski proof goes from uniformity in another generalizing direction, namely to strong nearnesses).

In addition, we also discuss the role played by closedness in extension of (regular) Cauchy points in the Cauchy completion.

After necessary Preliminaries (for the reader's convenience, somewhat more extensive in explaining the entourage technique) we discuss in Section 2 the complete lifting of uniform embeddings. Section 3 is devoted to the question of closed uniform sublocales of complete uniform frames. Learning that these are precisely the complete sublocales we learn that a Cauchy map on a closed sublocale is necessarily a homomorphism because it is extendable to a Cauchy map on the bigger space and this being complete makes it a homomorphism. This naturally leads to a question discussed in the last section: namely, a Cauchy map is a generalization of a Cauchy point (a Cauchy map into the two-point Boolean algebra); can a (regular) Cauchy point on a closed uniform sublocale of a uniform frame L be extended to a (regular) Cauchy point on the whole of L? The answer is in the affirmative; hence, in consequence, a closed uniform sublocale of a Cauchy complete uniform frame is Cauchy complete.

1. Preliminaries

For general background on frames and locales we refer to [6] or [11]. In this section we list the basic terminology, notions and facts used in the paper.

1.1. Frames and locales. A *frame* is a complete lattice L satisfying the distributive law

$$a \wedge (\bigvee B) = \bigvee \{a \wedge b \mid b \in B\} \tag{dist}$$

for all $a \in L$ and all subsets $B \subseteq L$. A *frame homomorphism* $h\colon L \to M$ preserves all joins (including the void one, the bottom 0) and all finite meets (including the top 1).

A typical frame is the lattice $\Omega(X)$ of all open sets of a topological space X; if $f\colon X \to Y$ is a continuous map then $\Omega(f) = (U \mapsto f^{-1}[U])\colon \Omega(Y) \to \Omega(X)$ is a frame homomorphism. Thus one has a contravariant functor $\Omega\colon \mathbf{Top} \to \mathbf{Frm}$ (where **Top** is the category of topological spaces and continuous maps and **Frm** the category of frames and frame homomorphisms).

Setting $\mathbf{Loc} = \mathbf{Frm}^{\mathrm{op}}$ one has the *category of locales*. Then Ω becomes a covariant functor $\mathbf{Top} \to \mathbf{Loc}$ (restricted to the subcategory of sober spaces it is a full embedding). It is useful to view **Loc** as a concrete category with the arrow opposite to a frame homomorphism $h\colon L \to M$ represented by its right Galois adjoint $f = h_*\colon M \to L$ (uniquely determined by the fact that h preserves all joins). Such f's will be referred to as *localic maps*.

With L a frame and $a \in L$, the map $a \wedge (\cdot)\colon L \to L$ preserves arbitrary joins and thus has a right adjoint $a \to (\cdot)\colon L \to L$ determined by $c \leq a \to b$ iff $a \wedge c \leq b$. Thus, $a \to b = \bigvee\{c \in L \mid a \wedge c \leq b\}$.

1.2. Sublocales. Subspaces of locales (viewed as generalized spaces) are represented by *sublocales*. A subset S of a frame L is a *sublocale* of L if, for any $A \subseteq S$, $x \in L$ and $a \in S$, we have $\bigwedge A \in S$ (in particular, $1 \in S$) and $x \to a \in S$. Sublocales are precisely such subsets for which the embedding map $j\colon S \to L$ is a (one-to-one) localic map — in fact, extremal monomorphisms in **Loc**.

The set $\mathcal{S}(L)$ of all sublocales of L, under inclusion, forms a *coframe* (i.e., a complete lattice satisfying the dual of (dist)), in which arbitrary infima coincide with intersections, $\{1\}$ is the bottom element and L is the top element.

For any $a \in L$, the sets

$$\mathfrak{c}(a) = {\uparrow}a \quad \text{and} \quad \mathfrak{o}(a) = \{a \to b \mid b \in L\}$$

are special sublocales of L (called, respectively *closed* and *open* since in the spatial case they correspond to closed and open subspaces). They are complements of each other in $\mathcal{S}(L)$.

1.3. Density. A frame homomorphism $h\colon L \to M$ is *dense* if

$$h(x) = 0 \;\Rightarrow\; x = 0,$$

and a sublocale $S \subseteq L$ is *dense* if $\overline{S} = L$. Here, of course, the closure $\overline{S}$, the smallest closed sublocale containing S, is ${\uparrow}(\bigwedge S)$ (since $\bigwedge S \in S$); thus $\overline{S} = L$ iff $0 \in S$.

Consider the localic map $f = h_*\colon M \to L$ associated with h. Then, obviously, h is dense iff $f(0) = 0$ iff $f[M]$ is dense in L.

1.4. Quotients of a frame. For a relation $R \subseteq L \times L$ on a frame L, call an $s \in L$ *saturated* (more precisely, *R-saturated*) if

$$aRb \quad \Rightarrow \quad \forall c,\ a \wedge c \leq s \ \text{ iff } \ b \wedge c \leq s.$$

A meet of saturated elements is saturated, we have a monotone mapping $\kappa = (x \mapsto \kappa(x)) = \bigwedge\{s \mid x \leq s,\ s \text{ saturated}\}$ satisfying $x \leq \kappa(x)$, $\kappa\kappa(x) = \kappa(x)$ and, moreover, $\kappa(x \wedge y) = \kappa(x) \wedge \kappa(y)$ (the *nucleus* of R), and if we set

$$L/R = \{x \mid x = \kappa(x)\}$$

we obtain a frame homomorphism $\kappa' = (x \mapsto \kappa(x))\colon L \to L/R$ satisfying (see e.g. [11]):

(1.4.1) $xRy \ \Rightarrow \ \kappa'(x) = \kappa'(y)$,

(1.4.2) if for $h\colon L \to K$ in **Frm** one has $xRy \Rightarrow h(x) = h(y)$, then there is precisely one frame homomorphism $\overline{h}\colon L/R \to K$ such that $\overline{h} \cdot \kappa' = h$; moreover, for $x \in L/R$, $\overline{h}(x) = h(x)$.

1.5. Down-set frame. The *down-set frame* $\mathfrak{D}L$ of a frame L is the frame $\{U \subseteq L \mid \emptyset \neq U = {\downarrow}U\}$ (where ${\downarrow}U = \{x \mid x \leq a \in U\}$, as usual) with meets and joins given respectively by intersections and unions. The mapping

$$\lambda' = (a \mapsto {\downarrow}a)\colon L \to \mathfrak{D}L$$

is a localic map; it is the right adjoint to $v = (U \mapsto \bigvee U)\colon \mathfrak{D}L \to L$.

1.6. Cover uniformities. A *cover* of a frame L is a subset $U \subseteq L$ such that $\bigvee U = 1$. A cover U is a *refinement* of a cover V (written, $U \leq V$) if

$$\forall u \in U\ \exists v \in V \quad \text{such that} \quad u \leq v.$$

For covers U, V we have the largest common refinement

$$U \wedge V = \{u \wedge v \mid u \in U, v \in V\}.$$

If $U \subseteq L$ is a cover and $a \in L$ we set $Ua = \bigvee\{u \in U \mid u \wedge a \neq 0\}$ and for covers U, V define $UV = \{Uv \mid v \in V\}$.

Finally, for a set of covers $\mathcal{U}$ define the relation

$$b \lhd_{\mathcal{U}} a \quad \equiv \quad \text{there is a } U \in \mathcal{U} \text{ such that} \quad Ub \leq a.$$

$\mathcal{U}$ is said to be *admissible* if

$$\forall a \in L, \quad a = \bigvee\{b \mid b \lhd_{\mathcal{U}} a\}.$$

A *cover-uniformity* ([14]) (briefly, *c-uniformity*) on a frame L is an admissible non-empty system of covers $\mathcal{U}$ such that

(U1) if $U \in \mathcal{U}$ and $U \leq V$ then $V \in \mathcal{U}$,
(U2) if $U, V \in \mathcal{U}$ then $U \wedge V \in \mathcal{U}$,
(U3) for every $U \in \mathcal{U}$ there is a $V \in \mathcal{U}$ such that $VV \leq U$.

A *c-uniform frame* (or *c-uniform locale*) is a pair $(L, \mathcal{U})$ where $\mathcal{U}$ is a c-uniformity on L. A frame homomorphism $h\colon L \to M$ is a *uniform homomorphism* $(L, \mathcal{U}) \to (M, \mathcal{V})$ if

$$\forall U \in \mathcal{U}, \quad h[U] \in \mathcal{V}.$$

1.7. Binary frame coproducts. Let L be a frame. Take the Cartesian product $L \times L$ as a poset and consider the frame $\mathfrak{D}(L \times L)$. Call $U \in \mathfrak{D}(L \times L)$ a *coproduct ideal* (shortly, a *cp-ideal*) if

(CP1) for any $A \subseteq L$ and $b \in L$, if $A \times \{b\} \subseteq U$ then $(\bigvee A, b) \in U$, and
(CP2) for any $a \in L$ and $B \subseteq L$, if $\{a\} \times B \subseteq U$ then $(a, \bigvee B) \in U$.

Intersections of cp-ideals are obviously cp-ideals and thus we have a complete lattice, which is in fact a frame. Note, however, that unions of cp-ideals are not necessarily cp-ideals and their joins (suprema) in this frame are typically bigger.

The sets A and B in (CP1)-(CP2) can be void; hence, in particular, each cp-ideal contains as a subset $\mathsf{n} = \{(0, b), (a, 0) \mid a, b \in L\}$. It is easy to check that for each $(a, b) \in L \times L$,

$$a \oplus b = {\downarrow}(a, b) \cup \mathsf{n}$$

is a cp-ideal.

The coproduct $L \oplus L$ in **Frm** (product in **Loc**) is just the frame

$$L \oplus L = \{U \in \mathfrak{D}(L \times L) \mid U \text{ is a cp-ideal}\}$$

with coproduct injections

$$\iota_1 = (a \mapsto a \oplus 1)\colon L \to L \oplus L, \quad \iota_2 = (b \mapsto 1 \oplus b)\colon L \to L \oplus L.$$

Note that for each $U \in L \oplus L$, $U = \bigvee\{a \oplus b \mid (a, b) \in U\}$.

Using the symbol $L \times L$ when speaking of $L \oplus L$ as a product in the category **Loc** would probably obscure the matter. Therefore, we will keep the notation $L \oplus L$ also in **Loc**. We only have to keep in mind that then the injections ι become projections $p_1, p_2\colon L \oplus L \to L$.

By the universal property of coproducts, for each frame homomorphism $h\colon L \to M$ there is a (unique) frame homomorphism $h \oplus h\colon L \oplus L \to M \oplus M$ such that $(h \oplus h)\iota_i = \iota_i h$. Clearly,

$$(h \oplus h)(E) = \bigvee_{(a,b)\in E} (h(a) \oplus h(b)).$$

Its right adjoint $(h \oplus h)_*$ in **Loc** is the unique localic map $f \times f\colon M \oplus M \to L \oplus L$ such that $p_i(f \times f) = fp_i$ for $i = 1, 2$.

1.8. Entourage uniformities. An *entourage* of a frame L is an $E \in L \oplus L$ such that $\{a \mid a \oplus a \leq E\}$ is a cover of L.

For each $x \in L$ and $E \in L \oplus L$, let

$$Ex = \bigvee\{a \in L \mid (a, a) \in E,\ a \wedge x \neq 0\}.$$

If E is an entourage then $x \leq Ex$.

For entourages E, F of L set

$$E \circ F = \bigvee\{a \oplus c \mid \exists b \neq 0,\ \ a \oplus b \leq E \text{ and } b \oplus c \leq F\}.$$

Further, for an entourage E set

$$E^{-1} = \{(a, b) \mid (b, a) \in E\}$$

(which is obviously an entourage again).

If E is an entourage (resp. $\mathcal{E}$ a set of entourages) write

$$b \lhd_E a \ \text{ if } \ E \circ (b \oplus b) \leq a \oplus a, \quad \text{and} \quad b \lhd_{\mathcal{E}} a \ \text{ if } \ \exists E \in \mathcal{E},\ b \lhd_E a.$$

A set of entourages $\mathcal{E}$ is said to be *admissible* if

$$\forall a \in L, \quad a = \bigvee\{b \mid b \lhd_{\mathcal{E}} a\}.$$

An *entourage-uniformity* [9, 10] (briefly, *e-uniformity*) on a frame L is an admissible set of entourages $\mathcal{E}$ such that

(E1) if $E \in \mathcal{E}$ and $E \leq F$ then $F \in \mathcal{E}$,
(E2) if $E, F \in \mathcal{E}$ then $E \cap F \in \mathcal{E}$,
(E3) for every $E \in \mathcal{E}$ there is an $F \in \mathcal{E}$ such that $F \circ F \leq E$, and
(E4) if $E \in \mathcal{E}$ then E^{-1} is in $\mathcal{E}$.

An *e-uniform frame* (or *e-uniform locale*) is a pair $(L, \mathcal{E})$ where $\mathcal{E}$ is an e-uniformity on L.

1.9. Non-symmetric uniformities. For a system $\mathcal{E}$ of entourages satisfying (E1), (E2) and (E3) define, first, $\overline{\mathcal{E}}$ as the filter of entourages generated by $\mathcal{E} \cup \{E^{-1} \mid E \in \mathcal{E}\}$ (of course, if $\mathcal{E}$ is a uniformity, $\overline{\mathcal{E}} = \mathcal{E}$). If $\overline{\mathcal{E}}$ is admissible we speak of $\mathcal{E}$ as of a *quasi-uniformity* and of $(L, \mathcal{E})$ as of a *quasi-uniform frame* ([10]). For any pair of quasi-uniform (resp. uniform) frames $(L, \mathcal{E}), (M, \mathcal{F})$, a frame homomorphism $h\colon L \to M$ is a *uniform homomorphism* $(L, \mathcal{E}) \to (M, \mathcal{F})$ if

$$\forall E \in \mathcal{E}, \quad (h \oplus h)(E) \in \mathcal{F}.$$

The two approaches to uniformities (via covers or entourages) are equivalent in the sense that the corresponding categories of uniform frames and uniform homomorphisms are isomorphic (see e.g. [12]).

2. Completion and surjectivity

2.1. In classical uniformity theory, complete uniform spaces X are characterized by the fact that they are a closed subset in every uniform embedding $X \subseteq Y$ in an arbitrary uniform space Y. This can be adopted for a definition of completeness in a point-free context:

A uniform frame $(L, \mathcal{E})$ is said to be complete if in every uniform embedding $j\colon (L, \mathcal{E}) \subseteq (M, \mathcal{F})$, L is a closed sublocale of M.

Equivalently,

A uniform frame $(L, \mathcal{E})$ is complete iff each dense uniform embedding homomorphism $h\colon (M, \mathcal{F}) \to (L, \mathcal{E})$ is an isomorphism.

(For the concept of uniform embedding and uniform embedding homomorphism see 2.1.2 below.)

2.1.1. One also has a counterpart with the behavior of the standard Cauchy structure.

Recall the *Cauchy points* from classical completions of uniform spaces $(X, \mathcal{E})$, that is, filters F in $L = \Omega(X)$ such that

$$F \cap \{a \mid (a, a) \in \mathcal{E}\} \neq \emptyset.$$

Taking their characteristic maps we have Cauchy points represented as bounded meet semilattice homomorphisms $\phi\colon (L, \mathcal{E}) \to \mathbf{2} = \{0, 1\}$ such that

(1) for all $E \in \mathcal{E}$, $\bigvee\{\phi(a) \mid a \oplus a \leq E\} = 1$, and

(2) for every $a \in L$, $\phi(a) = \bigvee\{\phi(b) \mid b \lhd_{\mathcal{E}} a\}$.

More generally, let M be a frame. A *Cauchy map* $\phi\colon (L,\mathcal{E}) \to M$ is a bounded meet homomorphism $\phi\colon L \to M$ such that

(C1) for all $E \in \mathcal{E}$, $\bigvee\{\phi(a) \mid a \oplus a \le E\} = 1$, and

(C2) for each $a \in L$, $\phi(a) = \bigvee\{\phi(b) \mid b \lhd_{\mathcal{E}} a\}$.

Thus, Cauchy maps are Cauchy points with a general frame M instead of $\mathbf{2}$.

Classically, a uniform space X is complete iff each Cauchy point is a neighbourhood system of an $x \in X$. In the pont-free uniformity theory one can prove that

$(L,\mathcal{E})$ is complete iff each Cauchy map $\phi\colon (L,\mathcal{E}) \to M$ is a frame homomorphism

(see, e.g., [11]).

For more information on the uniform completion in the point-free setting consult [1, 2, 3, 8].

2.1.2. Let $j\colon L \to M$ be an embedding in **Loc** with adjoint onto frame homomorphism $h = j^*\colon M \to L$. Since regular monomorphisms are preserved by products, $j \times j$ is also one-to-one.

By a *uniform embedding* $j\colon (L,\mathcal{E}) \to (M,\mathcal{F})$ we mean a one-to-one uniform map such that

$$\mathcal{E} = \{(h \oplus h)(F) \mid F \in \mathcal{F}\}.$$

The adjoint frame homomorphism is referred to as a *uniform embedding homomorphism*.

In terms of the covering approach this amounts to saying that $\mathcal{E}$ consists precisely of the $h[C]$'s with $C \in \mathcal{F}$.

2.2. Completion. Consider a quasi-uniform frame $(L,\mathcal{E})$. On the down-set frame $\mathfrak{D}L$ consider the relation

$$R_{\mathcal{E}} = \{(\bigcup\{\downarrow x \mid x \lhd_{\overline{\mathcal{E}}} y\}, \downarrow y), (\bigcup\{\downarrow a \mid a \oplus a \le E\}, L) \mid y \in L, E \in \mathcal{E}\},$$

and the resulting quotient (recall 1.4)

$$\mathbf{C}(L,\mathcal{E}) = \mathfrak{D}(L)/R_{\mathcal{E}}.$$

It is easy to check that the saturated sets constituting this frame (they will be referred to as *Cauchy ideals* in $(L,\mathcal{E})$) are the $U \in \mathfrak{D}(L)$ such that

(R1) if $\{x \mid x \lhd_{\overline{\mathcal{E}}} y\} \subseteq U$ then $y \in U$, and

(R2) if for some $E \in \mathcal{E}$, $E \wedge \{x\} = \{a \wedge x \mid a \oplus a \le E\} \subseteq U$, then $x \in U$.

From this it is easy to prove that the restriction of λ' (from 1.5) to a mapping $L \to \mathbf{C}(L, \mathcal{E})$, denoted by $\lambda_{(L,\mathcal{E})}$, is a Cauchy map and that it is universal in the sense that

for every Cauchy map $\phi\colon (L, \mathcal{E}) \to M$ there is precisely one frame homomorpism $h\colon \mathbf{C}(L, \mathcal{E}) \to M$ such that $\phi = h\lambda_{(L,\mathcal{E})}$. (See [13].)

For us it is important that $\mathbf{C}(L, \mathcal{E})$ endowed with the quasi-uniformity (res. uniformity) induced by $\mathcal{E}$ via $\lambda_{(L,\mathcal{E})}$, that is, the (quasi-)uniformity generated by all $(\lambda_{(L,\mathcal{E})} \times \lambda_{(L,\mathcal{E})})(E)$, $E \in \mathcal{E}$ [13, Prop. 6.2.1], constitutes a completion

$$\lambda_{(L,\mathcal{E})}\colon (L, \mathcal{E}) \to \mathbf{C}(L, \mathcal{E}),$$

a dense uniform embedding with the adjoint uniform embedding homomorphism

$$v_{(L,\mathcal{E})} = (U \mapsto \bigvee U)$$

that is, that for every dense uniform embedding $j\colon (L, \mathcal{E}) \to (M, \mathcal{F})$ there is a (dense) uniform embedding $k\colon \mathbf{C}(L, \mathcal{E}) \to (M, \mathcal{F})$ such that $j = k\lambda_{(L,\mathcal{E})}$. (See, e.g., [13]).

Moreover, the construction $\mathbf{C}$ is functorial; indeed, $(\mathbf{C}, \lambda)$ constitutes a reflection of the category of quasi-uniform (resp. uniform) frames with uniform homomorphisms into the corresponding subcategory of complete ones.

2.3. The following fact was proved for the case of (symmetric) uniformities (indeed for strong nearnesses) by Banaschewski in [1]. We present a proof using different techniques allowing to extend the result to quasi-uniformities.

2.3.1. Theorem. *Let $(L, \mathcal{E})$ be a quasi-uniform frame, let S be a sublocale of L and let $h\colon (L, \mathcal{E}) \to (S, \mathcal{F})$ be a uniform embedding homomorphism. Then $g = \mathbf{C}(h)\colon \mathbf{C}(L, \mathcal{E}) \to \mathbf{C}(S, \mathcal{F})$ is a uniform embedding.*

Proof: We will write just L for $(L, \mathcal{E})$ and S for $(S, \mathcal{F})$. Consider the diagram

$$\begin{tikzcd}
\mathbf{C}(L) \arrow[r, shift left, "g"] \arrow[d, shift right, "v_L"'] & \mathbf{C}(S) \arrow[l, shift left, "g^*"] \arrow[d, shift right, "v_S"'] \\
L \arrow[r, shift left, "h"] \arrow[u, shift right, "\lambda_L"'] & S \arrow[l, shift left, "j=\subseteq"] \arrow[u, shift right, "\lambda_S"']
\end{tikzcd}$$

where g_* is the right adjoint of g.

In the symmetric case we will work with $\mathcal{E},\mathcal{F}$ in the cover representation, in the non-symmetric case they will be systems of entourages like in 1.9. We have

$$\begin{aligned}&v\lambda = \mathrm{id}, \quad \lambda v \geq \mathrm{id},\\ &hj = \mathrm{id}, \quad jh \geq \mathrm{id}, \text{ in particular } h(x) \geq x \text{ for all } x \in L,\\ &gg_* \leq \mathrm{id} \quad \text{and} \quad g_*g \geq \mathrm{id}.\end{aligned}$$

We immediately obtain

$$g(\lambda_L(b)) = g\lambda_L j(b) = gg_*\lambda_S(b) \leq \lambda_S(b). \tag{$*$}$$

(I) *Symmetric case.* Since λ is a Cauchy map we have

$$\lambda_S(b) = \bigvee\{\lambda_S(x) \mid x \lhd_{\mathcal{F}} b\}.$$

Consider an $x \lhd_{\mathcal{F}} b$ and a cover $C \in \mathcal{F}$ such that $Cx \leq b$. Because h is a uniform embedding, $C = h[B]$ for a $B \in \mathcal{E}$ and since $h(u) \geq u$ for all $u \in L$, $C \geq B$ and hence $C \in \mathcal{E}$. By the first Cauchy property, $\bigvee\{\lambda_L(c) \mid c \in C\} = 1$ so that $\bigvee\{g\lambda_L(c) \mid c \in C\} = 1$. Hence

$$\lambda_S(x) = \bigvee\{\lambda_S(x) \wedge g\lambda_L(c) \mid c \in C\}.$$

Let $x \wedge c = 0$ for a $c \in C$. Then, as $C \subseteq S$

$$v_S(\lambda_S(x) \wedge g\lambda_L(c)) = x \wedge v_S g\lambda_L(c)) = x \wedge hv_L\lambda_L(c) = x \wedge c = 0$$

and by density of v_S, $\lambda_S(x) \wedge g\lambda_L(c) = 0$. Hence, as $Cx \leq b$,

$$\lambda_S(x) = \bigvee\{\lambda_S(x) \wedge g\lambda_L(c) \mid c \in C, c \wedge x \neq 0\} \leq g\lambda_L(b)$$

and together with $(*)$ and using the identity $b = \bigvee\{x \mid x \lhd_{\mathcal{F}} b\}$ we obtain that $\lambda_S(b) = g\lambda_L(b)$.

Since the elements $\lambda_S(b) = g\lambda_L(b)$, $b \in S$, join generate $\mathbf{C}(S,\mathcal{F})$ we conclude that g is onto.

To see that g is a uniform embedding homomorphism, take any basic element of the uniformity of $\mathbf{C}(S)$ of the form $\lambda_S[C]$ with $C = h[B] \in \mathcal{F}$. Again, since $jh \geq \mathrm{id}$, $B \leq C$ and $C \in \mathcal{E}$, and finally $g[\lambda_L[B]] \leq \lambda_S[C]$.

(II) *Non-symmetric case.* Consider an $x \lhd_{\overline{\mathcal{F}}} b$ and an entourage $F \in \overline{\mathcal{F}}$ such that $Fx \leq b$. Let $F_1, F_2 \in \mathcal{F}$ such that $F_1 \cap F_2^{-1} \subseteq F$. Since h is a uniform embedding, $F_1 = (h \oplus h)(E_1)$ and $F_2 = (h \oplus h)(E_2)$ for some $E_1, E_2 \in \overline{\mathcal{E}}$. Then

$$F_2^{-1} = (h \oplus h)(E_2)^{-1} = (h \oplus h)(E_2^{-1}) \quad ([13, §3.6]).$$

Therefore

$$F \supseteq (h \oplus h)(E_1) \cap (h \oplus h)(E_2^{-1}) = (h \oplus h)(E)$$

for $E = E_1 \cap E_2^{-1} \in \overline{\mathcal{E}}$ and thus

$$F \supseteq \bigvee_{(a,b)\in E} (h(a) \oplus h(b)) \supseteq E$$

since $h(u) \geq u$ for all $u \in L$. Hence $F \in \overline{\mathcal{E}}$. By the first Cauchy property, $(\lambda_L \oplus \lambda_L)(F)$ is an entourage so that $(g\lambda_L \oplus g\lambda_L)(F)$ is also an entourage, that is, $\bigvee\{g\lambda_L(c) \mid (c,c) \in F\} = 1$. Hence

$$\lambda_S(x) = \bigvee\{\lambda_S(x) \wedge g\lambda_L(c) \mid (c,c) \in F\}. \tag{$**$}$$

Let $x \wedge c = 0$ for some $(c,c) \in F \in S \oplus S$. Then

$$v_S(\lambda_S(x) \wedge g\lambda_L(c)) = x \wedge v_S g\lambda_L(c)) = x \wedge hv_L\lambda_L(c) = x \wedge c = 0$$

and by density of v_S, $\lambda_S(x) \wedge g\lambda_L(c) = 0$. Hence, from $(**)$ we get

$$\lambda_S(x) = \bigvee\{\lambda_S(x) \wedge g\lambda_L(c) \mid (c,c) \in F, c \wedge x \neq 0\} \leq g\lambda_L(b)$$

as $c \leq b$ for every such c (since $Fx \leq b$). So we have proved that

$$x \lhd_{\overline{\mathcal{F}}} b \;\Rightarrow\; \lambda_S(x) \leq g\lambda_L(b).$$

Therefore $\lambda_S(b) = \bigvee\{\lambda_S(x) \mid x \lhd_{\overline{\mathcal{F}}} b\} \leq g\lambda_L(b)$ and together with $(*)$ we obtain that $\lambda_S(b) = g\lambda_L(b)$.

Again, since the elements $\lambda_S(b) = g\lambda_L(b)$, $b \in S$, join generate $\mathbf{C}(S,\mathcal{F})$ we conclude that g is onto.

To see that g is a uniform embedding homomorphism, take any basic element of the quasi-uniformity of $\mathbf{C}(S)$ of the form

$$(\lambda_S \times \lambda_S)(F)$$

with $F = (h \oplus h)(E) \in \mathcal{F}$ for some $E \in \mathcal{E}$. Again, since $jh \geq \mathrm{id}$, $E \leq F$ and $F \in \mathcal{E}$. Finally,

$$(g \oplus g)(\lambda_L \times \lambda_L)(E) \leq (g \oplus g)(\lambda_L \times \lambda_L)(F)$$

and, by [13, Prop. 4.3.1],

$$\begin{aligned}(g \oplus g)(\lambda_L \times \lambda_L)(F) &= \bigvee_{(a,b)\in F} (g\lambda_L(a) \oplus g\lambda_L(b)) \\ &= \bigvee_{(a,b)\in F} (\lambda_S(a) \oplus \lambda_S(b)) = (\lambda_S \times \lambda_S)(F). \quad \blacksquare\end{aligned}$$

3. Completion and closedness

3.1. Theorem. *A uniform sublocale of a complete quasi-uniform locale is complete if and only if it is closed.*

Proof: $\Leftarrow$: Consider the following diagram in which S is a closed uniform sublocale of L (we omit the symbols for the uniformities). Since L is complete, λ_L and v_L are mutually inverse isomorphisms.

$$\begin{tikzcd}
\mathbf{C}(L) \arrow[rr, shift left, "\mathbf{C}(h)=g"] \arrow[dd, shift right, "\text{iso } v_L"'] & & \mathbf{C}(S) \arrow[ll, shift left, "g^*"] \arrow[dl, "\text{iso } \alpha"', "u\mapsto v_L g^*(u)"] \arrow[dd, shift right, "v_S"'] \\
& T = v_L g^* g \lambda_L[\mathbf{C}(S)] \arrow[dl, "\text{inclusion}"'] & \\
L \arrow[uu, shift right, "\lambda_L \text{ iso}"'] \arrow[rr, shift left, "h"] & & S \arrow[ll, shift left, "j=\text{inclusion}"] \arrow[uu, shift right, "\lambda_S"'] \arrow[ul, "\text{inclusion}", "\beta"']
\end{tikzcd}$$

We have $\alpha\lambda_S(x) = v_L g^* \lambda_S(x) = v_L \lambda_L j(x) = x$, hence $\beta = \alpha\lambda_L$ is an inclusion and since λ_S is dense we have $\lambda_S(0_S) = 0_T$ and $T \subseteq \overline{S}$. Since $S = \overline{S}$ we conclude that $S = \overline{S} \subseteq T \subseteq \overline{S}$, and $S = T$.

$\Rightarrow$: Suppose S is a complete uniform sublocale of a complete uniform locale L. We have uniform embeddings $S \subseteq \overline{S} \subseteq L$ and since the former is dense and S is complete, it is an isomorphism, that is, $S = \overline{S}$. ∎

3.1.1. Note. For (symmetric) uniformities, indeed for strong nearnesses, this fact was proved by Banaschewski in [1].

3.2. In a certain analogy with the classical case, what makes a closed sublocale S of a complete $(L, \mathcal{E})$ complete is that a Cauchy map ("generalized Cauchy point") on S can be extended to a Cauchy map on $(L, \mathcal{E})$, which makes it, because of the completeness of the latter, a homomorphism ("generalized Cauchy point"). The mechanism of the extension, however, is not really very transparent. In the following Section we will discuss the case of Cauchy and spectrum points; there the extension will be explicit.

4. Extending Cauchy filters

In this section we will work with the cover uniformities as in 1.6 above. The necessary modifications for the non-symmetric case would unduly obscure the procedures.

Recall that a uniform frame is *Cauchy complete* if each Cauchy point F is a (spectrum) point, that is, if $\bigvee a_i \in F$ only if $a_j \in F$ for some j (*complete primeness*, see [4, 11]).

4.1. Consider a uniform frame $(L, \mathcal{E})$ and a closed sublocale $S = \uparrow s$ of L endowed with the uniformity making it a uniform sublocale, that is, since the homomorphism adjoint with the embedding $S \subseteq L$ is given by $x \mapsto x \vee s$,

$$\mathcal{E}_S = \{A_S \mid A \in \mathcal{E}\}, \quad A_S = \{a \vee s \mid a \in A\}.$$

4.2. For $A \in \mathcal{E}$ and $u \in L$ set

$$\phi(u, A) = \bigvee\{a \in A \mid a \wedge u \nleq s\}.$$

4.2.1. Lemma.
(1) $\varphi(u, A) \leq Au$.
(2) $\varphi(u \vee s, A) = \varphi(u, A)$.
(3) *If* $u \leq v$ *then* $\varphi(u, A) \leq \varphi(v, A)$.
(4) *If* $A \leq B$ *then* $\varphi(u, A) \leq \varphi(u, B)$.
(5) $\varphi(u \wedge v, A \wedge B) \leq \varphi(u, A) \wedge \varphi(v, B)$.
(6) $A\varphi(u, A) \leq \varphi(u, AA)$.

Proof: Properties (1)-(5) are trivial.
(6) Let $a \in A$ such that $a \wedge \varphi(u, A) \neq 0$. This means there is some $b \in A$ such that $b \wedge a \neq 0$ (and thus $a \leq Ab$) and $b \wedge u \nleq s$ (and thus $Ab \wedge u \nleq s$). Then $a \leq Ab \leq \varphi(u, AA)$ since $Ab \in AA$ and $Ab \wedge u \nleq s$. ■

4.3. Let F be a filter on S. By (4) above, $\{\phi(u, A) \mid u \in F, a \in A\}$ is down directed in L and hence we have a filter

$$G = \{x \in L \mid x \geq \phi(u, A),\ u \in F,\ a \in A\}.$$

4.3.1. Proposition. *If F is a weak Cauchy filter on S then G is a Cauchy filter on L.*

Proof: Let $A \in \mathcal{E}$ and take $B \in \mathcal{E}$ such that $BB \leq A$. By hypothesis, $F \cap A_S \neq \emptyset$, that is, there is some $v \in B$ for which $v \vee s \in F$. Hence $\varphi(v \vee s, B) \in G$ and, by properties (2) and (1), $\varphi(v \vee s, B) = \varphi(v, B) \leq Bv$ and since Bv refines some $u \in A$, this shows that $G \cap A \neq \emptyset$.

Let $a \in G$ with $a \geq \varphi(u, A)$ for some $u \in F$ and $A \in \mathcal{E}$. Take $B \in \mathcal{E}$ such that $BB \leq A$. By (6) $B\varphi(u, B) \leq \varphi(u, BB) \leq \varphi(u, A) \leq a$ and thus $\varphi(u, B) \lhd_{\mathcal{E}} a$. ■

4.3.2. Lemma. *For every $A \in \mathcal{E}$ and $u \in S$, $A_S u = \varphi(u, A) \vee s$.*

Proof: Since $A_S u = \bigvee\{a \vee s \mid a \in A, (a \vee s) \wedge u \neq s\}$ and $(a \vee s) \wedge u \neq s$, that is, $(a \wedge u) \vee (u \wedge s) \neq s$, is clearly equivalent to $a \wedge u \nleq s$, then $A_S u = \varphi(u, A) \vee s$. ■

4.3.3. Theorem. *If F is a Cauchy filter then $F = G \cap S$. Thus, every Cauchy filter on a closed uniform sublocale of L can be extended to a Cauchy filter on the whole of L.*

Proof: Let $a \geq \varphi(u, A)$ in $G \cap S$. By the lemma, $u \leq A_S u = \varphi(u, A) \vee s \leq a$ and thus $a \in F$.

Conversely, let $u \in F$. By the regularity of F, there is some $v \in F$ such that $v \lhd_{\mathcal{E}_S} u$. Take $A \in \mathcal{E}$ such that $A_S v \leq u$. Again by the lemma, $u \geq \varphi(v, A) \vee s \in G \cap S$. ■

4.3.4. Corollary. *A closed uniform sublocale of a Cauchy complete uniform frame is Cauchy complete.*

Proof: We need to proof that each Cauchy point F of S is completely prime. If $\bigvee a_i \in F = G \cap S$ then $\bigvee a_i \in F = G$, hence $a_j \in G$ for some j, but as $F \subseteq S$, $a_j \in F$. ■

References

[1] B. Banaschewski, Uniform completion in pointfree topology, in: *Topological and Algebraic Structures in Fuzzy Sets*, pp. 19–56. Trends Log. Stud. Log. Libr., vol. 20, Kluwer Academic Publishers, Boston, Dordrecht, London, 2003.

[2] B. Banaschewski, S. S. Hong and A. Pultr, On the completion of nearness frames, *Quaest. Math.* 21 (1998) 19–37.

[3] B. Banaschewski and A. Pultr, Samuel compactification and completion of uniform frames, *Math. Proc. Cambridge Phil. Soc.* 108 (1990) 63–78.

[4] B. Banaschewski and A. Pultr, Cauchy points of uniform and nearness frames, *Quaest. Math.* 19 (1996) 101–127.

[5] J. R. Isbell, *Uniform Spaces*, Amer. Math. Soc., Providence, 1964.

[6] P. T. Johnstone, *Stone Spaces*, Cambridge University Press, Cambridge, 1982.

[7] J. L. Kelley, *General Topology*, D. Van Nostrand Co., 1955.

[8] I. Kříž, A direct description of uniform completion in locales and a characterization of LT-groups, *Cahiers Topologie Géom. Différentielle Catég.* 27 (1986) 19–34.

[9] J. Picado, Weil uniformities for frames, *Comment. Math. Univ. Carolinae* 36 (1995) 357–370.

[10] J. Picado, Structured frames by Weil entourages, *Appl. Categ. Struct.* 8 (2000) 351–366.
[11] J. Picado and A. Pultr, *Frames and locales: Topology without points*, Frontiers in Mathematics, vol. 28, Springer, Basel, 2012.
[12] J. Picado and A. Pultr, Entourages, covers and localic groups, *Appl. Categ. Struct.* 21 (2013) 49–66.
[13] J. Picado and A. Pultr, *Entourages, density, Cauchy structure, and completion*, DMUC Preprint 18-19 (`http://www.mat.uc.pt/preprints/ps/p1819.pdf`), 2018.
[14] A. Pultr, Pointless uniformities I, *Comment. Math. Univ. Carolinae* 25 (1984) 91–104.

Dharmanand Baboolal
School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal, Private Bag X54001, Durban 4000, South Africa
E-mail address: `baboolald@ukzn.ac.za`

Jorge Picado
CMUC, Department of Mathematics, University of Coimbra, 3001-501 Coimbra, Portugal
E-mail address: `picado@mat.uc.pt`

Aleš Pultr
Department of Applied Mathematics and CE-ITI, MFF, Charles University, Malostranské nám. 24, 11800 Praha 1, Czech Republic
E-mail address: `pultr@kam.mff.cuni.cz`