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SCHRÖDINGER'S TRIDIAGONAL MATRIX

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Abstract: In the third part of his famous 1926 paper 'Quantisierung als Eigenwertproblem', Schrödinger conjectured the eigenvalues of a certain parametrized family of tridiagonal matrices. His conjecture, solved here, seems to have remained open in spite of a 1991 paper suggesting otherwise.

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1. Introduction

Erwin Schrödinger won the Nobel prize for physics mainly due to the paper 'Quantisierung als Eigenwertproblem' which appeared in 1926 in four parts in the first of which, the Schrödinger equation is announced. In the third part, [S, p. 481] , he investigates spectra using his wave mechanics and in particular is led to consider a determinant which he writes as

$$\begin{vmatrix} -\varepsilon & \varepsilon_{m+1,m} & & & & \\ \varepsilon_{m+1,m} & -\varepsilon & \varepsilon_{m+2,m} & & & \\ & \varepsilon_{m+2,m} & -\varepsilon & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & -\varepsilon & \varepsilon_{l-1,m} \\ & & & & \varepsilon_{l-1,m} & -\varepsilon \end{vmatrix},$$

defining $\varepsilon_{n,m} = -6lg\sqrt{\frac{(l^2-n^2)(n^2-m^2)}{4n^2-1}}$, see [S, p. 480]. Here g is a constant and l, m, n are positive integers, so that the matrix is $(l-m)\times(l-m)$ and n may assume the values $m, m+1, ..., l-1$; m the values $1, ..., l-1$.

Schrödinger then says: 'If one divides each term by the common factor $6lg$ of the ε_{nm} and views $k^* = -\frac{\varepsilon}{6lg}$ as unknown, then the equation [determinant=0] has the roots $k^* = \pm(l-m-1), \pm(l-m-3), \pm(l-m-5), ...$ which series stops with ±1 or with 0 (inclusively) depending on whether the degree

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$l-m$ is even or odd. The proof of this, unfortunately, you do *not* find in the mathematical appendix, since I did not succeed in finding it ...' (author's translation; emphasis by Schrödinger).

The 1991 paper [TT, p. 342] claims that Schrödinger unwittingly considered in his famous paper determinants known as Sylvester Kac determinants. This claim is from time to time repeated; see e.g. [BR, p.408]. The Sylvester Kac determinant is given as

$$\begin{aligned}\mathrm{SK}_n(x) &= \begin{vmatrix} x & 1 & & & \\ n-1 & x & 2 & & \\ & n-2 & \ddots & \ddots & \\ & & \ddots & x & n-1 \\ & & & 1 & x \end{vmatrix} \\ &= (-n+1+x)(-n+3+x)\cdots(n-3+x)(n-1+x).\end{aligned}$$

According to the history reported in Taussky and Todd [TT], this formula was first stated without proof by Sylvester in 1854 in a half page long paper reprinted in [TT], but a proof was provided only in 1866 by F. Mazza; in 1947 Mark Kac [K] and in 1957 P. Rósza [R] independently, not knowing about the earlier results found other proofs; all these are reviewed in [TT] . But perhaps the best proof this author knows of can be found in Edelman and Kostlan's paper [EK, p.18]. Note that depending on whether n is odd or even, 0 is or is not a root; that is an eigenvalue of the underlying zero-axial matrix.

It seems not at all obvious how to deduce from the results known about the Sylvester Kac matrix the value of the Schrödinger determinant and in spite of a thorough search through the literature on tridiagonal matrices we have not found any paper that would have Schrödinger's conjecture as a topic of investigation. The claim in [TT] seems due to a confusion not free of peculiarities: At about the same time Schrödinger wrote his famous series of papers, he also published together with F. Kohlrausch a short paper [KS] on Boltzmann's H-theorem which the present author had the possibility to see* and in that paper indeed stumbled in the second section over a linear system whose matrix (never written down) would be essentially that of the Sylvester Kac matrix. Now the 1968 appendix of Mark Kac's Chauvenet prize-winning

*Thanks to magnificent libraries Coimbra University possesses.

paper [K],[A], on Brownian motion to which [TT] refers, contains in its references [KS] but not [S], while [TT] has in its references [S] but not [KS]. Furthermore, as we saw, for the $n \times n$ Schrödinger determinants the values of the $n \times n$ Sylvester Kac determinant are predicted. For example, we have after substituting $-\varepsilon$ by x and putting

$$\varepsilon_{nm} = \sqrt{\frac{(l^2 - n^2)(n^2 - m^2)}{4n^2 - 1}}$$

for $(l, m) = (10, 5)$ and $(l, m) = (11, 6)$, respectively, the values $\varepsilon_{m+1,m} = 8/\sqrt{13}$ and $\varepsilon_{m+1,m} = 2\sqrt{6/5}$ and get the determinants

$$\begin{vmatrix} x & \frac{8}{\sqrt{13}} & & & \\ \frac{8}{\sqrt{13}} & x & 2\sqrt{\frac{102}{65}} & & \\ & 2\sqrt{\frac{102}{65}} & x & 6\sqrt{\frac{13}{85}} & \\ & & 6\sqrt{\frac{13}{85}} & x & 2\sqrt{\frac{14}{17}} \\ & & & 2\sqrt{\frac{14}{17}} & x \end{vmatrix}, \quad \begin{vmatrix} x & 2\sqrt{\frac{6}{5}} & & & \\ 2\sqrt{\frac{6}{5}} & x & 2\sqrt{\frac{133}{85}} & & \\ & 2\sqrt{\frac{133}{85}} & x & 30\sqrt{\frac{2}{323}} & \\ & & 30\sqrt{\frac{2}{323}} & x & \frac{8}{\sqrt{19}} \\ & & & \frac{8}{\sqrt{19}} & x \end{vmatrix},$$

respectively. Both these determinants are equal to the 5×5 Sylvester Kac determinant which is

$$\begin{vmatrix} x & 1 & & & \\ 4 & x & 2 & & \\ & 3 & x & 3 & \\ & & 2 & x & 4 \\ & & & 1 & x \end{vmatrix} = (-4 + x)(-2 + x)(x)(2 + x)(4 + x).$$

Is this so in general?

If we define $n = l - m$, then we have $l^2 - (m + i)^2 = (n - i)(2m + n + i)$, and Schrödinger's $n \times n$ determinants can be written

$$\mathrm{Sch}_n(x) = \begin{vmatrix} x & \varepsilon_{1,2} & & & & \\ \varepsilon_{1,2} & x & \varepsilon_{2,3} & & & \\ & \varepsilon_{2,3} & x & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & x & \varepsilon_{n-1,n} \\ & & & & \varepsilon_{n-1,n} & x \end{vmatrix},$$

where $i = 1, 2, ..., n-1$, $m \in \mathbb{Z}$, is a free parameter which we mostly will suppress in notation and the $\varepsilon_{i,1+i} = \varepsilon_{i,1+i}(m) \geq 0$ are defined by

$$\varepsilon_{i,1+i}^2 = \frac{i(n-i)(2m+i)(2m+n+i)}{(2m+2i-1)(2m+2i+1)}.$$

From this formula and an argument in Section 2 it is clear that

$$\text{as } m \to \infty, \text{ we have } \varepsilon_{i,1+i} \to i(n-i)$$

and so whatever the Schrödinger determinant is, it converges to the Sylvester Kac determinant of the same size. Schrödinger's conjecture $\mathrm{Sch}_n(x) = \mathrm{SK}_n(x)$ says of course much more: namely that the polynomial in x, $\mathrm{Sch}_n(x)$, has coefficients that do not depend on m. From observations on partial fraction decomposition it will follow in Section 3 that the coefficients of the polynomial $\mathrm{Sch}_n(x)$ must be $\mathbb{Q}$-linear combinations of the fractions

$$1/(2m+2i-1),\ i = 1, 2, ..., n-1.$$

Hence we shall have to show that the coefficients of these fractions are all 0. To prove this is the topic of sections 4 and 6. Section 5 proves some special cases, partly to cover limiting cases not transparently covered by the arguments for the typical case, partly to prepare the reader's mind for the somewhat technical section that follows.

Since any of the underlying zero axial Schrödinger matrices of given size n has the same n distinct eigenvalues that the zero axial Sylvester Kac matrix of size n has, these matrices are similar to each other. The author has not worked on the problem to find parametrized similarity transforms that diagonalize the Schrödinger matrices of given size in dependence of m or that transforms them into Sylvester Kac matrices. It might be hard to find such transforms but at the other hand they might yield a more elegant proof. A deeper understanding of Schrödinger's determinant might also come from the relation existing between orthogonal polynomials and tridiagonal matrices via Favard's theorem; see e.g. [Ch]. We should also note that a number of generalizations of Sylvester Kac determinants are known (see e.g. [MM] or papers on Cayley continuants and others relating to Favard's theorem) but as yet this author was not able to use them in a productive way to tackle Schrödinger's conjecture.

2. Translation to a conjecture for coefficients

Chapter XIII of the famous treatise by Muir enlarged by Metzler, [MM], is completely dedicated to 'continuants', nowadays better known as determinants of tridiagonal matrices. [MM] defines the compact notation

$$K\begin{pmatrix} & b_1\ b_1\ \cdots\ b_{n-1} & \\ a_1\ a_2\ \cdots a_{n-1}\ a_n \\ & c_1\ c_2\ \cdots\ c_{n-1} & \end{pmatrix} := \begin{vmatrix} a_1 & b_1 & & & & \\ c_1 & a_2 & b_2 & & & \\ & c_2 & a_3 & b_3 & & \\ & & c_3 & \ddots & \ddots & \\ & & & \ddots & \ddots & b_{n-1} \\ & & & & c_{n-1} & a_n \end{vmatrix},$$

which if context allows [MM] abbreviates to $K(1,n)$.

After some combinatorial reasoning [MM, §545] concludes that the determinant $K(1,n)$ can be formed from $a_1a_2...a_n$ by replacing in all possible ways 0,1,2, or more pairs of consecutive as by the signed product of bs and c that have the same index as the first of the consecutive as. In other words for each pair a_ra_{1+r} replaced we have to write $-b_rc_r$.

Thus for example

$$K\begin{pmatrix} & b_1\ b_2\ b_3\ b_4 & \\ a_1\ a_2\ a_3\ a_4\ a_5 \\ & c_1\ c_2\ c_3\ c_4 & \end{pmatrix} = \begin{cases} a_1a_2a_3a_4a_5 + (-b_1c_1)a_3a_4a_5 + a_1(-b_2c_2)a_4a_5 + \\ a_1a_2(-b_3c_3)a_5 + a_1a_2a_3(-b_4c_4) + \\ (-b_1c_1)(-b_3c_3)a_5 + (-b_1c_1)a_3(-b_4c_4) + \\ a_1(-b_2c_2)(-b_4c_4). \end{cases}$$

An important obvious consequence is

Lemma 2.1. *Assume $b_1, c_1, ..., b_{n-1}, c_{n-1}$ and $b'_1, c'_1, ..., b'_{n-1}, c'_{n-1}$ are complex numbers such that for $i = 1, ..., n-1$ there holds $b_ic_i = b'_ic'_i$. Then*

$$K\begin{pmatrix} & b_1\ b_2\ \cdots\ b_{n-1} & \\ a_1\ a_2\ \cdots a_{n-1}\ a_n \\ & c_1\ c_2\ \cdots\ c_{n-1} & \end{pmatrix} = K\begin{pmatrix} & b'_1\ b'_2\ \cdots\ b'_{n-1} & \\ a_1\ a_2\ \cdots a_{n-1}\ a_n \\ & c'_1\ c'_2\ \cdots\ c'_{n-1} & \end{pmatrix}. \quad \blacksquare$$

We write from now on

$$e_i = \varepsilon_{i,1+i}^2 \quad \text{and} \quad c_i = i(n-i).$$

By the previous lemma, the $n \times n$ Schrödinger determinant then is

$$\mathrm{Sch}_n(x) = K\begin{pmatrix} & e_1 & e_2\ ...\ e_{n-1} & \\ x & x & & x \\ & 1 & 1\ ...\ 1 & \end{pmatrix},$$

while the $n \times n$ Sylvester Kac determinant is

$$\mathrm{SK}_n(x) = K\begin{pmatrix} & c_1 & & c_2 \dots c_{n-1} & \\ x & & x & & x \\ & 1 & & 1 \dots\; 1 & \end{pmatrix}.$$

According to the previous arguments, a tridiagonal $n \times n$ matrix whose diagonal consists entirely of $x$es must have as determinant a monic polynomial in x of degree n with coefficients of $x^{n-1}, x^{n-3}, \dots$ equal to 0. Hence we have for certain reals coef_ν and coef'_ν developments

$$\mathrm{Sch}_n(x) = \sum_{i=0}^{\lfloor n/2 \rfloor} \mathrm{coef}_{n-2i}\, x^{n-2i} \quad \text{and} \quad \mathrm{SK}_n(x) = \sum_{i=0}^{\lfloor n/2 \rfloor} \mathrm{coef}'_{n-2i}\, x^{n-2i}.$$

From this it follows that for n odd, $x|\mathrm{Sch}_n(x)$ so that 0 is an eigenvalue of the underlying zero axial matrix. To keep notation simple, the Muir's rule suggests to define $a \prec b$ to say that $b - a \geq 2$. With this we can write the nonzero coefficients of $\mathrm{Sch}_n(x)$ and of $\mathrm{SK}_n(x)$ as follows.

$$\begin{array}{rcc} \mathrm{coef}_n & = & 1 \\ \mathrm{coef}_{n-2} & = & -\sum\limits_{1\leq i\leq n-1} e_i \\ \mathrm{coef}_{n-4} & = & +\sum\limits_{1\leq i\prec j\leq n-1} e_i e_j \\ \vdots & \vdots & \vdots \\ \mathrm{coef}_{n-2\mu} & = & (-1)^\mu \sum\limits_{1\leq i_1 \prec i_2 \dots \prec i_\mu \leq n-1} e_{i_1} e_{i_2} \cdots e_{i_\mu} \\ \vdots & \vdots & \vdots \end{array} \qquad \begin{array}{rcc} \mathrm{coef}'_n & = & 1 \\ \mathrm{coef}'_{n-2} & = & -\sum\limits_{1\leq i\leq n-1} c_i \\ \mathrm{coef}'_{n-4} & = & +\sum\limits_{1\leq i\prec j\leq n-1} c_i c_j \\ \vdots & \vdots & \vdots \\ \mathrm{coef}'_{n-2\mu} & = & (-1)^\mu \sum\limits_{1\leq i_1 \prec i_2 \dots \prec i_\mu \leq n-1} c_{i_1} c_{i_2} \cdots c_{i_\mu} \\ \vdots & \vdots & \vdots. \end{array}$$

Thus the conjecture is equivalent to:

Conjecture 2.2. *For all* $\mu = 0, 1, 2, \dots, \lfloor n/2 \rfloor$ *there holds* $\mathrm{coef}_{n-2\mu} = \mathrm{coef}'_{n-2\mu}$, *that is,*

$$\sum_{1\leq i_1 \prec i_2 \dots \prec i_\mu \leq n-1} e_{i_1} e_{i_2} \cdots e_{i_\mu} = \sum_{1\leq i_1 \prec i_2 \dots \prec i_\mu \leq n-1} c_{i_1} c_{i_2} \cdots c_{i_\mu}.$$

After skimming over Proposition 3.1 below, the reader will have no difficulty to follow the proofs of some of these equalities, namely the cases $\mu = 0, 1$, and n even $, \mu = \lfloor n/2 \rfloor$, given in Section 5.

3. Notation and auxiliary results

Our proof of Schrödinger's conjecture unfortunately makes necessary a number of abbreviations in order to obtain manageable expressions. Also, we avoid repetitions giving certain letters a fixed meaning.

Throughout the rest of the paper, the letters n, T, μ, E have the following meanings.
n is a fixed integer ≥ 3; it indicates the size of the determinant.
μ is a fixed integer $\in \{0, 1, ..., \lfloor n/2 \rfloor\}$; it refers to the μ in Corollary 2.2.
T is a fixed integer $\in \{1, ..., n\}$. Its significance is explained below.
x will replace $2m$ as the parameter in e_i. (x is not used anymore in the previous sense.)

The symbols $c_i, d_i, t_i, e_i, e_{i_1 i_2 \ldots i_\mu}$ and E are defined as follows:

$$\begin{array}{rcll} c_i & := & i(-i+n) & i \in \mathbb{Z} \\ d_i & := & 1/(x+2i-1) & i = 1, ..., n \\ t_i & := & -2^{-1} c_{-1+i} c_i d_i & i = 1, ..., n \\ e_i & := & c_i \dfrac{(x+i)(x+n+i)}{(x+2i-1)(x+2i+1)} & i = 1, ..., n-1 \\ e_{i_1 i_2 \ldots i_\mu} & := & e_{i_1} e_{i_2} \cdots e_{i_\mu} & \\ E & := & \displaystyle\sum_{1 \leq i_1 \prec i_2 \ldots \prec i_\mu \leq n-1} e_{i_1} e_{i_2} \cdots e_{i_\mu} & \end{array}$$

Proposition 3.1. *Among the c_i, d_i, t_i, and e_i there hold the following relations*

$$\begin{array}{rcll} e_i & = & c_i + t_i - t_{1+i} & \\ d_i d_j & = & 2^{-1}(j-i)^{-1}(d_i - d_j) & i \neq j \\ d_i t_j & = & -2^{-2}(j-i)^{-1} c_{-1+j} c_j (d_i - d_j) & i \neq j \\ t_i t_j & = & 2^{-3}(j-i)^{-1} c_{-1+i} c_i c_{-1+j} c_j (d_i - d_j) & i \neq j \\ 0 & = & c_{-2+i} - 2c_i + c_{2+i} + 8 & i \in \mathbb{Z} \end{array}$$

and for $i \neq j, j+1$:

$$d_i e_j = 2^{-2} c_j (4d_i - c_{-1+j}(j-i)^{-1}(d_i - d_j) + c_{1+j}(1+j-i)^{-1}(d_i - d_{1+j})).$$

Proof: The identities claimed can all be verified by direct computation. For example, the computation

$$\begin{array}{rcl} c_i + t_i - t_{1+i} & = & c_i - 2^{-1} c_{-1+i} c_i d_i + 2^{-1} c_i c_{1+i} d_{1+i} \\ & = & c_i (1 - 2^{-1} c_{-1+i} d_i + 2^{-1} c_{1+i} d_{1+i}) \\ & = & c_i (1 - \frac{2^{-1}(-1+i)(1-i+n)}{x+2i-1} + \frac{2^{-1}(1+i)(-1-i+n)}{x+2i+1}) \\ & = & c_i \frac{(x+i)(x+n+i)}{(x+2i-1)(x+2i+1)} \end{array}$$

proves the identity given for e_i. ■

Of particular importance in these formulas is that $d_i d_j$ (for $i \neq j$) is a linear combination of d_i and d_j obtained by partial fraction decomposition. The fractions $1, d_1, ..., d_n$ are as elements of the rational function field $\mathbb{Q}(x)$ evidently linearly independent over $\mathbb{Q}$. Since the t_i are rational multiples of the d_i we have by iterative use of the multiplication formulae for $d_i d_j$ the following important fact.

Corollary 3.2. *Every polynomial in $\mathbb{Q}[d_1, ..., d_n, t_1, ..., t_n]$ in which the d_i and t_i occur in degrees 0 or 1 only and in a product never with the same indices can be written uniquely as a $\mathbb{Q}$-linear combination of $1, d_1, ..., d_n$.*

Example. $3d_1d_3 - 2d_1d_2d_3 = \frac{1}{2}d_1 + \frac{1}{2}d_2 - d_3$, that is

$$\frac{3}{(x+1)(x+5)} - \frac{2}{(x+1)(x+3)(x+5)} = \frac{1}{2(x+1)} + \frac{1}{2(x+3)} - \frac{1}{(x+5)}.$$

Consequently if $p \in \mathbb{Q}[d_1, ..., d_n, t_1, ..., t_n]$ is such a square free polynomial, then we can define $\mathrm{cf}(p)$ as the rational coefficient of d_T in the presentation of p as a $\mathbb{Q}-$linear combination of the $1, d_1, d_2, ..., d_n$. Also, by speaking of 'the coefficient of d_T in p' or 'the cf of p', we shall mean $\mathrm{cf}(p)$. Thus if p is the polynomial of the example and $T = 2$, then $\mathrm{cf}(p) = 1/2$; if $T = 3$, then $\mathrm{cf}(p) = -1$.

It is easy to see that $E \in \mathbb{Q}[t_1, ..., t_n]$ is a square free polynomial in the said sense.

Corollary 3.3. *If $T \neq j, j + 1$, then*

$$\mathrm{cf}(d_T e_j) = 2^{-2}c_j(4 - c_{-1+j}(j - T)^{-1} + c_{1+j}(1 + j - T)^{-1}).$$

Proof: This follows from the formula found for $d_i e_j$ putting $i = T$ and extracting the coefficient of d_T. ■

Lemma 3.4. *Assume p, q are polynomials in $d_1, ..., d_n$, in which each d_i has degree at most 1 and q is d_T-free, i.e. does not have t_T or d_T as a variable, Then* $$\mathrm{cf}(pq) = \mathrm{cf}(p)\mathrm{cf}(d_T q).$$

Proof: After developing p into a linear combination of the d_i, we can write $p = \mathrm{cf}(p)d_T + r$, where r is d_T-free. Then $\mathrm{cf}(qp) = \mathrm{cf}(q\,\mathrm{cf}(p)d_T + qr) = \mathrm{cf}(q\,\mathrm{cf}(p)d_T) + \mathrm{cf}(qr) = \mathrm{cf}(p)\mathrm{cf}(qd_T) + 0$. ■

Convention. Whenever we write a sum of the form $\sum e_{l_1,\ldots,l_\mu}$, where the l_i can be partially fixed (by context), it will always be assumed that

$$1 \leq l_1 \prec \ldots \prec l_\mu \leq -1+n.$$

Furthermore we allow notations of the form $\sum\{s_i : i \in I\}$ instead of $\sum_{i\in I} s_i$, whenever the description of the index set over which summations have to be done is complicated.

Indeed, later we have to consider sums of the form

$$\sum\{e_{i_1,\ldots,i_s,i_t,\ldots,i_\mu} : {}^{1\leq i_1\prec\ldots\prec i_{s-i}\prec i_s\leq a,}_{b\leq i_t\prec i_{1+t}\prec\ldots\prec i_\mu\leq -1+n}\}$$

which will often be abbreviated to

$$\sum\{e_{i_1,\ldots,i_s,i_t,\ldots,i_\mu} : {}^{i_s\leq a}_{b\leq i_t}\} \quad \text{or even to} \quad \sum\{{}^{i_s\leq a}_{b\leq i_t}\}.$$

Example. Assume, say, $n = 15, T = 4$. Then the sum

$$\sum\{e_{i_1,T,i_3,i_4,i_5} : {}^{i_3\leq 8}_{11\leq i_4}\}$$

consists of the $2\times3\times3 = 18$ summands associated to the 5-uples $i_1, 4, i_3, i_4, i_5$ for which $i_1 \in \{1,2\}$, $i_3 \in \{6,7,8\}$, $(i_4, i_5) \in \{(11,13),(11,14),(12,14)\}$.

Sums whose conditions are unsatisfiable, are of course to be considered 0. Thus if in the example above e.g. $T = 2$, then the sum would be 0.

4. Definition of E_{red}

If we recall that $e_i = c_i + t_i - t_{1+i}$, then it is obvious that the polynomial at the left hand side of Conjecture 2.2, when expanded, has as the subsum of its $t_{\ldots}$-free terms the right hand side. Thus we need to prove that for an arbitrary $T \in \{1, 2, \ldots, n\}$, the coefficient of d_T in E is 0. Given that we consider T fixed we have to show simply that $\mathrm{cf}(E) = 0$. But E in expanded form has many terms that cannot contribute to the coefficient of d_T. So we begin by focusing on the important ones. The summation conventions made in Section 3 are full in place.

Proposition 4.1. *Assume* $\mu \geq 2$. *Define the polynomial* $E_{\rm red}$, *by the following sum of* $1+2(\mu-2)+1 = 2\mu-2$ *sums :*

$$\begin{array}{rcl} E_{\rm red} = \sum e_{-1+T,1+T,i_3,\dots,i_\mu} & + & \\ \sum e_{i_1,-1+T,1+T,i_4,\dots,i_\mu} & + & \sum e_{-2+T,T,i_3,\dots,i_\mu} \\ \sum e_{i_1,i_2,-1+T,1+T,i_5,\dots,i_\mu} & + & \sum e_{i_1,-2+T,T,i_4,\dots,i_\mu} \\ \sum e_{i_1,i_2,i_3,-1+T,1+T,i_6,\dots,i_\mu} & + & \sum e_{i_1,i_2,-2+T,T,i_5,\dots,i_\mu} \\ & \vdots & \\ \sum e_{i_1,i_2,\dots,i_{\mu-2},-1+T,1+T} & + & \sum e_{i_1,i_2,\dots,i_{\mu-3},-2+T,T,i_\mu} \\ & + & \sum e_{i_1,i_2,\dots,i_{\mu-3},i_{\mu-2},-2+T,T}, \end{array}$$

Then

$$\mathrm{cf}(E_{\rm red}) = \mathrm{cf}(E).$$

Proof: Evidently a necessary condition for $e_{i_1,i_2,\dots,i_\mu}$ to contribute terms in which occurs d_T, is that t_T occurs in $e_{i_1,i_2,\dots,i_\mu}$. Since by definition $e_{i_1 i_2 \dots i_\mu} = \prod_{\nu=1}^{\mu}(c_{i_\nu} + t_{i_\nu} - t_{1+i_\nu})$, for the occurrence it is necessary that $T \in \{i_1, 1+i_1, \dots, i_\mu, 1+i_\mu\}$. By the definition of '$\prec$' , it can happen only for at most one $\nu \in \{1,2,\dots,\mu\}$, that $T \in \{i_\nu, 1+i_\nu\}$. Hence $\mathrm{cf}(E)$ equals the cf of

$$\sum e_{-1+T,i_2,\dots,i_\mu} + \sum e_{T,i_2,\dots,i_\mu} + \sum e_{i_1,-1+T,i_3,\dots,i_\mu} + \sum e_{i_1,T,i_3,\dots,i_\mu}$$
$$+ \cdots + \sum e_{i_1,i_2,i_3,\dots,i_{\mu-1},-1+T} + \sum e_{i_1,i_2,i_3,\dots,i_{\mu-1},T}.$$

We shrink this sum further. We look first at the case $T \in \{i_1, 1+i_1\}$ and compare

$$e_{-1+T,i_2,\dots,i_\mu} = (c_{-1+T} + t_{-1+T} - t_T)\prod_{\nu=2}^{\mu}(c_{i_\nu} + t_{i_\nu} - t_{1+i_\nu})$$

with

$$e_{T,i_2,\dots,i_\mu} = (c_T + t_T - t_{1+T})\prod_{\nu=2}^{\mu}(c_{i_\nu} + t_{i_\nu} - t_{1+i_\nu}).$$

Note that, whenever $T \prec i_2$ then also $-1+T \prec i_2$ and then the cfs of $e_{-1+T,i_2,\dots,i_\mu}$ and $e_{T,i_2,\dots,i_\mu}$ are simply the cfs of $-t_T\prod_{\nu=2}^{\mu}(c_{i_\nu} + t_{i_\nu} - t_{1+i_\nu})$ and $t_T\prod_{\nu=2}^{\mu}(c_{i_\nu} + t_{i_\nu} - t_{1+i_\nu})$, respectively , and hence cancel each other. Therefore the coefficient of d_T in $\sum e_{-1+T,i_2,\dots,i_\mu} + \sum e_{T,i_2,\dots,i_\mu}$ equals this coefficient in the surviving terms containing t_T and these pertain to the $i_2 i_3 \dots . i_\mu$ for which $i_2 = 1+T$. So the coefficient of d_T of the sum of the two sums equals $\mathrm{cf}(\sum e_{-1+T,1+T,i_3,\dots,i_\mu})$.

Next we compare, supposing $2 \leq l \leq \mu - 1$,

$$e_{i_1,\dots,i_{l-1},-1+T,i_{l+1},\dots,i_\mu} = (c_{-1+T} + t_{-1+T} - t_T)\cdot\prod_{\nu\neq l}(c_{i_\nu} + t_{i_\nu} - t_{1+i_\nu})$$

with

$$e_{i_1,\ldots,i_{l-1},T,i_{l+1},\ldots,i_\mu} = (c_T + t_T - t_{1+T}) \cdot \prod_{\nu\neq l}(c_{i_\nu} + t_{i_\nu} - t_{1+i_\nu}).$$

Whenever $i_{l-1} \prec -1+T$ and $T \prec i_{1+l}$, then both of above es occur in the sum E but the coefficients of their respective d_T are opposed and so cancel. The strings of indices subordinated to es that do not cancel are those with $i_{l-1} = -2+T$ and those with $i_{l+1} = 1+T$. So we see that the coefficient of d_T in $\sum e_{i_1,\ldots,i_{l-1},-1+T,i_{l+1},\ldots,i_\mu} + \sum e_{i_1,\ldots,i_{l-1},T,i_{l+1},\ldots,i_\mu}$ equals the coefficient of d_T in $\sum e_{i_1,\ldots,i_{l-2},-2+T,T,i_{l+1},\ldots,i_\mu} + \sum e_{i_1,\ldots,i_{l-1},-1+T,1+T,i_{l+1},\ldots,i_\mu}$.

Finally we look at the coefficient of d_T in $\sum e_{i_1,\ldots,i_{\mu-1},-1+T} + \sum e_{i_1,\ldots,i_{\mu-1},T}$. If $i_{\mu-1} \prec -1+T$, then $i_{\mu-1} \prec T$. By analogous reasoning as in the cases before, it follows that the coefficient of d_T of the present sum is that of $\sum e_{i_1,\ldots,i_{\mu-2},-2+T,T}$. This proves the proposition. ■

Lemma 4.2. *There hold the following equalities, by which we also define nonzero reals C_*.*
a. $\mathrm{cf}(d_T e_{-1-i+T}) = \mathrm{cf}(d_T e_{i+T}) =: C_i$ *(* $i \neq -1, 0$*)*
b. $\mathrm{cf}(e_{-1+T,1+T}) = -\mathrm{cf}(e_{-2+T,T}) = -2^{-4}c_{-2+T}c_{-1+T}c_T c_{1+T} =: -C_0$

Proof: a. We have by Corollary 3.3 the equalities

$$\begin{aligned}
\mathrm{cf}(d_T e_{-1-i+T}) &= c_{-1-i+T}(1 + 2^{-2}(1+i)^{-1}c_{-2-i+T} - 2^{-2}i^{-1}c_{-i+T}) \\
\mathrm{cf}(d_T e_{i+T}) &= c_{i+T}(1 - 2^{-2}i^{-1}c_{-1+i+T} + 2^{-2}(1+i)^{-1}c_{1+i+T}).
\end{aligned}$$

One now proves by a lengthy but straightforward computation that the quantities at the right hand side are equal.

b. Since e_{1+T} has t_T not as a variable, we find by Lemma 3.4 that

$$\mathrm{cf}(e_{-1+T}e_{1+T}) = \mathrm{cf}(e_{-1+T})\mathrm{cf}(d_T e_{1+T}).$$

Now $\mathrm{cf}(e_{-1+T}) = 2^{-1}c_{-1+T}c_T$, while by Corollary 3.2 and Proposition 3.1, $\mathrm{cf}(d_T e_{1+T}) = 2^{-3}c_{1+T}(8 - 2c_T + c_{2+T}) = -2^{-3}c_{-2+T}c_{1+T}$. Hence follows part of the claim. The other part follows by very similar computations. ■

Corollary 4.3. *Assume $\mu \geq 2$. With the definitions*

$$\begin{aligned}
C_0 &:= 2^{-4}c_{-2+T}c_{-1+T}c_T c_{1+T}, \\
I(\nu) &:= \sum \left\{ e_{i_1,\ldots,i_{\nu-1},i_{\nu+2},\ldots,i_\mu} : {}^{i_{\nu-1}\leq -3+T}_{3+T\leq i_{\nu+2}} \right\} \quad (\nu = 1, \ldots, -1+\mu) \\
II(\nu) &:= \sum \left\{ e_{i_1,\ldots,i_{\nu-1},i_{\nu+2},\ldots,i_\mu} : {}^{i_{\nu-1}\leq -4+T}_{2+T\leq i_{\nu+2}} \right\} \quad (\nu = 1, \ldots, -1+\mu)
\end{aligned}$$

there holds

$$\mathrm{cf}(E_{\mathrm{red}}) = C_0 \sum_{\nu=1}^{-1+\mu} \mathrm{cf}(d_T(-I(\nu) + II(\nu))).$$

Observation. In the cases $\nu = 1, \nu = -1 + \mu$ occur in $I(\nu), II(\nu)$ formally conditions involving i_0 or $i_{1+\mu}$. These should be discarded.

Proof: Recall that a typical sum occurring in the left column in the definition of E_{red} in Proposition 4.1 has the implicit requirement $i_{\nu-1} \prec -1 + T \prec 1 + T \prec i_{\nu+2}$. This is taken care of via the specifications introduced in the definition of $I(\nu)$. A similar remark holds for the sums of the right column in E_{red} and $II(\nu)$. By the multiplicative definition of $e_{\dots}$ we can put in the left column of the proposition the products $e_{-1+T,1+T}$ into evidence and in the right column the products $e_{-2+T,T}$. So we get

$$E_{\mathrm{red}} = \sum_{\nu=1}^{-1+\mu} e_{-1+T,1+T} I(\nu) + \sum_{\nu=1}^{-1+\mu} e_{-2+T,T} II(\nu).$$

Using that $I(\nu), II(\nu)$ are d_T-free polynomials, lemmas 3.4 and 4.2 yield

$$\begin{aligned}
\mathrm{cf}(E_{\mathrm{red}}) &= \sum_{\nu=1}^{-1+\mu} \mathrm{cf}(e_{-1+T,1+T} I(\nu)) + \sum_{\nu=1}^{-1+\mu} \mathrm{cf}(e_{-2+T,T} II(\nu)), \\
&= \sum_{\nu=1}^{-1+\mu} \mathrm{cf}(e_{-1+T,1+T}) \mathrm{cf}(d_T I(\nu)) + \sum_{\nu=1}^{-1+\mu} \mathrm{cf}(e_{-2+T,T}) \mathrm{cf}(d_T II(\nu)) \\
&= \sum_{\nu=1}^{-1+\mu} -C_0 \mathrm{cf}(d_T I(\nu)) + \sum_{\nu=1}^{-1+\mu} C_0 (\mathrm{cf}(d_T II(\nu)) \\
&= C_0 \sum_{\nu=1}^{-1+\mu} \mathrm{cf}(d_T(-I(\nu) + II(\nu))).
\end{aligned}$$

This analysis refers to the 'typical' case that $2 \le \nu \le -2 + \mu$ holds. The general formulation bears the vestiges of this case. If $\nu = 1$ the natural translation for E_{red} given in Proposition 4.1 to the formula for E_{red} given above requires evidently the interpretation $I(1) := \sum \left\{ e_{i_3,\dots,i_\mu} : 3 + T \le i_3 \right\}$. So either one discards the condition $i_0 \le -3 + T$ or, equivalently, assumes it automatically satisfied. Similar remarks hold of course for $I(1 + \mu)$. See also the treatment of the case $\mu = 5$ in the next section. ■

5. The equation of Conjecture 2.2 for $\mu = 0, 1, 2, 3, 5, n/2$

We prove the equation of conjecture 2.2 for the cases $\mu = 1$ and n even, $\mu = n/2$ in a direct manner using for these simple proofs very little from the general developments above. The cases $\mu = 2, 3, 5$ are also treated here. Their study will ease the digestion of the proof for general μ in the next section since they rely on showing $\mathrm{cf}(E_{\mathrm{red}}) = 0$.

Case $\mu = 0$. The equation of conjecture 2.2 says for $\mu = 0$ the triviality that the leading term of $\mathrm{Sch}_n(x)$ as well as that of $\mathrm{SK}_n(x)$ is x^n. ■

Case $\mu = 1$. Then we have to prove

$$\sum_{1 \le i \le -1+n} e_i = \sum_{1 \le i \le -1+n} c_i.$$

Since $e_i = c_i + t_i - t_{i+1}$, the sum at the left yields

$$\sum_{1 \le i \le -1+n} c_i + \sum_{1 \le i \le -1+n} (t_i - t_{i+1}) = \sum_{1 \le i \le \mu} c_i + (t_1 - t_n),$$

by telescoping. But $t_1 = -2^{-1} c_0 c_1 d_1$ and $t_n = -2^{-1} c_{-1+n} c_n d_n$ and $c_0 = c_n = 0$. Done. ■

Case $\mu = 2$. Then

$$E = \sum_{1 \le i \prec j \le -1+n} e_{ij} \quad \text{and} \quad E_{\mathrm{red}} = e_{-1+T,1+T} + e_{-2+T,T}.$$

Thus $\mathrm{cf}(E_{\mathrm{red}}) = \mathrm{cf}(e_{-1+T,1+T}) + \mathrm{cf}(e_{-2+T,T}) = 0$ by Lemma 4.2b. ■

Case $\mu = 3$. Then

$$E = \sum_{1 \le i \prec j \prec k \le -1+n} e_{ijk},$$

and

$$\begin{aligned} E_{\mathrm{red}} &= \sum e_{-1+T,1+T,k} + \sum e_{-2+T,T,k} + \sum e_{i,-1+T,1+T} + \sum e_{i,-2+T,T} \\ &= e_{-1+T,1+T}\left(\sum\{e_k : 3+T \le k\} + \sum\{e_i : i \le -3+T\}\right) \\ &\quad + e_{-2+T,T}\left(\sum\{e_k : 2+T \le k\} + \sum\{e_i : i \le -4+T\}\right) \\ &= e_{-1+T,1+T}(I(1) + I(2)) + e_{-2+T,T}(II(1) + II(2)). \end{aligned}$$

Again by Lemma 4.2b, we have that $\mathrm{cf}(e_{-1+T,1+T}) = -C_0$, while $\mathrm{cf}(e_{-2+T,T}) = C_0$. Expressions $I(.)$ and $II(.)$ are d_T-free polynomials. Also note that the set of k for which $3+T \leq k$ differs from the set of k for which $2+T \leq k$ by the single element $2+T$. Hence

$$-I(1)+II(1) = -\sum\{e_k : 3+T \leq k\} + \sum\{e_k : 2+T \leq k\} = e_{2+T}.$$

Similar considerations can be made for $-I(2)+II(2)$ and are underlying many of the equations we later write down. Therefore by the lemmas 3.4 and 4.2a, putting there $i=2$, we find

$$\begin{aligned}
\mathrm{cf}(E_{\mathrm{red}}) &= -C_0\mathrm{cf}(d_T(I(1)+I(2)) + C_0\mathrm{cf}(d_T(II(1)+II(2)) \\
&= C_0\mathrm{cf}(d_T(-I(1)+II(1)) + C_0\mathrm{cf}(d_T(-I(2)+II(2)). \\
&= C_0\mathrm{cf}(d_T\, e_{2+T}) + C_0\mathrm{cf}(d_T \cdot -e_{-3+T}) \\
&= C_0C_2 - C_0C_2 \\
&= 0 \quad \blacksquare
\end{aligned}$$

We now jump to the case $\mu = 5$ since here the phenomena relevant in the general case can be better seen than in case $\mu = 4$.

Case $\mu = 5$. Here

$$E = \sum\{e_{ijklm} : 1 \leq i \prec j \prec k \prec l \prec m \leq n-1\}$$

and hence

$$\begin{aligned}
E_{\mathrm{red}} &= \sum e_{-1+T,1+T,k,l,m} + \sum e_{-2+T,T,k,l,m} + \sum e_{i,-1+T,1+T,l,m} + \sum e_{i,-2+T,T,l,m} \\
&\quad + \sum e_{i,j,-1+T,1+T,m} + \sum e_{i,j,-2+T,T,m} + \sum e_{i,j,k,-1+T,1+T} + \sum e_{i,j,k,-2+T,T} \\
&= e_{-1+T,1+T}(I(1)+I(2)+I(3)+I(4)) + \\
&\qquad + e_{-2+T,T}(II(1)+II(2)+II(3)+II(4));
\end{aligned}$$

and so, for similar reasons as in case explained before,

$$\mathrm{cf}(E_{\mathrm{red}}) = C_0\mathrm{cf}(d_T(-I(1)+II(1)-I(2)+II(2)-I(3)+II(3)-I(4)+II(4)).$$

Hence we can write

$$
\begin{aligned}
\mathrm{cf}(E_{\mathrm{red}}) \overset{*_1}{=}\ & C_0\mathrm{cf}(d_T\times \\
& \quad (-\sum\left\{e_{klm} : \scriptstyle 3+T\leq k\right\} + \sum\left\{e_{klm} : \scriptstyle 2+T\leq k\right\} \\
& \quad -\sum\left\{e_{ilm} : \substack{i\leq -3+T \\ 3+T\leq l}\right\} + \sum\left\{e_{ilm} : \substack{i\leq -4+T \\ 2+T\leq l}\right\} \\
& \quad -\sum\left\{e_{ijm} : \substack{j\leq -3+T \\ 3+T\leq m}\right\} + \sum\left\{e_{ijm} : \substack{j\leq -4+T \\ 2+T\leq m}\right\} \\
& \quad -\sum\left\{e_{ijk} : \scriptstyle k\leq -3+T\right\} + \sum\left\{e_{ijk} : \scriptstyle k\leq -4+T\right\})) \\
\overset{*_2}{=}\ & C_0\mathrm{cf}(d_T\times \\
& \quad (\qquad e_{2+T}\sum\{e_{lm} : 4+T\leq l\} \\
& \quad -e_{-3+T}\sum\{e_{lm} : 3+T\leq l\} + e_{2+T}\sum\left\{e_{im} : \substack{i\leq -4+T \\ 4+T\leq m}\right\} \\
& \quad -e_{-3+T}\sum\left\{e_{im} : \substack{i\leq -5+T \\ 3+T\leq m}\right\} + e_{2+T}\sum\{e_{ij} : j\leq -4+T\} \\
& \quad -e_{-3+T}\sum\{e_{ij} : j\leq -5+T\} \qquad)) \\
\overset{*_3}{=}\ & C_0C_2\mathrm{cf}(d_T\times \\
& \quad (\sum\{e_{lm} : 4+T\leq l\} - \sum\{e_{lm} : 3+T\leq l\} \\
& \quad +\sum\left\{e_{im} : \substack{i\leq -4+T \\ 4+T\leq m}\right\} - \sum\left\{e_{im} : \substack{i\leq -5+T \\ 3+T\leq m}\right\} \\
& \quad +\sum\left\{e_{ij} : \scriptstyle j\leq -4+T\right\} - \sum\{e_{ij} : j\leq -5+T\})) \\
\overset{*_4}{=}\ & C_0C_2\mathrm{cf}(d_T\times \\
& \quad (-e_{3+T}\sum\left\{e_m : \scriptstyle 5+T\leq m\right\} \\
& \quad -e_{3+T}\sum\left\{e_i : \scriptstyle i\leq -5+T\right\} + e_{-4+T}\sum\left\{e_m : \scriptstyle 4+T\leq m\right\} \\
& \quad e_{-4+T}\sum\left\{e_i : \scriptstyle i\leq -6+T\right\})) \\
\overset{*_5}{=}\ & C_0C_2C_3\mathrm{cf}(d_T\times \\
& \quad (-\sum\left\{e_m : \scriptstyle 5+T\leq m\right\} + \sum\left\{e_m : \scriptstyle 4+T\leq m\right\} \\
& \quad -\sum\left\{e_i : \scriptstyle i\leq -5+T\right\} + \sum\left\{e_i : \scriptstyle i\leq -6+T\right\})) \\
\overset{*_6}{=}\ & C_0C_2C_3\mathrm{cf}(d_T\times \\
& \quad (\qquad e_{4+T} \\
& \quad -e_{-5+T} \qquad)) \\
\overset{*_7}{=}\ & C_0C_2C_3(C_4 - C_4)) \\
=\ & 0.
\end{aligned}
$$

Here in '$\overset{*_1}{=}$' we used the definitions of $I(\nu), II(\nu)$ for the case $\mu = 5$ and transcribed them to the notation $ijklm$ in place of $i_1i_2i_3i_4i_5$. In '$\overset{*_2}{=}$' we carried through line for line the additions of the lines of the previous block. Consider for example

$$-I(2) + II(2) = -\sum\left\{e_{ilm} : {\scriptstyle i\leq -3+T \atop 3+T\leq l}\right\} + \sum\left\{e_{ilm} : {\scriptstyle i\leq -4+T \atop 2+T\leq l}\right\}$$

Whenever ilm is so that $i \leq -4 + T \& 3 + T \leq l$ (supposing as always $i \prec l \prec m$) then e_{ilm} will occur in both sums and hence cancel. There remain thus the triples ilm with $i = -3+T$ in the left sum and those with $l = 2+T$ in the right sum. Putting them into evidence the net result for $-I(2)+II(2)$ is given as indicated by the second line of the second block as

$$-e_{-3+T}\sum\{e_{lm} : 3 + T \leq l\} + e_{2+T}\sum\left\{e_{im} : {\scriptstyle i\leq -4+T \atop 4+T\leq m}\right\}.$$

To obtain '$\overset{*_3}{=}$' note that the block after the '$\times$' in '$\overset{*_2}{=}$' represents a d_T-free polynomial which comes as a sum in the form $-e_{-3+T}q_1+e_{2+T}q_2$ (with evident q_1, q_2). So, using Lemma 3.4 with the pairs $(p, q) = (d_T \cdot -e_{-3+T}, q_1)$ and $(d_T \cdot e_{2+T}, q_2)$ and Lemma 4.2a we get

$$\begin{aligned}\mathrm{cf}(d_T(-e_{-3+T}q_1 + e_{2+T}q_2)) &= \mathrm{cf}(d_T - e_{-3+T})\mathrm{cf}(d_Tq_1) + \mathrm{cf}(d_Te_{2+T})\mathrm{cf}(d_Tq_2)\\ &= C_2\mathrm{cf}(d_Tq_2) - C_2\mathrm{cf}(d_Tq_1) = C_2(\mathrm{cf}(d_T(q_2 - q_1)).\end{aligned}$$

This yields the block introduced by $*_3$. The block $*_4$ is obtained from block $*_3$ similarly as block $*_2$ is obtained from $*_1$. Block $*_5$ from block $*_4$ similarly as $*_3$ from $*_2$ using that $\mathrm{cf}(d_Te_{3+T}) = \mathrm{cf}(d_Te_{-4+T}) = C_3$. Block $*_6$ comes from block $*_5$ just as block $*_4$ is obtained from block $*_3$. Finally block $*_7$ comes from $*_6$ using $\mathrm{cf}(d_Te_{4+T}) = \mathrm{cf}(d_Te_{-5+T}) = C_4$. ∎

Remark. Note that in case it happens that one of the $C_i = 0$, then the desired fact $\mathrm{cf}(E_{\mathrm{red}}) = 0$ follows at an earlier instance in this arguments; that is in this case one needs not the full reasoning here presented.

Case n even and $\mu = n/2$. In this case the equation boils down to the claim $e_1e_3\cdots e_{n-3}e_{n-1} = c_1c_3\cdots c_{n-1}$. Now working directly with the definitions of the e_i we find

$$e_1e_3\cdots e_{n-3}e_{n-1} = c_1c_3\cdots c_{n-3}c_{n-1}\cdot\prod_{\substack{i=1\\ i\equiv_2 1}}^{-1+n}\frac{(x+i)(x+n+i)}{(x+2i-1)(x+2i+1)}$$

Evidently the first of the products at the right is independent of x. The second product can be rewritten by substituting the admitted product index i by $2i+1$ and ranging with i from 0 to $\frac{n}{2}-1$. We then get that this second product equals

$$\prod_{i=0}^{(n/2)-1} \frac{(x+2i+1)(x+n+2i+1)}{(x+4i+1)(x+4i+3)}.$$

The product of the first factors in the numerator equals

$$(x+1)(x+3)\cdots(x+n-1)$$

while the product of the second factors equals

$$(x+n+1)(x+n+3)\cdots(x+2n-1).$$

The product of these products is easily seen to be the product of the denominators. So the product $\prod_{i=0}^{(n/2)-1} \ldots = 1$. ∎

6. Proof of Conjecture 2.2

We start from the equation for $\mathrm{cf}(E_{\mathrm{red}})$ found in Corollary 4.3:

$$\mathrm{cf}(E_{\mathrm{red}}) = C_0 \sum_{\nu=1}^{-1+\mu} \mathrm{cf}(d_T(-I(\nu)+II(\nu))).$$

We have

$$\begin{aligned}
&-I(\nu)+II(\nu)\\
&= -\sum\left\{e_{i_1,\ldots,i_{\nu-1},i_{\nu+2},\ldots,i_\mu} : {}^{i_{\nu-1}\leq -3+T}_{3+T\leq i_{\nu+2}}\right\} + \sum\left\{e_{i_1,\ldots,i_{\nu-1},i_{\nu+2},\ldots,i_\mu} : {}^{i_{\nu-1}\leq -4+T}_{2+T\leq i_{\nu+2}}\right\}\\
&= -e_{-3+T}\sum\left\{{}^{i_{\nu-2}\leq -5+T}_{3+T\leq i_{\nu+2}}\right\} + e_{2+T}\sum\left\{{}^{i_{\nu-1}\leq -4+T}_{4+T\leq i_{\nu+3}}\right\}.
\end{aligned}$$

The explanation is a generalisation of one given in Section 5, for the case $\mu = 5$. Whenever $i_1, \ldots, i_{\nu-1}, i_{\nu+2}, \ldots, i_\mu$ is an uple so that $i_{\nu-1} \leq -4+T \,\&\, 3+T \leq i_{\nu+2}$ holds, then $e_{i_1,\ldots,i_{\nu-1},i_{\nu+2},\ldots,i_\mu}$ occurs in both the sums of the first line but with opposite signs and therefore cancels. Noting

$$\mathbb{Z}_{\leq -3+T} = \{-3+T\} \uplus \mathbb{Z}_{\leq -4+T} \text{ and } \mathbb{Z}_{\leq 3+T} \uplus \{2+T\} = \mathbb{Z}_{\leq 2+T},$$

one sees that the signed $e_{i_1,\ldots,i_{\nu-1},i_{\nu+2},\ldots,i_\mu}$ which do not cancel are those of the form $-e_{i_1,\ldots,i_{\nu-2},-3+T,,i_{\nu+2},\ldots,i_\mu}$ and those of the form $e_{i_1,\ldots,i_{\nu-1},2+T,i_{\nu+3}\ldots,i_\mu}$.

Putting e_{-3+T} and e_{2+T} into evidence, this and the abbreviation conventions introduced earlier explain the second equation above.

Now by Lemma 4.2 we find $\mathrm{cf}(d_T e_{-3+T}) = \mathrm{cf}(d_T e_{2+T}) = C_2$, and by Lemma 3.4 and using that the inner sums in $*_1$ below are d_T-free, we find $\mathrm{cf}(E_{\mathrm{red}}) = 0$ iff the following holds:

$$*_1 : \mathrm{cf}(d_T(\sum_{\nu=1}^{-1+\mu}(-\sum\{{}^{\,i_{\nu-2}\leq -5+T}_{\,3+T\leq i_{\nu+2}}\} + \sum\{{}^{\,i_{\nu-1}\leq -4+T}_{\,4+T\leq i_{\nu+3}}\}))) = 0.$$

This is the beginning of an inductive procedure in which the sum outer sum $\sum_{\nu=1}^{\cdots}$ extends over less and less summands till it vanishes. We define more generally the claim $*_l$ as follows and and show how to deduce $*_{1+l}$:

$$*_l : \ \ \mathrm{cf}(d_T\left(\sum_{\nu=1}^{-l+\mu}(-1)^l\underbrace{\sum\{{}^{\,i_{\nu-2}\leq -4-l+T}_{\,2+l+T\leq i_{\nu+l+1}}\}}_{q_\nu} + (-1)^{1+l}\underbrace{\sum\{{}^{\,i_{\nu-1}\leq -3-l+T}_{\,3+l+T\leq i_{\nu+l+2}}\}}_{q'_\nu}\right)) = 0,$$

The first step is to write $\sum_{\nu=1}^{-l+\mu}\ldots$ as

$$\begin{aligned}\sum_{\nu=1}^{-l+\mu}(-^l q_\nu + -^{1+l}q'_\nu) &= -^l q_1 + \sum_{\nu=1}^{-1-l+\mu}(-^{1+l}q'_\nu + -^l q_{1+\nu}) + -^{1+l}q'_{-l+\mu}\\ &= -^l\sum_{\nu=1}^{-1-l+\mu}(-q'_\nu + q_{1+\nu}),\end{aligned}$$

where we used $q_1 = 0$ for it involves an undefined i_{-1} in its conditions, and $q'_{-l+\mu} = 0$, since it involves $i_{\mu+2}$ in its conditions.

Now

$$\begin{aligned}-q'_\nu + q_{1+\nu} &= -\sum\{e_{i_1\ldots i_{\nu-1},i_{\nu+l+2}\ldots i_\mu} : {}^{\,i_{\nu-1}\leq -l-3+T}_{\,3+l+T\leq i_{\nu+l+2}}\} +\\ &\qquad\qquad + \sum\{e_{i_1\ldots i_{\nu-1},i_{\nu+l+2}\ldots i_\mu} : {}^{\,i_{\nu-1}\leq -4-l+T}_{\,2+l+T\leq i_{\nu+l+2}}\}\\ &= -\sum\{e_{i_1\ldots i_{\nu-2},-3-l+T,i_{\nu+l+2}\ldots i_\mu} : {}^{\,i_{\nu-1}\leq -l-5+T}_{\,3+l+T\leq i_{\nu+l+2}}\} +\\ &\qquad\qquad + \sum\{e_{i_1\ldots i_{\nu-1},2+l+T,i_{\nu+l+2}\ldots i_\mu} : {}^{\,i_{\nu-1}\leq -l-4+T}_{\,4+l+T\leq i_{\nu+l+2}}\}\\ &= -e_{-3-l+T}\underbrace{\sum\{{}^{\,i_{\nu-2}\leq -l-5+T}_{\,3+l+T\leq i_{\nu+l+2}}\}}_{\tilde{q}'_\nu} + e_{2+l+T}\underbrace{\sum\{{}^{\,i_{\nu-1}\leq -l-4+T}_{\,4+l+T\leq i_{\nu+l+2}}\}}_{\tilde{q}_{1+\nu}}.\end{aligned}$$

Here we used that the e's with indices $i_1 ... i_{\nu-1}, i_{\nu+l+2} ... i_\mu$ for which

$$(i_{\nu-1} \leq -l-4+T) \;\;\&\;\; (3+l+T \leq i_{\nu+l+2})$$

hold occur in the two sums with opposite signs and thus cancel. The e's that do not cancel are those with $i_{\nu-1} = -3-l+T$ in the left hand sum and those with $i_{\nu+l+2} = 2+l+T$ in the right hand sum. Thus the sum of the two sums can be rewritten as in the second equality.

So the left hand side of claim $*_l$ can be processed as follows:

$$\begin{aligned}
\mathrm{lhs}(*_l) &= \mathrm{cf}(d_T \cdot -^l \sum_{\nu=1}^{-1-l+\mu} (-q'_\nu + q_{1+\nu})) \\
&= -^l \sum_{\nu=1}^{-1-l+\mu} \mathrm{cf}(d_T(-e_{-3-l+T} \cdot \tilde{q}'_\nu + e_{2+l+T} \cdot \tilde{q}_{1+\nu})) \\
&= -^l \sum_{\nu=1}^{-1-l+\mu} (\mathrm{cf}(d_T \cdot -e_{-3-l+T})\mathrm{cf}(d_T\tilde{q}'_\nu) + \mathrm{cf}(d_T e_{2+l+T})\mathrm{cf}(d_T\tilde{q}_{1+\nu})) \\
&= -^l \sum_{\nu=1}^{-1-l+\mu} (-C_{2+l})\mathrm{cf}(d_T\tilde{q}'_\nu) + C_{2+l}\mathrm{cf}(d_T\tilde{q}_{1+\nu})) \\
&= C_{2+l} \sum_{\nu=1}^{-1-l+\mu} (-^{1+l}\mathrm{cf}(d_T\tilde{q}'_\nu) + -^l\mathrm{cf}(d_T\tilde{q}_{1+\nu})) \\
&= C_{2+l}\mathrm{cf}(d_T \cdot \sum_{\nu=1}^{-1-l+\mu} (-^{1+l}\tilde{q}'_\nu + -^l\tilde{q}_{1+\nu})).
\end{aligned}$$

Therefore if $C_{2+l} \neq 0$, then by the definitions of $\tilde{q}'_\nu, \tilde{q}_\nu$ we can conclude

$$\mathrm{cf}(d_T(\sum_{\nu=1}^{-1-l+\mu} (-^{1+l} \sum \{ {}^{i_{\nu-2} \leq -l-5+T}_{3+l+T \leq i_{\nu+l+2}} \} + -^l \sum \{ {}^{i_{\nu-1} \leq -l-4+T}_{4+l+T \leq i_{\nu+l+2}} \}))) = 0.$$

In other words we have proved that if $C_{2+l} \neq 0$, then $*_l$ is equivalent to $*_{1+l}$. That $C_{2+l} \neq 0$ can be assumed will follow at the end by a similar argument as the one given in the final remark for the case $\mu = 5$ in the previous section.

We finally infer the truth of these claims by verifying claim $*_{\mu-3}$. Indeed for $l = \mu - 3$ the sum

$$\sum_{\nu=1}^{-1-l+\mu} (-q'_\nu + q_{1+\nu})$$

is

$$
\begin{aligned}
\sum_{\nu=1}^{2}(-q'_\nu + q_{1+\nu}) &= -q'_1 + q_2 - q'_2 + q_3 \\
&= -\sum\left\{ {}^{i_0\le -\mu+T}_{\mu+T\le i_\mu} \right\} + \sum\left\{ {}^{i_0\le -1-\mu+T}_{-1+\mu+T\le i_\mu} \right\} - \\
&\qquad\qquad - \sum\left\{ {}^{i_1\le -\mu+T}_{\mu+T\le i_{\mu+1}} \right\} + \sum\left\{ {}^{i_1\le -1-\mu+T}_{1+\mu+T\le i_{\mu+1}} \right\} \\
&= -\sum\{e_{i_\mu} : {\scriptstyle \mu+T\le i_\mu}\} + \sum\{e_{i_\mu} : {\scriptstyle -1+\mu+T\le i_\mu}\} - \\
&\qquad\qquad - \sum\{e_{i_1} : {\scriptstyle i_1\le -\mu+T}\} + \sum\{e_{i_1} : {\scriptstyle i_1\le -1-\mu+T}\} \\
&= e_{-1+\mu+T} - e_{-\mu+T}.
\end{aligned}
$$

Since

$$\mathrm{cf}(d_T e_{-1+\mu+T}) = \mathrm{cf}(d_T e_{-\mu+T})$$

claim $*_{\mu-3}$ follows and Schrödinger's conjecture is proved. ■

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