Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 18–31

SPHERICAL HELLINGER-KANTOROVICH GRADIENT FLOWS

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Abstract: We study nonlinear degenerate parabolic equations of Fokker-Planck type which can be viewed as gradient flows with respect to the recently introduced spherical Hellinger-Kantorovich distance. The driving entropy is not assumed to be geodesically convex. We prove solvability of the problem and the entropy-entropy production inequality, which implies exponential convergence to the equilibrium. As a corollary, we obtain some related results for the Wasserstein gradient flows. We also deduce transportation inequalities in the spirit of Talagrand, Otto and Villani for the spherical and conic Hellinger-Kantorovich distances.

Keywords: functional inequalities, Talagrand inequalities, optimal transport, Hellinger-Kantorovich distance, geodesic non-convexity .
Math. Subject Classification (2010): 26D10, 35Q84, 49Q20, 58B20.

1.Introduction

Unbalanced optimal transport [33, 29, 10, 32, 11, 39] is a recent variant of the Monge-Kantorovich transport which is relevant in the situations lacking the conservation of the total mass, such as processes involving reaction. Important objects in the field are the conic Hellinger-Kantorovich distance (also known as the Wasserstein-Fisher-Rao distance) on the set of Radon measures and the spherical Hellinger-Kantorovich distance on the set of probability measures, see Section 3.3 below for the definitions and references.

On both the conic and spherical Hellinger-Kantorovich spaces, some Otto calculus [36, 46] can be developed [29, 5], and it is easy to formally define the gradient flows. This paper considers the spherical gradient flows.

Our basic setting is as follows. Let Ω be either an open connected bounded domain in $\mathbb{R}^d$ with sufficiently smooth boundary or a flat torus

Received September 11, 2016.
The second author would like to thank J.A. Carrillo and W. Shi for helpful discussions. The research was partially supported by the Portuguese Government through FCT/MCTES and by the ERDF through PT2020 (projects CMUC UID/MAT/00324/2013, PTDC/MAT-PUR/28686/2017 and TUBITAK/0005/2014).

$\mathbb{T}^d$. Fix functions $E \in C(\overline{\Omega} \times [0,\infty))$, $f \in C^1(\overline{\Omega} \times (0,+\infty))$, and a probability density $m \in C(\overline{\Omega})$ satisfying

$$E(x,u) \geq 0, \quad (x,u) \in \overline{\Omega} \times [0,\infty); \tag{1.1}$$

$$m(x) > 0, \quad x \in \overline{\Omega}; \tag{1.2}$$

$$E(x,m(x)) = 0, \quad x \in \Omega; \tag{1.3}$$

$$E_u(x,u) = -f(x,u), \quad (x,u) \in \Omega \times (0,+\infty); \tag{1.4}$$

$$f_u(x,u) < 0, \quad (x,u) \in \overline{\Omega} \times (0,+\infty). \tag{1.5}$$

Here we opted to fix E, f, m satisfying some hypotheses, but it is possible to state all the assumptions in terms of f only, and then reconstruct E and m in a relevant way, see Section 3.1. The function

$$\mathcal{E}(u) = \int_\Omega E(x,u(x))\,dx. \tag{1.6}$$

will be called the *relative entropy*.

We are interested in the formal gradient flow

$$\partial_t u = -\operatorname{grad} \mathcal{E}(u), \tag{1.7}$$

where the gradient is taken w.r.t. the spherical Hellinger-Kantorovich structure on the set of probability measures on Ω. More specifically, we study the problem

$$\partial_t u = -\operatorname{div}(u\nabla f) + u\Big(f - \int_\Omega uf\,dx\Big), \qquad (x,t) \in \Omega \times (0,\infty), \tag{1.8}$$

$$u\frac{\partial f}{\partial \nu} = 0, \qquad (x,t) \in \partial\Omega \times (0,\infty), \tag{1.9}$$

$$u = u^0, \qquad (x,t) \in \Omega \times 0, \tag{1.10}$$

$$u \geq 0, \ \int_\Omega u\,dx = 1, \qquad (x,t) \in \Omega \times (0,\infty). \tag{1.11}$$

We refer to Remark 1.1 concerning the relation between (1.7) and this problem. The model (1.8)–(1.11) can be viewed as a reactive nonlinear equation of Fokker-Planck type, in the spirit of [20], with conservation of mass. Reaction-diffusion problems with conservation of mass were studied in [38, 25, 41, 42, 1, 24, 16], see also the references therein. On the other hand, after a change of variables, our problem fits into the framework of fitness-driven models of population dynamics, and might be applicable to some human societies. In Remark 1.3 we discuss this issue in detail.

Remark 1.1. The right-hand sides of (1.7) and (1.8) formally coincide when Ω is a torus or is convex, see [31, 5]. In the case of non-convex Ω we will still refer to (1.8)–(1.11) as to a gradient flow, although this is sloppy.

Remark 1.2. For the metric gradient flows like (1.7), the geodesic convexity of the driving entropy functional (or at least semi-convexity, i.e., λ-convexity with a negative constant λ) makes a difference [36, 3, 45, 46]. The presence of convexity allows one to apply minimizing movement schemes [3, 28] to construct solutions to the gradient flow. Moreover, λ-convexity with λ strictly positive enables the Bakry-Emery procedure [4] which usually yields the exponential convergence of the relative entropy to zero. Minimizing movement schemes for conic Hellinger-Kantorovich gradient flows of geodesically convex functionals and for related reaction-diffusion equations were suggested in [22, 21].

Under our assumptions, the entropy, generally speaking, does not possess neither geodesic convexity nor semi-convexity with respect to either the spherical or conic Hellinger-Kantorovich structure, or even to the classical Wasserstein one, cf. [30, 29].

Remark 1.3. The fitness-driven models [34, 13, 14, 23] of population dynamics assume that the dispersal strategy is determined by a local intrinsic characteristic of organisms called *fitness*. The fitness manifests itself as a growth rate, and simultaneously affects the dispersal as the species move along its gradient towards the most favorable environment. In terms of the PDEs, this can be expressed [30] in the following manner:

$$\partial_t U = -\operatorname{div}(U\nabla F) + UF, \qquad (x,t) \in \Omega \times (0,\infty), \tag{1.12}$$

$$U\frac{\partial F}{\partial \nu} = 0, \qquad (x,t) \in \partial\Omega \times (0,\infty). \tag{1.13}$$

$$U = U^0, \qquad (x,t) \in \Omega \times 0. \tag{1.14}$$

Here $U(x,t)$ is the nonnegative density of individuals, and F is the fitness which depends on x and U in a certain way. Namely, we assume that

$$F(x,t) = f\left(x, \frac{U(x,t)}{\int_\Omega U(\xi,t)\,d\xi}\right). \tag{1.15}$$

The direct dependence on x expresses the spatial inhomogeneity of the resources. The dependence on the normalized population density (in contrast with [34, 13, 14, 15, 30] and the references therein, where the fitness depends on the density U itself) models the phenomenon that the individuals compare the quality of their life with the ones of the other members of the society, and their fitness is determined by their relative success in comparison with the others. This model seems to be specifically relevant for those human societies where the population growth (which depends on various factors including fertility, ability of kids to survive, longevity etc.) is an increasing function of the quality of life. The problem (1.12)–(1.14) resembles a conic Hellinger-Kantorovich gradient flow, cf. [30], but this guess is wrong. The reason is that (1.15) is not an L^2 variation of any functional. Setting

$$M := \int_\Omega U\,dx,\ u := \frac{U}{M},\ M^0 := \int_\Omega U^0\,dx,\ u^0 := \frac{U^0}{M^0},$$

we recast (1.12), (1.13) in the form

$$\partial_t u = -\operatorname{div}(u\nabla f) + u\Big(f - \frac{d(\log M)}{dt}\Big), \qquad (x,t) \in \Omega \times (0,\infty), \tag{1.16}$$

$$u\frac{\partial f}{\partial \nu} = 0, \qquad (x,t) \in \partial\Omega \times (0,\infty). \tag{1.17}$$

$$u = u^0, \qquad (x,t) \in \Omega \times 0, \tag{1.18}$$

$$u \ge 0,\ \int_\Omega u\,dx = 1, \qquad (x,t) \in \Omega \times (0,\infty). \tag{1.19}$$

Since $u(t)$ is a probability distribution, we at least formally infer that

$$\frac{d(\log M)}{dt} = \int_\Omega uf\,dx, \tag{1.20}$$

arriving at (1.8)–(1.11). On the other hand, given U^0 (and thus u^0 and M^0) and a solution u to (1.8)–(1.11), we can recover the mass $M(t)$ from (1.20), and $U = Mu$ solves (1.12)– (1.14).

In what follows, d_{HK}, d_{HKS}, and W_2 stand for the Hellinger-Kantorovich distance (which will be also referred to as the conic distance), spherical Hellinger-Kantorovich distance and the quadratic Wasserstein distance. Observe that

$$d_{HK} \leq d_{HKS} \leq W_2 \tag{1.21}$$

for probability measures (see Section 3.3 below), although d_{HK} is of course defined for Radon measures of any mass.

In this paper, we prove solvability (Section 3.1) and the entropy-entropy production inequality (Section 2) for the spherical Hellinger-Kantorovich gradient flow (1.7), and derive a related transportation inequality in the spirit of Talagrand, Otto and Villani. We also deduce some results of this kind for the Wasserstein and the conic Hellinger-Kantorovich gradient flows. As was already anticipated, we do not assume geodesic convexity of the driving entropies of the gradient flows. In order to better illustrate our results and compare them with the existing ones, let us formally write down the conceivable inequalities.

The following four inequalities are expected to hold under the assumption $\int_\Omega u = 1$:

$$\mathcal{E}(u) \lesssim \int_\Omega u|\nabla f|^2, \tag{1.22}$$

$$\mathcal{E}(u) \lesssim \int_\Omega u\left(f - \int_\Omega uf\right)^2 + \int_\Omega u|\nabla f|^2, \tag{1.23}$$

$$W_2^2(u,m) \lesssim \mathcal{E}(u), \tag{1.24}$$

$$d_{HKS}^2(u,m) \lesssim \mathcal{E}(u). \tag{1.25}$$

The next two inequalities do not require that $\int_\Omega u = 1$:

$$\mathcal{E}(u) \lesssim \int_\Omega uf^2 + \int_\Omega u|\nabla f|^2, \tag{1.26}$$

$$d_{HK}^2(u,m) \lesssim \mathcal{E}(u). \tag{1.27}$$

Inequalities (1.22),(1.23), (1.26) are the entropy-entropy production inequalities for the Wasserstein, spherical Hellinger-Kantorovich and conic Hellinger-Kantorovich gradient flows, respectively. Inequalities

(1.24),(1.25), (1.27) are the transportation (Talagrand) inequalities in those spaces. Note that (1.22) implies (1.23), and (1.23) yields (1.26) since

$$\int_\Omega u\left(f - \int_\Omega uf\right)^2 = \int_\Omega uf^2 - \left(\int_\Omega uf\right)^2.$$

However, the last implication is only valid for probability distributions u, whereas (1.26) would not be a consequence of (1.23) for u of arbitrary mass. These three inequalities can be used to derive exponential convergence to the equilibrium m for the corresponding gradient flows, see [45, 46, 30] as well as Theorems 3.9 and 3.11 below.

Due to (1.21), inequality (1.24) implies (1.25), and (1.25) yields (1.27) for probability distributions. Generally speaking, (1.27) is not a corollary of (1.25) (cf. Remark 3.18 below).

Inequality (1.22) was proved in [8] via the Bakry-Emery approach provided the entropy is geodesically convex w.r.t. the Wasserstein structure (displacement convex). It may be viewed as a generalized log-Sobolev inequality. The classical log-Sobolev corresponds to the case $f = -\log u$. Inequality (1.23) will be proved in Section 2 without assuming any kind of geodesic convexity. This inequality can be used to derive (1.22) for geodesically non-convex entropies (see Section 3.2) provided u satisfies the Poincaré inequality (this is true for instance when u is a Muckenhoupt weight [18]). Inequality (1.26) was established in [30] and will be used in the proof of (1.23). Inequality (1.24) was proved in [44, 37, 9, 12] (mainly for the case $\Omega = \mathbb{R}^d$) for displacement convex entropies. Inequalities (1.25) and (1.27) will be proved in Section 3.3, again without assuming any geodesic convexity.

2.Spherical inequality

Let Ω be an open connected bounded domain in $\mathbb{R}^d$ with sufficiently smooth boundary. The results of the section remain valid for the torus $\Omega = \mathbb{T}^d$. Throughout the section, we will work with functions $E \in C(\overline{\Omega}\times[0,\infty))$,

$f \in C^1(\overline{\Omega} \times (0,+\infty))$, and a probability density $m \in C(\overline{\Omega})$ satisfying

$$E(x,u) \geq 0, \quad (x,u) \in \overline{\Omega} \times [0,\infty); \tag{2.1}$$

$$m(x) > 0, \quad x \in \overline{\Omega}; \tag{2.2}$$

$$E(x,m(x)) = 0, \quad x \in \Omega; \tag{2.3}$$

$$E_u(x,u) = -f(x,u), \quad (x,u) \in \Omega \times (0,+\infty); \tag{2.4}$$

$$f_u(x,u) < 0, \quad (x,u) \in \overline{\Omega} \times (0,+\infty). \tag{2.5}$$

In what follows, bare f stands for $f(x,u(x))$, where $u \in U$ is given; likewise, ∇f stands for the full gradient of $f(x,u(x))$ with respect to x.

The following theorem states the main result.

Theorem 2.1. *Assume* (2.1)–(2.5). *Let U be a uniformly integrable set of smooth probability measures on $\overline{\Omega}$. Then*

$$\int_\Omega E(x,u(x))\,dx \\ \leq C\left[\int_\Omega u(x)(f(x,u(x))-a)^2\,dx + \int_\Omega u(x)|\nabla f(x,u(x))|^2\,dx\right], \tag{2.6}$$

where the constant C may depend on U but is independent of $u \in U$ and $a \in \mathbb{R}$.

Our strategy of the proof of Theorem 2.1 consists in proving the inequality

$$\int_\Omega u(f-a)^2\,dx + \int_\Omega u|\nabla f|^2\,dx \geq \varkappa a^2 \tag{2.7}$$

with a constant $\varkappa > 0$ independent of u ranging over a uniformly integrable set U. Indeed, by [30, Theorem 2.9], we have the inequality

$$\int_\Omega E\,dx \leq C_1 \int_\Omega u(f^2 + |\nabla f|^2)\,dx$$

(we can apply the theorem because uniform integrability ensures that no sequence in U converges to 0 in measure). Setting

$$\bar{f} = \int uf\,dx$$

and recalling that u is a probability measure, we see that

$$\int_\Omega uf^2\,dx = \int_\Omega u(f-\bar{f})^2\,dx + \bar{f}^2,$$

so if we had (2.7), we would apply it for $a = \bar{f}$ obtaining

$$\int_\Omega u f^2\,dx \le (1+\varkappa^{-1})\int_\Omega u(f-\bar{f})^2\,dx + \varkappa^{-1}\int_\Omega u|\nabla f|^2\,dx,$$

and thus,

$$\int_\Omega E\,dx \le C\left[\int_\Omega u(f-\bar{f})^2\,dx + \int_\Omega u|\nabla f|^2\,dx\right].$$

This particular case of (2.6) actually implies (2.6), as

$$\int_\Omega u(f-\bar{f})^2\,dx = \min_{a\in\mathbb{R}}\int_\Omega u(f-a)^2\,dx,$$

which is a consequence of the following instance of the Pythagorean theorem in $L^2(du)$:

$$\int_\Omega u(f-a)^2\,dx = \int_\Omega u(f-\bar{f})^2\,dx + (\bar{f}-a)^2.$$

Actually we will prove a slightly stronger inequality than (2.7), as stated in the following lemma.

Lemma 2.2. *Let U be a uniformly integrable set of smooth probability measures on $\overline{\Omega}$; then there exist $\varkappa > 0$ and $\sigma > 0$ such that*

$$\int_{[u\ge\sigma]} u(x)\Big((f(x,u(x))-a)^2 + |\nabla f(x,u(x))|^2\Big)\,dx \ge \varkappa a^2 \tag{2.8}$$

for all $u \in U$ and $a \in \mathbb{R}$.

The proof is carried out in the subsequent lemmas.

Given a set M of integrable functions on Ω, let

$$\omega_M(\delta) = \sup\left\{\int_A |u|\,dx\colon\ u\in M, A\subset\Omega, |A|\le\delta\right\}$$

be the modulus of integrability of M. Clearly, $\omega_M\colon [0,\infty)\to[0,\infty]$ is a nondecreasing function. Denote by

$$\omega^-(t) = \inf\{\delta\ge 0\colon\ \omega_M(\delta)\ge t\}$$

its generalized inverse, cf. [17]. Obviously,

$$M \text{ is uniformly integrable} \Leftrightarrow \lim_{\delta\to+0}\omega_M(\delta) = 0 \Leftrightarrow \forall t>0\colon\ \omega_M^-(t)>0.$$

Remark 2.3. Suppose that $f \to -\infty$ as $u \to \infty$ uniformly in x. Then if the entropy is bounded on U, the set U is uniformly integrable. This can be shown using a simple de la Vallée-Poussin argument. First of all, note that by L'Hôpital's rule we have

$$\lim_{u\to\infty} \frac{E(x,u)}{u} = \lim_{u\to\infty}(-f(x,u)) = \infty,$$

where the limits are uniform in x. Given $\varepsilon > 0$ take $k > 0$ such that $u \leq \varepsilon E(x,u)$ whenever $u \geq k$ and assume that $|A| \leq \varepsilon$; then for any $u \in U$ we have

$$\int_A u(x)\,dx \leq k|A| + \varepsilon \int_\Omega E(x,u(x))\,dx \leq \left(k + \sup_{u\in U} \mathcal{E}(u)\right)\varepsilon$$

proving the uniform integrability.

Given c, the equation

$$f(x,\xi) = c$$

defines a positive function $m_c \in C(\overline{\Omega})$, at least if c is sufficiently close to 0. Clearly, $[u \geq m_c]$=$[f \leq c]$, and similarly for other comparisons.

Remark 2.4. If m_c exists for some $c > 0$, then $m_{c'}$ exists whenever $0 < c' \leq c$; similarly, if m_c exists for some $c < 0$, then $m_{c'}$ exists whenever $c < c' < 0$.

Remark 2.5. It follows easily from the Mean Value Theorem that if m_c exists for some $c > 0$, then

$$\inf_\Omega(m - m_c) \geq \frac{c}{\sup\limits_{m_c(x)\leq\xi\leq m(x)} |f_u(x,\xi)|}, \tag{2.9}$$

and if m_c exists for some $c < 0$, then

$$\inf_\Omega(m_c - m) \geq -\frac{c}{\sup\limits_{m(x)\leq\xi\leq m_c(x)} |f_u(x,\xi)|}. \tag{2.10}$$

Clearly, the suprema in (2.9) and (2.10) are finite.

Remark 2.6. Note that

$$\inf_{u>m}(uf)_u < 0. \tag{2.11}$$

Indeed, one only needs to observe that $(uf)_u = f + uf_u$ is uniformly negative both as $u \to m$ (since m is uniformly positive and $f_u\Big|_{u=m}$ is uniformly negative) and as $u \to \infty$ (since so is f).

Lemma 2.7. *Suppose that m_c exists for some $c > 0$; then*

$$\int_{[m_c<u<m]} (m-u)\,dx \le \frac{1}{\inf\limits_{m_c(x)\le\xi\le m(x)} |f_u(x,\xi)|} \int_{[m_c<u<m]} f\,dx; \tag{2.12}$$

likewise, if m_c exists for some $c < 0$, then

$$\int_{[m<u<m_c]} (u-m)\,dx \le \frac{1}{\inf\limits_{m(x)\le\xi\le m_c(x)} |f_u(x,\xi)|} \int_{[m<u<m_c]} f\,dx. \tag{2.13}$$

Proof: Both inequalities are easy consequences of the Mean Value Theorem if we take into account that $f(x,\xi) = 0$ when $\xi = m(x)$. ■

Lemma 2.8. *Suppose that m_c is defined for some $c > 0$; then for any $u \in U$ we have*

$$\big|[u > m]\big| \ge \omega_U^-\Big(\inf_\Omega (m-m_c)\big|[u \le m_c]\big|\Big). \tag{2.14}$$

Proof: We have:

$$\begin{aligned} 1 &= \int_{[u\le m_c]} u\,dx + \int_{[m_c<u\le m]} u\,dx + \int_{[u>m]} u\,dx \\ &\le \int_{[u\le m_c]} m_c\,dx + \int_{[m_c<u\le m]} m\,dx + \int_{[u>m]} u\,dx \\ &= \int_{[u>m]} (u-m)\,dx - \int_{[u\le m_c]} (m-m_c)\,dx + \int_\Omega m\,dx. \end{aligned}$$

The last integral equals 1, so

$$\int_{[u>m]} (u-m)\,dx \ge \int_{[u\le m_c]} (m-m_c)\,dx \ge \inf_\Omega (m-m_c)\big|[u\le m_c]\big|.$$

Now using the positivity of m we deduce

$$\omega_U\Big(\big|[u>m]\big|\Big) \ge \int_{[u>m]} u\,dx \ge \int_{[u>m]} (u-m)\,dx \ge \inf_\Omega (m-m_c)\big|[u\le m_c]\big|,$$

and (2.14) follows, observing that $\omega_U^-(\omega_U(s)) \le s$. ■

Lemma 2.9. *Suppose that m_c is defined for some $c < 0$; then for any $u \in U$ we have*

$$\big|[u < m]\big| \ge \frac{\inf_\Omega (m_c - m)}{\sup_\Omega m}\big|[u \ge m_c]\big|. \tag{2.15}$$

Proof: Mimicking the proof of Lemma 2.8, we obtain

$$\int_{[u<m]} (m-u)\,dx \geq \int_{[u\geq m_c]} (m_c - m)\,dx \geq \inf_\Omega (m_c - m)\big|[u \geq m_c]\big|.$$

On the other hand, as u is nonnegative, we have

$$\int_{[u<m]} (m-u)\,dx \leq \sup_\Omega m\big|[u < m]\big|,$$

and (2.15) follows. ■

Lemma 2.10. *Let $c_0 < c_1$ and suppose that m_{c_1} is defined; then for any $u \in U$ we have*

$$\int_{[c_0<f<c_1]} u|\nabla f|^2\,dx \\ \geq \frac{C_\Omega^2}{|\Omega|}(c_1 - c_0)^2 \inf_\Omega m_{c_1} \min\Big(\big|[f \leq c_0]\big|, \big|[f \geq c_1]\big|\Big)^{2(d-1)/d}. \quad (2.16)$$

Proof: By monotonicity of f we have $u \geq m_{c_1}$ on $[c_0 < f < c_1]$, so

$$\int_{[c_0<f<c_1]} u|\nabla f|^2\,dx \geq \inf_\Omega m_{c_1} \int_{[c_0<f<c_1]} |\nabla f|^2\,dx \\ \geq |\Omega|^{-1} \inf_\Omega m_{c_1} \left(\int_{[c_0<f<c_1]} |\nabla f|\,dx\right)^2. \quad (2.17)$$

In what follows, we use some basic results and concepts from the geometric measure theory, which can be found in [35]. Using the coarea formula, we have

$$\int_{[c_0<f<c_1]} |\nabla f|\,dx = \int_{-\infty}^{\infty} P\Big([f<t]; [c_0 < f < c_1]\Big)\,dt \\ \geq \int_{c_0}^{c_1} P\Big([f<t]; [c_0 < f < c_1]\Big)\,dt.$$

The support of the Gauss–Green measure $\mu_{[f<t]}$ is contained in the topological boundary of the set $[f < t]$, so if $c_0 < t < c_1$, we see that the intersection of the support with Ω lies in $[c_0 < f < c_1]$. Consequently, we can take relative perimeter with respect to Ω and proceed using the relative isoperimetric inequality as follows:

$$\int_{[c_0<f<c_1]} |\nabla f|\,dx \ge \int_{c_0}^{c_1} P\Big([f<t];\Omega\Big)\,dt$$
$$\ge C_\Omega \int_{c_0}^{c_1} \min\Big(\big|[f<t]\big|,\big|[f\ge t]\big|\Big)^{(d-1)/d}\,dt.$$

The integrand can be estimated using the obvious inclusions

$$[f<t]\supset[f\le c_0],\quad [f\ge t]\supset[f\ge c_1]\quad (c_0<t<c_1),$$

and thus

$$\int_{[c_0<f<c_1]} |\nabla f|\,dx \ge C_\Omega(c_1-c_0)\min\Big(\big|[f\le c_0]\big|,\big|[f\ge c_1]\big|\Big)^{(d-1)/d}.$$

Combining this with (2.17), we obtain (2.16). ∎

Lemma 2.11. *Let $c_0<0$ and $c_1>0$ and suppose that m_{c_i} $(i=0,1)$ are defined; then for any $u\in U$ we have*

$$\int_{[0<f<c_1]} f\,dx$$
$$\ge \inf_{m_{c_1}(x)\le\xi\le m(x)} |f_u(x,\xi)| \left(-\frac{c_0\big|[u\ge m_{c_0}]\big|}{\sup\limits_{m(x)\le\xi\le m_{c_0}(x)} |f_u(x,\xi)|} - \sup_\Omega m\big|[u\le m_{c_1}]\big|\right) \quad (2.18)$$

Proof: Since u and m are probability measures, we have

$$\int_{[u>m]} (u-m)\,dx = \int_{[u<m]} (m-u)\,dx, \qquad (2.19)$$

Let us estimate the sides of (2.19).

For the left-hand side, we have

$$\begin{aligned}\int_{[u>m]}(u-m)\,dx &\geq \int_{[u\geq m_{c_0}]}(u-m)\,dx\\ &\geq \inf_{\Omega}(m_{c_0}-m)\big|[u\geq m_{c_0}]\big|\\ &\geq -\frac{c_0\big|[u\geq m_{c_0}]\big|}{\sup\{|f_u(x,\xi)|:\ m(x)\leq\xi\leq m_{c_0}(x)\}},\end{aligned}$$

where we have used (2.10); for the right-hand side we have

$$\begin{aligned}\int_{[u<m]}(m-u)\,dx &= \int_{[u\leq m_{c_1}]}(m-u)\,dx + \int_{[m_{c_1}<u<m]}(m-u)\,dx\\ &\leq \sup_{\Omega} m\big|[u\leq m_{c_1}]\big|\\ &\quad+\frac{1}{\inf\limits_{m_{c_1}(x)\leq\xi\leq m(x)}|f_u(x,\xi)|\}}\int_{[m_{c_1}<u<m]} f\,dx,\end{aligned}$$

where we have used (2.12). Comparing the estimates, we arrive at (2.18). ■

Now we are in the position to prove Lemma 2.2 for small negative a.

Lemma 2.12. *Suppose that m_c exists for $|c|\leq\delta$; then there exist $a_\delta\in(-\delta,0)$ and $\varkappa_\delta>0$ such that* (2.8) *holds for all $a\in(a_\delta,0)$ and $u\in U$ with $\varkappa=\varkappa_\delta$ and any positive $\sigma\leq\inf_\Omega m_\delta$.*

Proof: Fix $u\in U$, $\sigma\leq\inf_\Omega m_\delta$, and $a\in(a_\delta,0)$, the constant a_δ to be defined below. We examine the possible alternatives and in each of them, we find a suitable value for $\varkappa_\delta$.

Observe that in Ω,

$$f<\delta\Leftrightarrow u>m_\delta\Rightarrow u>\sigma.$$

Consider the following partition of Ω:

$$\Omega=[f\geq\delta]\cup[a/2<f<\delta]\cup[f\leq a/2]. \tag{2.20}$$

Clearly, at least one set on the right-hand side has volume $\geq|\Omega|/3$.

If $\big|[f \geq \delta]\big| \geq |\Omega|/3$, it follows from Lemma 2.8 that $\big|[f \leq 0]\big| \geq \sigma_\delta$ with $\sigma_\delta > 0$ independent of u and a. Then Lemma 2.10 guarantees the estimate

$$\int_{[u>\sigma]} u|\nabla f|^2\,dx \geq \int_{[0<f<\delta]} u|\nabla f|^2\,dx \geq C_\delta \geq \frac{C_\delta}{\delta^2}a^2$$

with $C_\delta > 0$ independent of u and a, so (2.8) holds with $\varkappa = \varkappa'_\delta := C_\delta/\delta^2$.

If $\big|[a/2 < f < \delta]\big| \geq |\Omega|/3$, we have the following simple lower bound on the first term on the right-hand side of (2.8):

$$\begin{aligned}\int_{[u>\sigma]} u(f-a)^2\,dx &\geq \int_{[m_\delta<u<m_{a/2}]} u(f-a)^2\,dx \\ &\geq \frac{\inf_\Omega m_\delta}{|\Omega|}\left(\int_{[a/2<f<\delta]}(f-a)\,dx\right)^2 \\ &\geq \frac{\inf_\Omega m_\delta}{4|\Omega|}\big|[a/2<f<\delta]\big|^2 a^2 \\ &\geq \frac{|\Omega|\inf_\Omega m_\delta}{36}a^2 =: \varkappa''_\delta a^2,\end{aligned}$$

so (2.8) holds with $\varkappa = \varkappa''_\delta$.

It remains to assume that $\big|[f \leq a/2]\big| \geq |\Omega|/3$ and $s := \big|[f \geq \delta]\big| < |\Omega|/3$. Using Lemma 2.10 with $c_1 = \delta$ and $c_0 = a/2$, we obtain

$$\int_{[a/2<f<\delta]} u|\nabla f|^2\,dx \geq C_\delta s^{2(d-1)/d}.$$

Of course, the right-hand side is a lower bound for the left-hand side of (2.8), so if $s \geq |a|^{d/(d-1)}$, the inequality holds with $\varkappa = \varkappa'''_\delta = C_\delta$.

Thus, assume that

$$s < |a|^{d/(d-1)}.$$

Now we evoke Lemma 2.11 with $c_0 = a/2$ and $c_1 = \delta$. Taking the supremum and infimum of $|f_u|$ on the right-hand side of (2.18) over the larger set $\Omega \times [-\delta \leq f \leq \delta]$, we ensure that these extreme values are independent of a and the inequality still holds, i. e. we have

$$\int_{[0<f<\delta]} f\,dx \geq A_\delta a - B_\delta s \geq \left(A_\delta - B_\delta |a|^{1/(d-1)}\right)|a| \geq \frac{A_\delta}{2}|a|$$

given that $|a| < -a_\delta := \min((A_\delta/(2B_\delta))^{d-1}, \delta)$. Then the first term on the left-hand side of (2.8) admits the estimate

$$\begin{aligned}\int_{[u>\sigma]} u(f-a)^2\,dx &\geq \inf_\Omega m_\delta \int_{[m_\delta<u<m]} (f-a)^2\,dx\\ &\geq \frac{\inf_\Omega m_\delta}{|\Omega|}\left(\int_{[0<f<\delta]} f\,dx\right)^2\\ &\geq \frac{A_\delta^2 \inf_\Omega m_\delta}{4|\Omega|} a^2 =: \varkappa_\delta'''' a^2.\end{aligned}$$

To complete the proof, it suffices to take $\varkappa_\delta = \min(\varkappa_\delta', \varkappa_\delta'', \varkappa_\delta''', \varkappa_\delta'''')$. ∎

Lemma 2.13. *Let $a \geq 0$ and $c > 0$, and suppose that m_c exists; then for any $u \in U$ we have*

$$\int_{[u>m]} u(f-a)^2 \geq \left(\frac{\inf_{u>m}(uf)_u}{\sup_{m_c\leq u\leq m}|f_u|}\right)^2 c^2\big|[f>c]\big|^2. \tag{2.21}$$

Proof: Let us again estimate both sides of (2.19).

On one hand, we have

$$\begin{aligned}\int_{[u<m]} (m-u)\,dx &\geq \int_{[u<m_c]} (m-u)\,dx\\ &\geq \inf_\Omega (m-m_c)\big|[u<m_c]\big|\\ &\geq \frac{c\big|[u<m_c]\big|}{\sup\limits_{m_c(x)\leq\xi\leq m(x)} |f_u(x,\xi)|},\end{aligned}$$

where we take advantage of (2.9).

Before estimating the right-hand side of (2.19), observe that if $\xi > m$, we can use the Mean Value Theorem and get

$$\xi|f(x,\xi)| = |\xi f(x,\xi) - m(x) f(x,m(x))| \geq \left|\inf_{u>m}(uf)_u\right|(\xi - m(x)),$$

where the modulus of the infimum is uniformly positive by Remark 2.6. Now, setting $\xi = u(x)$, we have

$$\begin{aligned}\int_{[u>m]} (u-m)\,dx &\leq \left|\left(\inf_{u>m}(uf)_u\right)\right|^{-1} \int_{[u>m]} u|f|\,dx \\ &\leq \left|\left(\inf_{u>m}(uf)_u\right)\right|^{-1} \int_{[u>m]} u|f-a|\,dx \\ &= \left|\left(\inf_{u>m}(uf)_u\right)\right|^{-1} \int_{\Omega} u|f-a|1_{[u>m]}(x)\,dx \\ &\leq \left|\left(\inf_{u>m}(uf)_u\right)\right|^{-1} \left(\int_{\Omega} u(f-a)^2 1_{[u>m]}(x)\,dx\right)^{\frac{1}{2}} \\ &= \left|\left(\inf_{u>m}(uf)_u\right)\right|^{-1} \left(\int_{[u>m]} u(f-a)^2\,dx\right)^{\frac{1}{2}},\end{aligned}$$

since u is a probability measure. Comparing this with the above estimate of the left-hand side of (2.19), we recover (2.21). ■

Now we prove Lemma 2.2 for small positive a.

Lemma 2.14. *Suppose that $\delta > 0$ is such that $m_{\delta/2}$ is defined; then there exists $\varkappa_\delta > 0$ such that inequality* (2.8) *holds with $\varkappa = \varkappa_\delta$ and any positive $\sigma \leq \inf_\Omega m_{\delta/2}$ for all $u \in U$ and $a \in (0, \delta)$.*

Proof: Fix $\sigma \leq \inf_\Omega m_{\delta/2}$, $u \in U$, and $a \in (0, \delta)$. Observe that in Ω,

$$f < \frac{\delta}{2} \Leftrightarrow u > m_{\delta/2} \Rightarrow u > \sigma.$$

By Remark 2.4, $m_{a/2}$ is defined. Consider the partition

$$\Omega = \left[f > \frac{a}{2}\right] \cup \left[f \leq \frac{a}{2}\right].$$

Obviously, at least one of the sets on the right-hand side has volume $\geq |\Omega|/2$.

Suppose that

$$\left|\left[f > \frac{a}{2}\right]\right| \geq \frac{|\Omega|}{2}.$$

Using into account inequality (2.21) for $c = a/2$ and observing that

$$\sup_{m_{a/2} \le u \le m} |f_u| \ge \sup_{m_{\delta/2} \le u \le m} |f_u|$$

with the right-hand side independent of a, we obtain

$$\int_{[u>\sigma]} u(f-a)^2\,dx \ge \int_{[u>m]} u(f-a)^2\,dx \ge \varkappa'_\delta a^2$$

with some constant $\varkappa'_\delta$ independent of a and u.

If, on the other hand, we have

$$\left|\left[f \le \frac{a}{2}\right]\right| \ge \frac{|\Omega|}{2},$$

then

$$\begin{aligned}
\int_{[u>\sigma]} u(f-a)^2\,dx &\ge \int_{[f\le a/2]} u(f-a)^2\,dx \\
&\ge \left(\frac{1}{4}\left|\left[f \le \frac{a}{2}\right]\right| \inf_\Omega m_{a/2}\right) a^2 \\
&\ge \left(\frac{1}{8}|\Omega| \inf_\Omega m_{\delta/2}\right) a^2 =: \varkappa''_\delta a^2
\end{aligned}$$

with $\varkappa''_\delta$ independent of u and a.

To complete the proof, it suffices to take $\varkappa_\delta = \min(\varkappa'_\delta, \varkappa''_\delta)$. ■

Lemma 2.15. *Suppose that $\delta > 0$ is such that m_δ is defined; then there exists $\varkappa_\delta > 0$ such that inequality (2.8) holds with $\varkappa = \varkappa_\delta$ and any positive $\sigma \le \inf_\Omega m_\delta$ for all $u \in U$ and $a < -2\delta$.*

Proof: Given $a < -2\delta$ and $u \in U$, write

$$\begin{aligned}
|\Omega| &= \left|\left[f \le \frac{a}{2}\right]\right| + \left|\left[\frac{a}{2} < f \le 0\right]\right| + \left|[0 < f < \delta]\right| + \left|[f \ge \delta]\right| \\
&=: s_1 + s_2 + s_3 + s_4.
\end{aligned}$$

Clearly,

$$\max s_i \ge \frac{|\Omega|}{4}. \tag{2.22}$$

It follows from Lemmas 2.8 and 2.9 that a lower bound on $\left|[f \ge \delta]\right| = s_4$ yields a lower bound on $\left|[f < 0]\right| \le s_1 + s_2$ and a lower bound on $s_1 = \big|[f \le$

$a/2]\big| \le \big|[f \le -\delta]\big|$ yields a lower bound on $\big|[f > 0]\big| = s_3 + s_4$. Together with (2.22) this implies that at least one of the following inequalities hold:

$$s_2 \ge \frac{|\Omega|}{4}, \quad s_3 \ge \frac{|\Omega|}{4},$$
$$\min(s_1 + s_2, s_4) \ge 2c_\delta, \quad \min(s_3 + s_4, s_1) \ge 2c_\delta,$$

where $c_\delta > 0$ is independent of u and a. Assuming for definiteness that $c_\delta < |\Omega|/4$, we easily check that either

$$\min\left(\left|\left[f \le \frac{a}{2}\right]\right|, \big|[f \ge \delta]\big|\right) = \min(s_1, s_4) \ge c_\delta \tag{2.23}$$

or

$$\left|\left[\frac{a}{2} < f < \delta\right]\right| = s_2 + s_3 \ge c_\delta. \tag{2.24}$$

On the set $[a/2 < f < \delta]$ we clearly have $u > \sigma$. Thus, if (2.23) is true, using Lemma 2.10 we obtain

$$\int_{[u>\sigma]} u|\nabla f|^2\,dx \ge \int_{[a/2<f<\delta]} u|\nabla f|^2\,dx \ge 4\varkappa'_\delta\left(\delta - \frac{a}{2}\right)^2 \ge \varkappa'_\delta a^2.$$

If, on the other hand, (2.24) holds, note that $a/2 < f < \delta$ implies $f - a > -a/2 > 0$, and estimate

$$\begin{aligned}\int_{[u>\sigma]} u(f-a)^2\,dx &\ge \int_{[a/2<f<\delta]} u(f-a)^2\,dx \\ &\ge \frac{a^2}{4}\inf_\Omega m_\delta \left|\left[\frac{a}{2} < f < \delta\right]\right| \\ &\ge \varkappa''_\delta a^2.\end{aligned}$$

Thus, one can take $\varkappa_\delta = \min(\varkappa'_\delta, \varkappa''_\delta)$. ■

Lemma 2.16. *Suppose that $\delta > 0$ is such that m_δ is defined; then there exists $\varkappa_\delta > 0$ such that inequality* (2.8) *holds with $\varkappa = \varkappa_\delta$ and any $\sigma \le \inf m_\delta$ for all $u \in U$ and $a \ge 2\delta$.*

Proof: The partition

$$\Omega = [f < \delta] \cup [f \ge \delta]$$

ensures that either $\big|[f < \delta]\big| \ge |\Omega|/2$ or $\big|[f \ge \delta]\big| \ge |\Omega|/2$. In the latter case Lemma 2.8 guarantees a lower bound on $\big|[f \le 0]\big|$ and hence on $\big|[f < \delta]\big|$.

Either way, we can write

$$\Big|[f<\delta]\Big| \geq s_\delta,$$

where s_δ is independent of a and u.

As $f<\delta$ implies $u>\sigma$ and $f-a<\delta-a\leq -a/2$, we have the estimate

$$\begin{aligned}\int_{[u>\sigma]} u(f-a)^2\,dx &\geq \int_\Omega u(f-a)^2 1_{[f<\delta]}(x)\,dx\\ &\geq \left(\int_\Omega u|f-a|1_{[f<\delta]}(x)\,dx\right)^2\\ &= \left(\int_{[u>m_\delta]} u|f-a|\,dx\right)^2\\ &\geq \left(\frac{1}{4}s_\delta \inf_\Omega m_\delta\right)^2 a^2\end{aligned}$$

and (2.8) follows. ∎

Now we can assemble the proof of Lemma 2.2 from established particular cases.

Proof of Lemma 2.2: Take $\delta_1>0$ such that m_c exists whenever $|c|\leq\delta_1$. By Lemma 2.12, there exist $\varkappa_1>0$, $\sigma_1>0$, and $a_1\in(-\delta_1,0)$ such that (2.8) holds with $\varkappa=\varkappa_1$ and $\sigma=\sigma_1$ for all $u\in U$ and $a\in(a_1,0)$. Set $\delta_2=-a_1$. This is a suitable value of δ for Lemma 2.14, so we conclude that (2.8) holds with $\varkappa=\varkappa_2$ and $\sigma=\sigma_2$ for $u\in U$ and $a\in(-\delta_2,\delta_2)$ and, moreover, m_c is defined whenever $|c|\leq\delta_2$. Now in order to find $\varkappa$ and σ such that (2.8) holds for all $u\in U$ and all real a, it suffices to evoke Lemmas 2.15 and 2.16 with $\delta=\delta_2/3$. ∎

3. Applications

3.1. Spherical gradient flows. Let Ω be an open connected bounded domain in $\mathbb{R}^d$ with sufficiently smooth boundary and let ν be the outward unit normal along $\partial\Omega$. We are interested in nonnegative solutions of

$$\partial_t u = -\operatorname{div}(u\nabla f) + u\left(f - \int_\Omega uf\,dx\right), \qquad (x,t) \in \Omega \times (0,\infty), \tag{3.1}$$

$$u\frac{\partial f}{\partial \nu} = 0, \qquad (x,t) \in \partial\Omega \times (0,\infty), \tag{3.2}$$

$$u = u^0, \qquad (x,t) \in \Omega \times 0, \tag{3.3}$$

$$u \geq 0, \int_\Omega u\,dx = 1, \qquad (x,t) \in \Omega \times (0,\infty). \tag{3.4}$$

Here u is the unknown function and $f = f(x, u(x,t))$ is a known nonlinear scalar function of x and u. The initial data u^0 is a probability density.

For the sake of brevity we will denote

$$\bar{f} = \int_\Omega uf\,dx.$$

Remark 3.1. The Neumann boundary condition (3.2) can be substituted with the space-periodic one without affecting the validity of the results of this section.

Throughout Section 3.1, we make the following assumptions about the nonlinearity f. Some of the results do not require all of these assumptions: it will be explicitly indicated where relevant.

$$f \in C^2(\overline{\Omega} \times (0,\infty)) \cap L^1_{\mathrm{loc}}(\overline{\Omega} \times [0,\infty)), \tag{3.5}$$

$$uf, uf_x \in C(\overline{\Omega} \times [0,+\infty)), \tag{3.6}$$

$$f_u < 0, \tag{3.7}$$

$$|f(x,u)| \leq g_1(u) \quad \text{a. a. } u > 0;\ g_1 \in L^1_{\mathrm{loc}}[0,\infty), \tag{3.8}$$

$$u|f_u(x,u)| + u|f_{xu}(x,u)| \leq g_2(u) \text{ a. a. } u > 0;\ g_2 \in L^1_{\mathrm{loc}}[0,\infty), \tag{3.9}$$

$$(uf_x)\Big|_{u=0} = 0, \tag{3.10}$$

$$\text{either } f_x = 0 \text{ for large } u \quad \text{or} \quad \lim_{u\to\infty} f(x,u) = -\infty\ \forall x \in \overline{\Omega}, \tag{3.11}$$

$$\text{either } f_x = 0 \text{ for small } u \quad \text{or} \quad \lim_{u\to+0} f(x,u) = \infty\ \forall x \in \overline{\Omega}, \tag{3.12}$$

$$u\left[f_x^2 + (uf_{xu})^2 + (uf_u)^2\right] = O(1),\ u \to 0 \text{ uniformly in } x \in \Omega, \tag{3.13}$$

$$uf_{uu} = O(f_u) \quad \text{as } u \to 0 \text{ uniformly in } x \in \Omega. \tag{3.14}$$

Assumption (3.7) ensures non-strict parabolicity of the problem. The remaining assumptions are technical. It is easy to check (see [30, Remark 3.4]) that (3.11) and (3.12) ensure that given $v \in L^\infty_+(\Omega)$ bounded away from 0, there exist m_{c_1} and m_{c_2} (this notation was introduced in the beginning of Section 2) such that $m_{c_1} \le v \le m_{c_2}$ a. e. in Ω. In particular, taking $v \equiv \frac{2}{|\Omega|}$ and $v \equiv \frac{1}{2|\Omega|}$ in this observation, we infer existence of m_{c_1}, m_{c_2} such that

$$\int_\Omega m_{c_1}\, dx \le \frac{1}{2}, \ \int_\Omega m_{c_2}\, dx \ge 2.$$

This implies (cf. Remark 2.4) existence and uniqueness of a C^2-smooth probability density $m\colon \overline{\Omega} \to (0,\infty)$ such that $f(x, m(x))$ is constant on $\overline{\Omega}$. Since problem (3.1)–(3.4) does not change after adding constants to f, without loss of generality we will assume that

$$f(x, m(x)) = 0. \tag{3.15}$$

Let us introduce the energy and entropy functionals for equation (3.1) as well as the notion of weak solution.

Bound (3.9) ensures that

$$\Phi(x,u) = -\int_0^u \xi f_u(x,\xi)\, d\xi, \quad \Psi(x,u) = \int_0^u \Phi(x,\xi)\, d\xi$$

are well defined and belong to $C^1(\overline{\Omega} \times [0,\infty))$, whereas

$$\begin{aligned} \Phi(x,0) &= \Psi(x,0) = 0, & \Phi_u &= -u f_u, \\ \Phi_x &= -\int_0^u \xi f_{xu}(x,\xi)\, d\xi, & \Psi_u &= \Phi, \\ \Phi_{uu} &= -(u f_u)_u, & \Phi_{xu} &= -u f_{xu}. \end{aligned}$$

Note that both Φ and Ψ are nonnegative and strictly increase with respect to u.

Observe that the superposition operator $L^\infty_+ \to L^\infty$ associated with Φ is bounded, i. e. if u is a nonnegative function of x and, possibly, t, then an L^∞-bound on u is translated into an L^∞-bound on $\Phi(\cdot, u(\cdot))$. The same is true of Φ_u, Φ_{uu}, Ψ_{xu}, and Ψ.

In accordance with [30], we call the functional

$$\mathcal{W}(u) = \int_\Omega \Psi(x, u(x))\, dx$$

the *energy* of problem (3.1)–(3.4).

Define

$$E(x,u) = -\int_{m(x)}^{u} f(x,\xi)\,d\xi. \tag{3.16}$$

It follows from (3.8) that E is well-defined and continuous on $\overline{\Omega}\times[0,\infty)$. Moreover, $E \geq 0$ and $E(x,u) = 0$ if and only if $u = m(x)$, and the superposition operator associated with E is bounded in $L^\infty_+ \to L^\infty_+$. Thus, for $u \in L^\infty_+(\Omega)$ we can define the relative *entropy* of equation (3.1) as follows:

$$\mathcal{E}(u) = \int_\Omega E(x,u(x))\,dx. \tag{3.17}$$

Lemma 3.2. *Let u be a classical solution of* (3.1)–(3.4) *on* $[0,T]$. *Then u satisfies*

(1) *the* energy identity

$$\partial_t \mathcal{W}(u) = -\int_\Omega |\nabla\Phi|^2\,dx + \int_\Omega (\Phi_x + u f_x)\cdot\nabla\Phi\,dx + \int_\Omega u(f-\bar f)\Phi\,dx \quad t>0; \tag{3.18}$$

(2) *the* entropy dissipation identity

$$\partial_t \mathcal{E}(u) = -\int_\Omega u((f-\bar f)^2 + |\nabla f|^2)\,dx \quad t>0; \tag{3.19}$$

(3) *the bounds*

$$\inf_\Omega f(x,u^0(x)) \leq f(x,u(x,t)) \leq \sup_\Omega f(x,u^0(x)) \quad (x,t)\in\Omega\times(0,\infty). \tag{3.20}$$

Proof: Straightforward computation proves (i) and (ii).

Let us prove the first inequality in (3.20). Assume that the infimum is finite, because otherwise there is nothing to prove; denote it by c. It follows from (3.11) that the function $m_c\colon \overline{\Omega}\to\mathbb{R}$ satisfying $f(x,m_c(x)) \equiv c$ is defined. We have

$$\partial_t \int_\Omega (u-m_c)_+\,dx = \int_\Omega \theta(u-m_c)\partial_t u\,dx,$$

where

$$\theta(s) = \begin{cases} 1 & \text{if } s > 0, \\ 0 & \text{if } s \le 0 \end{cases}$$

is the Heaviside step function. Substituting the right-hand side of the equation for $\partial_t u$, we obtain

$$\begin{aligned} \partial_t \int_\Omega (u - m_c)_+ \, dx &= -\int_\Omega \theta(u - m_c) \operatorname{div}(u \nabla f) \, dx \\ &\quad + \int_\Omega \theta(u - m_c) u (f - \bar{f}) \, dx \\ &=: -I_1 + I_2. \end{aligned}$$

Writing

$$I_1 = \int_\Omega \theta(u - m_c) \operatorname{div}(u \nabla f - m_c \nabla f(x, m_c(x))) \, dx,$$

we can use [30, Lemma 3.1] and conclude that $I_1 \ge 0$ (though the lemma is proved for C^∞ functions, it holds for C^2 functions by density).

Now, if

$$\int_{[u \ge m_c]} u \, dx = 0,$$

we have $u \le m_c$ a. e. in Ω and consequently, $I_2 = 0$. Otherwise,

$$I_2 = \int_{[u \ge m_c]} u \, dx \left(\frac{\int_{[u \ge m_c]} u f \, dx}{\int_{[u \ge m_c]} u \, dx} - \bar{f} \right) \ge 0,$$

since the average of f with weight u over the set $[u \ge m_c] = [f \le c]$ is no greater than the weighted average over the whole Ω.

Thus, we see that

$$\partial_t \int_\Omega (u - m_c)_+ \, dx \le 0,$$

and as this integral equals 0 at $t = 0$, it equals 0 for any t, which is equivalent to $u \le m_c$ and to the first inequality in (3.20).

The second inequality in (3.20) is proved in the same way. ■

The integral on the right-hand side of (3.19) is called the *entropy production*. We denote it by $D\mathcal{E}(u)$, so that (3.19) can be written as

$$\partial_t \mathcal{E}(u) = -D\mathcal{E}(u). \tag{3.21}$$

Remark 3.3. We can extend the definition of the entropy production to functions $u \in L^\infty_+(\Omega)$ such that $\Phi(\cdot, u(\cdot)) \in H^1(\Omega)$ by the formula

$$D\mathcal{E}(u) = \int_\Omega u(f-\bar{f})^2\,dx + \int_{[u>0]} \frac{1}{u}|-\nabla\Phi + \Phi_x + u f_x|^2\,dx,$$

where the second integral on the right-hand side may be infinite. This is relevant for the weak solutions which will be introduced in Definition 3.6.

Lemma 3.4. *If u is a classical solution of* (3.1)–(3.4) *on* $[0,T]$ *satisfying*

$$\|u\|_{L^\infty(Q_T)} \le R,$$

then

$$\|\partial_t \Phi(u)\|_{[C(0,T;(W^{1,\infty}(\Omega))]^*} \le C(R,T)$$

with $C(R,T) > 0$ independent of u.

Proof: For a given test function $\psi \in C([0,T]; W^{1,\infty}(\Omega))$ we have

$$\begin{aligned}
|\langle \partial_t\Phi(u), \psi\rangle| &= \left| \int_{Q_T} \psi \Phi_u \partial_t u\,dx\,dt \right| \\
&= \left| \int_{Q_T} \psi\Phi_u(-\operatorname{div}(u\nabla f) + u(f-\bar{f}))\,dx\,dt \right| \\
&\le \|\psi\|_{C([0,T];W^{1,\infty}(\Omega))} \Bigg(\int_{Q_T} u|\nabla\Phi_u||\nabla f|\,dx\,dt \\
&+ \int_{Q_T} u|\Phi_u||\nabla f|\,dx\,dt \\
&\qquad + \int_{Q_T} u|\Phi_u||f-\bar{f}|\,dx\,dt \Bigg) \\
&= \|\psi\|_{C([0,T];W^{1,\infty}(\Omega))}(I_1 + I_2 + I_3).
\end{aligned}$$

Our goal is to show that the integrals I_k are bounded from above.

By (3.13), (3.14) there exist $C \geq 0$ and $\varepsilon > 0$ both independent of u such that

$$u|f_x|^2 \leq C, \tag{3.22}$$

$$u^3|f_{xu}|^2 \leq C, \tag{3.23}$$

$$u|\Phi_u|^2 \leq C, \tag{3.24}$$

$$|\Phi_{uu}| \leq C|f_u| \tag{3.25}$$

whenever $0 < u < \varepsilon$. Moreover, if we allow C to depend on T, we can assume that (3.22)–(3.24) hold on $\overline{\Omega} \times (0,T]$, since the left-hand sides are continuous and $|f_u|$ is positive.

For I_1 we have

$$\begin{aligned}
I_1 &= \int_{Q_T} u\Big(|\Phi_{uu}||\nabla u| + |\Phi_{xu}|\Big)|\nabla f|\,dx\,dt \\
&\leq \int_{Q_T} u\Big(C|f_u||\nabla u| + u|f_{xu}|\Big)|\nabla f|\,dx\,dt \\
&\leq \int_{Q_T} u\Big(C|f_u\nabla u + f_x| + C|f_x| + u|f_{xu}|\Big)|\nabla f|\,dx\,dt \\
&\leq C\int_{Q_T} u|\nabla f|^2\,dx\,dt \\
&\qquad + \left(2\int_{Q_T} (Cu|f_x|^2 + u^3|f_{xu}|^2)\,dx\,dt\right)^{1/2}\left(\int_{Q_T} u|\nabla f|^2\,dx\,dt\right)^{1/2} \\
&\leq C'\int_0^T \Big(D\mathcal{E}(u) + \sqrt{D\mathcal{E}(u)}\Big)\,dt \\
&\leq C'\int_0^T D\mathcal{E}(u)\,dt + C'\sqrt{T}\left(\int_0^T D\mathcal{E}(u)\,dt\right)^{1/2}.
\end{aligned}$$

As we assume an upper bound on u, the integral

$$\int_0^T D\mathcal{E}(u)\,dt = \mathcal{E}(0) - \mathcal{E}(T)$$

is bounded, so we see that I_1 is bounded uniformly in u.

Further, we have

$$I_2 + I_3 \le \left(\int_{Q_T} u|\Phi_u|^2 \, dx\, dt \right)^{1/2} \left(2 \int_{Q_T} u(|\nabla f|^2 + |f - \bar{f}|^2) dx\, dt \right)^{1/2}$$
$$\le C'' \left(\int_0^T D\mathcal{E}(u)\, dt \right)^{1/2},$$

where the last term is bounded. ■

Lemma 3.5. *For any smooth probability density $u^0 \colon \overline{\Omega} \to (0, \infty)$ satisfying the non-flux boundary condition, problem* (3.1)–(3.4) *has a classical solution.*

Proof: Equation (3.1) can be cast in the form

$$\partial_t u = -u f_u \Delta u - \nabla u \cdot (f_x + f_u \nabla u) - u(f_{xx} + 2 f_{xu} \cdot \nabla u + f_{uu} |\nabla u|^2 - f + \bar{f}). \quad (3.26)$$

Since the initial data u^0 is strictly positive, any classical solution u is *a priori* bounded away from 0 and ∞. Indeed, evoking [30, Remark 3.4], we can find m_{c_1} and m_{c_2} strictly positive such that $c_2 \le 0 \le c_1$ and

$$m_{c_1}(x) \le u^0(x) \le m_{c_2}(x) \quad (x \in \Omega).$$

Then (3.20) and (3.7) yield

$$m_{c_1}(x) \le u(x,t) \le m_{c_2}(x), \quad (x,t) \in \Omega \times (0, \infty).$$

Hence we can avoid degeneracies or singularities in (3.26) and apply [2, Theorem 13.1] to secure existence and uniqueness of a maximal weak solution $\tilde{u}$ in the sense of Amann. This solution is global in time provided we can control its norm in a certain Sobolev space. Viewing

$$\bar{f}(t) := \int_\Omega \tilde{u}(x,t) f(x, \tilde{u}(x,t))\, dx$$

as a given coefficient, we “desactivate” the nonlocal term in (3.26). Bootstrapping and employing the results of [2, Sections 14 and 15], we can improve the regularity of $\bar{f}$ (as a function of time) and that of $\tilde{u}$ (as a function of time and space). We conclude that $\tilde{u}$ is actually a global smooth solution to (3.1)–(3.4). ■

Definition 3.6. Let $u^0 \in L^\infty(\Omega)$ be a probability density. A function $u \in L^\infty(Q_T)$ is called a *weak solution* of (3.1)–(3.4) on $[0,T]$ if $\int_\Omega u(x,t)\, dx = 1$

for a.a. $t \in (0,T)$, $\Phi(\cdot, u(\cdot)) \in L^2(0,T; H^1(\Omega))$, and

$$\int_0^T \int_\Omega (u\partial_t\varphi + (-\nabla\Phi + \Phi_x + uf_x)\cdot\nabla\varphi + (f-\bar{f})u\varphi)\,dx\,dt = \int_\Omega u^0(x)\varphi(x,0)\,dx \quad (3.27)$$

for any function $\varphi \in C^1(\overline{\Omega}\times[0,T])$ such that $\varphi(x,T) = 0$. A function $u \in L^\infty_{\mathrm{loc}}([0,\infty); L^\infty(\Omega))$ is called a *weak solution* of (3.1)–(3.4) on $[0,\infty)$ if for any $T > 0$ it is a weak solution on $[0,T]$.

Theorem 3.7 (Existence of weak solutions). *Suppose that f satisfies* (3.5)–(3.14). *Then for any probability density $u^0 \in L^\infty_+(\Omega)$ there exists a weak solution $u \in L^\infty_+(\Omega\times(0,\infty))$ of problem* (3.1)–(3.4) *enjoying the following properties:*

(1) *u satisfies the energy inequality*

$$\partial_t\mathcal{W}(u) \le \int_\Omega \Big(-|\nabla\Phi|^2 + (\Phi_x + uf_x)\cdot\nabla\Phi + u(f-\bar{f})\Phi\Big)\,dx \quad (3.28)$$

in the sense of measures and

$$\operatorname*{ess\,lim\,sup}_{t\to+0} \mathcal{W}(u(t)) \le \mathcal{W}(u^0); \quad (3.29)$$

(2) *u satisfies the entropy dissipation inequality*

$$\partial_t\mathcal{E}(u) \le -D\mathcal{E}(u) \quad (3.30)$$

in the sense of measures and

$$\operatorname*{ess\,sup}_{t>0} \mathcal{E}(u(t)) \le \mathcal{E}(u^0). \quad (3.31)$$

Proof: It is easy to see that we can approximate the initial data u^0 by smooth and strictly positive probability densities u^0_n satisfying the boundary condition in such a way that

$$\|u^0_n\|_{L^\infty(\Omega)} \le C, \quad (3.32)$$

$$u^0_n \to u^0 \quad \text{weakly}^* \text{ in } L^\infty(\Omega) \text{ and a.e. in } \Omega, \quad (3.33)$$

$$\mathcal{W}(u^0_n) \to \mathcal{W}(u^0), \quad (3.34)$$

$$\mathcal{E}(u^0_n) \to \mathcal{E}(u^0). \quad (3.35)$$

The last two convergences can be secured using the Lebesgue dominated convergence theorem. Let u_n be the classical solution starting from u_n^0.
Put

$$\begin{aligned} f_n &= f(x, u_n(x,t)), & f_{xn} &= f_x(x, u_n(x,t)),\\ \Phi_n &= \Phi(x, u_n(x,t)), & \Phi_{xn} &= \Phi_x(x, u_n(x,t)),\\ \Psi_n &= \Psi(x, u_n(x,t)), & E_n &= E(x, u_n(x,t)). \end{aligned}$$

Given $T > 0$, by Lemma 3.2 the sequence $\{u_n\}$ is bounded in $L^\infty(Q_T)$, and so are the sequences $\{u_n f_n\}$, $\{u_n f_{xn}\}$, $\{\Phi_n\}$, $\{\Phi_{xn}\}$, $\{\Psi_n\}$, and $\{E_n\}$. It follows from the energy identity (3.18) that

$$\partial_t \mathcal{W}(u_n) \le -\frac{1}{2}\int_\Omega |\nabla \Phi_n|^2\, dx + C, \tag{3.36}$$

whence the integral

$$\int_{Q_T} |\nabla \Phi_n|^2\, dx \le 2\Big(\mathcal{W}(u_n^0) - \mathcal{W}(u_n(T)) + CT\Big)$$

is bounded, i. e. $\{\Phi_n\}$ is bounded in $L^2(0,T;H^1(\Omega))$. By Lemma 3.4 the derivatives $\{\partial_t \Phi_n\}$ are bounded in $C(0,T;(W^{1,\infty}(\Omega))]^*$. Hence, [40, Corollary 7.9] implies that $\{\Phi_n\}$ is compact in $L^2(Q^T)$. This is true for any T, so $\{\Phi_n\}$ is compact in $L^2_{\mathrm{loc}}([0,\infty);L^2(\Omega))$ and there is no loss of generality that $\Phi_n \to \phi$ in this space and a. e. in $\Omega \times (0,\infty)$.
Fix $(x,t) \in \Omega \times (0,\infty)$ such that

$$\Phi(x, u_n(x,t)) = \Phi_n(x,t) \to \phi(x,t).$$

Assuming that $\|u_n\|_{L^\infty(\Omega\times(0,\infty))} \le R$ and taking into account that Φ increases in u, we have $\Phi_n(x,t) \le \Phi(x,R)$ and so $0 \le \phi(x,t) \le \Phi(x,R)$. As Φ is continuous in u, there exists a unique $u(x,t) \in [0,R]$ such that $\Phi(x,u(x,t)) = \phi(x,t)$, and as the inverse of Φ with respect to u is continuous in u as well, we have $u_n(x,t) \to u(x,t)$. Thus, we have defined a function $u \in L^\infty_+(\Omega\times(0,\infty))$

such that for any $T > 0$ we have

$$\left.\begin{aligned} u_n &\to u \\ u_n f_n &\to u f \\ u_n f_{xn} &\to u f_x \\ \Phi_n &\to \Phi \\ \Phi_{xn} &\to \Phi_x \\ \Psi_n &\to \Psi \end{aligned}\right\} \begin{array}{l} \text{a. e. in } Q_T, \\ \text{strongly in any } L^p(Q_T),\ 1 \le p < \infty, \\ \text{weakly* in } L^\infty(Q_T), \\ \text{and in the sense of distributions,} \end{array} \tag{3.37}$$

$$\bar{f}_n \to \bar{f} \tag{3.38}$$

$$\nabla\Phi_n \to \nabla\Phi \quad \text{weakly in } L^2(Q_T). \tag{3.39}$$

where we write Φ for $\Phi(\cdot, u(\cdot))$, etc.

The function u is a weak solution of (3.1)–(3.4) on $[0,T]$ as it follows from (3.33) and (3.37)–(3.39) that one can pass to the limit in the weak setting for the approximate solution

$$\int_0^T \int_\Omega (u_n \partial_t \varphi + (-\nabla\Phi_n + \Phi_{xn} + u_n f_{xn}) \cdot \nabla\varphi + (f_n - \bar{f}_n) u_n \varphi)\, dx\, dt \\ = \int_\Omega u_n^0(x) \varphi(x, 0)\, dx, \tag{3.40}$$

where φ is an admissible test function.

In order to show that u satisfies the energy inequality on $[0,T]$ in the sense of measures, we take a smooth nonnegative test function $\chi \in C^\infty(\mathbb{R})$ vanishing outside of $[0,T]$ and write the energy inequality in the sense of measures for the approximate solutions:

$$-\int_{Q_T} \Psi_n \chi'(t)\, dx\, dt \le -\int_{Q_T} |\nabla\Phi_n|^2 \chi(t)\, dx\, dt \\ + \int_{Q_T} \chi(t)(\Phi_{xn} + u_n f_{xn}) \cdot \nabla\Phi_n\, dx\, dt + \int_{Q_T} u_n (f_n - \bar{f}_n) \Phi_n \chi(t)\, dxcdt$$

Here one can use convergences (3.37) to pass to the limit in all the terms but for the first one on the right-hand side. Further, (3.39) implies that $\sqrt{\chi}\,\nabla\Phi_n \to \sqrt{\chi}\,\nabla\Phi$ weakly in $L^2(Q_T)$, so

$$\int_{Q_T} \chi |\nabla\Phi|^2\, dx\, dt \le \liminf_{n\to\infty} \int_{Q_T} \chi |\nabla\Phi_n|^2\, dx\, dt,$$

and the energy inequality follows.

Let us check (3.29). By (3.36), the approximate solutions satisfy

$$\operatorname*{ess\,sup}_{t\in(0,\varepsilon)} \mathcal{W}(u_n(t)) \le \mathcal{W}(u_n^0) + C\varepsilon.$$

It follows from (3.37) that

$$\mathcal{W}(u_n) \to \mathcal{W}(u) \quad \text{weakly* in } L^\infty(0,\varepsilon),$$

so we get

$$\begin{aligned}\operatorname*{ess\,sup}_{t\in(0,\varepsilon)} \mathcal{W}(u(t)) &\le \liminf_{n\to\infty} \operatorname*{ess\,sup}_{t\in(0,\varepsilon)} \mathcal{W}(u_n(t)) \\ &\le \lim_{n\to\infty} \mathcal{W}(u_n^0) + C\varepsilon \\ &= \mathcal{W}(u^0) + C\varepsilon.\end{aligned}$$

Now sending $\varepsilon \to 0$ we recover (3.29).

Let us show that u satisfies the entropy dissipation inequality on $[0,T]$ in the sense of measures. Let $\chi \in C^\infty$ be a smooth nonnegative test function vanishing outside of $[0,T]$. The approximate solutions satisfy the entropy dissipation inequality in the sense of measures, so we have

$$\begin{aligned}-\int_{Q_T} E_n \chi'(t)\,dx\,dt \le -\int_{Q_T} \chi(t) u_n (f_n - \bar f_n)^2\,dx\,dt \\ -\int_{u_n>0} \frac{\chi(t)}{u_n} |-\nabla\Phi_n + \Phi_{xn} + u_n f_{xn}|^2\,dx\,dt.\end{aligned}$$

Consequently, for any $\delta > 0$ we have

$$\begin{aligned}-\int_{Q_T} E_n \chi'(t)\,dx\,dt \le -\int_{Q_T} \frac{\chi(t)}{\max(u_n,\delta)} (u_n(f_n - \bar f_n))^2\,dx\,dt \\ -\int_{Q_T} \frac{\chi(t)}{\max(u_n,\delta)} |-\nabla\Phi_n + \Phi_{xn} + u_n f_{xn}|^2\,dx\,dt. \quad (3.41)\end{aligned}$$

Observe that

$$\frac{\chi(t)}{\max(u_n,\delta)} \to \frac{\chi(t)}{\max(u,\delta)} \quad \begin{array}{l}\text{a. e. in } Q_T,\\ \text{strongly in any } L^p,\ 1 \le p < \infty,\\ \text{and weakly* in } L^\infty(Q_T),\end{array} \quad (3.42)$$

$$v_n := -\nabla\Phi_n + \Phi_{xn} + u_n f_{xn} \to -\nabla\Phi + \Phi_x + u f_x \quad \text{weakly in } L^2(\Omega). \quad (3.43)$$

In [30, claim (3.24)] it was proved that

$$\int_{Q_T} \frac{\chi(t)}{\max(u,\delta)} |-\nabla\Phi + \Phi_x + u f_x|^2 \, dx\, dt$$
$$\leq \liminf_{n\to\infty} \int_{Q_T} \frac{\chi(t)}{\max(u_n,\delta)} |-\nabla\Phi_n + \Phi_{xn} + u_n f_{xn}|^2 \, dx\, dt \quad (3.44)$$

and using (3.37), we pass to the limit in (3.41) obtaining

$$-\int_{Q_T} E\chi'(t)\, dx\, dt \leq -\int_{Q_T} \frac{\chi(t)}{\max(u,\delta)} (u(f-\bar{f}))^2 \, dx\, dt$$
$$-\int_{Q_T} \frac{\chi(t)}{\max(u,\delta)} |-\nabla\Phi + \Phi_x + u f_x|^2 \, dx\, dt.$$

On the set $\{(x,t) \in Q_T\colon u(x,t) = 0\}$ we have $uf_x = 0$ (by virtue of (3.10)), $\Phi_x = 0$ and $\Phi = 0$, whence also $\nabla\Phi = 0$ a. e. on this set. Thus, we can write

$$-\int_{Q_T} E\chi'(t)\, dx\, dt \leq -\int_{Q_T} \frac{\chi(t)}{\max(u,\delta)} (u(f-\bar{f}))^2 \, dx\, dt$$
$$-\int_{u>0} \frac{\chi(t)}{\max(u,\delta)} |-\nabla\Phi + \Phi_x + u f_x|^2 \, dx\, dt$$

Letting $\delta \to 0$, by Beppo Levi's theorem we obtain the entropy inequality.

Inequality (3.31) is proved in the same way as (3.29) given that it holds for the approximate solutions. ■

Theorem 3.8 (Entropy-entropy production inequality). *Suppose that f satisfies* (3.5)–(3.8). *Assume that the second of the alternatives in* (3.11) *holds, and the limit is uniform w.r.t. x. Let $U \subset L^\infty(\Omega)$ be a set of probability densities such that for any $u \in U$, we have $\Phi(\cdot, u(\cdot)) \in H^1(\Omega)$ and*

$$\sup_{u\in U} \mathcal{E}(u) < \infty. \quad (3.45)$$

Then there exists C_U such that

$$\mathcal{E}(u) \leq C_U D\mathcal{E}(u) \quad (u \in U). \quad (3.46)$$

Proof: Let us show that (2.6) holds with U merely satisfying the hypotheses of Theorem 3.8. According to Remark 2.3, condition (3.45) ensures the uniform integrability of U. As explained before Lemma 2.2, it suffices to ensure that inequality (2.7) holds for U.

Given $u \in U$, we use the construction presented in the proof of [30, Theorem 1.7] and approximate the function $\Phi(\cdot, u(\cdot))$ with smooth functions Φ_n in such a way that

$$\Phi_n \to \Phi(\cdot, u(\cdot)) \quad \text{in } H^1 \text{and a. e. in } \Omega,$$

while the functions $u_n \in C^2(\Omega)$ satisfying $\Phi(x, u_n(x)) = \Phi_n(x)$ are well-defined and

$$\left.\begin{aligned} &\|u_n\|_{L^\infty} \leq C, \\ &u_n \to u \quad \text{a. e. in } \Omega. \end{aligned}\right\} \tag{3.47}$$

There is no loss of generality in assuming that u_n are probability measures, since we can normalize them taking into account that

$$\|u_n\|_{L^1(\Omega)} \to \|u\|_{L^1(\Omega)} = 1.$$

By Lemma 2.2, we have

$$\int_{[u_n \geq \sigma]} u_n \left((f_n - \bar{f}_n)^2 + |\nabla f_n|^2 \right) dx \geq \varkappa \bar{f}_n^2 \tag{3.48}$$

with $\sigma > 0$ and $\varkappa > 0$ independent of n, where as usual f_n stands for $f(x, u_n(x))$, etc. Inequality (3.48) can be written as

$$\int_\Omega \left(1_{[u_n \geq \sigma]} u_n (f_n - \bar{f}_n)^2 + \frac{1_{[u_n \geq \sigma]}}{u_n} |-\nabla \Phi_n + \Phi_{xn} + u_n f_{xn}|^2 \right) dx \\ \geq \varkappa \bar{f}_n^2.$$

As the integrand vanishes whenever $u_n < \sigma$, one can pass to the limit as $n \to \infty$ (cf. [30]). Observing that

$$\limsup_{n \to \infty} 1_{[u_n \geq \sigma]}(x) \leq 1_{[u \geq \sigma]}(x) \quad \text{a. e. in } \Omega,$$

we obtain

$$\int_\Omega \left(1_{[u \geq \sigma]} u (f - \bar{f})^2 + \frac{1_{[u \geq \sigma]}}{u} |-\nabla \Phi + \Phi_x + u f_x|^2 \right) dx \geq \varkappa \bar{f}^2,$$

which is stronger than (2.7). ■

Theorem 3.9 (Convergence to equilibrium)**.** *Suppose that f satisfies* (3.5)–(3.8)*. Assume that the second of the alternatives in* (3.11) *holds, and the limit is uniform w.r.t. x. Let u be a weak solution of* (3.1)–(3.4) *with the initial data $u^0 \in L^\infty_+(\Omega)$, $\int_\Omega u^0 = 1$. Suppose that u satisfies the entropy dissipation* *inequality* (3.30) *and inequality* (3.31). *Then* u *exponentially converges to* m *in the sense of entropy:*

$$\mathcal{E}(u(t)) \leq \mathcal{E}(u^0)\mathrm{e}^{-\gamma t} \quad a.\ a.\ t > 0, \tag{3.49}$$

where $\gamma > 0$ *can be chosen uniformly over initial data satisfying*

$$\mathcal{E}(u^0) \leq C \tag{3.50}$$

with some $C > 0$.

Proof: As the entropy decreases along the solution, the set

$$U = \left\{ u \in L^\infty_+(\Omega) : \int_\Omega u = 1,\ \mathcal{E}(u) \leq C \right\}$$

is invariant under the flow generated by the problem: more precisely, $u(t) \in U$ for a. a. $t \geq 0$. Let C_U be correspondent constant in the entropy-entropy production inequality granted by Theorem 3.8. Combining the entropy dissipation and entropy-entropy production inequalities for a given solution u, we obtain

$$\partial_t \mathcal{E}(u(t)) \leq -C_U^{-1}\mathcal{E}(u(t)) \quad \text{a. a. } t > 0.$$

Letting $e(t) = \mathcal{E}(u(t))\mathrm{e}^{C_U^{-1}t}$, we see that $\partial_t e(t) \leq 0$ in the sense of measures, whence e a. e. coincides with a nonincreasing function. Moreover,

$$\operatorname*{ess\,sup}_{t>0} e(t) = \operatorname*{ess\,lim\,sup}_{t\to 0} e(t) = \operatorname*{ess\,lim\,sup}_{t\to 0} \mathcal{E}(u(t))\mathrm{e}^{C_U^{-1}t} \leq \mathcal{E}(u^0)$$

yielding (3.49) with $\gamma = C_U^{-1}$. ∎

3.2. Nonlinear Fokker-Planck equations and generalized log-Sobolev inequalities. Let us return for a moment to the setting (2.1)-(2.5). Note that we still do not assume any displacement convexity. Theorem 2.1 immediately implies

Corollary 3.10 (Generalized log-Sobolev)**.** *Let* U *be a uniformly integrable set of smooth probability measures on* $\overline{\Omega}$, *which satisfy the weighted Poincaré inequality*

$$\int_\Omega u(x)\left(g(x) - \int_\Omega ug\right)^2 dx \leq c\int_\Omega u(x)|\nabla g(x)|^2\,dx \tag{3.51}$$

with a uniform constant c independent of $u \in U$ and $g \in C^1(\overline{\Omega})$. Then

$$\int_\Omega E(x,u(x))\,dx \le C \int_\Omega u(x)|\nabla f(x,u(x))|^2\,dx, \tag{3.52}$$

where the constant C may depend on U but is independent of $u \in U$.

Consider the nonlinear Fokker-Planck equation

$$\partial_t u = -\operatorname{div}(u\nabla f), \qquad (x,t) \in \Omega \times (0,\infty), \tag{3.53}$$

$$u\frac{\partial f}{\partial \nu} = 0, \qquad (x,t) \in \partial\Omega \times (0,\infty), \tag{3.54}$$

$$u = u^0, \qquad (x,t) \in \Omega \times 0, \tag{3.55}$$

$$u \ge 0, \int_\Omega u\,dx = 1, \qquad (x,t) \in \Omega \times (0,\infty). \tag{3.56}$$

Here u is the unknown function and $f = f(x,u(x))$ is a known nonlinear scalar function of x and u, satisfying (3.5), (3.7). The initial data u^0 is a probability density. As in Remark 3.1, (3.54) can be replaced by the periodic boundary conditions.

For simplicity, assume that u^0 is bounded away from 0 and ∞. Then the behaviour of f at $u = 0,\infty$ is not important, and we do not lose any generality in assuming existence and uniqueness of a C^2-smooth probability density $m\colon \overline{\Omega} \to (0,\infty)$ such that $f(x,m(x)) = 0$ (cf. Section 3.1). Define the relative entropy $\mathcal{E}$ by (3.16), (3.17). The existence of a unique classical solution (which is smooth for $t > 0$) for such initial data is straightforward.

Theorem 3.11 (Convergence to equilibrium without reaction)**.** *Assume* (3.5), (3.7). *Let u be a solution of* (3.53)–(3.56) *with the initial data $u^0 \in L^\infty_+(\Omega)$, $\int_\Omega u^0 = 1$, $\kappa_1 \le u^0 \le \kappa_2$ a.e. in Ω with some $\kappa_1,\kappa_2 > 0$. Then u exponentially converges to m in the sense of entropy:*

$$\mathcal{E}(u(t)) \le \mathcal{E}(u^0)\mathrm{e}^{-\gamma t}, \tag{3.57}$$

where $\gamma = \gamma(\kappa_1,\kappa_2) > 0$ is independent of u^0.

Remark 3.12. A particular case of Theorem 3.11 when $f(x,u) = \frac{\rho(x)}{u^{r+1}}$, $\rho(x)$ is a given function bounded away from 0 and ∞, $r = cst > 0$, with Ω being a torus or a bounded convex domain, has recently been established in [26, 27]. The corresponding Wasserstein gradient flow is related to the problem of quantisation for probability measures. In this situation it is even possible to prove the exponential convergence merely if certain Lebesgue norms of u^0 and $\frac{1}{u^0}$ are finite, since under this hypothesis any solution instantaneously [27] becomes bounded away from 0 and ∞. This assumption at least visually resembles the definition of the Muckenhoupt weights [43], which are known [18] to satisfy the Poincaré inequality. In view of Corollary 3.10, it is plausible that similar exponential convergence results hold for general entropies when u^0 is, for instance, merely a Muckenhoupt weight.

Let us sketch the proof of Theorem 3.11. Since the behaviour of f at $u = 0, \infty$ is not relevant, we may assume (3.11) and (3.12). Using [30, Remark 3.4], we find m_{c_1} and m_{c_2} strictly positive such that $c_2 \leq 0 \leq c_1$ and

$$m_{c_1}(x) \leq \kappa_1 \leq \kappa_2 \leq m_{c_2}(x) \quad (x \in \Omega).$$

Now observe that problem (3.53)– (3.55) (without fixing the mass to be 1) admits a comparison principle: $u_1^0(x) \leq u_2^0(x)$ a.e. in Ω implies $u_1(x,t) \leq u_2(x,t)$, $t > 0$. This follows from [30, Lemma 3.1] by mimicking the proof of [30, Lemma 3.2]. Hence, the set U of smooth probability measures satisfying $m_{c_1} \leq u \leq m_{c_2}$ is invariant under the flow generated by this problem. Corollary 3.11 guarantees that (3.52) holds for this U. A standard Wasserstein entropy-entropy production argument [45] yields (3.57).

3.3. Unbalanced transportation inequalities. For simplicity, here we restrict ourselves to the spatially periodic setting, although everything seems to work for bounded convex domains. Let $\mathcal{M}^+$ and $\mathcal{P}$ be the sets of Radon and probability measures, resp., on the flat torus $\mathbb{T}^d$. The Hellinger-Kantorovich distance, cf. [29, 32, 33, 10, 11, 39], on $\mathcal{M}^+$ and the spherical Hellinger-Kantorovich distance, cf. [31, 5], on $\mathcal{P}$ can be introduced as follows.

Definition 3.13 (Conic distance)**.** Given two Radon measures $\rho_0, \rho_1 \in \mathcal{M}^+$ we define

$$d_{HK}^2(\rho_0, \rho_1) = \inf_{\mathcal{A}(\rho_0,\rho_1)} \int_0^1 \left(\int_{\mathbb{T}^d} (|v_t|^2 + |\alpha_t|^2)\, d\rho_t \right) dt, \tag{3.58}$$

where the admissible set $\mathcal{A}(\rho_0,\rho_1)$ consists of all $(\rho_t,\alpha_t,v_t)_{t\in[0,1]}$ such that

$$\begin{cases} \rho \in \mathcal{C}_w([0,1];\mathcal{M}^+), \\ \rho|_{t=0} = \rho_0; \quad \rho|_{t=1} = \rho_1, \\ (u,v) \in L^2(0,T;L^2(d\rho_t)\times L^2(d\rho_t)^d), \\ \partial_t\rho_t + \operatorname{div}(\rho_t v_t) = \rho_t\alpha_t \quad \text{in the weak sense.} \end{cases}$$

Definition 3.14 (Spherical distance)**.** Given probability measures $\rho_0,\rho_1 \in \mathcal{P}$ we define

$$d^2_{HKS}(\rho_0,\rho_1) = \inf_{\mathcal{A}_1(\rho_0,\rho_1)} \int_0^1 \left(\int_{\mathbb{T}^d} (|v_t|^2 + |\alpha_t|^2)\, d\rho_t \right) dt, \tag{3.59}$$

where the admissible set $\mathcal{A}_1(\rho_0,\rho_1)$ consists of all $(\rho_t,\alpha_t,v_t)_{t\in[0,1]}$ such that

$$\begin{cases} \rho \in \mathcal{C}_w([0,1];\mathcal{P}), \\ \rho|_{t=0} = \rho_0; \quad \rho|_{t=1} = \rho_1, \\ (u,v) \in L^2(0,T;L^2(d\rho_t)\times L^2(d\rho_t)^d), \\ \partial_t\rho_t + \operatorname{div}(\rho_t v_t) = \rho_t\alpha_t \quad \text{in the weak sense.} \end{cases}$$

The relation between the two distances is given by the fact that $(\mathcal{M}^+, d_{HK})$ is a metric cone over $(\mathcal{P}, d_{HKS})$ [31, 5] (see, e.g., [7, 6] for the abstract definition of a metric cone). The definitions above and the classical Benamou-Brenier formula immediately imply that

$$d_{HK}(\rho_0,\rho_1) \le d_{HKS}(\rho_0,\rho_1) \le W_2(\rho_0,\rho_1) \tag{3.60}$$

for all $\rho_0,\rho_1 \in \mathcal{P}(\mathbb{T}^d)$, where W_2 stands for the quadratic Wasserstein distance.

The conventional transportation inequality (1.24) (also known as Talagrand's inequality [37, 9, 12]) estimates the Wasserstein distance by displacement convex relative entropies. Here we present similar inequalities for the spherical distance d_{HKS} and the conic distance d_{HK}, but for a much wider class of entropies. In view of (3.60), our results are interesting merely for the entropies which are not geodesically convex in the Wasserstein space.

Remark 3.15. In Section 3.1 we defined the relative entropy $\mathcal{E}(u)$ for bounded probability distributions, but we can actually use any absolutely continuous probability measure u, although the entropy may become infinite. Moreover, the relative entropy can be defined in the same way for distributions of any mass, and without assuming that the implicit function m defined by (3.15) is a probability measure (cf. [30]).

Theorem 3.16 (Spherical Talagrand inequality). *Suppose that f satisfies* (3.5)–(3.8). *Assume that the second of the alternatives in* (3.11) *holds, and the limit is uniform w.r.t. $x \in \mathbb{T}^d$. Let $u^0 \in L^1(\mathbb{T}^d)$ be an absolutely continuous probability density with $\mathcal{E}(u^0) < \infty$. Then*

$$d_{HKS}^2(u^0, m) \leq C\,\mathcal{E}(u^0), \tag{3.61}$$

with C independent of u^0.

Proof: The proof is an adaptation of the Otto-Villani strategy [37]. We first observe that it suffices to prove the theorem when u^0 is smooth and strictly positive. Indeed, every $u^0 \in L^1(\mathbb{T}^d)$ with finite entropy can be approximated with bounded (from above and below) functions $\chi_k \circ u^0$, where $\chi_k(s) = \max(k^{-1}, \min(s,k))$. Since both d_{HK} and W_2 metrize the weak topology of $\mathcal{P}(\mathbb{T}^d)$, (3.60) implies that d_{HKS} metrizes the same topology. This fact and Beppo Levi's theorem imply that both sides of (3.61) are continuous w.r.t. our approximation. Each of the $\chi_k \circ u^0$ can be approximated by smooth bounded (from above and below) functions, cf. the proof of Theorem 3.7, so that both sides of (3.61) are continuous w.r.t. the approximation. The claim follows by a diagonal argument with renormalization of the masses in order to have an approximating sequence of probability distributions.

Since the left-hand side is always bounded by π^2 [5], we only need to consider the case when $\mathcal{E}(u^0)$ is bounded, say, by 1. Consider the classical solution u to problem (3.1), (3.3), (3.4) on $\mathbb{T}^d$ (cf. Lemma 3.5 and Remark 3.1), and let $f = f(x, u(x,t))$. As in the proof of Theorem 3.9, with the help of Theorem 3.8 we can find a constant C_1 such that

$$\mathcal{E}(u_t) \leq C_1 D\mathcal{E}(u_t),\ t \geq 0. \tag{3.62}$$

A simple scaling observation shows that the triple

$$(u_{s+th}, h(f_{s+th} - \overline{f}_{s+th}), h\nabla f_{s+th})$$

belongs to the admissible set $\mathcal{A}_1(u_s, u_{s+h})$, $s \geq 0$, $h > 0$. By the definition of the distance,

$$d_{HKS}(u_s, u_{s+h})$$
$$\leq h\sqrt{\int_0^1 \left(\int_{\mathbb{T}^d} (|f_{s+th} - \overline{f}_{s+th}|^2 + |\nabla f_{s+th}|^2) u_{s+th}\, dx\right) dt}.$$

As $h \to 0$, the square root on the right-hand side converges to $D\mathcal{E}(u_s)$, and we infer

$$\frac{d}{dh}\Big|^{+}_{h=0} d_{HKS}(u_s, u_{s+h}) \le \sqrt{D\mathcal{E}(u_s)}. \tag{3.63}$$

Consequently,

$$\begin{aligned}\frac{d}{ds}\Big|^{+} d_{HKS}(u_t, u_s) &= \limsup_{h\to 0} \frac{d_{HKS}(u_t, u_{s+h}) - d_{HKS}(u_t, u_s)}{h} \\ &\le \limsup_{h\to 0} \frac{d_{HKS}(u_s, u_{s+h})}{h} \le \sqrt{D\mathcal{E}(u_s)},\ t \le s. \end{aligned} \tag{3.64}$$

Consider the function

$$\phi(s) := 2\sqrt{C_1\mathcal{E}(u_s)} + d_{HKS}(u_t, u_s),\ s \ge t.$$

By (3.21), (3.62) and (3.64),

$$\frac{d}{ds}\Big|^{+} \phi(s) \le \left[-\sqrt{\frac{C_1 D\mathcal{E}(u_s)}{\mathcal{E}(u_s)}} + 1\right]\sqrt{D\mathcal{E}(u_s)} \le 0.$$

Therefore

$$d_{HKS}(u_t, u_s) \le \phi(s) \le \phi(t) = 2\sqrt{C_1\mathcal{E}(u_t)} \le 2\sqrt{C_1 e^{-\gamma t}\mathcal{E}(u^0)}. \tag{3.65}$$

The cone $(\mathcal{M}^+, d_{HK})$ is a complete metric space (cf. [29]), hence [6] the sphere $(\mathcal{P}, d_{HKS})$ is also complete. Now (3.65) yields existence of $u_\infty \in \mathcal{P}$ such that $u_t \to u_\infty$ as $t \to \infty$ in $(\mathcal{P}, d_{HKS})$ and thus weakly as probability measures. Fix $c > 0$ such that there exists m_{-c} (actually any $c > 0$ would work since the second alternative in (3.11) is assumed). Observing that $E_u = -f > c$ for $u > m_{-c}(x)$ we can deduce existence of a continuous function $a : \mathbb{T}^d \to \mathbb{R}$ such that

$$E(x,u) > a(x) + cu. \tag{3.66}$$

Taking into account that $E_{uu} > 0$ and using the results of [19, Subsection 6.4.5] we infer that the entropy functional $\mathcal{E}$ is lower-semicontinuous w.r.t. the weak convergence , whence $\mathcal{E}(u_\infty) = 0$, and $u_\infty = m$. Letting $t = 0$ and $s \to +\infty$ in (3.65), we get the claim (3.61). ∎

Using a similar argument and the entropy-entropy production inequality obtained in [30, Theorem 2.9] for the Hellinger-Kantorovich gradient flows, we can get a transportation inequality for the conic distance. From now on we do not assume that that the implicit function m defined by (3.15) has mass 1 (cf. Remark 3.15).

Theorem 3.17 (Conic Talagrand inequality)**.** *Suppose that f satisfies* (3.5)–(3.8)*. Let $u^0 \in L^1(\mathbb{T}^d)$, $\mathcal{E}(u^0) < \infty$. Then*

$$d_{HK}^2(u^0, m) \le C\,\mathcal{E}(u^0), \tag{3.67}$$

with C independent of u^0.

Proof: As in the previous proof, we may assume that u^0 is smooth and strictly positive. In the case when $\mathcal{E}(u^0) < \mathcal{E}(0)$ the proof mimicks the previous one, basically substituting the objects related to the spherical Hellinger-Kantorovich distance with the conic ones. Let us merely describe the small differences that show up. Consider the classical solution u to the conic Hellinger-Kantorovich gradient flow [30]. The condition (3.11) is not needed because the conic entropy-entropy production inequality [30, Theorem 2.9] does not require it. However, in order to apply that theorem we need to find a set U containing the trajectory u_t of the conic gradient flow starting from u^0 such that no sequence in U converges to 0 in the sense of measures. An argument involving Lebesgue's dominated convergence theorem shows that we can simply take

$$U = \left\{ u \in L_+^\infty(\Omega) :\ \mathcal{E}(u) \le \mathcal{E}(u^0) < \mathcal{E}(0) \right\}.$$

It remains to treat the case $\mathcal{E}(u^0) \ge \mathcal{E}(0)$. Since $\mathcal{E}(0)$ is a positive constant, it suffices to prove the inequality

$$d_{HK}^2(u^0, m) \le C\,(1 + \mathcal{E}(u^0)). \tag{3.68}$$

We recall [10, 31, 5] the upper bound for the Hellinger-Kantorovich distance in terms of the masses,

$$d_{HK}^2(u^0, m) \le 4\left(\int_{\mathbb{T}^d} u^0 + \int_{\mathbb{T}^d} m \right).$$

Consequently, it is enough to show

$$\int_{\mathbb{T}^d} u^0 \le C\,(1 + \mathcal{E}(u^0)). \tag{3.69}$$

Let c be a small positive constant such that the implicit function m_{-c} exists. As in the previous proof, we can deduce (3.66) with c just defined and some function $a(x)$ independent of u. Hence,

$$\int_{\mathbb{T}^d} u^0 \leq C + c^{-1} \int_{\mathbb{T}^d} E(x, u^0(x))\, dx,$$

proving (3.69). ■

Remark 3.18. Inequality (3.67) follows from (3.61) and (3.60) provided u^0 and m are probability measures. However, when the masses of u^0 and m do not coincide, (3.67) is not an immediate consequence of (3.61).

Conflict of interest statement. We have no conflict of interest to declare.

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