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VARIATIONS OF THE SHIFTING LEMMA AND GOURSAT CATEGORIES

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Abstract: We prove that Mal'tsev and Goursat categories may be characterised through stronger variations of the Shifting Lemma, that is classically expressed in terms of three congruences R, S and T, and characterises congruence modular varieties. We first show that a regular category is a Mal'tsev category if and only if the Shifting Lemma holds for reflexive relations on the same object in $\mathbb{C}$. Moreover, we prove that a regular category $\mathbb{C}$ is a Goursat category if and only if the Shifting Lemma holds for a reflexive relation S and reflexive and positive relations R and T in $\mathbb{C}$. In particular this provides a new characterisation of 2-permutable and 3-permutable varieties and quasi-varieties of universal algebras.

Keywords: Mal'tsev categories, Goursat categories, Shifting Lemma, congruence modular varieties, 3-permutable varieties.
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Introduction

For a variety $\mathbb{V}$ of universal algebras, Gumm's *Shifting Lemma* [9] is stated as follows. Given congruences R, S and T on the same algebra X in $\mathbb{V}$ such that $R \wedge S \leqslant T$, whenever x, y, u, v are elements in X with $(x, y) \in R \wedge T$, $(x, u) \in S$, $(y, v) \in S$ and $(u, v) \in R$, it then follows that $(u, v) \in T$. We picture the lemma by drawing the given relations in solid lines

$$\begin{tikzcd} x \arrow[r, dash, "S"] \arrow[d, dash, "R"] \arrow[d, dash, bend right=60, "T"'] & u \arrow[d, dash, "R"'] \arrow[d, dash, dashed, bend left=60, "T"] \\ y \arrow[r, dash, "S"'] & v \end{tikzcd} \tag{1}$$

and the induced relation with the dashed line.

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A variety $\mathbb{V}$ of universal algebras satisfies the Shifting Lemma precisely when it is congruence modular [9], this meaning that the lattice of congruences on any algebra in $\mathbb{V}$ is modular. In particular, since any 3-permutable variety is congruence modular [12], the Shifting Lemma always holds in this context. Recall that a variety $\mathbb{V}$ is 3-*permutable* when, given any congruences R and S on a same algebra X, we have the equality

$$RSR = SRS, \tag{2}$$

where

$$RSR = \{(x,u) \in X \times X \mid \exists y \in X, z \in X \text{ with } (x,y) \in R, (y,z) \in S, (z,u) \in R\}$$

and

$$SRS = \{(x,u) \in X \times X \mid \exists y \in X, z \in X \text{ with } (x,y) \in S, (y,z) \in R, (z,u) \in S\}$$

are the usual composites of congruences. Such varieties are characterised by the existence of two ternary terms $r(x,y,z)$ and $s(x,y,z)$ [10] satisfying the identities

$$\begin{aligned} r(x,y,y) &= x \\ r(x,x,y) &= s(x,y,y) \\ s(x,x,y) &= y. \end{aligned}$$

Any 2-permutable variety [19] is necessarily 3-permutable, and there are known examples of varieties that are 3-permutable, but fail to be 2-permutable, such as the varieties of right complemented semigroups [10] and the one of implication algebras [17].

The notions of 2-permutability and 3-permutability can be extended from varieties to *regular* categories by replacing *congruences* with *internal equivalence relations*, allowing one to explore some interesting new (non-varietal) examples. A categorical version of the Shifting Lemma (stated differently from the original formulation recalled above) may be considered in any finitely complete category, and this leads to the notion of a *Gumm category* [3, 4]. In a regular context one can use set-theoretical terms thanks to Barr's embedding theorem [1] (see also Metatheorem A.5.7 in [2]), so that the property given in diagram (1) may still be expressed by using generalised elements. In a regular context one can show that the property of 3-permutability still implies the modularity of the lattice of equivalence relations [5], and that this latter property implies that the Shifting Lemma holds. However, the converse is false, as it was shown by G. Janelidze in Example 12.5 in [11].

Corollary 3.7. *Let $\mathbb{V}$ be a quasi-variety. The following conditions are equivalent:*

(i) *$\mathbb{V}$ is 3-permutable;*
(ii) *The Shifting Lemma holds in $\mathbb{V}$ when S is a reflexive compatible relation, and R and T are reflexive and positive compatible relations.*

References

[1] M. Barr, P. A. Grillet and D. H. van Osdol, *Exact categories and categories of sheaves*, Lecture Notes in Math. 236, Springer-Verlag (1971).
[2] F. Borceux and D. Bourn, *Mal'cev, protomodular, homological and semi-abelian categories*, Kluwer, (2004).
[3] D. Bourn, M. Gran, *Categorical Aspects of Modularity*, Galois Theory, Hopf Algebras and Semiabelian Categories, Fields Instit. Commun., 43, Amer. Math. Soc., Providence RI (2004) 77-100.
[4] D. Bourn, M. Gran, *Normal sections and direct product decompositions*, Comm. Algebra. 32, no. 10 (2004) 3825-3842.
[5] A. Carboni, G.M. Kelly, M.C. Pedicchio, *Some remarks on Maltsev and Goursat categories*, Appl. Categ. Structures 1, no.4 (1993) 385-421.
[6] A. Carboni, J. Lambek, M.C. Pedicchio, *Diagram chasing in Mal'cev categories*, J. Pure Appl. Algebra 69, no. 3 (1991) 271-284.
[7] A. Carboni, M.C. Pedicchio, N. Pirovano, *Internal graphs and internal groupoids in Mal'cev categories*, Category theory 1991 (Montreal, PQ, 1991), 97-109, CMS Conf. Proc., 13, Amer. Math. Soc., Providence, RI, 1992.
[8] M. Gran, F. Sterck, J. Vercruysse, *A semi-abelian extension of a theorem by Takehuchi*, preprint, arXiv:1808.04998 (2018).
[9] H.P. Gumm, *Geometrical methods in congruence modular algebras*, Mem. Amer. Math. Soc. 45 (1983), no. 286.
[10] J. Hagemann and A. Mitschke, *On n-permutable congruences*, Algebra Universalis 3 (1973), 8-12.
[11] G. Janelidze, *A history of selected topics in categorical algebra I: From Galois theory to abstract commutators and internal groupoids*, Categories and General Algebraic Structures with Applications 5, no. 1, (2016) 1-54.
[12] B. Jónnsson, *On the representation of lattices*, Math. Scand. 1 (1953) 193–06.
[13] K. Kearnes and R. McKenzie, *Commutator theory for relatively modular quasivarieties*, Trans. Amer. Math. Soc. 331 (1992) 465–502.
[14] E. W. Kiss, *Three remarks on the modular commutator*, Algebra Universalis 29 (1992) 455476.
[15] A.I. Mal'cev, *On the general theory of algebraic systems*, Mat. Sbornik N.S. 35 (1954), 3-20.
[16] N. Martins-Ferreira, D. Rodelo, T. Van der Linden, *An observation on n-permutability*, Bull. Belg. Math. Soc. Simon Stevin 21 (2014), no. 2, 223–230.
[17] A. Mitschke, *Implication algebras are 3-permutable and 3-distributive*, Algebra Universalis 1 (1971), 182-186.
[18] M.C. Pedicchio and E.M. Vitale, *On the abstract characterization of quasi-varieties* , Algebra Universalis 43 (2000), 269-278.
[19] J.D.H. Smith, *Mal'cev varieties*, Lecture Notes in Math. 554, Springer-Verlag (1976).
[20] S. Tull, *Conditions for an n-permutable category to be Mal'tsev*, Cah. Topol. Géom. Différ. Catég. 58 (2017), no. 3 & 4. 189-194

Remark 3.5. The Shifting Lemma when S is a reflexive relation and R and T are equivalence relations, as stated in Proposition 3.4, is the categorical version of the Shifting Principle recalled in the Introduction. First, assuming that $R \wedge S \leqslant T$ is equivalent to assuming that $R \wedge S \leqslant T \leqslant R$ for the property of diagram (1) to hold (take $R \wedge T$, which is such that $R \wedge S \leqslant R \wedge T \leqslant R$, for the non-obvious implication). Second, going carefully through the proofs in [9], one may check that the symmetry of S is not necessary. So the Shifting Principle could equivalently be stated by asking that S is just a reflexive and compatible relation.

We now use Propositions 3.3 and 3.4 to obtain the characterisation of Goursat categories through a variation of the Shifting Lemma:

Theorem 3.6. *Let $\mathbb{C}$ be a regular category. The following conditions are equivalent:*

(i) $\mathbb{C}$ *is a Goursat category;*
(ii) *The Shifting Lemma holds in $\mathbb{C}$ when S is a reflexive relation and R and T are reflexive and positive relations.*

Proof: (i) $\Rightarrow$ (ii): this implication follows from the fact that reflexive and positive relations are necessarily equivalence relations in the Goursat context (Proposition 3.3) and from Proposition 3.4.
(ii) $\Rightarrow$ (i): we shall prove that for any reflexive relation E on X in $\mathbb{C}$, $EE^\circ = E^\circ E$ (see Theorem 3.2 (iv)). Suppose that $(x, y) \in EE^\circ$. Then, for some z in X, one has that $(z, x) \in E$ and $(z, y) \in E$. Consider the reflexive and positive relations $R = EE^\circ$ and $T = E^\circ E$, and the reflexive relation E on X. From the reflexivity of E, we get $E \leqslant EE^\circ$ and $E \leqslant E^\circ E$; thus $EE^\circ \wedge E = E \leqslant E^\circ E$. We may apply our assumption to the following relations given in solid lines:

$$\begin{tikzcd}
z \arrow[r, dash, "E"] \arrow[d, dash, "EE^\circ"] \arrow[d, dash, bend right=60, "E^\circ E"'] & x \arrow[d, dash, "EE^\circ"'] \arrow[d, dash, dashed, bend left=60, "E^\circ E"] \\
z \arrow[r, dash, "E"'] & y
\end{tikzcd}$$

to conclude that $(x, y) \in E^\circ E$. The proof that $E^\circ E \leqslant EE^\circ$ is similar. ∎

The fact that any quasi-variety is a regular categories [18] implies the following

Regular categories that are 2-permutable, or 3-permutable, are usually called *Mal'tsev categories* [6] and *Goursat categories* [5], respectively. As examples of regular Mal'tsev categories that are not (finitary) varieties of algebras we list: $\mathbb{C}$*-algebras, compact groups, topological groups [5], torsion-free abelian groups, reduced commutative rings, cocommutative Hopf algebras over a field [8], the dual of the category of abelian groups, and the dual of any topos [5].

In the varietal context, H.-P. Gumm has also considered a slight variation of the Shifting Lemma called the *Shifting Principle* [9]: given congruences R and T and a reflexive, symmetric and compatible relation S on the same algebra X such that $R \wedge S \leqslant T \leqslant R$, whenever x, y, u, v are elements in X with $(x, y) \in R \wedge T$, $(x, u) \in S$, $(y, v) \in S$ and $(u, v) \in R$ as in (1), it then follows that $(u, v) \in T$. The Shifting Principle, although apparently stronger, turns out to be equivalent to the Shifting Lemma in the varietal case.

With this observation in mind, it seems reasonable to expect that considering variations on the assumptions on the relations R, S or T appearing in the Shifting Lemma might provide characterisations of other types of categories. The variations we have in mind for R, S and T are to make them weaker, so that they give stronger versions of the Shifting Lemma. This idea comes from the well known characterisation of Mal'tsev categories through the fact that reflexive relations are equivalence relations [6, 7], and from a more recent one of Goursat categories recalled in Proposition 3.3: a regular category is a Goursat category if and only if any reflexive and positive relation (i.e. of the form $U^\circ U$, for some relation U) is an equivalence relation. The main results of the paper show that stronger versions of the Shifting Lemma characterise regular categories that are Mal'tsev categories (Theorems 2.2 and 2.3), and those ones that are Goursat categories (Theorem 3.6). These results apply in particular to 2-permutable and 3-permutable quasi-varieties, since these latter categories are known to be regular.

1. Regular categories and relations

A morphism in a category $\mathbb{C}$ is a regular epimorphism if it is a coequaliser of a pair of parallel morphisms in $\mathbb{C}$. A category $\mathbb{C}$ with finite limits is called *regular* if any morphism f admits a unique (up to isomorphism) factorisation $f = mr$, where r is a regular epimorphism and m is a monomorphism, and

these factorisations are pullback stable. Since monomorphisms are always pullback stable, this latter property can be equivalently expressed by asking that in any pullback

$$\begin{tikzcd} E \times_B A \arrow[r, two heads, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\ E \arrow[r, two heads, "p"'] & B \end{tikzcd}$$

the arrow π_2 is a regular epimorphism whenever p is a regular epimorphism.

Example 1.1. Any variety of universal algebras is a regular category, where regular epimorphisms are surjective homomorphisms, and finite limits (in particular, pullbacks) are computed as in the category of sets. The same is true for any quasi-variety of algebras (see [18], for example). The factorisation of any homomorphism $f\colon X \to Y$ as a regular epimorphism followed by a monomorphism is simply the factorisation $X \to f(X) \to Y$, where $f(X) = \{f(x) \mid x \in X\}$ is the direct image of f. A thorough study of commutator theory for quasi-varieties with modular lattice of congruences can be found in [13].

In this article, we mainly work in a regular category, thus the proofs are partially given in set-theoretical terms (see Metatheorem $A.5.7$ in [2], for instance).

An (internal) *relation* R from X to Y is a subobject $\langle r_1, r_2\rangle\colon R \rightarrowtail X \times Y$. The opposite relation of R, denoted R°, is the relation from Y to X given by the subobject $\langle r_2, r_1\rangle\colon R \rightarrowtail Y \times X$. A relation R from X to X is called a relation on X. Given two relations $R \rightarrowtail X \times Y$ and $S \rightarrowtail Y \times Z$ in a regular category, their relational composite can be defined and is written $SR \rightarrowtail X \times Z$ (see [5], for instance).

A relation $R \rightarrowtail X \times X$ on X is called *reflexive* when $1_X \leqslant R$, where 1_X denotes the relation $\langle 1_X, 1_X\rangle\colon X \rightarrowtail X \times X$. It is called *symmetric* when $R^\circ \leqslant R$ (or, equivalently $R^\circ = R$), and *transitive* when $RR \leqslant R$. A reflexive, symmetric and transitive relation is called an *equivalence* relation (it then follows that $RR = R$). When the regular category is a variety $\mathbb{V}$, an equivalence relation in $\mathbb{V}$ is simply a congruence.

A relation $D \rightarrowtail X \times Y$ is called *difunctional* if $(x, y) \in D$ whenever $(x, v) \in D, (u, v) \in D, (u, y) \in D$. In a finitely complete category this property can be expressed by the equality $DD^\circ D = D$. A relation $P \rightarrowtail X \times X$ on X is called *positive* when it is of the form $P = U^\circ U$, for some relation $U \rightarrowtail X \times Y$. In

Given the equivalence relations $R\square S$, $\mathrm{Eq}(s_2)$ and W on S, Remark 3.1 yields the following description of the supremum of $R\square S \wedge \mathrm{Eq}(s_2)$ and W as equivalence relations on S:

$$\begin{aligned} (\, R\square S \wedge \mathrm{Eq}(s_2)\,) \vee W &= (\, R\square S \wedge \mathrm{Eq}(s_2)\,)\, W\, (\, R\square S \wedge \mathrm{Eq}(s_2)\,) \\ &= W\, (\, R\square S \wedge \mathrm{Eq}(s_2)\,)\, W. \end{aligned}$$

Since

$$R\square S \wedge \mathrm{Eq}(s_2) \leqslant (\, R\square S \wedge \mathrm{Eq}(s_2)\,) \vee W$$

we may apply the Shifting Lemma to the following diagram

$$\begin{tikzcd} xSu \arrow[r, dash, "\mathrm{Eq}(s_2)"] \arrow[d, dash, "R\square S"] \arrow[d, dash, bend right=60, "(\, R\square S \wedge \mathrm{Eq}(s_2)\,) \vee W \geqslant W"'] & uSu \arrow[d, dash, "R\square S"'] \arrow[d, dash, dashed, bend left=60, "(\, R\square S \wedge \mathrm{Eq}(s_2)\,) \vee W"] \\ ySv \arrow[r, dash, "\mathrm{Eq}(s_2)"'] & vSv. \end{tikzcd}$$

Note that, uSu and vSv by the reflexivity of S. We then obtain

$$(uSu, vSv) \in (\, R\square S \wedge \mathrm{Eq}(s_2)\,) \vee W.$$

Using

$$(\, R\square S \wedge \mathrm{Eq}(s_2)\,) \vee W = (\, R\square S \wedge \mathrm{Eq}(s_2)\,)\, W\, (\, R\square S \wedge \mathrm{Eq}(s_2)\,)$$

this means that

$$(uSu)\, (R\square S \wedge \mathrm{Eq}(s_2))\, (aSu) W (bSv)\, (R\square S \wedge \mathrm{Eq}(s_2))\, (vSv),$$

for some a, b in X, i.e.

$$\begin{array}{ccc} u & S & u \\ R & & R \\ a & S & u \\ T & & R \\ b & S & v \\ R & & R \\ v & S & v. \end{array}$$

Since aRu (R is symmetric), aSu and $R \wedge S \leqslant T$, it follows that aTu; similarly bTv. From uTa (T is symmetric), aTb and bTv, we conclude that uTv (T is transitive), as desired. ■

If $\mathbb{C}$ is a Goursat category, then the lattice of equivalence relations on a same object is modular [5] and, consequently, the Shifting Lemma holds. Moreover, the Shifting Lemma still holds when S is just a reflexive relation, as we show next. The following result is partly based on Lemma 2.2 in [14], and it gives a first step towards the characterisation we aim to obtain for Goursat categories.

Proposition 3.4. *In any regular Goursat category $\mathbb{C}$, the Shifting Lemma holds when S is a reflexive relation and R and T are equivalence relations.*

Proof: Let R and T be equivalence relations and let S be a reflexive relation on an object X such that $R \wedge S \leqslant T$. Suppose that we have $(x, y) \in R \wedge T$, $(x, u) \in S$, $(y, v) \in S$ and $(u, v) \in R$ as in (1). We consider the two equivalence relations on S determined by the following pullbacks:

$$\begin{tikzcd} R\square S \arrow[r, tail, "\langle \pi_{12},\pi_{34}\rangle"] \arrow[d, tail, "\langle \pi_{13},\pi_{24}\rangle"'] \arrow[dr, phantom, "\lrcorner", very near start] & S\times S \arrow[d, "\langle s_1,s_2\rangle\times\langle s_1,s_2\rangle"] \\ R\times R \arrow[r, tail, "\langle r_1\times r_1, r_2\times r_2\rangle"'] & X\times X\times X\times X \end{tikzcd}$$

and

$$\begin{tikzcd} W \arrow[r, tail, "\langle \pi_{12},\pi_{34}\rangle"] \arrow[d, tail, "\langle \pi_{13},\pi_{24}\rangle"'] \arrow[dr, phantom, "\lrcorner", very near start] & S\times S \arrow[d, "\langle s_1,s_2\rangle\times\langle s_1,s_2\rangle"] \\ T\times R \arrow[r, tail, "\langle t_1\times r_1, t_2\times r_2\rangle"'] & X\times X\times X\times X. \end{tikzcd}$$

We have $(aSb, cSd) \in R\square S$ if and only if

$$\begin{array}{ccc} a & S & b \\ R & & R \\ c & S & d \end{array}$$

and $(aSb, cSd) \in W$ if and only if

$$\begin{array}{ccc} a & S & b \\ T & & R \\ c & S & d. \end{array}$$

Note that they are in fact equivalence relations on S since R and T are both equivalence relations.

set-theoretical terms: $(x, x') \in P$ whenever $(x, y) \in U$ and $(x', y) \in U$, for some relation U. It is easy to see that any positive relation is symmetric, and that any equivalence relation R is positive since $R = R^\circ R$.

2. Regular Mal'tsev categories and the Shifting Lemma

A finitely complete category $\mathbb{C}$ is called a *Mal'tsev category* if every reflexive relation in $\mathbb{C}$ is an equivalence relation [6, 7]. These categories are also characterised by other properties on relations, as follows:

Theorem 2.1. [7] *Let $\mathbb{C}$ be a finitely complete category. The following conditions are equivalent:*

(i) *$\mathbb{C}$ is a Mal'tsev category;*
(ii) *every relation $D \rightarrowtail X \times Y$ in $\mathbb{C}$ is difunctional;*
(iii) *every reflexive relation E in $\mathbb{C}$ is symmetric: $E^\circ = E$.*

Regular Mal'tsev categories may be characterised by the fact that they are 2-permutable: for any pair of equivalence relations R, S on a same object, $RS = SR$ [6]. If $\mathbb{C}$ is a regular Mal'tsev category, then the lattice of equivalence relations on a same object is modular [5] and, consequently, the Shifting Lemma holds.

Using the fact that in a Mal'tsev category reflexive relations coincide with equivalence relations, or with symmetric relations, we are now going to show that regular Mal'tsev categories may be characterised through a stronger version of the Shifting Lemma where, in the assumption, the equivalence relations are replaced by reflexive relations. Note that, for a diagram as (1) where R, S or T are not equivalence relations, the relations are always to be considered from left to right and from top to bottom.

Theorem 2.2. *Let $\mathbb{C}$ be a regular category. The following conditions are equivalent:*

(i) *$\mathbb{C}$ is a Mal'tsev category;*
(ii) *The Shifting Lemma holds in $\mathbb{C}$ when R, S and T are reflexive relations.*

Proof: (i) $\Rightarrow$ (ii): this implication follows from the fact that reflexive relations are necessarily equivalence relations and the Shifting Lemma holds in any regular Mal'tsev category.
(ii) $\Rightarrow$ (i): we shall prove that every reflexive relation $\langle e_1, e_2\rangle \colon E \rightarrowtail X \times X$

is symmetric (Theorem 2.1(iii)). Suppose that $(x,y) \in E$, and consider the following reflexive relations T and R on E defined by the following pullbacks:

$$\begin{tikzcd} T \arrow[r] \arrow[d, tail, "\langle t_1, t_2 \rangle"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, tail, "\langle e_1, e_2 \rangle"] \\ E \times E \arrow[r, "e_1 \times e_2"'] & X \times X \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} R \arrow[rr] \arrow[d, tail, "\langle r_1, r_2 \rangle"'] \arrow[drr, phantom, "\lrcorner", very near start] & & E \arrow[d, tail, "\langle e_1, e_2 \rangle"] \\ E \times E \arrow[r, "e_2 \times e_1"'] & X \times X \arrow[r, "\cong", "\langle \pi_2, \pi_1 \rangle"'] & X \times X, \end{tikzcd}$$

where $\pi_1 \colon X \times X \to X$ and $\pi_2 \colon X \times X \to X$ are the product projections. We have $(aEb, cEd) \in T$ if and only if $(a,d) \in E$, and $(aEb, cEd) \in R$ if and only if $(c,b) \in E$.

The third reflexive relation we consider on E is $\mathrm{Eq}(e_2)$

$$\begin{tikzcd} \mathrm{Eq}(e_2) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "e_2"] \\ E \arrow[r, "e_2"'] & X \end{tikzcd}$$

the *kernel pair* of e_2, which is actually an equivalence relation. It is easy to check that $\mathrm{Eq}(e_2) \leqslant R$ and $\mathrm{Eq}(e_2) \leqslant T$, so that $R \wedge \mathrm{Eq}(e_2) = \mathrm{Eq}(e_2) \leqslant T$. We may apply the assumption to the following relations given in solid lines

$$\begin{tikzcd} xEy \arrow[r, dash, "\mathrm{Eq}(e_2)"] \arrow[d, dash, "R"] \arrow[d, dash, bend right=60, "T"'] & yEy \arrow[d, dash, "R"'] \arrow[d, dash, dashed, bend left=60, "T"] \\ xEx \arrow[r, dash, "\mathrm{Eq}(e_2)"'] & xEx \end{tikzcd}$$

(xEx and yEy by the reflexivity of the relation E). We conclude that $(yEy, xEx) \in T$ and, consequently, that $(y,x) \in E$. ■

In the proof of the implication (ii) ⇒ (i) we only used two "true" reflexive relations R and T. This observation gives:

Theorem 2.3. *Let $\mathbb{C}$ be a regular category. The following conditions are equivalent:*

(i) *$\mathbb{C}$ is a Mal'tsev category;*
(ii) *The Shifting Lemma holds in $\mathbb{C}$ when R, S and T are reflexive relations;*
(iii) *The Shifting Lemma holds in $\mathbb{C}$ when R and T are reflexive relations and S is an equivalence relation.*

3. Goursat categories and the Shifting Lemma

A regular category $\mathbb{C}$ is called a *Goursat* category [5] when it is 3-permutable, i.e. for any pair of equivalence relations R, S on a same object in $\mathbb{C}$ one has that

$$RSR = SRS.$$

Remark 3.1. As already observed in [6], for any pair of equivalence relations R and S on a same object X in a Goursat category, one has that RSR is an equivalence relation, that is then the supremum of R and S as equivalence relations on X

$$R \vee S = RSR.$$

The 3-permutable version of Theorem 2.1 is:

Theorem 3.2. [5] *Let $\mathbb{C}$ be a regular category. The following conditions are equivalent:*

(i) *$\mathbb{C}$ is a Goursat category;*
(ii) *for any relation $D \rightarrowtail X \times Y$ in $\mathbb{C}$, $DD^\circ DD^\circ = DD^\circ$;*
(iii) *for any reflexive relation E in $\mathbb{C}$, EE° is an equivalence relation;*
(iv) *for any reflexive relation E in $\mathbb{C}$, $EE^\circ = E^\circ E$.*

To adapt our approach from Mal'tsev categories to Goursat categories, we need a suitable property on relations guaranteeing that they are equivalence relations, where this property should be useful to characterise for Goursat categories. A first attempt would be to use the fact that, in a Goursat context, all reflexive and transitive relations are equivalence relations [16]. However, such a property on relations holds for any n-permutable category, $n \geqslant 2$, so it would not provide the needed characterisation. The right property we were looking for was discovered after reading [20], and is given in the next proposition:

Proposition 3.3. *A regular category $\mathbb{C}$ is a Goursat category if and only if any reflexive and positive relation in $\mathbb{C}$ is an equivalence relation.*

Proof: Suppose that $\mathbb{C}$ is a Goursat category and consider a reflexive and positive relation $1_X \leqslant P = U^\circ U$. Then P is symmetric since $P^\circ = (U^\circ U)^\circ = U^\circ U = P$. As for the transitivity of P, we have $PP = U^\circ U U^\circ U = U^\circ U$, by Theorem 3.2(ii). It follows that $PP = P$.

Conversely, let E be a reflexive relation on X. Then EE° is a reflexive and positive relation, thus an equivalence relation by assumption. It follows that $\mathbb{C}$ is a Goursat category by Theorem 3.2 (iii). ■