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SYMBOLIC DYNAMICS AND SEMIGROUP THEORY

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ABSTRACT: The correspondence $\mathcal{X} \mapsto B(\mathcal{X})$, associating to each symbolic dynamical system $\mathcal{X}$ the formal language $B(\mathcal{X})$ of its blocks, entails a connection between symbolic dynamics and semigroup theory. In this article we survey some developments on this connection.

1. Introduction

A major motivation for the development of semigroup theory was, and still is, its applications to the study of formal languages. Therefore, it is not surprising that the correspondence $\mathcal{X} \mapsto B(\mathcal{X})$, associating to each symbolic dynamical system $\mathcal{X}$ the formal language $B(\mathcal{X})$ of its blocks, entails a connection between symbolic dynamics and semigroup theory. In this article we survey some developments on this connection, since when it was noticed in an article by Almeida, published in the CIM bulletin, in 2003 [2].

2. Symbolic dynamics

A *topological dynamical system* is a pair (X, T) consisting of a topological space X and a continuous self-map $T\colon X \to X$. It is useful to think of X as representing a sort of "space", where each point x is moved to $T(x)$ when a unit of time has passed. A morphism between two topological dynamical systems (X_1, T_1) and (X_2, T_2) is a continuous map $\varphi\colon X_1 \to X_2$ such that $\varphi \circ T_1 = T_2 \circ \varphi$. In this way, topological dynamical systems form a category, if we take the identity on X as the local identity at (X, T). In this category, an isomorphism is called a *conjugacy*, and isomorphic objects are said to be *conjugate*.

We focus on a special class of topological dynamical systems, the symbolic ones. Their applications in the study of general topological dynamical systems frequently stem from the following procedure: use symbols to mark a finite number of regions of the underlying space, and register, with a string of those symbols, the regions visited by a orbit. In the next paragraph we give a brief formal introduction to symbolic systems. For a more developed

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introduction, we refer to the book [26]. Also, the book review [33] is an excellent short introduction to the field and its ramifications.

Consider a finite nonempty set A, whose elements are called *symbols*, and the set $A^{\mathbb{Z}}$ of sequences $(x_i)_{i\in\mathbb{Z}}$ of symbols of A indexed by $\mathbb{Z}$. One should think of an element $x = (x_i)_{i\in\mathbb{Z}}$ of $A^{\mathbb{Z}}$ as a biinfinite string

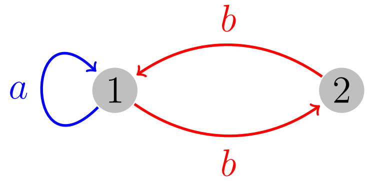
$$\ldots x_{-3}x_{-2}x_{-1}.x_0x_1x_2x_3\ldots,$$

with the dot marking the reference position. A *block* of x is a finite string appearing in x: a finite sequence of the form $x_kx_{k+1}\ldots x_{k+\ell}$, with $k\in\mathbb{Z}$ and $\ell\geq 0$, also denoted $x_{[k,k+\ell]}$. Of special relevance are the central blocks $x_{[-k,k]}$, as one endows $A^{\mathbb{Z}}$ with the topology induced by the metric $d(x,y) = 2^{-r(x,y)}$ such that $r(x,y)$ is the minimum $k\geq 0$ for which $x_{[-k,k]}\neq y_{[-k,k]}$. Hence, two elements of $A^{\mathbb{Z}}$ are "close" if they have a "long" common central block. The *shift mapping* $\sigma\colon A^{\mathbb{Z}}\to A^{\mathbb{Z}}$, defined by $\sigma(x) = (x_{i+1})_{i\in\mathbb{Z}}$, shifts the dot to the right. A *symbolic dynamical system*, also called *subshift*, is a pair $(\mathcal{X},\sigma_{\mathcal{X}})$ formed by a nonempty closed subspace $\mathcal{X}$ of $A^{\mathbb{Z}}$, for some A, such that $\sigma(\mathcal{X}) = \mathcal{X}$, and by the restriction $\sigma_{\mathcal{X}}$ of σ to $\mathcal{X}$. As it is clear what self-map is considered, $(\mathcal{X},\sigma_{\mathcal{X}})$ is identified with $\mathcal{X}$. When $\mathcal{X} = A^{\mathbb{Z}}$, the system is a *full shift*. The *sliding block code* from the subshift $\mathcal{X}\subseteq A^{\mathbb{Z}}$ to the subshift $\mathcal{Y}\subseteq B^{\mathbb{Z}}$, with *block map* $\Phi\colon A^{m+n+1}\to B$, *memory* m and *anticipation* n, is the map $\varphi\colon\mathcal{X}\to\mathcal{Y}$ defined by $\varphi(x) = (\Phi(x_{[i-m,i+n]}))_{i\in\mathbb{Z}}$. It follows from the definition of the metric on a full shift that the morphisms between subshifts are precisely the sliding block codes.

3. Formal languages

Given a set A of symbols, the set of finite nonempty strings of elements of A is denoted by A^+. In the jargon of formal languages, A is said to be an *alphabet*, the elements of A and those of A^+ are respectively called *letters* and *words*, and the subsets of A^+ are *languages*. Moreover, A^+ is viewed as a semigroup for the operation "concatenation of words". For example, in $\{a,b\}^+$, the product of *aba* and *bab* is *ababab*. In fact, A^+ is the *free semigroup generated by* A, since, for every semigroup S, every mapping $A\to S$ extends uniquely to a homomorphism $A^+\to S$. Concerning semigroups, formal languages, and their interplay, we give [4] as a source of detailed references and as a very convenient guide, since, in this sort of introductory text, connections with symbolic dynamics are also highlighted.

If $\mathcal{X}$ is a subshift of $A^{\mathbb{Z}}$, we let $B(\mathcal{X})$ be the language over the alphabet A such that $u \in B(\mathcal{X})$ if and only if u is a block of some element x of $\mathcal{X}$. As a concrete example, consider the subshift $\mathcal{E}$, known as the *even shift*, formed by the biinfinite sequences of a's and b's that have no odd number of b's between two consecutive a's, that is, the biinfinite paths in the following labeled graph:



A language L is *factorial* if, for each $u \in L$, every factor of u belongs to L. A factorial language over A is *prolongable* if $u \in L$ implies $aub \in L$ for some $a, b \in A$. It is easy to see that the languages of the form $B(\mathcal{X})$, with $\mathcal{X}$ a subshift of $A^{\mathbb{Z}}$, are precisely the factorial prolongable languages over A. Moreover, the correspondence $\mathcal{X} \mapsto B(\mathcal{X})$ is a bijection between subshifts and factorial prolongable languages. Moreover, one has $\mathcal{X} \subseteq \mathcal{Y}$ if and only if $B(\mathcal{X}) \subseteq B(\mathcal{Y})$. In view of this bijection, symbolic dynamics may be regarded as a subject of formal language theory.

Semigroups appear in the study of formal languages via the concept of *recognition*. In the labeled graph of the figure above, letters a and b may be seen as the binary relations $a = \{(1,1)\}$ and $b = \{(1,2),(2,1)\}$. Let $S(\mathcal{E})$ be the semigroup of binary relations, on the vertices 1 and 2, generated by a and b. For example, ab is the binary relation $\{(1,2)\}$. The words in $B(\mathcal{E})$ are precisely the words that in $S(\mathcal{E})$ are not the empty relation $\emptyset$. Formally, given a semigroup homomorphism $\varphi\colon A^+ \to S$, a language $L \subseteq A^+$ is *recognized* by φ if $L = \varphi^{-1}(P)$ for some subset P of S. Note that $B(\mathcal{E})$ is recognized by the homomorphism $\varphi\colon \{a,b\}^+ \to S(\mathcal{E})$ such that $\varphi(a) = \{(1,1)\}$ and $\varphi(b) = \{(1,2),(2,1)\}$, since $B(\mathcal{E}) = \varphi^{-1}(S(\mathcal{E}) \setminus \{\emptyset\})$.

A language over A is *recognized by the semigroup* S when recognized by a homomorphism from A^+ into S. It is said to be *recognizable* if it is recognized by a finite semigroup. Recognizable languages constitute one of the main classes of languages, as they describe "finite-like" properties of words, captured by finite devices. Frequently the devices are finite automata, which are labeled graphs with a distinguished set of initial vertices and final vertices. These devices recognize the words labeling the paths from the initial to the final vertices. Recognition by a finite automaton is the same as recognition by a finite semigroup, because in fact an automaton may be seen as a semigroup with generators acting on its vertices.

Another reason why recognizable languages matter is Kleene's theorem (1956) [22], stating that the recognizable languages of A^+, with A finite, are precisely the *rational* languages of A^+, that is, the languages which can be obtained from subsets of A by applying finitely many times the Boolean operations, concatenation of languages, and the operation that associates to each nonempty language L the subsemigroup L^+ of A^+ generated by L. The rational languages obtainable using only the first two of these three sets of operations, the *plus-free* languages*, are precisely the languages recognized by finite *aperiodic* semigroups [31]. This characterization, due to Schützenberger and dated from 1965, is one of the first important applications of semigroups to languages (for the reader unfamiliar with the concept: a semigroup is aperiodic if all its subgroups, i.e., subsemigroups that have a group structure, are trivial). Eilenberg, later on (1976), provided the framework for several results in the spirit of that of Schützenberger on aperiodic semigroups, by establishing a natural correspondence between *pseudovarieties of semigroups* (classes of finite semigroups closed under taking homomorphic images, subsemigroups and finitary products) and the types of classes of languages recognized by their semigroups, called *varietes of languages* [17].

4. Classification of subshifts

The correspondence $\mathcal{X} \mapsto B(\mathcal{X})$ provides ways of classifying subshifts in special classes with "static" definitions in terms of $B(\mathcal{X})$ that, from a semigroup theorist viewpoint, may be more convenient than the alternative definitions of a more "dynamical" flavor.

As a first example, consider the *irreducible* subshifts: these are the subshifts $\mathcal{X}$ such that, for every $u, v \in B(\mathcal{X})$, one has $uwv \in B(\mathcal{X})$ for some word w. The dynamical characterization is that a subshift is irreducible when it has a dense forward orbit.

In the same spirit, a subshift $\mathcal{X}$ is *minimal* (for the inclusion) if and only if $B(\mathcal{X})$ is *uniformly recurrent*, the latter meaning that for every $u \in B(\mathcal{X})$, there is a natural number N_u such that u is a factor of every word of $B(\mathcal{X})$ of length N_u. Note that uniform recurrence implies irreducibility. A procedure

*Actually, Schützenberger's result is usually formulated in terms of finite aperiodic monoids and languages admitting the empty word, with *star-free* languages in place of *plus-free* languages.

for building minimal subshifts, with a semigroup-theoretic flavor that was useful for getting results mentioned in the final section, is as follows. Consider a *primitive substitution* $\varphi\colon A^+ \to A^+$, i.e., a semigroup endomorphism φ of A^+ such that every letter of A appears in $\varphi^n(a)$, for all $a \in A$ and all sufficiently large n: if φ is not the identity in an one-letter alphabet, then the language of factors of words of the form $\varphi^k(a)$, with $k \geq 1$ and $a \in A$, is factorial and prolongable, thus defining a subshift $\mathcal{X}_\varphi$, and in fact this subshift is minimal.

A subshift $\mathcal{X}$ is *sofic* when $B(\mathcal{X})$ is recognizable. Hence, the even subshift is sofic. Sofic and minimal subshifts are arguably the most important big realms of subshifts, with only periodic subshifts in the intersection. Every subshift $\mathcal{X}$ of $A^{\mathbb{Z}}$ is characterized by a set F of *forbidden blocks*, a language $F \subseteq A^+$ such that $x \in \mathcal{X}$ if and only if no element of F is a block of x. We write $\mathcal{X} = \mathcal{X}_F$ for such a set F. It turns out that $\mathcal{X}$ is sofic if and only if F can be chosen to be rational. A subshift $\mathcal{X}$ is of *finite type* if there is a finite set of forbidden blocks F such that $\mathcal{X} = \mathcal{X}_F$. The class of finite type subshifts is closed under conjugacy and is contained in the class of sofic subshifts. The inclusion is strict: the even subshift is not a finite type subshift.

The most important open problem in symbolic dynamics consists in classifying (irreducible) finite type subshifts up to conjugacy. A related problem is the classification of (irreducible) sofic subshifts up to *flow equivalence*. In few words, two subshifts are flow equivalent when they have equivalent mapping tori, a description that is somewhat technical, when made precise. Next is an alternative characterization (from [29]), more prone to a semigroup theoretical approach. Take $\alpha \in A$ and a letter $\diamond$ not in A. Consider the homomorphism $E_\alpha\colon A^+ \to (A \cup \{\diamond\})^+$ that replaces α by $\alpha\diamond$ and leaves the remaining letters of A unchanged. The symbol expansion of a subshift $\mathcal{X} \subseteq A^{\mathbb{Z}}$ with respect to α is the subshift whose blocks are factors of words in $E_\alpha(B(\mathcal{X}))$. Flow equivalence is the least equivalence relation between subshifts that contains the conjugacy relation and the symbol expansions. A symbol expansion on α represents a time dilation when reading α in a biinfinite string, thus flow equivalence preserves "shapes" of orbits, but not in a "rigid" way. Finite type subshifts have been completely classified up to flow equivalence [18]. The strictly sofic case remains open. In [10] one finds recent developments.

5. The Karoubi envelope of a subshift

Let L be a language over A. Two words u and v of A^+ are *syntactically equivalent* in L if they share the contexts in which they appear in words of L. Formally, the *syntactic congruence* $\equiv_L$ is defined by $u \equiv_L v$ if and only if the equivalence $xuv \in L \Leftrightarrow xvy \in L$ holds, for all (possibly empty) words x, y over A. The quotient $S(L) = A^+/{\equiv_L}$ is the *syntactic semigroup* of L. The quotient homomorphism $\eta_L \colon A^+ \to A^+/{\equiv_L}$ is minimal among the onto homomorphisms recognizing L: if the onto homomorphism $\varphi \colon A^+ \to S$ recognizes L, then there is a unique onto homomorphism $\theta \colon S \to S(L)$ such that the diagram

$$\begin{tikzcd} A^+ \arrow[r, "\varphi"] \arrow[dr, "\eta_L"'] & S \arrow[d, "\theta"] \\ & S(L) \end{tikzcd}$$

commutes. In particular, L is recognizable if and only if $S(L)$ is finite. More generally, L is recognized by a semigroup of a pseudovariety V if and only if $S(L)$ belongs to V. For example, a language is plus-free if and only if $S(L)$ is an aperiodic semigroup, in view of Schützenberger's characterization of plus-free languages. Since $S(L)$ is computable if L is adequately described (e.g, by an automaton), this gives an algorithm to decide if a rational language is plus-free. This example illustrates why syntactic semigroups and pseudovarieties are important for studying rational languages.

Let S be a semigroup, and denote by $E(S)$ the set of idempotents of S. The *Karoubi envelope* of S is the small category $\mathsf{Kar}(S)$ such that

- the set of objects is $E(S)$;
- an arrow from e to f is a triple (e, s, f) such $s \in S$ and $s = esf$;
- composition of consecutive arrows is given by $(e, s, f)(f, t, g) = (e, st, g)$ (we compose on the opposite direction adopted by category theorists);
- the unit at vertex e is (e, e, e).

This construction found an application in finite semigroup theory in the *Delay Theorem* [32]. Avoiding details, this result concerns a certain correspondence $\mathsf{V} \mapsto \mathsf{V}'$ between semigroup pseudovarieties, with one of the formulations of the Delay Theorem stating that a finite semigroup S belongs to V' if and only if $\mathsf{Kar}(S)$ is the quotient of a category admitting a faithful functor into a monoid in V. Interestingly, the variety of languages corresponding in Eilenberg's sense to V' is, roughly speaking, determined by the inverse images of languages recognized by semigroups of V via block maps of sliding block codes. Hence, it is natural to relate the Karoubi envelope with subshifts. This was done in the paper [15], of which we highlight some results in the next paragraphs.

The *syntactic semigroup* $S(\mathcal{X})$ *of a subshift* $\mathcal{X}$ is the syntactic semigroup of $B(\mathcal{X})$. One finds this object in some papers [20, 21, 8, 9, 12, 13, 11], namely for (strictly) sofic subshifts. Several invariants encoded in $S(\mathcal{X})$ were deduced. The *Karoubi envelope of* $\mathcal{X}$, denoted $\mathsf{Kar}(\mathcal{X})$, is the Karoubi envelope of $S(\mathcal{X})$. Conjugate subshifts do not need to have isomorphic syntactic semigroups, but the Karoubi envelope of a subshift is invariant in the sense of the following result from [15].

Theorem 5.1. *If* $\mathcal{X}$ *and* $\mathcal{Y}$ *are flow equivalent subshifts, then the categories* $\mathsf{Kar}(\mathcal{X})$ *and* $\mathsf{Kar}(\mathcal{Y})$ *are equivalent.*

For some classes of subshifts, the Karoubi envelope is of no use. For example, all irreducible finite type subshifts have equivalent Karoubi envelopes. But in the strictly sofic case, the Karoubi envelope does bring meaningful information, as testified by several examples given in [15]. We already mentioned the previous existence in the literature of several (flow equivalence) invariants encoded in $S(\mathcal{X})$. It turns out that the Karoubi envelope is the best possible syntactic invariant for flow equivalence of sofic subshifts: indeed, the main result in [15], which we formulate precisely below, states that all syntactic invariants of flow equivalence of sofic subshifts are encoded in the Karoubi envelope. First, it is convenient to formalize what a syntactic flow invariant is. An equivalence relation ϑ on the class of sofic subshifts is: an *invariant of flow equivalence* if $\mathcal{X}\ \vartheta\ \mathcal{Y}$ whenever $\mathcal{X}$ and $\mathcal{Y}$ are flow equivalent; a *syntactic invariant* if $\mathcal{X}\ \vartheta\ \mathcal{Y}$ whenever $S(\mathcal{X})$ and $S(\mathcal{Y})$ are isomorphic; a *syntactic invariant of flow equivalence* if it satisfies the two former properties.

Theorem 5.2. *If* ϑ *is a syntactic invariant of flow equivalence of sofic subshifts and* $\mathcal{X}$ *and* $\mathcal{Y}$ *are sofic shifts such that* $\mathsf{Kar}(\mathcal{X})$ *is equivalent to* $\mathsf{Kar}(\mathcal{Y})$*, then* $\mathcal{X}\ \vartheta\ \mathcal{Y}$*.*

Outside the sofic realm, the Karoubi envelope was successfully applied in [15] to what is arguably an almost complete classification of the *Markov-Dyck subshifts*, a class of subshifts introduced by Krieger [23]. Loosely speaking, a Markov-Dyck subshift D_G is formed by biinfinite strings of several types of parentheses, subject to the usual parenthetic rules, and to additional restrictions defined by a graph G. The edges of G are the opening parentheses, and consecutive opening parentheses appearing in an element of D_G correspond to consecutive edges, with a natural symmetric rule for closing parentheses also holding. Flow invariance of $\mathsf{Kar}(D_G)$, together with a characterization of $S(D_G)$, implicit in [19], gives the following result (a different and independent proof appears in [24]).

Theorem 5.3. *Let G and H be finite graphs. If each vertex of G or of H has out-degree not equal to one and in-degree at least one, then D_G and D_H are flow equivalent if and only if G and H are isomorphic.*

6. Free profinite semigroups

We already looked at the importance of (pseudovarieties of) finite semigroups in the study of (varieties of) rational languages. It is well known that free algebras (e.g., free groups, free Abelian groups, free semigroups, etc.) are crucial for the study of varieties of algebras, but for pseudovarieties, a difficulty arises: there is no universal object within the category of *finite* semigroups. To cope with this difficulty, an approach successfully followed by semigroup theorists, since the 1980's, was to enlarge the class of finite semigroups, by considering profinite semigroups. We pause to define the latter, giving [4] as a supporting reference.

A *profinite semigroup* is a compact semigroup (i.e., one with a compact Hausdorff topology for which the semigroup operation is continuous) that is *residually finite*, in the sense that every pair of distinct elements s, t of S admits a continuous homomorphism φ from S onto a finite semigroup F such that $\varphi(s) \neq \varphi(t)$, where finite semigroups get the discrete topology.

Assuming A is finite, consider in A^+ the metric $d(u,v) = 2^{-r(u,v)}$ such that $r(u,v)$ is the least possible size of the image of a homomorphism $\psi\colon A^+ \to S$ satisfying $\psi(u) \neq \varphi(v)$. The completion $\widehat{A^+}$ of A^+ with respect to d is a profinite semigroup. Moreover, each map $\varphi\colon A \to S$ from A into a profinite semigroup S has a unique extension to a continuous homomorphism $\widehat{\varphi}\colon \widehat{A^+} \to S$. Hence, $\widehat{A^+}$ is the *free profinite semigroup generated by A*. The next theorem gives a glimpse of why free profinite semigroups matter [1]. This theorem identifies the free profinite semigroup as the Stone dual of the Boolean algebra of recognizable languages.

Theorem 6.1. *The recognizable languages of A^+, are the traces in A^+ of the clopen subsets of $\widehat{A^+}$: if $L \subseteq A^+$ is recognizable, then $\overline{L}$ is clopen in $\widehat{A^+}$, and, conversely, if K is clopen in $\widehat{A^+}$, then $K \cap A^+$ is recognizable.*

The elements of $\widehat{A^+}$ constitute a sort of generalization of the words in A^+, and for that reason they are often named *pseudowords*. The elements in $\widehat{A^+}\, n\, A^+$ are the *infinite* pseudowords over A. While the algebraic-topological structure of A^+ is poor, that of $\widehat{A^+}$ is very rich: for example, A^+ has no subgroups, while $\widehat{A^+}$ contains all finitely generated free profinite groups when $|A| \geq 2$, and actually many more groups [30]. The structure of $\widehat{A^+}$ is nowadays less mysterious than it was fifteen years ago, symbolic dynamics having been very useful for achieving that. Our goal until the end of the text is to give examples of such utility.

Most connections between symbolic dynamics and free profinite semigroups developed over Almeida's idea of considering, for each subshift $\mathcal{X}$ of $A^{\mathbb{Z}}$, the topological closure $\overline{B(\mathcal{X})}$ of $B(\mathcal{X})$ in $\widehat{A^+}$ [2, 4].

In a semigroup S, the quasi-order $\leq_{\mathcal{J}}$ is defined by $s \leq_{\mathcal{J}} t$ if and only if t is a factor of s. The equivalence relation on S induced by $\leq_{\mathcal{J}}$ is denoted by $\mathcal{J}$. By standard compactness arguments, when $\mathcal{X}$ is an irreducible subshift there is a $\leq_{\mathcal{J}}$-minimum $\mathcal{J}$-class of $\widehat{A^+}$ among the $\mathcal{J}$-classes contained in $\overline{B(\mathcal{X})}$ (equivalently, intersecting $\overline{B(\mathcal{X})}$), as explained in [14]. This $\mathcal{J}$-class is denoted $J(\mathcal{X})$. The proof of the existence of $J(\mathcal{X})$ also entails that it is a *regular* $\mathcal{J}$-class, that is, one that contains idempotents. One has $J(\mathcal{X}) = J(\mathcal{Y})$ if and only if $\mathcal{X} = \mathcal{Y}$, and so $J(\mathcal{X})$ contains all information about $\mathcal{X}$. Something more holds: one has $\mathcal{X} \subseteq \mathcal{Y}$ if and only if $J(\mathcal{Y}) \leq_{\mathcal{J}} J(\mathcal{X})$. For the next statement, have in mind that an infinite pseudoword u of $\widehat{A^+}$ is a *$\leq_{\mathcal{J}}$-maximal infinite pseudoword* if every factor of u either belongs to A^+ or is $\mathcal{J}$-equivalent to u.

Theorem 6.2 ([3])**.** *An element u of $\widehat{A^+}$ is a $\mathcal{J}$-maximal infinite pseudoword if and only if $u \in J(\mathcal{X})$ for some minimal subshift $\mathcal{X}$ of $A^{\mathbb{Z}}$.*

The next theorem states that, in a natural sense, $\widehat{A^+}$ is very "large" and very "high" (a weaker version appears in [16], with a harder proof). Its proof is a good example of the potential of symbolic dynamics in the study of free profinite semigroups. A *regular* pseudoword is one that is $\mathcal{J}$-equivalent to an idempotent.

Theorem 6.3. *Let A be an alphabet with at least two letters. For the relation $<_{\mathcal{J}}$ in $\widehat{A^+}$, there are both chains and anti-chains with $2^{\aleph_0}$ regular elements.*

Proof: On the one hand, $A^{\mathbb{Z}}$ contains $2^{\aleph_0}$ minimal subshifts (cf. [27, Chapter 2]), and minimal subshifts clearly form an anti-chain for the inclusion. On the other hand, $A^{\mathbb{Z}}$ contains a chain of $2^{\aleph_0}$ irreducible subshifts [34, Section 7.3]. Hence, the theorem follows immediately from the equivalence $\mathcal{X} \subseteq \mathcal{Y} \Leftrightarrow J(\mathcal{Y}) \leq_{\mathcal{J}} J(\mathcal{X})$ for irreducible subshifts. ■

Since $J(\mathcal{X})$ is regular, it contains a maximal subgroup, which is a profinite group for the induced topology. Because all maximal subgroups in a regular $\mathcal{J}$-class are isomorphic, we may consider the abstract profinite maximal subgroup $G(\mathcal{X})$ of $J(\mathcal{X})$. The group $G(\mathcal{X})$ was called in [5] the *Schützenberger group of* $\mathcal{X}$. This group is a conjugacy invariant (see [12] for a proof). We collect other facts about $G(\mathcal{X})$.

- In [3] it was shown that $G(\mathcal{X})$ is a free profinite group of rank k if $\mathcal{X}$ is a subshift over a k-letter alphabet that belongs to an extensively studied class of minimal subshifts, called Arnoux-Rauzy subshifts. On the other hand, also in [3], it was shown that the substitution φ defined by $\varphi(a) = ab$ and $\varphi(b) = a^3b$ is such that $G(\mathcal{X}_\varphi)$ is not a free profinite group. This was the first example of a non-free maximal subgroup of a free profinite semigroup. More generally, profinite presentations for $G(\mathcal{X}_\psi)$ were obtained in [5], for all primitive substitutions ψ.
- If $\mathcal{X}$ is a nonperiodic irreducible sofic subshift, then $G(\mathcal{X})$ is a free profinite group of rank $\aleph_0$ [14].
- A sort of geometrical interpretation for $G(\mathcal{X})$ was obtained in [6], when $\mathcal{X}$ is minimal. It was shown that $G(\mathcal{X})$ is an inverse limit of the profinite completions of the fundamental groups of a certain sequence of finite graphs. The n-th graph in this sequence captures information about the blocks of $\mathcal{X}$ with length $2n + 1$.

While free profinite semigroups are interesting *per se*, it is worthy mentioning that some of the achievements on the Schützenberger group of a minimal subshift were used in the technical report [25] to obtain results on code theory, whose statement may appear to have nothing to do with profinite semigroups. These results were incorporated and further developed in [7].

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