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ON HEREDITARY PROPERTIES OF EXTREMALLY DISCONNECTED FRAMES AND NORMAL FRAMES

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Dedicated to Aleš Pultr on the occasion of his eightieth birthday

Abstract: Hereditary extremal disconnectedness of frames and its equivalent form of complete extremal disconnectedness are the topic of this paper. We study them in parallel with the corresponding normality properties of frames. Among several characterizations, we show that a frame is hereditarily normal [resp., hereditarily extremally disconnected] if and only if all its open *and* dense sublocales [resp., closed *and* dense sublocales] are normal [resp., extremally disconnected]. Some of the presented results are new for the traditional topological spaces. Furthermore, we provide such a general setting that permits us to treat several variants of the concepts under study in a unified way.

Keywords: Frame, locale, sublocale, normality, extremal disconnectedness, hereditary normality, complete normality, hereditary extremal disconnectedness, complete extremal disconnectedness, density.

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1. Introduction and initial observations

The purpose of this paper is to study hereditary extremal disconnectedness of frames and its equivalent form of complete extremal disconnectedness vis-à-vis to hereditary normality.

There is a specific duality between topological normality (N) and extremal disconnectedness (ED) that relies on the circumstance that a space is N [resp., ED] if and only if every two disjoint *closed* [resp., *open*] sets are separated by disjoint *open* [resp., *closed*] sets. We refer to [20, 21, 22], and especially to the recent paper [16], for more details. A similar duality holds in a pointfree context (cf. Definition 1.4).

Both N and ED fail to be hereditary properties. They are, respectively, closed-hereditary and open-hereditary — i.e. each closed [resp., open] subspace of a normal space [resp., extremally disconnected space] is normal [resp., extremally disconnected]. The topological cases are well-known. For

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the pointfree cases we refer to [16] and [14]. It may be worth mentioning that hereditary ED is useful in modal logic (cf. [2]).

It is well known that a topological space X is hereditarily normal (HN) if and only if it is *completely normal* (CN) — i.e. if every two separated subsets of X are separated by disjoint open subsets of X (recall that $A, B \subseteq X$ are *separated* if $\overline{A} \cap B = \varnothing = A \cap \overline{B}$).

Bearing in mind the duality between N and ED, one can consider the concept of a *completely extremally disconnected* (CED) space as one in which every two separated subsets are separated by disjoint closed subsets. This is of course equivalent to the statement that every two separated sets have disjoint closures. Without being named, such a concept was investigated in [3] (also cf. [7]) and proved to be equivalent to hereditary ED. We record the following:

Proposition 1.1. ([3]). *The following are equivalent for any topological space X:*

(1) *X is hereditarily extremally disconnected.*
(2) *Any two separated subsets of X have disjoint closures.*
(3) *X is completely extremally disconnected.*

A characterization of CN solely in terms of the lattice $\mathcal{O}(X)$ of all open sets of X is given in [27]:

Proposition 1.2. ([27, 10]) *The following are equivalent for any topological space X:*

(1) *X is completely normal.*
(2) *For every open sets $A, B \subseteq X$, there exist open sets $U, V \subseteq X$ such that $U \cap V = \varnothing$, $A \cup U \supseteq B$ and $B \cup V \supseteq A$.*

When phrased in terms of the dual lattice $\mathcal{O}(X)^{op}$, condition (2) of Proposition 1.2 becomes a characterization of CED. Indeed, the following holds:

Proposition 1.3. *The following are equivalent for any topological space X:*

(1) *X is completely extremally disconnected.*
(2) *For any open sets $A, B \subseteq X$, $\overline{A - B} \cap \overline{B - A} = \varnothing$.*
(3) *For any open sets $A, B \subseteq X$, there exist open sets $U, V \subseteq X$ such that $U \cup V = X$, $A \cap U \subseteq B$ and $B \cap V \subseteq A$.*

Proof: (1)⇒(2): Let A and B be open in X. The result follows immediately from Proposition 1.1 and the fact that $A - B$ and $B - A$ are separated.

Indeed, $(A-B)\cap\overline{B-A}\subseteq A\cap\overline{X-A}=\varnothing$ and dually.

(2)⇒(3): Let A and B be open in X and take $U=X-\overline{A-B}$ and $V=X-\overline{B-A}$. By hypothesis $\overline{A-B}\cap\overline{B-A}=\varnothing$ and thus $U\cup V=X$. On the other hand, $A-B\subseteq\overline{A-B}=X-U$ and so $A\cap U\subseteq B$. Dually, we have that $B\cap V\subseteq A$.

(3)⇒(1): We prove that X satisfies condition (2) in Proposition 1.1. Let $A,B\subseteq X$ be separated. Consider $U=X-\overline{A}$ and $V=X-\overline{B}$. By hypothesis there exist open sets $U',V'\subseteq X$ such that $U'\cup V'=X$, $U\cap U'\subseteq V$ and $V\cap V'\subseteq U$. Since $B\subseteq X-\overline{A}=U$ and $B\subseteq X-V$ it follows that

$$B\subseteq U-V\subseteq U-(U\cap U')=U-U'\subseteq X-U'.$$

Dually, $\overline{A}\subseteq X-V'$. Hence $\overline{A}\cap\overline{B}\subseteq X-(U'\cup V')=\varnothing$. ■

Following [30], a topological property P is said to be *lattice-invariant* if, when X has P and the lattices $\mathcal{O}(X)$ and $\mathcal{O}(Y)$ are isomorphic, the space Y has property P too. In particular, Propositions 1.2 and 1.3 show that both complete N and complete ED are lattice-invariant.

Moreover, Propositions 1.2 and 1.3 also indicate how to formulate complete N and complete ED in an arbitrary (bounded) lattice. We thus have the following definitions:

Definitions 1.4. Let L be a (bounded) lattice with bounds 0 and 1. Then:

(1) L is *normal* if

$$\forall a,b\in L\ \ [a\vee b=1\ \Rightarrow\ \exists u,v\in L\colon u\wedge v=0,\ a\vee u=1=b\vee v]. \tag{N}$$

(2) L is *extremally disconnected* if the dual lattice L^{op} is normal — i.e.

$$\forall a,b\in L\ \ [a\wedge b=0\ \Rightarrow\ \exists u,v\in L\colon u\vee v=1,\ a\wedge u=0=b\wedge v]. \tag{ED}$$

(3) L is *completely normal* if

$$\forall a,b\in L\ \ \exists u,v\in L\colon\ u\wedge v=0,\ a\vee u\geq b,\ \ b\vee v\geq a. \tag{CN}$$

(4) L is *completely extremally disconnected* if L^{op} is completely normal — i.e.

$$\forall a,b\in L\ \ \exists u,v\in L\colon\ u\vee v=1,\ a\wedge u\leq b,\ \ b\wedge v\leq a. \tag{CED}$$

Remark 1.5. Condition (CN) appears in [1, 2.4] under the name of *relative normality*). Note that (CN) is really stronger than (N), for if $a\vee b=1$, then $a\vee u\geq b$ and $b\vee v\geq a$ imply $a\vee u=b\vee v=1$. By a dual argument, (CED) implies (ED).

Fact 1.6. *Any completely normal and extremally disconnected lattice is completely extremally disconnected. Dually, any normal and completely extremally disconnected lattice is completely normal.*

Proof: Let $a, b \in L$. By complete normality there exist $u, v \in L$ such that $u \wedge v = 0$, $a \vee u \geq b$ and $b \vee v \geq a$. Since L is extremally disconnected there exist $x, y \in L$ such that $x \vee y = 1$, $x \wedge v = 0$ and $y \wedge u = 0$. Hence $a \wedge x \leq (b \vee v) \wedge x \leq b$ and $b \wedge y \leq (a \vee u) \wedge y \leq a$. ■

Fact 1.7. *The following are equivalent for any lattice* L:

(1) L *is completely normal and extremally disconnected.*
(2) L *is normal and completely extremally disconnected.*
(3) L *is completely normal and completely extremally disconnected.*

In particular, if $L = \mathcal{O}(X)$, then X is HN and ED if and only if X is N and HED (cf. [12, 6R] and [23]) if and only if X is HN and HED.

2. Some background on pointfree topology

We recall some basic notions and facts about frames and locales. For further information see [25].

The category Frm of *frames* has as objects those complete lattices L in which

$$a \wedge \bigvee B = \bigvee\{a \wedge b \mid b \in B\} \tag{2.1}$$

for all $a \in L$ and $B \subseteq L$. Morphisms, called *frame homomorphisms*, are those maps between frames that preserve arbitrary joins (in particular, the top element 1) and finite meets (in particular, the bottom element 0).

The lattice $\Omega(X)$ of open subsets of a space X is the typical example of a frame. For any continuous map $f\colon X \to Y$, the mapping $\Omega(f)\colon \Omega(Y) \to \Omega(X)$ defined by $\Omega(f)(U) = f^{-1}[U]$ is a frame homomorphism. The category of *locales* is the opposite category of Frm.

Remark 2.1. Note that in the case of a frame L, conditions (CN) and (CED) are, by Propositions 1.2 and 1.3, both conservative extensions of their topological counterparts — i.e. a topological space X is CN (resp., ED) if and only if the frame $L = \mathcal{O}(X)$ satisfies CN (resp., CED).

With L a frame and $a \in L$, the map $a \wedge (\cdot)\colon L \to L$ preserves arbitrary joins and thus has a right (Galois) adjoint $a \to (\cdot)\colon L \to L$ determined by

$$a \wedge c \leq b \ \text{ iff } \ c \leq a \to b. \tag{2.2}$$

Thus, $a \to b = \bigvee\{c \in L \mid a \wedge c \le b\}$. The *pseudocomplement* of $a \in L$ is $a^* = a \to 0$. Then: $a \wedge a^* = 0$, $a \le a^{**}$, $a^{***} = a^*$ and $(\bigvee A)^* = \bigwedge_{a\in A} a^*$ for all $A \subseteq L$ (but the dual de Morgan law is not true in general; we only have $(\bigwedge A)^* \ge \bigvee_{a\in A} a^*$). In particular, $(\cdot)^*$ is order-reversing.

An $S \subseteq L$ is a *sublocale* of L if, for any $A \subseteq S$, $a \in L$ and $b \in S$, we have $\bigwedge A \in S$ (in particular, $1 \in S$) and $a \to b \in S$. The set $\mathcal{S}(L)$ of all sublocales of L forms a *coframe* (i.e. a complete lattice satisfying the dual of (2.1)) under inclusion. Arbitrary infima coincide with intersections, $\{1\}$ is the bottom element and L is the top element. Regarding suprema, there is the formula

$$\bigvee_{i\in I} S_i = \{\bigwedge A \mid A \subseteq \bigcup_{i\in I} S_i\} \tag{2.3}$$

for every $\{S_i \mid i \in I\} \subseteq \mathcal{S}(L)$.

Since $\mathcal{S}(L)$ is the dual of a complete Heyting algebra, it has co-pseudocomplements, usually called *pseudodifferences* or *remainders* (see [11] for more information). We shall denote the pseudodifference of an S in $\mathcal{S}(L)$ by $L\smallsetminus S$ and will use the formula

$$L\smallsetminus S = \bigcap\{R \in \mathcal{S}(L) \mid R \vee S = L\}.$$

In case S is complemented, the pseudodifference $L\smallsetminus S$ is the complement and in that case we simply refer to it as S^{c}.

For any $a \in L$, the sets

$$\mathfrak{c}(a) = {\uparrow} a \quad \text{and} \quad \mathfrak{o}(a) = \{a \to x \mid x \in L\} = \{x \in L \mid a \to x = x\}$$

are special sublocales of L referred to, respectively as *closed* and *open* sublocales. They are complements of each other in $\mathcal{S}(L)$. Furthermore, the map $a \mapsto \mathfrak{o}(a)$ is a lattice embedding $L \hookrightarrow \mathcal{S}(L)$ that preserves arbitrary joins. Therefore, denoting by $\mathfrak{o}[L]$ the sublattice of $\mathcal{S}(L)$ consisting of all open sublocales, L and $\mathfrak{o}[L]$ are isomorphic frames.

On the other hand,

$$a \le b \ \text{ iff } \ \mathfrak{c}(a) \supseteq \mathfrak{c}(b), \quad \mathfrak{c}(a) \vee \mathfrak{c}(b) = \mathfrak{c}(a \wedge b) \ \text{ and } \ \bigcap_{i\in I} \mathfrak{c}(a_i) = \mathfrak{c}(\bigvee_{i\in I} a_i).$$

In particular, this means that the set $\mathfrak{c}[L]$ of closed sublocales of L is a sub-coframe of $\mathcal{S}(L)$.

In the following, given any $A \subseteq L$, we shall denote by $\mathfrak{o}[A]$ and $\mathfrak{c}[A]$ the sets

$$\mathfrak{o}[A] = \{\mathfrak{o}(a) \mid a \in A\} \quad \text{and} \quad \mathfrak{c}[A] = \{\mathfrak{c}(a) \mid a \in A\}.$$

Given a sublocale T of L, its *closure* is defined by

$$\overline{T} = \bigcap\{\mathfrak{c}(a) \mid T \subseteq \mathfrak{c}(a)\} = \mathfrak{c}(\bigwedge T).$$

In particular, $\overline{\mathfrak{o}(a)} = \mathfrak{c}(a^*)$.

A sublocale T is *dense* whenever $\overline{T} = L$, that is, $0 = \bigwedge T \in T$. An important distinctive feature of pointfree topology, known as the *Isbell's Density Theorem* [25], is the existence in any frame of a least dense sublocale, namely its *Booleanization*

$$\mathfrak{B}_L = \{x \to 0 \mid x \in L\}.$$

For any sublocale T of L, $\mathcal{S}(T) \subseteq \mathcal{S}(L)$ (note that, furthermore, if S and T are sublocales of L with $S \subseteq T$, then $S \in \mathcal{S}(T)$). Non-empty infima in $\mathcal{S}(T)$, being given by intersection, coincide with infima in $\mathcal{S}(L)$. On the other hand, formula (2.3) shows that arbitrary suprema in $\mathcal{S}(T)$ also coincide with suprema in $\mathcal{S}(L)$: for any $A_i \in \mathcal{S}(T)$, $i \in I$, we have

$$\bigvee_{i\in I}^{\mathcal{S}(L)} A_i = \{\bigwedge R \mid R \subseteq \bigcup_{i\in I} A_i\} \subseteq T$$

and thus

$$\bigvee_{i\in I}^{\mathcal{S}(L)} A_i = \bigvee_{i\in I}^{\mathcal{S}(T)} A_i. \tag{2.4}$$

Given a sublocale T of L let $\rho_T \colon L \to T$ be the surjective frame homomorphism defined by

$$\rho_T(a) = \bigwedge\{t \in T \mid t \geq a\} \in T. \tag{2.5}$$

It is easy to verify that the open sublocales of T (that we denote as $\mathfrak{o}_T(t)$, $t \in T$) are precisely the intersections $\mathfrak{o}(a) \cap T$ ($a \in L$): indeed, $\mathfrak{o}_T(t) = \mathfrak{o}(t) \cap T$ for any $t \in T$, and for any $a \in L$, $\mathfrak{o}(a) \cap T = \mathfrak{o}_T(\rho_T(a))$. Analogously for the closed sublocales. For each $A \subseteq T$ we shall denote by $\mathfrak{o}_T[A]$ and $\mathfrak{c}_T[A]$ the sets

$$\mathfrak{o}_T[A] = \{\mathfrak{o}_T(a) \mid a \in A\} \quad \text{and} \quad \mathfrak{c}_T[A] = \{\mathfrak{c}_T(a) \mid a \in A\}.$$

3. Complete extremal disconnectedness

Complete normality formulated in frames appeared for the first time in the literature with [18] (see also [28]), in the following form: a frame is completely normal if every pair S, T of separated sublocales of L (i.e. such that $S \cap \overline{T} = \{1\} = \overline{S} \cap T$) is separated by open sublocales, that is, there exist open sublocales U and V of L such that $U \cap V = \{1\}$, $S \subseteq U$ and $T \subseteq V$. As proved in [10, Proposition 3.3], this is equivalent to condition (CN) above. In what follows we will show a dual characterization for condition (CED).

We start with the following easy characterization:

Proposition 3.1. *A frame L is completely extremally disconnected if and only if it satisfies the strong De Morgan's law*

$$(a \to b) \vee (b \to a) = 1 \tag{SDM}$$

for all $a, b \in L$.

Proof: Sufficiency is obvious since $a \wedge (a \to b) = a \wedge b$ for any $a, b \in L$. Regarding necessity, let $a, b \in L$. By hypothesis, there are $u, v \in L$ such that $u \vee v = 1$, $a \wedge u \leq b$ and $b \wedge v \leq a$. Hence $1 = u \vee v \leq (a \to b) \vee (b \to a)$ by adjunction (2.2). ■

Next characterization is the pointfree counterpart of a characterization of hereditary ED presented in [2, Proposition 2.1]. In order to prove it, we need first some technical observations concerning Booleanizations of closed sublocales.

First, it should be remarked that any sublocale S of a frame L is a frame itself with the same meets as in L, and since the Heyting operation $\to$ depends on the meet structure only, with the same Heyting operation. Consequently, for each $a \in L$, the Booleanization of $\mathfrak{c}(a)$ is given by

$$\mathfrak{B}_{\mathfrak{c}(a)} = \{x \to a \mid x \in \mathfrak{c}(a)\} = \{x \to a \mid x \in L\}.$$

Note that this is the smallest sublocale of L containing a. Hence $\mathfrak{B}_{\mathfrak{c}(a)} = \mathfrak{B}_{\mathfrak{c}(b)}$ if and only if $a = b$.

Lemma 3.2. *For any $a, b \in L$ we have:*

(a) $\mathfrak{B}_{\mathfrak{c}(a \to b)} = \mathfrak{o}(a) \cap \mathfrak{B}_{\mathfrak{c}(b)}$.

(b) $\mathfrak{c}(a) \cap \mathfrak{B}_{\mathfrak{c}(b)} = \{1\}$ *iff* $a \to b = b$ *iff* $b \in \mathfrak{o}(a)$.

(c) $\mathfrak{c}(a \to b) = \overline{\mathfrak{o}(a) \cap \mathfrak{c}(b)}$.

Proof: (a) If $y = x \to (a \to b) \in \mathfrak{B}_{\mathfrak{c}(a \to b)}$ then, using some basic properties of the Heyting operator, we may conclude that

$$y = a \to (x \to b) \in \mathfrak{o}(a) \quad \text{and} \quad y = (x \wedge a) \to b \in \mathfrak{B}_{\mathfrak{c}(b)};$$

conversely, if $y \in \mathfrak{o}(a) \cap \mathfrak{B}_{\mathfrak{c}(b)}$ then $y = a \to y = x \to b$ for some $x \in L$, hence

$$y = a \to (x \to b) = x \to (a \to b) \in \mathfrak{B}_{\mathfrak{c}(a \to b)}.$$

(b) It follows immediately from (1) since

$$\mathfrak{c}(a)\cap\mathfrak{B}_{\mathfrak{c}(b)}=\{1\} \text{ iff } \mathfrak{B}_{\mathfrak{c}(b)}\subseteq\mathfrak{o}(a) \text{ iff } \mathfrak{B}_{\mathfrak{c}(a\to b)}=\mathfrak{B}_{\mathfrak{c}(b)} \text{ iff } a\to b=b \text{ iff } b\in\mathfrak{o}(a).$$

(c) From $a\to b\in\mathfrak{o}(a)\cap\mathfrak{c}(b)$ it follows that $\mathfrak{c}(a\to b)\subseteq\overline{\mathfrak{o}(a)\cap\mathfrak{c}(b)}$. On the other hand, $\mathfrak{c}(b)\subseteq\mathfrak{c}(a\wedge b)=\mathfrak{c}(a)\vee\mathfrak{c}(a\to b)$ and thus $\mathfrak{o}(a)\cap\mathfrak{c}(b)\subseteq\mathfrak{c}(a\to b)$. Hence $\overline{\mathfrak{o}(a)\cap\mathfrak{c}(b)}\subseteq\mathfrak{c}(a\to b)$. ■

Proposition 3.3. *The following are equivalent for a frame L:*

(1) *L is completely extremally disconnected.*
(2) *For any open sublocales A,B of L, $\overline{A\cap B^{\mathsf{c}}}\cap\overline{B\cap A^{\mathsf{c}}}=\{1\}$.*
(3) *For any open sublocales A,B of L, there exist open sublocales U,V such that $U\vee V=L$, $A\cap U\subseteq B$ and $B\cap V\subseteq A$.*

Proof: (1)⇒(2): Let $A=\mathfrak{o}(a)$ and $B=\mathfrak{o}(b)$ be open sublocales. Then it follows from Lemma 3.2 (c) and (SDM) that

$$\overline{A\cap B^{\mathsf{c}}}\cap\overline{B\cap A^{\mathsf{c}}}=\mathfrak{c}(a\to b)\cap\mathfrak{c}(b\to a)=\mathfrak{c}((a\to b)\vee(b\to a))=\{1\}.$$

(2)⇒(3): Let $A=\mathfrak{o}(a)$ and $B=\mathfrak{o}(b)$ and consider $U=\mathfrak{o}(a\to b)$ and $V=\mathfrak{o}(b\to a)$. By hypothesis (and Lemma 3.2 (c)) we conclude that

$$\{1\}=\overline{A\cap B^{\mathsf{c}}}\cap\overline{B\cap A^{\mathsf{c}}}=\mathfrak{c}(a\to b)\cap\mathfrak{c}(b\to a)$$

and thus $U\vee V=\mathfrak{o}(a\to b)\vee\mathfrak{o}(b\to a)=L$. On the other hand,

$$A\cap U=\mathfrak{o}(a)\cap\mathfrak{o}(a\to b)=\mathfrak{o}(a\wedge b)\subseteq B.$$

Dually, we have that $B\cap V\subseteq A$.

(3)⇒(1): Let $a,b\in L$. By hypothesis there exist open sublocales $U=\mathfrak{o}(u)$ and $V=\mathfrak{o}(v)$ such that $U\vee V=L$, $\mathfrak{o}(a)\cap U\subseteq\mathfrak{o}(b)$ and $\mathfrak{o}(b)\cap V\subseteq\mathfrak{o}(a)$. Then $u,v\in L$ satisfy (CED). ■

Now, recall that a sublocale S of L is *nowhere dense* in L if $S\cap\mathfrak{B}_L=\{1\}$ ([19]). We say that two sublocales S and T of L are *almost disjoint* if $S\cap T$ is a nowhere dense sublocale in both sublocales S and T, i.e.

$$S\cap T\cap\mathfrak{B}_S=\{1\}=S\cap T\cap\mathfrak{B}_T.$$

Next characterization is the pointfree extension of a characterization of completely ED spaces from [3].

Proposition 3.4. *The following are equivalent for a frame L:*

(1) *L is completely extremally disconnected.*
(2) *Any two almost disjoint closed sublocales of L are disjoint.*

(3) *Any two separated sublocales of L have disjoint closures.*

Proof: (1)⇒(2): Let $\mathfrak{c}(a)$, $\mathfrak{c}(b)$ be a pair of almost disjoint closed sublocales. This means that

$$\mathfrak{c}(b)\cap\mathfrak{B}_{\mathfrak{c}(a)}=\mathfrak{c}(a)\cap\mathfrak{c}(b)\cap\mathfrak{B}_{\mathfrak{c}(a)}=\{1\}$$

and

$$\mathfrak{c}(a)\cap\mathfrak{B}_{\mathfrak{c}(b)}=\mathfrak{c}(a)\cap\mathfrak{c}(b)\cap\mathfrak{B}_{\mathfrak{c}(b)}=\{1\}.$$

Therefore, by Lemma 3.2 (b), $a\to b=b$ and $b\to a=a$. Finally, by (SDM) we conclude that $a\vee b=1$, that is, $\mathfrak{c}(a)\cap\mathfrak{c}(b)=\{1\}$.

(2)⇒(3): Let S, T be a pair of separated sublocales and $\overline{S}=\mathfrak{c}(a)$ and $\overline{T}=\mathfrak{c}(b)$. Then $\overline{S}\cap T=\{1\}$, that is, $T\subseteq\mathfrak{o}(a)$. Moreover, since $b=\bigwedge T\in T\subseteq\mathfrak{o}(a)$, it follows from Lemma 3.2 (b) that

$$\overline{S}\cap\overline{T}\cap\mathfrak{B}_{\overline{T}}\subseteq\overline{S}\cap\mathfrak{B}_{\overline{T}}=\mathfrak{c}(a)\cap\mathfrak{B}_{\mathfrak{c}(b)}=\{1\}.$$

Analogously, $\overline{S}\cap\overline{T}\cap\mathfrak{B}_{\overline{S}}=\{1\}$. Consequently, $\overline{S}$ and $\overline{T}$ are almost disjoint and hence disjoint, by hypothesis.

(3)⇒(1): Let $a,b\in L$ and $S=\mathfrak{o}(a)\cap\mathfrak{c}(b)$ and $T=\mathfrak{o}(b)\cap\mathfrak{c}(a)$. Then $\overline{S}\cap T\subseteq\mathfrak{c}(b)\cap\mathfrak{o}(b)=\{1\}$ and $S\cap\overline{T}\subseteq\mathfrak{o}(a)\cap\mathfrak{c}(a)=\{1\}$, hence S and T are separated. By the hypothesis and Lemma 3.2 (c), $\overline{S}=\mathfrak{c}(a\to b)$ and $\overline{T}=\mathfrak{c}(b\to a)$ are disjoint, that is, $(a\to b)\vee(b\to a)=1$. ■

4. Complete extremal disconnectedness vs. complete normality

The duality expressed in 1.4 may be formulated in a more extended setting permitting a simultaneous treatment of several variants of normality and extremal disconnectedness (as proposed in [16]). In what follows we shall speak about a lattice L being normal (resp. extremally disconnected) with respect to some fixed $A\subseteq L$. In this terminology, L is *A-normal* or *completely A-normal* (A-N or A-CN for short) if it satisfies condition (N) or (CN) of Definition 1.4 with $a,b,u,v\in A$. Analogously, L is *extremally A-disconnected* or *completely extremally A-disconnected* (A-ED or A-CED for short) if it satisfies condition (ED) or (CED) of Definition 1.4 with $a,b,u,v\in A$.

Examples 4.1. The standard example for A is L; then, the A-notions are just the standard notions. In the case of a frame L, the case we are most interested, there are other interesting examples (cf. [16]):

(1) $A = \operatorname{Reg} L$: in this case, A-normal = mildly normal and extremally A-disconnected = extremally disconnected.
(Recall that the *regular part* of L, denoted $\operatorname{Reg} L$, is the set of all regular elements of L, that is, the elements $a \in L$ such that $a^{**} = a$.)

(2) $A = \delta$-$\operatorname{Reg} L$: A-normal = δ-normal and extremally A-disconnected = extremally δ-disconnected.
(An element a of L is δ*-regular* whenever $a = \bigvee_{n\in\mathbb{N}} a_n$ with $a_n \prec a$, i.e., $a_n^* \vee a = 1$, and the set of all δ-regular elements is denoted by δ-$\operatorname{Reg} L$.)

(3) $A = \operatorname{Coz} L$: in this case, any frame is A-normal while the class of extremally A-disconnected frames is precisely the class of F-frames.
(The *cozero part* of L, denoted $\operatorname{Coz} L$, is the set of all cozero elements of L, that is, the elements $a \in L$ such that $a = \bigvee_n a_n$ for some $a_n \prec\!\prec a$, $n = 1, 2, \cdots$, where $x \prec\!\prec a$ expresses the familiar relation that x is really inside, or completely below, a. This is the largest interpolative relation contained in $\prec$.)

However, we cannot apply the dualizing process 1.4 directly in frames since the dual lattice of a frame is not a frame in general.

Yet, if all elements in A are complemented in L then, for the set A^{c} of complements of all elements in A, A^{c}-normality is precisely extremal A-disconnectedness while extremal A^{c}-disconnectedness is A-normality, and the two notions may be treated simultaneously.

Therefore the familiar result that the sublocale lattice $\mathcal{S}(L)$ is isomorphic to the coframe obtained by freely adjoining to L^{op} a complement for each $a \in L$ provides a way of dealing with the duality in frames ([16]). Indeed, it suffices to embed L in the coframe $\mathcal{S}(L)$ via the isomorphism

$$L \cong \mathfrak{o}[L] \subseteq \mathcal{S}(L).$$

Then L will be (completely) A-normal iff the *lattice* $\mathcal{S}(L)$ is (completely) $\mathfrak{o}[A]$-normal. But now, any element of $\mathfrak{o}[A]$ is complemented, hence L will be (completely) A-normal iff $\mathcal{S}(L)$ is (completely) extremally $\mathfrak{c}[A]$-disconnected.

Similarly, L will be (completely) extremally A-disconnected iff the lattice $\mathcal{S}(L)$ is (completely) extremally $\mathfrak{o}[A]$-disconnected, that is, (completely) $\mathfrak{c}[A]$-normal.

To illustrate this idea we have a first result that characterizes completely A-normal frames and completely extremally A-disconnected frames (for any $A \subseteq L$) at once, with a single proof. Before presenting it, we need to introduce some terminology and notation.

For $\mathcal{A} = \mathfrak{o}[A]$ or $\mathcal{A} = \mathfrak{c}[A]$ let $\mathcal{A}^{\mathsf{c}}$ be the set of complements of all elements of $\mathcal{A}$. We say that two sublocales S and T of L are $\mathcal{A}$-*separated* if there exist $F, G \in \mathcal{A}^{\mathsf{c}}$ such that

$$S \subseteq F,\ T \subseteq G \ \text{ and } \ S \cap G = \{1\} = T \cap F.$$

Further, we say that S and T are *separated by* $\mathcal{A}$-*sublocales* if there exist sublocales U and V in $\mathcal{A}$ such that

$$U \cap V = \{1\},\ S \subseteq U \ \text{ and } \ T \subseteq V.$$

Remark 4.2. We may speak further about the A-*closure* of a sublocale T, that is,

$$\mathrm{cl}_A(T) = \bigcap\{\mathfrak{c}(a) \mid T \subseteq \mathfrak{c}(a), a \in A\}$$

(for $A = L$ this is just the standard closure). Whenever A is closed under joins, $\mathrm{cl}_A(T)$ belongs to $\mathfrak{c}[A]$ and it is of course the smallest sublocale in $\mathfrak{c}[A]$ that contains T. Then it is easy to check that sublocales S and T are $\mathfrak{o}[A]$-separated if and only if

$$S \cap \mathrm{cl}_A(T) = \{1\} = \mathrm{cl}_A(S) \cap T.$$

The selection $A = L$ recovers this way the original definition in [18].

Proposition 4.3. *For any frame L and any $A \subseteq L$, the following are equivalent for $\mathcal{A} = \mathfrak{o}[A]$ or $\mathcal{A} = \mathfrak{c}[A]$:*

(1) *$\mathcal{S}(L)$ is completely $\mathcal{A}$-normal.*
(2) *Every pair of $\mathcal{A}$-separated sublocales of L is separated by $\mathcal{A}$-sublocales.*

Proof: (1)$\Rightarrow$(2): Let S and T be $\mathcal{A}$-separated. Then there exist $F, G \in \mathcal{A}^{\mathsf{c}}$ such that $S \subseteq F$, $T \subseteq G$ and $S \cap G = \{1\} = T \cap F$. Of course, $F^{\mathsf{c}}, G^{\mathsf{c}} \in \mathcal{A}$ and therefore, by hypothesis, there exist $U, V \in \mathcal{A}$ such that $U \cap V = \{1\}$, $G^{\mathsf{c}} \subseteq F^{\mathsf{c}} \vee U$ and $F^{\mathsf{c}} \subseteq G^{\mathsf{c}} \vee V$. Moreover, $\{1\} = S \cap G \supseteq S \cap F \cap U^{\mathsf{c}} = S \cap U^{\mathsf{c}}$, from which it follows that $S \subseteq U$, and $\{1\} = T \cap F \supseteq T \cap G \cap V^{\mathsf{c}} = T \cap V^{\mathsf{c}}$, from which it follows that $T \subseteq V$.
(2)$\Rightarrow$(1): Let $F, G \in \mathcal{A}$ and $S = F^{\mathsf{c}} \cap G$, $T = F \cap G^{\mathsf{c}}$. Clearly, $S \subseteq F^{\mathsf{c}}$, $T \subseteq G^{\mathsf{c}}$ and $S \cap G^{\mathsf{c}} = \{1\} = F \cap T^{\mathsf{c}}$. By hypothesis, there exist $U, V \in \mathcal{A}$ satisfying $U \cap V = \{1\}$, $S \subseteq U$ and $T \subseteq V$. This means that

$$F^{\mathsf{c}} \cap G = S \subseteq U \Leftrightarrow F^{\mathsf{c}} \cap G \cap U^{\mathsf{c}} = \{1\} \Leftrightarrow G \subseteq F \vee U \qquad \text{and}$$
$$F \cap G^{\mathsf{c}} = T \subseteq V \Leftrightarrow F \cap G^{\mathsf{c}} \cap V^{\mathsf{c}} = \{1\} \Leftrightarrow F \subseteq G \vee V. \qquad \blacksquare$$

Corollary 4.4. *For any frame L and any $A \subseteq L$, the following are equivalent*:

(1) *L is completely A-normal.*
(2) *Every pair of $\mathfrak{o}[A]$-separated sublocales of L is separated by $\mathfrak{o}[A]$-sublocales.*

Corollary 4.5. *For any frame L and any $A \subseteq L$, the following are equivalent:*

(1) *L is completely A-disconnected.*
(2) *Every pair of $\mathfrak{c}[A]$-separated sublocales of L is separated by $\mathfrak{c}[A]$-sublocales.*

Proposition 4.3 can be expanded to the next result which extends Proposition 3.3 of [10].

Proposition 4.6. *For any frame L and any $A \subseteq L$, the following are equivalent for $\mathcal{A} = \mathfrak{o}[A]$ or $\mathcal{A} = \mathfrak{c}[A]$:*

(1) *$\mathcal{S}(L)$ is completely $\mathcal{A}$-normal.*
(2) *For every $S, T \in \mathcal{S}(L)$ satisfying $S \subseteq F \subseteq T$ and $S \subseteq G^{\mathsf{c}} \subseteq T$ for some $F, G \in \mathcal{A}$, there exist $U, V \in \mathcal{A}$ such that $S \subseteq V \subseteq U^{\mathsf{c}} \subseteq T$.*
(3) *For every $S, T \in \mathcal{S}(L)$ satisfying $S \vee F = L = G \vee T$ for some $F, G \in \mathcal{A}$ such that $F \subseteq T$ and $G \subseteq S$, there exist $U, V \in \mathcal{A}$ such that $U \cap V = \{1\}$, $V^{\mathsf{c}} \subseteq S$ and $U^{\mathsf{c}} \subseteq T$.*

Proof: (1)⇒(2): Let $S \subseteq F \subseteq T$ and $S \subseteq G^{\mathsf{c}} \subseteq T$. By (1), there are $U, V \in \mathcal{A}$ such that $U \cap V = \{1\}$, $G \subseteq F \vee U$ and $F \subseteq G \vee V$. Of course, $V \subseteq U^{\mathsf{c}}$. Moreover,

$$\begin{aligned} &S \subseteq F \cap G^{\mathsf{c}} \subseteq (G \vee V) \cap G^{\mathsf{c}} = V \cap G^{\mathsf{c}} \subseteq V \qquad \text{and} \\ &U^{\mathsf{c}} = (U^{\mathsf{c}} \cap F) \vee (U^{\mathsf{c}} \cap F^{\mathsf{c}}) \subseteq (U^{\mathsf{c}} \cap F) \vee G^{\mathsf{c}} \subseteq T. \end{aligned}$$

(2)⇒(3): Let $S \vee F = L = G \vee T$ as in the hypothesis. Then

$$L \smallsetminus S \subseteq F \subseteq T \quad \text{and} \quad L \smallsetminus S \subseteq G^{\mathsf{c}} \subseteq T.$$

By (2) there exist $U, V \in \mathcal{A}$ such that $L \smallsetminus S \subseteq V \subseteq U^{\mathsf{c}} \subseteq T$. Clearly, $U \cap V = \{1\}$, $V^{\mathsf{c}} \subseteq L \smallsetminus (L \smallsetminus S) \subseteq S$ and $U^{\mathsf{c}} \subseteq T$.

(3)⇒(1): Let $F, G \in \mathcal{A}$ and consider $S = F^{\mathsf{c}} \vee G$ and $T = F \vee G^{\mathsf{c}}$. Clearly, $S \vee F = L = G \vee T$, $F \subseteq T$ and $G \subseteq S$. Consequently, by (3), there are $U, V \in \mathcal{A}$ such that $U \cap V = \{1\}$, $V^{\mathsf{c}} \subseteq S$ and $U^{\mathsf{c}} \subseteq T$. Then, finally, $F \vee U \supseteq F \vee T^{\mathsf{c}} = F \vee G \supseteq G$ and $G \vee V \supseteq G \vee S^{\mathsf{c}} = G \vee F \supseteq F$. ■

Again, the case $\mathcal{A} = \mathfrak{c}[A]$ yields immediately the dual result for ED:

Corollary 4.7. *For any frame L and any $A \subseteq L$, the following are equivalent:*

(1) *L is completely A-disconnected.*
(2) *For every $S, T \in \mathcal{S}(L)$ satisfying $S \subseteq \mathfrak{c}(a) \subseteq T$ and $S \subseteq \mathfrak{o}(b) \subseteq T$ for some $a, b \in A$, there exist $u, v \in A$ such that $S \subseteq \mathfrak{c}(v) \subseteq \mathfrak{o}(u) \subseteq T$.*
(3) *For every $S, T \in \mathcal{S}(L)$ satisfying $S \vee \mathfrak{c}(a) = L = \mathfrak{c}(b) \vee T$ for some $a, b \in A$ such that $\mathfrak{c}(a) \subseteq T$ and $\mathfrak{c}(b) \subseteq S$, there exist $u, v \in A$ such that $u \vee v = 1$, $\mathfrak{o}(v) \subseteq S$ and $\mathfrak{o}(u) \subseteq T$.* ■

5. Heredity properties

Recall from Section 1 that an $L = \mathcal{O}(X)$ is hereditary ED if and only it is completely ED. In this section we prove that this equivalence extends to arbitrary frames by proving the equivalence between completely A-normal frames and hereditarily A-normal frames (for any sublattice A of L).

A frame L is called *hereditarily normal* [13] whenever every sublocale of L is normal. Dually, we say that L is called *hereditarily ED* if every sublocale of L is ED.

More generally, for a fixed $A \subseteq L$, we say that L is *hereditarily A-normal* in case every sublocale T of L is A_T-normal, for $A_T = \rho_T[A] \subseteq T$ (recall the mapping ρ_T). This is the same as saying that the lattice $\mathcal{S}(T)$ is $\mathfrak{o}_T[A_T]$-normal for every sublocale T of L. Dually, we say that L is *hereditarily A-disconnected* if every sublocale T of L is A_T-disconnected, that is, $\mathcal{S}(T)$ is $\mathfrak{c}_T[A_T]$-normal for every sublocale T of L.

For any sublocale T of L and $\mathcal{A} \subseteq \mathcal{S}(L)$ let

$$\mathcal{A}_T = \{S \cap T \mid S \in \mathcal{A}\} \subseteq \mathcal{S}(T).$$

Then, for the case of a sublattice $A \subseteq L$ we have:

Theorem 5.1. *For any frame L and any sublattice A of L, the following are equivalent for $\mathcal{A} = \mathfrak{o}[A]$ or $\mathcal{A} = \mathfrak{c}[A]$:*
(1) *$\mathcal{S}(L)$ is completely $\mathcal{A}$-normal.*
(2) *For each $T \in \mathcal{S}(L)$, $\mathcal{S}(T)$ is $\mathcal{A}_T$-normal.*
(3) *For each $T \in \mathcal{A}$, $\mathcal{S}(T)$ is $\mathcal{A}_T$-normal.*

Proof: (1)⇒(2): Let T be a sublocale of L and $F_T = F \cap T$, $G_T = G \cap T$ (with $F, G \in \mathcal{A}$) such that $F_T \vee G_T = T$, that is, $F \vee G \supseteq T$. By hypothesis there exist $U, V \in \mathcal{A}$ such that $U \cap V = \{1\}$, $F \vee U \supseteq G$ and $G \vee V \supseteq F$. Then

$$U_T = U \cap T,\ V_T = V \cap T \in \mathcal{A}_T$$

and $U_T \cap V_T = \{1\}$. On the other hand, by (2.4),

$$F_T \vee U_T = (F \vee U) \cap T \supseteq F_T \vee G_T = T$$

and, similarly, $G_T \vee V_T = T$. This shows that $\mathcal{S}(T)$ is $\mathcal{A}_T$-normal.

(2)⇒(3): It is obvious.

(3)⇒(1): We shall prove that $\mathcal{S}(L)$ is completely $\mathcal{A}$-normal using Proposition 4.3. So let S and T be $\mathcal{A}$-separated and $F, G \in \mathcal{A}^{\mathsf{c}}$ such that $S \subseteq F$, $T \subseteq G$ and $S \cap G = \{1\} = F \cap T$ and consider

$$R = F^{\mathsf{c}} \vee G^{\mathsf{c}} \in \mathcal{A}$$

(R is indeed in $\mathcal{A}$ since A being a sublattice of L means that both $\mathfrak{o}[A]$ and $\mathfrak{c}[A]$ are closed under finite joins). By hypothesis, $\mathcal{S}(R)$ is $\mathcal{A}_R$-normal. Since $F^{\mathsf{c}} = F^{\mathsf{c}} \cap R, G^{\mathsf{c}} = G^{\mathsf{c}} \cap R \in \mathcal{A}_R$ and $F^{\mathsf{c}} \vee G^{\mathsf{c}} = R$, there exist $U, V \in \mathcal{A}$ such that

$$(U \cap R) \cap (V \cap R) = \{1\} \quad \text{and} \quad F^{\mathsf{c}} \vee (U \cap R) = R = G^{\mathsf{c}} \vee (V \cap R).$$

Since A is a sublattice of L it follows that $\mathfrak{o}[A]$ and $\mathfrak{c}[A]$ are closed under finite meets and so $U' = U \cap G^{\mathsf{c}} \in \mathcal{A}$ and $V' = V \cap F^{\mathsf{c}} \in \mathcal{A}$. We have

$$U' \cap V' = (U \cap G^{\mathsf{c}}) \cap (V \cap F^{\mathsf{c}}) \subseteq U \cap V \cap R = \{1\}.$$

Moreover, from $F^{\mathsf{c}} \vee (U \cap R) = R$ it follows that $R \subseteq F^{\mathsf{c}} \vee U$. Then $S \cap U^{\mathsf{c}} \subseteq F \cap U^{\mathsf{c}} \subseteq R^{\mathsf{c}} = F \cap G \subseteq G$ and thus $S \cap (U^{\mathsf{c}} \vee G) \subseteq S \cap G = \{1\}$. Hence $S \subseteq U'$. Similarly, $T \subseteq V'$. ■

For $\mathcal{A} = \mathfrak{o}[A]$ we have

$$\mathcal{A}_T = \{\mathfrak{o}(a) \cap T \mid a \in A\} = \{\mathfrak{o}_T(\rho_T(a)) \mid a \in A\} = \mathfrak{o}_T(A_T)$$

and hence we get:

Corollary 5.2. *For any frame L and any sublattice A of L, the following are equivalent*:

(1) *L is completely A-normal.*
(2) *L is hereditarily A-normal.*
(3) *Each open sublocale of the form $\mathfrak{o}(a)$ with $a \in A$ is $A_{\mathfrak{o}(a)}$-normal.*

In particular, for $A = L$ we have $A_{\mathfrak{o}(a)} = \mathfrak{o}(a)$ and, therefore, both Theorem 3.7 of [10] and Proposition 3.3 of [13] follow:

Corollary 5.3. *The following are equivalent for any frame L*:

(1) *L is completely normal.*

(2) *L is hereditarily normal.*
(3) *Each open sublocale of L is normal.*

For $\mathcal{A} = \mathfrak{c}[A]$ we have

$$\mathcal{A}_T = \{\mathfrak{c}(a) \cap T \mid a \in A\} = \{\mathfrak{c}_T(\rho_T(a)) \mid a \in A\} = \mathfrak{c}_T(A_T)$$

and hence we also get:

Corollary 5.4. *For any frame L and any sublattice A of L, the following are equivalent*:

(1) *L is completely A-disconnected.*
(2) *L is hereditarily A-disconnected.*
(3) *Each closed sublocale of the form $\mathfrak{c}(a)$ with $a \in A$ is $A_{\mathfrak{c}(a)}$-normal.*

Finally, for $A = L$ we have $A_{\mathfrak{c}(a)} = \mathfrak{c}(a)$ and we get immediately the announced result for ED:

Corollary 5.5. *The following are equivalent for any frame L*:

(1) *L is completely extremally disconnected.*
(2) *L is hereditarily extremally disconnected.*
(3) *Each closed sublocale of L is extremally disconnected.*

Remark 5.6. Since a topological space may have more sublocales than subspaces, so far it was not immediately clear that the pointfree notions of hereditary normality and hereditary extremal disconnectedness are conservative. This follows now immediately from Corollaries 5.3 and 5.5 and Remark 2.1.

6. Heredity and density

A property P of frames is said to be *hereditary* if each sublocale of a frame satisfying P also satisfies P, and it is said to be *closed-hereditary* [resp., *open-hereditary*] if it is hereditary for closed [resp., open] sublocales.

A natural question arises: when checking hereditary normality of a frame L do we really need to check normality of *all* the *open* sublocales? We end the paper with the new observation that it suffices to check it for dense *and* open sublocales. One of possible arguments for it depends only on the fact that normality is closed-hereditary and its heredity is equivalent to its open-heredity (examples of the same situation include collectionwise normality and κ–collectionwise normality for frames [15]):

Proposition 6.1. *Let P be a property of frames which is closed-hereditary and such that heredity of P is equivalent to open-heredity. For L a frame the following are equivalent*:

(1) *Each sublocale of L has property P.*
(2) *Each dense sublocale of L has property P.*
(3) *Each open and dense sublocale of L has property P.*

Proof: (3)⇒(1): This is the only non-trivial implication that need to be shown. Consider an open sublocale $\mathfrak{o}(a)$ and its closure $\overline{\mathfrak{o}(a)} = \mathfrak{c}(a^*)$. The sublocale

$$S = \mathfrak{o}(a) \vee (L \smallsetminus \mathfrak{c}(a^*)) = \mathfrak{o}(a) \vee \mathfrak{o}(a^*) = \mathfrak{o}(a \vee a^*)$$

is also open. Further, S is dense in L:

$$\overline{S} = \mathfrak{c}((a \vee a^*)^*) = \mathfrak{c}(a^* \wedge a^{**}) = \mathfrak{c}(0) = L.$$

Thus, S has property P. Moreover, $S \cap \mathfrak{c}(a^*) = \mathfrak{o}(a) \cap \mathfrak{c}(a^*) \subseteq \mathfrak{o}(a)$, that is, $\mathfrak{o}(a) = \mathfrak{c}(a^*) \cap S$. Hence $\mathfrak{o}(a)$ is closed in S. Since P is closed-hereditary, then $\mathfrak{o}(a)$ has also the property and the conclusion follows from the assumption that heredity of P is equivalent to open-heredity. ■

Remark 6.2. It is interesting that when P = normality, we can prove the implication (3)⇒(1) without assuming that heredity of P is equivalent to open-heredity. This is a novelty even for the case of topological normality (see Corollary 6.4). For that, consider an open sublocale $\mathfrak{o}(a)$ of L and $u, v \in \mathfrak{o}(a)$ satisfying

$$1 = u \overset{\mathfrak{o}(a)}{\vee} v = a \to (u \vee v).$$

By the proof above $S = \mathfrak{o}(a) \vee \mathfrak{o}(a^*)$ is an open and dense sublocale of L hence normal. Moreover, $u, v \in \mathfrak{o}(a) \subseteq S$, $a \to (u \vee v) = 1$ and $a^* \to (u \vee v) = 1$ (since $a^* = \bigwedge \mathfrak{o}(a) \leq u \vee v$). Hence

$$u \overset{S}{\vee} v = (a \vee a^*) \to (u \vee v) = (a \to (u \vee v)) \wedge (a^* \to (u \vee v)) = 1.$$

Then, by the normality of S, there exist $x, y \in S$ such that $x \wedge y = 0_S = 0$ and

$$u \overset{S}{\vee} x = 1 = v \overset{S}{\vee} y.$$

Now consider $a \to x$ and $a \to y$ in $\mathfrak{o}(a)$. Notice that

$$(a \to x) \wedge (a \to y) = a \to (x \wedge y) = a \to 0 = 0_{\mathfrak{o}(a)}.$$

Moreover, $a \leq a \vee a^* \leq u \vee x \leq u \vee (a \to x)$, since

$$1 = u \overset{S}{\vee} x = (a \vee a^*) \to (u \vee x).$$

Hence

$$u \overset{\mathfrak{o}(a)}{\vee} (a \to x) = a \to (u \vee (a \to x)) = 1,$$

and it can be proved in a similar way that $v \overset{\mathfrak{o}(a)}{\vee} (a \to y) = 1$.

After Remark 6.2 we have a small contribution to general topology. The point of the proposition and the corollary which follow should have been known (but is not) since Urysohn told us in his celebrated 1925 paper that it is not necessary to check all subspaces for whether they are normal.

Mimicking the proof in 6.1 for a topological space X, an arbitrary open subspace A of X, and $S = A \cup (X \smallsetminus \overline{A})$, we get:

Proposition 6.3. *Let P be a topological property which is closed-hereditary and such that heredity of P is equivalent to open-heredity. For X a topological space the following are equivalent*:

(1) *Each subspace of X has property P.*
(2) *Each dense subspace of X has property P.*
(3) *Each open and dense subspace of X has the property P.*

Corollary 6.4. *For X a topological space the following are equivalent*:

(1) *X is hereditarily normal.*
(2) *Each dense subspace of X is normal.*
(3) *Each open and dense subspace of X is normal.*

Examples 6.5. Besides topological normality, among examples of topological properties P which are closed-hereditary and such that heredity of P is equivalent to open-heredity of P are the following:

(1) $P =$ collectionwise normality ([9, 5.1.C (a)] and [17, Lemma 1]),
(2) $P =$ compactness ([29, Theorem 1]),
(3) $P =$ Lindelöfness ([9, 3.8.4 and 3.8.A (b)]),
(4) $P =$ paracompactness ([9, 5.1.29 and 5.1.F (a)]).

This list is not exhaustive (cf. e.g. diagram 4.1 in [5]). Another way of extending it is to require cardinal restrictions. As an example, we mention κ-collectionwise normality (cf. [9, 3.8.A]). We note that hereditarily compact spaces are also called *Noetherian* spaces. They play an important role in algebraic geometry (cf. [4, Chap. II, § 4, Proposition 9]).

Since normality [resp., extremal disconnectedness] is closed-hereditary [resp., open-hereditary], and extremal disconnectedness is dense-hereditary (cf. [26, Theorem 6.2 (b)]), we have the following enrichment to the discussion of heredity in spaces which are both normal and extremally disconnected (cf. [7] and [8]).

Corollary 6.6. *For X a topological space the following are equivalent*:

(1) *Each subspace of X is normal and extremally disconnected.*
(2) *Each open [closed] subspace of X is normal and extremally disconnected.*
(3) *Each dense subspace of X is normal and extremally disconnected.*
(4) *Each open and dense subspace of X is normal and extremally disconnected.*

A subset of a topological space is dense in the space if its only closed superset is the whole space. Dually, a subset of a topological space is said to be *codense* if its only open superset is the whole space. The earliest references for the concept of topological codensity, we are aware of, goes back to [24] and [20].

The concept of codensity deads if points are closed. But it is quite vital otherwise, and gives rise to another interesting result: normality is codense hereditary.

Proposition 6.7. *A topological space is normal if and only if each its codense subspace is normal.*

Proof: Let X be a normal space and A be its codense subspace. Let K_1 and K_2 be closed and disjoint in A. Then K_1 and K_2 are traces on A of closed subsets F_1 and F_2 of X. Since

$$A \subseteq X - (F_1 \cap F_2),$$

hence $F_1 \cap F_2 = \varnothing$ on account of A being codense. Hence F_1 and F_2 have disjoint open neighborhoods in X whose intersections with A are disjoint open neighborhoods of K_1 and K_2 in A. Hence A is normal. The converse is obvious, for X is codense in X. ■

We end with the extension of this result to frames. In pointfree topology codense sublocales are considered in [6]: a sublocale T of a frame L is *codense* if $T^\circ = L$, where $T^\circ = \bigcap\{\mathfrak{o}(a) \mid T \subseteq \mathfrak{o}(a)\}$ is the *fitting closure* on sublocales. Of course, we may now speak about the $\mathcal{A}$-*closure* of a sublocale T for $\mathcal{A} =$

$\mathfrak{o}[A]$ or $\mathcal{A} = \mathfrak{c}[A]$ (and an arbitrary $A \subseteq L$), that is, the sublocale

$$\mathrm{cl}_{\mathcal{A}}(T) = \bigcap\{A \in \mathcal{A} \mid T \subseteq A\}$$

(that may not belong to $\mathcal{A}$, unless $\mathcal{A}$ is closed under intersections). We then say that S is $\mathcal{A}$-*dense* whenever $\mathrm{cl}_{\mathcal{A}}(T) = L$. The case when $\mathcal{A} = \mathfrak{c}[A]$, with $A = L$, covers the usual density while $\mathcal{A} = \mathfrak{o}[A]$ and $A = L$ covers codensity.

Proposition 6.8. *For any frame L and any sublattice A of L, the following are equivalent for $\mathcal{A} = \mathfrak{o}[A]$ or $\mathcal{A} = \mathfrak{c}[A]$:*

(1) *$\mathcal{S}(L)$ is $\mathcal{A}$-normal.*
(2) *For each $\mathcal{A}$-dense $T \in \mathcal{S}(L)$, $\mathcal{S}(T)$ is $\mathcal{A}_T$-normal.*

Proof: The implication (2)⇒(1) is trivial since L is $\mathcal{A}$-dense. Conversely, let T be an $\mathcal{A}$-dense sublocale of L. Consider $F_1 = G_1 \cap T$ and $F_2 = G_2 \cap T$ in $\mathcal{A}_T$ with $F_1 \vee F_2 = T$, that is, $T \subseteq G_1 \vee G_2 \in \mathcal{A}$ (since A is a sublattice of L) and $G_1 \vee G_2 = L$ (since T is $\mathcal{A}$-dense). Hence, by hypothesis, there exist $U_1, U_2 \in \mathcal{A}$ such that $U_1 \cap U_2 = \{1\}$ and $G_1 \vee U_1 = L = G_2 \vee U_2$. Let

$$V_1 = U_1 \cap T,\ V_2 = U_2 \cap T \in \mathcal{A}_T.$$

Clearly, $V_1 \cap V_2 = \{1\}$ and $F_1 \vee V_1 = T = F_2 \vee V_2$. ■

For $\mathcal{A} = \mathfrak{o}[L]$ and $\mathcal{A} = \mathfrak{c}[L]$ we get, respectively:

Corollary 6.9. (1) *A frame L is normal iff each codense sublocale of L is normal.*
(2) *A frame L is extremally disconnected iff each dense sublocale of L is extremally disconnected.*

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