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EIGENVALUES OF MATRICES RELATED TO THE OCTONIONS

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Abstract: A pseudo real matrix representation of an octonion, which is based on two real matrix representations of a quaternion, is considered. We study how some operations defined on the octonions change the set of eigenvalues of the matrix obtained if these operations are performed after or before the matrix representation. The established results could be of particular interest to researchers working on estimation algorithms involving such operations.

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1. Introduction

Due to nonassociativity, the real octonion division algebra is not algebraically isomorphic to a real matrix algebra. Despite this fact, pseudo real matrix representations of an octonion may be introduced, as in [1], through real matrix representations of a quaternion.

In this work, the left matrix representation of an octonion over $\mathbb{R}$, as called by Tian in [1], is considered. For the sake of completeness, some definitions and results, in particular on this pseudo representation, are recalled in Section 2.

Using the mentioned representation, results concerning eigenvalues of matrices related to the octonions are established in Section 3. Previous research on this subject, although not explicitly applying real matrix representations of a quaternion, can be seen in [2].

2. Real octonion division algebra

Consider the real octonion division algebra $\mathbb{O}$, that is, the usual real vector space $\mathbb{R}^8$, with canonical basis $\{e_0, \ldots, e_7\}$, equipped with the multiplication given by the relations

$$\boldsymbol{e_i e_j} = -\delta_{ij}\boldsymbol{e_0} + \varepsilon_{ijk}\boldsymbol{e_k},$$

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where δ_{ij} is the Kronecker delta, ε_{ijk} is a Levi-Civita symbol, i.e., a completely antisymmetric tensor with a positive value $+1$ when $ijk = 123$, 145, 167, 246, 275, 374, 365 and $\boldsymbol{e_0}$ is the identity. This element will be omitted whenever it is clear from the context.

Every element $\boldsymbol{o} \in \mathbb{O}$ can be written as

$$\boldsymbol{o} = \sum_{\ell=0}^{7} o_\ell \boldsymbol{e}_\ell = \mathrm{Re}(\boldsymbol{o}) + \mathrm{Im}(\boldsymbol{o}),\ o_\ell \in \mathbb{R},$$

where $\mathrm{Re}(\boldsymbol{o}) = o_0$ and $\mathrm{Im}(\boldsymbol{o}) \equiv \overrightarrow{\boldsymbol{o}} = \sum_{\ell=1}^{7} o_\ell \boldsymbol{e}_\ell$ are called the *real* part and the *imaginary* (or *vector*) part, respectively. The conjugate of $\boldsymbol{o}$ is defined as $\overline{\boldsymbol{o}} = \mathrm{Re}(\boldsymbol{o}) - \mathrm{Im}(\boldsymbol{o})$. The *norm* of $\boldsymbol{o}$ is defined by $|\boldsymbol{o}| = \sqrt{\overline{\boldsymbol{o}}\boldsymbol{o}} = \sqrt{\boldsymbol{o}\overline{\boldsymbol{o}}} = \sqrt{\sum_{\ell=0}^{7} o_\ell^2}$. The inverse of a non-zero octonion $\boldsymbol{o}$ is $\boldsymbol{o}^{-1} = \dfrac{\overline{\boldsymbol{o}}}{|\boldsymbol{o}|^2}$.

The multiplication of $\mathbb{O}$ can be written in terms of the Euclidean inner product and the vector cross product in $\mathbb{R}^7$, hereinafter denoted by $\cdot$ and $\times$, respectively. Concretely, as in [3], we have

$$\boldsymbol{ab} = a_0 b_0 - \overrightarrow{\boldsymbol{a}} \cdot \overrightarrow{\boldsymbol{b}} + a_0 \overrightarrow{\boldsymbol{b}} + b_0 \overrightarrow{\boldsymbol{a}} + \overrightarrow{\boldsymbol{a}} \times \overrightarrow{\boldsymbol{b}}.$$

Following [4], we recall that $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{O}$ are *perpendicular* if $\mathrm{Re}(\boldsymbol{a}\overline{\boldsymbol{b}}) = 0$. In particular, if $\mathrm{Re}(\boldsymbol{a}) = \mathrm{Re}(\boldsymbol{b}) = 0$, then $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{O}$ are *perpendicular* if $\overrightarrow{\boldsymbol{a}} \cdot \overrightarrow{\boldsymbol{b}} = 0$. Moreover, $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{O}$ are *parallel* if $\mathrm{Im}(\boldsymbol{a}\overline{\boldsymbol{b}}) = 0$. In particular, if $\mathrm{Re}(\boldsymbol{a}) = \mathrm{Re}(\boldsymbol{b}) = 0$, then $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{O}$ are *parallel* if $\overrightarrow{\boldsymbol{a}} \times \overrightarrow{\boldsymbol{b}} = 0$.

The elements of the basis of $\mathbb{O}$ can also be written as

$$\begin{gathered} \boldsymbol{e_0} = 1, \quad \boldsymbol{e_1} = \boldsymbol{i}, \quad \boldsymbol{e_2} = \boldsymbol{j}, \quad \boldsymbol{e_3} = \boldsymbol{ij}, \\ \boldsymbol{e_4} = \boldsymbol{k}, \quad \boldsymbol{e_5} = \boldsymbol{ik}, \quad \boldsymbol{e_6} = \boldsymbol{jk}, \quad \boldsymbol{e_7} = \boldsymbol{ijk}. \end{gathered}$$

The real octonion division algebra $\mathbb{O}$, of dimension 8, can be constructed from the real quaternion division algebra $\mathbb{H}$, of dimension 4, by the Cayley-Dickson doubling process where $\mathbb{O}$ contains $\mathbb{H}$ as a subalgebra. As a consequence, it is well known that any $\boldsymbol{o} \in \mathbb{O}$ can be written as

$$\boldsymbol{o} = q_1 + q_2 \boldsymbol{k}, \tag{1}$$

where $q_1, q_2 \in \mathbb{H}$ are of the form $a_0 + a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{ij}$, with $a_0, a_1, a_2, a_3 \in \mathbb{R}$.

The real quaternion division algebra $\mathbb{H}$ is algebraically isomorphic to the real matrix algebra of the matrices in (2), where $\phi(q)$ is a real matrix representation of a quaternion q.

Definition 1. [1] *Let* $q = q_0 + q_1\boldsymbol{i} + q_2\boldsymbol{j} + q_3\boldsymbol{ij} \in \mathbb{H}$. *Then*

$$\phi(q) \;=\; \begin{bmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & -q_3 & q_2 \\ q_2 & q_3 & q_0 & -q_1 \\ q_3 & -q_2 & q_1 & q_0 \end{bmatrix}. \tag{2}$$

Some important properties of the matrices in Definition 1 are recalled in Lemma 1.

Lemma 1. [1] *Let* $a, b \in \mathbb{H}$ *and* $\lambda \in \mathbb{R}$. *Then*
(a) $a = b \Longleftrightarrow \phi(a) = \phi(b)$.
(b) $\phi(a+b) = \phi(a) + \phi(b)$, $\phi(ab) = \phi(a)\phi(b)$, $\phi(\lambda a) = \lambda\phi(a)$, $\phi(1) = I_4$.
(c) $\phi(\overline{a}) = \phi^T(a)$.
(d) $\phi(a^{-1}) = \phi^{-1}(a)$, *if* $a \neq 0$.
(e) $\det[\phi(a)] = |a|^4$.

The real quaternion division algebra $\mathbb{H}$ is algebraically anti-isomorphic to the real matrix algebra of the matrices in (3), where $\tau(q)$ is another real matrix representation of a quaternion q.

Definition 2. [1] *Let* $q = q_0 + q_1\boldsymbol{i} + q_2\boldsymbol{j} + q_3\boldsymbol{ij} \in \mathbb{H}$. *Then*

$$\tau(q) = K_4\phi^T(q)K_4 \;=\; \begin{bmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & q_3 & -q_2 \\ q_2 & -q_3 & q_0 & q_1 \\ q_3 & q_2 & -q_1 & q_0 \end{bmatrix}, \tag{3}$$

where $K_4 = \mathrm{diag}(1, -1, -1, -1)$.

Some relevant properties of the matrices in Definition 2 are recalled in Lemma 2.

Lemma 2. [1] *Let* $a, b \in \mathbb{H}$ *and* $\lambda \in \mathbb{R}$. *Then*
(a) $a = b \Longleftrightarrow \tau(a) = \tau(b)$.
(b) $\tau(a+b) = \tau(a) + \tau(b)$, $\tau(ab) = \tau(b)\tau(a)$, $\tau(\lambda a) = \lambda\tau(a)$, $\tau(1) = I_4$.
(c) $\tau(\overline{a}) = \tau^T(a)$.
(d) $\tau(a^{-1}) = \tau^{-1}(a)$, *if* $a \neq 0$.
(e) $\det[\tau(a)] = |a|^4$.

Based on the previous real matrix representations of a quaternion, Tian introduced the following pseudo real matrix representation of an octonion.

Definition 3. [1] *Let* $\boldsymbol{a} = a' + a''\boldsymbol{k} \in \mathbb{O}$, *where* $a' = a_0 + a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{ij}$, $a'' = a_4 + a_5\boldsymbol{i} + a_6\boldsymbol{j} + a_7\boldsymbol{ij} \in \mathbb{H}$. *Then the* 8×8 *real matrix*

$$\omega(a) = \begin{bmatrix} \phi(a') & -\tau(a'')K_4 \\ \phi(a'')K_4 & \tau(a') \end{bmatrix}, \tag{4}$$

is called the left matrix representation of a *over* $\mathbb{R}$, *where*

$$K_4 = \mathrm{diag}(1, -1, -1, -1).$$

Even though there are $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{O}$ such that $\omega(\boldsymbol{a})\omega(\boldsymbol{b}) \neq \omega(\boldsymbol{ab})$, there are still some properties which hold. These are recalled in Theorem 1.

Theorem 1. [1] *Let* $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{O}$, $\lambda \in \mathbb{R}$. *Then*

(a) $\boldsymbol{a} = \boldsymbol{b} \Longleftrightarrow \omega(\boldsymbol{a}) = \omega(\boldsymbol{b})$.
(b) $\omega(\boldsymbol{a} + \boldsymbol{b}) = \omega(\boldsymbol{a}) + \omega(\boldsymbol{b}), \quad \omega(\lambda\boldsymbol{a}) = \lambda\omega(\boldsymbol{a}), \quad \omega(1) = I_8$.
(c) $\omega(\overline{\boldsymbol{a}}) = \omega^T(\boldsymbol{a})$.

3. Main Results

In this section, the left matrix representation of an octonion over $\mathbb{R}$ is considered. First of all, given an octonion, the eigenvalues of its left matrix representation are calculated.

Proposition 1. *Let* $\boldsymbol{a} = a_0 + \mathrm{Im}(\boldsymbol{a}) \in \mathbb{O}$. *Then the eigenvalues of the real matrix* $\omega(\boldsymbol{a})$ *are*

$$\lambda = a_0 \pm \boldsymbol{i}|\mathrm{Im}(\boldsymbol{a})|,$$

each with algebraic multiplicity 4.

Proof: Let $\boldsymbol{a} = a_0 + \mathrm{Im}(\boldsymbol{a}) \in \mathbb{O}$, where $\mathrm{Im}(\boldsymbol{a}) = a' + a''\boldsymbol{k}$ and $a', a'' \in \mathbb{H}$. The characteristic polynomial of $\omega(\boldsymbol{a})$ is

$$\det(\lambda I_8 - \omega(\boldsymbol{a})) = \det(K_8(\lambda I_8 - \omega(\boldsymbol{a}))K_8),$$

where K_8 is the orthogonal matrix $\mathrm{diag}(1,-1,-1,-1,1,1,1,1) = \mathrm{diag}(K_4, I_4)$. Hence,

$$\begin{aligned}
\det(\lambda I_8 - \omega(\boldsymbol{a})) &= \det\left(K_8 \begin{bmatrix} \phi(\lambda - a_0 - a') & \tau(a'')K_4 \\ -\phi(a'')K_4 & \tau(\lambda - a_0 - a') \end{bmatrix} K_8\right) \\
&= \det \begin{bmatrix} \tau(\lambda - a_0 + a') & \phi(\overline{a''}) \\ -\phi(a'') & \tau(\lambda - a_0 - a') \end{bmatrix} \\
&= \det\left(\tau(\lambda - a_0 + a')\tau(\lambda - a_0 - a') + \phi(\overline{a''})\phi(a''))\right) \\
&= \det\left(\tau\left((\lambda - a_0)^2 + |a'|^2\right) + \phi(|a''|^2)\right) \\
&= \det\left(((\lambda - a_0)^2 + |a'|^2)I_4 + |a''|^2 I_4\right) \\
&= \det\left(((\lambda - a_0)^2 + |\mathrm{Im}\,(\boldsymbol{a})|^2)I_4\right) \\
&= ((\lambda - a_0)^2 + |\mathrm{Im}\,(\boldsymbol{a})|^2)^4,
\end{aligned}$$

and the result follows. ■

The set of eigenvalues of $\omega(\boldsymbol{ab})$ is equal to the set of eigenvalues of $\omega(\boldsymbol{a})\omega(\boldsymbol{b})$ since the characteristic polynomials are equal as can easily be seen. However, if we add an extra octonion $\boldsymbol{c}$ the set of eigenvalues of $\omega(\boldsymbol{ab} + \boldsymbol{c})$ and $\omega(\boldsymbol{a})\omega(\boldsymbol{b}) + \omega(\boldsymbol{c})$ may differ.

We now study the eigenvalues of the matrix $\omega(\boldsymbol{a})\omega(\boldsymbol{b}) + \omega(\boldsymbol{c})$, given three given octonions $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$.

Let $\boldsymbol{a} = a_0 + \mathrm{Im}(\boldsymbol{a})$ and $\boldsymbol{b} = b_0 + \mathrm{Im}(\boldsymbol{b})$. Notice that $\mathrm{Im}(\boldsymbol{b})$ can be decomposed into two parts: a part parallel to $\overrightarrow{\boldsymbol{a}}$, denoted by $\overrightarrow{\boldsymbol{b}_{\boldsymbol{a}}}$; and a part perpendicular to $\overrightarrow{\boldsymbol{a}}$, denoted by $\overrightarrow{\boldsymbol{b}_{\perp}}$. Hence, $\overrightarrow{\boldsymbol{b}_{\boldsymbol{a}}} \in \mathrm{Span}(\overrightarrow{\boldsymbol{a}})$ and $\overrightarrow{\boldsymbol{b}_{\boldsymbol{a}}} \cdot \overrightarrow{\boldsymbol{b}_{\perp}} = 0$. Then

$$\begin{aligned}
\boldsymbol{ab} &= (a_0 + \mathrm{Im}\,(\boldsymbol{a}))\,(b_0 + \boldsymbol{b_a} + \boldsymbol{b}_\perp) \\
&= a_0 b_0 + \mathrm{Im}\,(\boldsymbol{a})\boldsymbol{b_a} + b_0\mathrm{Im}\,(\boldsymbol{a}) + a_0\boldsymbol{b_a} + a_0\boldsymbol{b}_\perp + \mathrm{Im}\,(\boldsymbol{a})\boldsymbol{b}_\perp,
\end{aligned}$$

where $a_0 b_0, \mathrm{Im}\,(\boldsymbol{a})\boldsymbol{b_a} \in \mathbb{R}$ and $b_0\mathrm{Im}\,(\boldsymbol{a}), a_0\boldsymbol{b_a} \in \mathrm{Span}(\mathrm{Im}\,(\boldsymbol{a}))$. Hence, it suffices to consider only the product $\mathrm{Im}\,(\boldsymbol{a})\boldsymbol{b}_\perp$ since the remaining terms can be added to $\boldsymbol{c}$.

Proposition 2. *Let $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathbb{O}$ such that $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular, and* $\mathrm{Re}\,(\boldsymbol{a}) = \mathrm{Re}\,(\boldsymbol{b}) = 0$. *Then the eigenvalues of the real matrix $\omega(\boldsymbol{a})\omega(\boldsymbol{b}) + \omega(\boldsymbol{c})$*

are

$$\operatorname{Re}(\boldsymbol{c}) \pm \boldsymbol{i}\sqrt{(|\boldsymbol{a}||\boldsymbol{b}| \pm |\boldsymbol{c}_\perp|)^2 + |\boldsymbol{c}_\parallel|^2}, \tag{5}$$

where $\boldsymbol{c}_\parallel$ is the projection of $\boldsymbol{c}$ onto Span$(\boldsymbol{a}, \boldsymbol{b})$ and $\boldsymbol{c}_\perp = \boldsymbol{c} - \boldsymbol{c}_\parallel$, each with algebraic multiplicity 2.

Proof: Without loss of generality, we consider $\boldsymbol{a} = a\boldsymbol{i}$ and $\boldsymbol{b} = b\boldsymbol{j}$. Hence,

$$\begin{aligned} \omega(\boldsymbol{a})\omega(\boldsymbol{b}) &= \begin{bmatrix} \phi(a\boldsymbol{i}) & 0 \\ 0 & \tau(a\boldsymbol{i}) \end{bmatrix} \begin{bmatrix} \phi(b\boldsymbol{j}) & 0 \\ 0 & \tau(b\boldsymbol{j}) \end{bmatrix} \\ &= \begin{bmatrix} \phi(a\boldsymbol{i})\phi(b\boldsymbol{j}) & 0 \\ 0 & \tau(a\boldsymbol{i})\tau(b\boldsymbol{j}) \end{bmatrix}. \end{aligned}$$

By Lemmas 1 and 2, we have

$$\omega(\boldsymbol{a})\omega(\boldsymbol{b}) = \begin{bmatrix} \phi(ab\boldsymbol{i}\boldsymbol{j}) & 0 \\ 0 & \tau(-ab\boldsymbol{i}\boldsymbol{j}) \end{bmatrix}. \tag{6}$$

Let $\boldsymbol{c} = c_0 + c_1\boldsymbol{i} + c_2\boldsymbol{j} + c_3\boldsymbol{i}\boldsymbol{j} + c''\boldsymbol{k}$, where $c'' \in \mathbb{H}$. Then

$$\omega(\boldsymbol{c}) = \begin{bmatrix} \phi(c_0 + c_1\boldsymbol{i} + c_2\boldsymbol{j} + c_3\boldsymbol{i}\boldsymbol{j}) & -\tau(c'')K_4 \\ \phi(c'')K_4 & \tau(c_0 + c_1\boldsymbol{i} + c_2\boldsymbol{j} + c_3\boldsymbol{i}\boldsymbol{j}) \end{bmatrix}. \tag{7}$$

Taking into account (6) and (7), we obtain

$$\omega(\boldsymbol{a})\omega(\boldsymbol{b}) + \omega(\boldsymbol{c}) = \begin{bmatrix} A & -\tau(c'')K_4 \\ \phi(c'')K_4 & B \end{bmatrix},$$

where $A = \phi(c_0 + c_1\boldsymbol{i} + c_2\boldsymbol{j} + (c_3 + ab)\boldsymbol{i}\boldsymbol{j})$ and $B = \tau(c_0 + c_1\boldsymbol{i} + c_2\boldsymbol{j} + (c_3 - ab)\boldsymbol{i}\boldsymbol{j})$.

The characteristic polynomial of $\omega(\boldsymbol{a})\omega(\boldsymbol{b}) + \omega(\boldsymbol{c})$ is

$$\begin{aligned} p(\lambda) &= \det\left(\omega(\boldsymbol{a})\omega(\boldsymbol{b}) + \omega(\boldsymbol{c}) - \lambda I_8\right) \\ &= \det(K_8(\omega(\boldsymbol{a})\omega(\boldsymbol{b}) + \omega(\boldsymbol{c}) - \lambda I_8)K_8), \end{aligned}$$

where K_8 is the orthogonal matrix $\operatorname{diag}(1, -1, -1, -1, 1, 1, 1, 1) = \operatorname{diag}(K_4, I_4)$. Hence,

$$p(\lambda) = \det \begin{bmatrix} \Gamma_1 & -\phi(\overline{c''}) \\ \phi(c'') & \Gamma_2 \end{bmatrix},$$

where $\Gamma_1 = \tau(c_0 - \lambda - c_1\boldsymbol{i} - c_2\boldsymbol{j} - (c_3 + ab)\boldsymbol{i}\boldsymbol{j})$ and $\Gamma_2 = \tau(c_0 - \lambda + c_1\boldsymbol{i} + c_2\boldsymbol{j} + (c_3 - ab)\boldsymbol{i}\boldsymbol{j})$, which results in

$$\begin{aligned} p(\lambda) = \det\big(\tau\left((c_0 - \lambda)^2 + c_1^2 + c_2^2 + c_3^2 - (ab)^2 + 2ab\,(c_2\boldsymbol{i} - c_1\boldsymbol{j} - \right.\\ \left. -(c_0 - \lambda)\boldsymbol{i}\boldsymbol{j})\right) + \phi(|c''|^2)\big), \end{aligned}$$

and, since $\phi(|c''|^2) = \tau(|c''|^2)$, gives

$$\begin{aligned} p(\lambda) &= \det\left(\tau\left((c_0-\lambda)^2 + |\boldsymbol{c}_\parallel|^2 + |\boldsymbol{c}_\perp|^2 - (ab)^2 + \right.\right. \\ &\quad \left.\left. +2ab(c_2\boldsymbol{i} - c_1\boldsymbol{j} - (c_0-\lambda)\boldsymbol{ij})\right)\right), \end{aligned}$$

where $\boldsymbol{c}_\parallel = c_1\boldsymbol{i} + c_2\boldsymbol{j}$ and $\boldsymbol{c}_\perp = c_3\boldsymbol{ij} + c''\boldsymbol{k}$. By Lemma 2, we have

$$\begin{aligned} p(\lambda) &= \left[((c_0-\lambda)^2 + |\boldsymbol{c}_\parallel|^2 + |\boldsymbol{c}_\perp|^2 - (ab)^2)^2 + 4(ab)^2\left(|\boldsymbol{c}_\parallel|^2 + \ + (c_0-\lambda)^2\right)\right]^2 \\ &= \left[((c_0-\lambda)^2 + |\boldsymbol{c}_\parallel|^2)^2 + 2((c_0-\lambda)^2 + |\boldsymbol{c}_\parallel|^2)(|\boldsymbol{c}_\perp|^2 - (ab)^2) + \right. \\ &\quad \left. +(|\boldsymbol{c}_\perp|^2 - (ab)^2)^2 + 4(ab)^2(|\boldsymbol{c}_\parallel|^2 + (c_0-\lambda)^2)\right]^2 \\ &= \left[((c_0-\lambda)^2 + |\boldsymbol{c}_\parallel|^2)^2 + 2((c_0+\lambda)^2 + |\boldsymbol{c}_\parallel|^2)(|\boldsymbol{c}_\perp|^2 - (ab)^2) + \right. \\ &\quad \left. +(|\boldsymbol{c}_\perp|^2 + (ab)^2)^2 - 4(ab)^2|\boldsymbol{c}_\perp|^2\right]^2 \\ &= \left[((c_0-\lambda)^2 + |\boldsymbol{c}_\parallel|^2 + |\boldsymbol{c}_\perp|^2 + (ab)^2)^2 - 4(ab)^2|\boldsymbol{c}_\perp|^2\right]^2 \\ &= \left[((c_0-\lambda)^2 + |\mathrm{Im}(\boldsymbol{c})|^2 + (ab)^2)^2 - 4(ab)^2|\boldsymbol{c}_\perp|^2\right]^2, \end{aligned}$$

and the result follows. ∎

Corollary 1.1. *Let $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathbb{O}$. Then*

$$\rho\left(\omega(\boldsymbol{ab} + \boldsymbol{c})\right) \leq \rho\left(\omega(\boldsymbol{a})\omega(\boldsymbol{b}) + \omega(\boldsymbol{c})\right), \tag{8}$$

where $\rho(\cdot)$ stands for the spectral radius.

Proof: By Proposition 2, we obtain

$$\begin{aligned} \rho^2\left(\omega(\boldsymbol{a})\omega(\boldsymbol{b}) + \omega(\boldsymbol{c})\right) &= \mathrm{Re}^2(\boldsymbol{c}) + (|\boldsymbol{a}||\boldsymbol{b}| + |\boldsymbol{c}_\perp|)^2 + |\boldsymbol{c}_\parallel|^2 \\ &= \mathrm{Re}^2(\boldsymbol{c}) + |\boldsymbol{a}|^2|\boldsymbol{b}|^2 + 2|\boldsymbol{a}||\boldsymbol{b}||\boldsymbol{c}_\perp| + |\boldsymbol{c}_\perp|^2 + |\boldsymbol{c}_\parallel|^2 \\ &= |\boldsymbol{c}|^2 + |\boldsymbol{a}|^2|\boldsymbol{b}|^2 + 2|\boldsymbol{a}||\boldsymbol{b}||\boldsymbol{c}_\perp|. \end{aligned}$$

Furthermore, the eigenvalues of $\omega(\boldsymbol{ab}+\boldsymbol{c})$ are all equal in modulus and satisfy

$$\begin{aligned} \rho^2(\omega(\boldsymbol{ab} + \boldsymbol{c})) &= (\boldsymbol{ab} + \boldsymbol{c})(\overline{\boldsymbol{ab} + \boldsymbol{c}}) \\ &= |\boldsymbol{a}|^2|\boldsymbol{b}|^2 + |\boldsymbol{c}|^2 + 2\mathrm{Re}((\boldsymbol{ab})\overline{\boldsymbol{c}}). \end{aligned}$$

Without loss of generality, we can consider $\boldsymbol{a} = a\boldsymbol{i}$ and $\boldsymbol{b} = b\boldsymbol{j}$. If $\boldsymbol{c} = c_0 + c_1\boldsymbol{i} + c_2\boldsymbol{j} + c_3\boldsymbol{ij} + c_4\boldsymbol{k} + c_5\boldsymbol{ik} + c_6\boldsymbol{jk} + c_7\boldsymbol{ijk}$. Hence, we arrive at

$$\begin{aligned}
\rho^2(\omega(\boldsymbol{ab}+\boldsymbol{c})) &= |\boldsymbol{a}|^2|\boldsymbol{b}|^2 + |\boldsymbol{c}|^2 + 2\mathrm{Re}((\boldsymbol{ab})\overline{\boldsymbol{c}}) \\
&= (ab)^2 + |\boldsymbol{c}|^2 + 2abc_3 \\
&\leq (ab)^2 + |\boldsymbol{c}|^2 + 2|ab|\sqrt{c_3^2+c_4^2+c_5^2+c_6^2+c_7^2} \\
&= (ab)^2 + |\boldsymbol{c}|^2 + 2|ab||\boldsymbol{c}_\perp| \\
&= \rho^2\left(\omega(\boldsymbol{a})\omega(\boldsymbol{b}) + \omega(\boldsymbol{c})\right),
\end{aligned}$$

and the result follows. ■

Example 3.1. Let

$$\begin{aligned}
&\boldsymbol{a} = 1 + \boldsymbol{i} + \boldsymbol{k} + \boldsymbol{ijk}, \\
&\boldsymbol{b} = -1 + 2\boldsymbol{ij} - \boldsymbol{ik} + 3\boldsymbol{ijk}, \\
&\boldsymbol{c} = 2 + \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k} - 5\boldsymbol{ik} + \boldsymbol{jk} - 12\boldsymbol{ijk}.
\end{aligned}$$

To apply (5), we rewrite $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ as

$$\boldsymbol{a} = a_0 + \overrightarrow{\boldsymbol{a}}, \boldsymbol{b} = b_0 + \overrightarrow{\boldsymbol{b}}, \boldsymbol{c} = \mathrm{Re}(\boldsymbol{c}) + \overrightarrow{\boldsymbol{c}},$$

where $\overrightarrow{\boldsymbol{a}}$, $\overrightarrow{\boldsymbol{b}}$ and $\overrightarrow{\boldsymbol{c}}$ are the imaginary parts of $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{c}$, respectively.

Calculating $\overrightarrow{\boldsymbol{b}_{\boldsymbol{a}}}$, the projection of $\overrightarrow{\boldsymbol{b}}$ onto $\overrightarrow{\boldsymbol{a}}$, we obtain $\overrightarrow{\boldsymbol{b}_{\boldsymbol{a}}} = \overrightarrow{\boldsymbol{a}}$. Thus, the orthogonal part $\overrightarrow{\boldsymbol{b}_\perp}$, is equal to $\overrightarrow{\boldsymbol{b}} - \overrightarrow{\boldsymbol{b}_{\boldsymbol{a}}} = -\boldsymbol{i} + 2\boldsymbol{ij} - \boldsymbol{k} - \boldsymbol{ik} + 2\boldsymbol{ijk}$.

The projections of $\overrightarrow{\boldsymbol{c}}$ onto $\overrightarrow{\boldsymbol{a}}$ and $\overrightarrow{\boldsymbol{b}_\perp}$ are, respectively $\overrightarrow{\boldsymbol{c}_{\boldsymbol{a}}} = -3\overrightarrow{\boldsymbol{a}}$ and $\overrightarrow{\boldsymbol{c}_{\boldsymbol{b}_\perp}} = 2\boldsymbol{i} - 4\boldsymbol{ij} + 2\boldsymbol{k} + 2\boldsymbol{ik} - 4\boldsymbol{ijk}$. Hence, the projection of $\overrightarrow{\boldsymbol{c}}$ on the space of $\overrightarrow{\boldsymbol{a}}$ and $\overrightarrow{\boldsymbol{b}_\perp}$ is $\overrightarrow{\boldsymbol{c}_{\boldsymbol{a},\boldsymbol{b}_\perp}} = \overrightarrow{\boldsymbol{c}_{\boldsymbol{a}}} + \overrightarrow{\boldsymbol{c}_{\boldsymbol{b}_\perp}} = -\boldsymbol{i} - 4\boldsymbol{ij} - \boldsymbol{k} + 2\boldsymbol{ik} - 7\boldsymbol{ijk}$. This implies that the orthogonal part we have $\overrightarrow{\boldsymbol{c}_\perp} = \overrightarrow{\boldsymbol{c}} - \overrightarrow{\boldsymbol{c}_{\boldsymbol{a},\boldsymbol{b}_\perp}} = 2\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k} - 14\boldsymbol{ik} + \boldsymbol{jk} + 7\boldsymbol{ijk}$.

Taking all together, we have

$$\begin{aligned}
\boldsymbol{ab} + \boldsymbol{c} &= (a_0 + \overrightarrow{\boldsymbol{a}})(b_0 + \overrightarrow{\boldsymbol{b}_{\boldsymbol{a}}} + \overrightarrow{\boldsymbol{b}_\perp}) + \mathrm{Re}(\boldsymbol{c}) + \overrightarrow{\boldsymbol{c}_{\boldsymbol{a},\boldsymbol{b}_\perp}} + \overrightarrow{\boldsymbol{c}_\perp} \\
&= \overrightarrow{\boldsymbol{a}}\overrightarrow{\boldsymbol{b}_\perp} + (\mathrm{Re}(\boldsymbol{c}) + a_0b_0 + \overrightarrow{\boldsymbol{a}}\overrightarrow{\boldsymbol{b}_{\boldsymbol{a}}}) + \Big(\overrightarrow{\boldsymbol{c}_{\boldsymbol{a},\boldsymbol{b}_\perp}} + a_0\overrightarrow{\boldsymbol{b}_{\boldsymbol{a}}} + \\
&\quad + b_0\overrightarrow{\boldsymbol{a}} + a_0\overrightarrow{\boldsymbol{b}_\perp}\Big) + \overrightarrow{\boldsymbol{c}_\perp},
\end{aligned}$$

where $\mathrm{Re}(\boldsymbol{c}) + a_0b_0 + \overrightarrow{\boldsymbol{a}}\overrightarrow{\boldsymbol{b}_{\boldsymbol{a}}}$ is real and $\overrightarrow{\boldsymbol{c}_{\boldsymbol{a},\boldsymbol{b}_\perp}} + a_0\overrightarrow{\boldsymbol{b}_{\boldsymbol{a}}} + b_0\overrightarrow{\boldsymbol{a}} + a_0\overrightarrow{\boldsymbol{b}_\perp} \in \mathrm{Span}(\overrightarrow{\boldsymbol{a}}, \overrightarrow{\boldsymbol{b}_\perp})$. So

$$\boldsymbol{ab} + \boldsymbol{c} = c_0 + \boldsymbol{c}_\parallel + \boldsymbol{c}_\perp, \tag{9}$$

where $c_0 = \mathrm{Re}(\boldsymbol{c}) + a_0 b_0 + \overrightarrow{\boldsymbol{a}}\overrightarrow{\boldsymbol{b_a}} = -2$, $\boldsymbol{c}_\| = \overrightarrow{\boldsymbol{c_{a,b_\perp}}} + a_0\overrightarrow{\boldsymbol{b_a}} + b_0\overrightarrow{\boldsymbol{a}} + a_0\overrightarrow{\boldsymbol{b}_\perp} = -\boldsymbol{i} - 4\boldsymbol{ij} - \boldsymbol{k} + 2\boldsymbol{ik} - 7\boldsymbol{ijk}$, and $\boldsymbol{c}_\perp = \overrightarrow{\boldsymbol{c}_\perp} = 2\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k} - 14\boldsymbol{ik} + \boldsymbol{jk} + 7\boldsymbol{ijk}$.

From Proposition 2, we obtain the eigenvalues of

$$\omega(\boldsymbol{a})\omega(\boldsymbol{b}) + \omega(\boldsymbol{c}) = \begin{bmatrix} -2 & 1 & 2 & 1 & 0 & 6 & 2 & 8 \\ -1 & -2 & -5 & 0 & 6 & 4 & 12 & -2 \\ -2 & 5 & -2 & -1 & -4 & -12 & 2 & 6 \\ -1 & 0 & 1 & -2 & -8 & 4 & 6 & 2 \\ 0 & -6 & 4 & 8 & -2 & 1 & -4 & -1 \\ -6 & -4 & 12 & -4 & -1 & -2 & 5 & 2 \\ -2 & -12 & -2 & -6 & 4 & -5 & -2 & -1 \\ -8 & 2 & -6 & -2 & 1 & -2 & 1 & -2 \end{bmatrix}$$

which are

$$\lambda_+ = -2 \pm \boldsymbol{i}\sqrt{\left(\sqrt{33} + \sqrt{105}\right)^2 + 38}$$

and

$$\lambda_- = -2 \pm \boldsymbol{i}\sqrt{\left(\sqrt{33} - \sqrt{105}\right)^2 + 38},$$

while the eigenvalues of $\omega(ab + c)$ are $\lambda = -2 \pm \boldsymbol{i}\sqrt{110}$.

As predicted by Corollary 1.1, $\rho(\omega(ab + c)) < \rho(\omega(a)\omega(b) + \omega(c))$, since $\rho(\omega(ab+c)) = |\lambda| = \sqrt{114}$ and $\rho(\omega(a)\omega(b) + \omega(c)) = |\lambda_+| = \sqrt{180 + 6\sqrt{385}}$. ■

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