Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 19–05

HEWITT'S IRRESOLVABILITY AND INDUCED SUBLOCALES IN SPATIAL FRAMES

DHARMANAND BABOOLAL, JORGE PICADO, PARANJOTHI PILLAY AND ALEŠ PULTR

Dedicated to the memory of Harold Simmons

ABSTRACT: Sublocales of frames, even those representing subspaces (induced sublocales), are typically not complemented in the lattice of all sublocales. We present a necessary and sufficient condition for an induced sublocale to be so, and prove that all induced sublocales are complemented if and only if the space in question is hereditarily irresolvable, a property slightly weaker than — and in a broad class of spaces equivalent with — scatteredness (under which condition, by Simmons' result *all* sublocales are complemented).

KEYWORDS: Frame, spatial frame, locale, sublocale, sublocale lattice, induced sublocale, T_D-axiom, scattered space, hereditarily irresolvable space.

AMS SUBJECT CLASSIFICATION (2010): 06D22, 54G12, 54G15.

Introduction

In [6] (1943), Hewitt introduced the concept of *irresolvability*: a space X is *resolvable* if there are two disjoint $A, B \subseteq X$ such that $\overline{A} = \overline{B} = X$, it is *irresolvable* in the opposite case. In the localized form one speaks of *hereditarily irresolvable* (briefly, HI) space if there is no non-empty resolvable $Y \subseteq X$, in other words if

$$\forall A, B \subseteq X, \quad \emptyset \neq \overline{A} = \overline{B} \quad \Rightarrow \quad A \cap B \neq \emptyset.$$

Since, as it is easy to see, every scattered space is hereditarily irresolvable, the question naturally arises what is the relation between the two notions.

Received February 2, 2019.

Support from MFF of Charles University (under PROGRES Q48), NRF, South Africa (under Competitive Research Grant No. 113840) and project MTM2015-63608-P (of the Ministry of Economy and Competitiveness of Spain) is gratefully acknowledged. The second author also acknowledges partial financial support from the Centre for Mathematics of the University of Coimbra (funded by the Portuguese Government through FCT/MEC and co-funded by the European Regional Development Fund through the Partnership Agreement PT2020).

Hewitt himself considered the relation of resolvability with the property to be dense-in-itself and proved the coincidence for a class of spaces including the metric ones, or the locally compact Hausdorff ones. This has proved to be covered by more extensive results on scatteredness. Thus in [4] the authors showed that HI is equivalent with scatteredness for a large class of spaces containing the already mentioned metric and locally compact Hausdorff ones and many more (namely, Alexandroff spaces, first countable spaces and spectral spaces), and on the other side presented examples of non-scattered hereditarily irresolvable spaces.

Point-free topology puts the relations into a new perspective. It is well known that there are typically more point-free subobjects of a space (sublocales of the associated frame $\Omega(X)$) than classical subspaces. The system of all sublocales $\mathsf{S}(\Omega(X))$ is a co-frame (see 1.4 below), a complete lattice typically bigger than the Boolean algebra of subspaces (subsets) of X. In [11] Simmons proved that

> every *sublocale* of X is complemented in $\mathsf{S}(\Omega(X))$ iff X is *scattered*

(more precisely, weakly scattered, but for the spaces we are interested in it is the same). In this paper we present a characteristics of the subspaces that are complemented in $\mathsf{S}(\Omega(X))$ and as a consequence obtain that

> every *subspace* of X is complemented in $\mathsf{S}(\Omega(X))$ iff X is *hereditarily irresolvable*.

Thus, using the results of [4] we learn that

- in a large class $\mathcal{C}$ of spaces, every *sublocale* is complemented (that is, $\mathsf{S}(\Omega(X))$ is Boolean) iff every *subspace* is complemented (and indeed if every subspace is complemented each sublocale is a subspace),
- in other words, a space X in $\mathcal{C}$ has a sublocale that is not a subspace iff it has a subspace that is not complemented,
- and on the other hand there exist spaces such that each of their subspaces is complemented in $\mathsf{S}(\Omega(X))$ while this coframe contains also non-complemented elements.

The paper is organized as follows. In Preliminaries we introduce some necessary definitions concerning frames and their sublocales. Section 2 is devoted to sublocales of (localic representations of) classical spaces, in particular of the T_D ones in which case the representation is precise in a natural sense. Also, the classical concept of scatteredness and Simmons' theorem on complementarity of sublocales is recalled. In Section 3 we present the main results: the characteristic of complemented induced sublocales and proving that a space is hereditary irresolvability in the Hewitt sense if and only if *all the induced sublocales* are complemented (as opposed to the complementedness of *all sublocales* in the scattered case). In the last section we present (known) examples of spaces that are hereditarily irresolvable but not scattered, and analyze one of them in some detail to elucidate the resulting phenomena concerning the behavior of sublocales.

1. Preliminaries

1.1. A join (supremum) of a subset A of a poset $(X, \leq)$, if it exists, will be denoted by $\bigvee A$, and we write $a \vee b$ for $\bigvee\{a, b\}$; similarly we write $\bigwedge A$ and $a \wedge b$ for meets (infima). A *complete lattice* is a poset $(X, \leq)$ in which all subsets have suprema and infima. It is *distributive* if $a\wedge(b\vee c) = (a\wedge b)\vee(a\wedge c)$ (or, equivalently, $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$).

The smallest element of a poset (the supremum $\bigvee \emptyset$) will be denoted by 0, and the largest one (the infimum $\bigwedge \emptyset$) will be denoted by 1.

A complement of an element a is a b such that $a \vee b = 1$ and $a \wedge b = 0$. In a distributive lattice there is at most one such b.

1.1.1. Adjoint maps. Monotone maps $f : X \to Y$ and $g : Y \to X$ between posets are *adjoint, f to the left and g to the right*, if

$$f(x) \leq y \Leftrightarrow x \leq g(y).$$

Recall that this is characterized by the pair of inequalities $fg(y) \leq y$ and $x \leq gf(x)$, and that f resp. g preserves all the existing suprema resp. infima. Furthermore, if X and Y are complete lattices then a monotone map $f : X \to Y$ preserves all suprema iff it is a left adjoint, and a monotone map $g : Y \to X$ preserves all infima iff it is a right adjoint.

1.1.2. Proposition. *Let L be a distributive lattice and let $a \in L$ be complemented. Then, for any supremum $\bigvee x_i$, we have $a \wedge \bigvee x_i = \bigvee(a \wedge x_i)$, and for any infimum $\bigwedge x_i$ we have $a \vee \bigwedge x_i = \bigwedge(a \vee x_i)$.*

(See e.g. [7, 9]; but the proof is very easy: If a' is the complement of a we easily check that $a \wedge x \leq b$ iff $x \leq a' \vee b$. Thus for any complemented a, $(x \mapsto a \wedge x)$ is a left adjoint and $(x \mapsto a \vee x)$ is a right adjoint. Use 1.1.1.)

1.2. Frames. A *frame* resp. *coframe* is a complete lattice L satisfying the distributivity rule

$$(\bigvee A) \wedge b = \bigvee\{a \wedge b \mid a \in A\} \tag{frm}$$

resp.

$$(\bigwedge A) \vee b = \bigwedge\{a \vee b \mid a \in A\} \tag{cofrm}$$

for all $A \subseteq L$ and $b \in L$. A *frame homomorphism* $h\colon L \to M$ preserves all joins and all finite meets. The category of frames and frame homomorphisms is denoted by **Frm**.

The equality (frm) states, in other words, that for every $b \in L$ the mapping $- \wedge b = (x \mapsto x \wedge b)\colon L \to L$ preserves all joins (suprema). Hence every $- \wedge b$ has a right Galois adjoint resulting in a *Heyting operation* $\to$ with

$$a \wedge b \leq c \;\Leftrightarrow\; a \leq b \to c.$$

Thus, each frame is a Heyting algebra. The operation $\to$ and some of its basic properties (e.g. $a \to a = 1$, $a \to b = 1$ iff $a \leq b$, $1 \to a = a$, and $a \to (b \to c) = (a \wedge b) \to c$) will be used in the sequel (see also [9, III.3.1.1]).

Similarly, a coframe is a co-Heyting algebra with the operation of *difference* $a \smallsetminus b$ satisfying

$$c \smallsetminus b \leq a \;\Leftrightarrow\; c \leq b \vee a.$$

Note that in a frame every element a has a *pseudocomplement* a^* (satisfying $x \leq a^*$ iff $x \wedge a = 0$), namely $a^* = a \to 0$, and similarly in a coframe we have the *supplements* $a^\# = 1 \smallsetminus a$, the smallest x such that $x \vee a = 1$. Since in a distributive lattice each complement is both a pseudocomplement and a supplement, we will use the symbol a^* also for a complement, if it exists.

1.3. The concrete category Loc. The functor $\Omega\colon \mathbf{Top} \to \mathbf{Frm}$ from the category of topological spaces and continuous maps into that of frames ($\Omega(f)$ sending an open set $U \subseteq Y$ to $f^{-1}[U]$ for a continuous map $f\colon X \to Y$ in **Top**) is a full embedding on an important and substantial part of **Top**, the subcategory of *sober* spaces. This justifies to regard frames as a natural generalization of spaces. Since Ω is contravariant, one introduces the *category of locales* **Loc** as the dual of the category of frames. Often one just considers the formal $\mathbf{Frm}^{\mathrm{op}}$ but it is of advantage to represent it as a concrete category with specific maps as morphisms. For this purpose one defines a *localic map* $f\colon L \to M$ as the (unique) right Galois adjoint of a frame homomorphism $h = f^*\colon M \to L$. This can be done since frame homomorphisms preserve suprema (but of course not every mapping preserving infima is a localic one; for more information about **Loc** see [9, 8]).

1.4. Sublocales. A *sublocale* of a frame L is a subset $S \subseteq L$ such that

(1) $M \subseteq S$ implies $\bigwedge M \in S$, and
(2) if $a \in L$ and $s \in S$ then $a \to s \in S$.

The system
$$\mathsf{S}(L)$$
of all sublocales of L is a co-frame, with the lattice operations
$$\bigwedge_{i\in J} S_i = \bigcap_{i\in J} S_i \quad \text{and} \quad \bigvee_{i\in J} S_i = \{\bigwedge A \mid A \subseteq \bigcup_{i\in J} S_i\}.$$
The top element of $\mathsf{S}(L)$ is L and the bottom is the sublocale $\mathsf{O} = \{1\}$ (the *empty sublocale*).

Note that the sublocales just defined naturally represent subobjects in the category of locales: S is a sublocale of L iff the embedding map $j\colon S \subseteq L$ is an extremal monomorphism in the category **Loc**.

1.4.1. Points. Recall that an element $p \neq 1$ in L is *prime* if $a \wedge b = p$ implies that either $a = p$ or $b = p$ (or, equivalently, if $a \wedge b \leq p$ implies that either $a \leq p$ or $b \leq p$). It is easy to check that a sublocale S has precisely two elements iff it is
$$\widetilde{p} = \{p, 1\} \quad \text{with } p \text{ prime.}$$
These sublocales will be referred as the *points* of L (note that this makes the points of L into a natural one-to-one correspondence with the points of the spectrum of L — see e.g. [9, II.5.3], where one speaks of *meet-irreducibles* instead of primes).

For more about frames and locales the reader may consult, e.g. [8, 9] or [7].

1.5. Scattered and weakly scattered spaces. A space X is said to be *scattered* if for every non-empty closed set A there is an *isolated point* $a \in A$, that is, there is an $a \in A$ and an open $U \ni a$ such that
$$U \cap A = \{a\}.$$

It is *weakly scattered* (or *corrupted* [11]), if for every non-empty closed set A there is an $a \in A$ and an open $U \ni a$ such that
$$U \cap A \subseteq \overline{\{a\}}.$$

2. T_D-spaces and induced sublocales

In this section, besides more definitions, we reproduce some facts from elsewhere.

2.1. T_D-spaces. A T_D-space satisfies the axiom

T_D: *for every $x \in X$ there is an open set $U \ni x$ such that $U \setminus \{x\}$ is still open*

(and hence $U \setminus \{x\} = U \setminus \overline{\{x\}}$). This axiom, strictly between T_0 and T_1 was introduced in [1] for purposes not connected with anything we are discussing here; but already in [12] it found its use in point-free topology.

From [3] we will need the following two facts.

2.1.1. Lemma. *Let X satisfy T_D. Then:*

(1) *every $(X \setminus \overline{\{x\}}) \cup \{x\}$ is open, and*

(2) *the primes $p = X \setminus \overline{\{x\}}$ are covered, that is, if $p = \bigwedge_{i \in J} U_i$ then $p = U_k$ for some $k \in J$ (for arbitrary J).*

Remark. It should be noted that the elements p such that $p = \bigwedge_{i \in J} x_i$ implies $p = x_i$ for some $i \in J$ were referred to in [3] as *completely prime*. That term, however, is generally taken to mean that $p \leq \bigwedge_{i \in J} x_i$ implies $p \leq x_i$ for some $i \in J$. Any completely prime p is clearly a covered prime, but not conversely: in the topology of a T_1-space X, any $X \setminus \{x\}$, $x \in X$, is obviously a covered prime but the complete primes are only the $X \setminus \{x\}$ with isolated $x \in X$.

2.2. Subspaces represented as sublocales (induced sublocales). Let $j \colon Y \subseteq X$ be an embedding of a subspace Y into a space X. We have the localic embedding

$$\iota_Y \colon \Omega(Y) \to \Omega(X)$$

adjoint to the (quotient) frame homomorphism

$$\Omega(j) = (U \mapsto U \cap Y) \colon \Omega(X) \to \Omega(Y).$$

Since $\Omega(j)$ is onto, ι_Y is an isomorphic embedding and we have an isomorphic imprint

$$S_Y = \iota_Y[\Omega(Y)] \text{ of } \Omega(Y) \text{ in } \Omega(X).$$

We speak of the S_Y as of the *sublocale induced by the subspace* Y.

One often speaks of an induced sublocale as of a *subspace* of the frame in question. It does not seem to create confusion.

A more explicit description of S_Y will be in fact also more intuitive. Denote by $p_{X,x}$ the prime $X \smallsetminus \overline{\{x\}}$ in $\Omega(X)$ and recall the notation $\widetilde{p} = \{p, 1\}$ from 1.4.1. By the formula for joins in $\mathsf{S}(\Omega(X))$ (see 1.4: from $U = \bigcap\{X \smallsetminus \overline{\{x\}} \mid x \notin U\}$) we obviously have

$$\Omega(X) = \bigvee_{x\in X} \widetilde{p}_{X,x}$$

("a spatial frame $\Omega(X)$ is the join of its points"). The adjoint localic map ι_Y above is given by the formula

$$\iota_Y(V) = \mathrm{int}((X \smallsetminus Y) \cup V)$$

(since $U \cap Y \subseteq V$ iff $U \subseteq (X \smallsetminus Y) \cap V$ and we use this equivalence for open U). Hence (denoting by $\overline{A}^Y$ the closure of A in Y)

$$\iota_Y(p_{Y,y}) = \mathrm{int}((X \smallsetminus Y) \cup (Y \smallsetminus \overline{\{y\}}^Y)) = \mathrm{int}(X \smallsetminus \overline{\{y\}}^Y) = X \smallsetminus \overline{\{y\}} = p_{X,y},$$

and since ι, as a right adjoint, preserves meets we see that

$$\begin{aligned}\iota[\Omega(Y)] = \iota[\{\bigwedge_{y\in A} p_{Y,y} \mid A \subseteq Y\}] = \{\iota(\bigwedge_{y\in A} p_{Y,y}) \mid A \subseteq Y\} = \\ = \{\bigwedge_{y\in A} \iota(p_{Y,y}) \mid A \subseteq Y\} = \{\bigwedge_{y\in A} p_{X,y} \mid A \subseteq Y\}.\end{aligned}$$

Thus we conclude that

2.2.1. $S_Y = \bigvee_{y\in Y} \widetilde{p}_{X,y}$.

2.3. As we have already mentioned, frames (locales) can be viewed as a natural generalization of (sober) topological spaces. In fact, if we wish to have also the representation of the structure of the system of subspaces correct, we should restrict ourselves to T_D-spaces. One has the following fact ([2, 9]).

2.3.1. Proposition. *Distinct subspaces Y, Z of a space X are represented by distinct sublocales S_Y, S_Z of $\Omega(X)$ if and only if X is a T_D-space.*

2.3.2. Note. In general, in a space one has more sublocales than subspaces. This is an agreeable fact of point-free topology, throwing light on some important phenomena. The trouble with the non-T_D spaces is that the sublocales are not able to distinguish the classical subspaces, not in the existence of non-spatial subobjects.

2.4. Convention. Because of 2.3.1 let us agree that we will restrict ourselves, in the sequel, to T_D-spaces. Another advantage for this restriction is the following fact (see e.g. [11]).

2.4.1. Proposition. *For T_D-spaces, the notions of scattered and weakly scattered coincide.*

In particular, the Simmons' Theorem ([11]) can be interpreted as follows.

2.4.2. Theorem. *If X is a T_D-space then $\mathsf{S}(\Omega(X))$ is Boolean, that is, all the sublocales of $\Omega(X)$ are complemented, if and only if X is scattered.*

3. Complemented subspaces and Hewitt's irresolvability

3.1. The system of subspaces of a space is, trivially, a (complete) Boolean algebra. As an induced sublocale, however, a subspace is typically not complemented (consider, e.g. any dense subspace of the real line). We will now, first, characterize those subspaces that are.

3.2.1. Lemma. *Let Y be a subspace of a T_D-space X. Then*

$$T = \bigvee\{\{X \smallsetminus \overline{\{x\}}, X\} \mid x \notin Y\}$$

is the supplement of $S = S_Y$.

Proof: Suppose not. Since obviously $T \vee S_Y = L = \Omega(X)$, and since the supplement $S^\#$ exists, we have $S^\# \subsetneq T$. Hence there is a $X \smallsetminus \overline{\{z\}} \notin S^\#$ such that $z \notin Y$. Since $S^\# \vee S = L$, $X \smallsetminus \overline{\{z\}} = U \cap V$ for some $U \in S^\#$ and $V \in S$. Now $X \smallsetminus \overline{\{z\}} \neq U$ and hence

$$X \smallsetminus \overline{\{z\}} = V = \bigcap\{X \smallsetminus \overline{\{x\}} \mid x \in A\}$$

for some $A \subseteq Y$. But our space is T_D and hence, by 2.1.1(2), $X \smallsetminus \overline{\{z\}} = X \smallsetminus \overline{\{x\}}$ for some $x \in Y$, that is, $z \in Y$, a contradiction. ∎

3.2.2. Subsets $A, B \subseteq X$ are said to be *equi-dense* if $\overline{A} = \overline{B}$.

3.2.3. Theorem. *A subspace Y is complemented as a sublocale (that is, S_Y is complemented in $\mathsf{S}(\Omega(X))$ iff there are no non-empty equi-dense sets A, B such that $A \subseteq Y$ and $B \subseteq X \smallsetminus Y$.*

Proof: By the Lemma, S is complemented iff $S \cap T = \mathsf{O}$. We have $S \cap T \neq \mathsf{O}$ iff there are $\emptyset \neq A \subseteq Y$ and $B \subseteq X \smallsetminus Y$ such that

$$\bigwedge\{X \smallsetminus \overline{\{x\}} \mid x \in A\} = \bigwedge\{X \smallsetminus \overline{\{y\}} \mid y \in B\},$$

that is,

$$\operatorname{int}\bigcap\{X \smallsetminus \overline{\{x\}} \mid x \in A\} = \operatorname{int}\bigcap\{X \smallsetminus \overline{\{y\}} \mid y \in B\},$$

that is,

$$X \smallsetminus \overline{\bigcup_{x\in A}\overline{\{x\}}} = X \smallsetminus \overline{A} = X \smallsetminus \overline{B} = X \smallsetminus \overline{\bigcup_{y\in B}\overline{\{y\}}},$$

that is, iff $\overline{A} = \overline{B}$. ■

3.3. As mentioned in the Introduction, Hewitt defined in [6] a space X as *irresolvable* if there are no disjoint dense $A, B \subseteq X$. Localizing this concept one then obtains the *hereditary irresolvability*

$$\forall A, B \subseteq X, \quad \emptyset \neq \overline{A} = \overline{B} \quad \Rightarrow \quad A \cap B \neq \emptyset. \tag{HI}$$

As an immediate consequence of 3.2.3 we get

3.3.1. Theorem. *Every subspace of a space X is complemented as a sublocale (that is, S_Y is complemented in $\mathsf{S}(\Omega(X))$ iff X is hereditarily irresolvable.*

3.4. The following is an immediate

Observation. *Every scattered space is hereditarily irresolvable.*
(Indeed: each isolated element of $\overline{A}$ is in A.)

The question naturally arises whether this can be reversed, that is, whether hereditarily irresolvable spaces are necessarily scattered. Already from [6] one can infer that this holds true e.g. for every metrizable, or every locally compact Hausdorff space. In [4] the authors proved (a.o.) the equivalence of HI with another interesting property (*Hausdorff-irreducibility*) and proved the desired reverse implication for a much broader class of spaces. We can recommend [4] as a source of many interesting relevant facts, and of literature on the subject.

To simplify the terminology let us speak of spaces in which the reverse implication holds as of *$\mathcal{H}$-spaces.* Thus, X is an $\mathcal{H}$-space if

either it is not hereditarily irresolvable or it is scattered.

(Let us note right away that non-$\mathcal{H}$-spaces exist; an example from the literature will be analyzed in the last section.)

From 2.4.2, 3.2.3 and 3.3.1 we obtain

3.4.1. Theorem. *For an $\mathcal{H}$-space X (in particular, for a metrizable or locally compact Hausdorff one) the following statements are equivalent.*

(1) *X is scattered.*
(2) *Every sublocale of $\Omega(X)$ is complemented.*

(3) *Every subspace* (*induced sublocale*) *of* $\Omega(X)$ *is complemented.*

3.4.2. Note. Recall the standard fact ([8, 9]) that

every complemented sublocale of $\Omega(X)$ is induced.

(This immediately follows from 1.1.2 and 2.2: If S is complemented then

$$S = \bigvee\{\{p,1\} \mid p \text{ prime in } \Omega(X)\} \cap S =$$
$$= \bigvee\{\{p,1\} \cap S \mid p \text{ prime in } \Omega(X)\} = \bigvee\{\{p,1\} \mid p \text{ prime in } S\}.\ )$$

Hence, in an $\mathcal{H}$-space,

- there exists a non-induced sublocale iff
- there exists a non-complemented induced sublocale iff
- there are non-empty disjoint A, B with $\overline{A} = \overline{B}$.

Thus, in the $\mathcal{H}$-context, non-induced sublocales appear only in connection with the Isbell's minimal density phenomenon (see e.g. [7] or [9]). For the non-$\mathcal{H}$-spaces, however, there are non-induced sublocales based on quite different principles.

4. A note on non-$\mathcal{H}$-spaces

We close with the description of a class of non-$\mathcal{H}$-spaces mentioned in [4, Example 2.5] (this example is originally due to El'kin [5]).

4.1. Submaximal spaces. In his 1943 paper, Hewitt also introduced submaximal spaces. A topological space X is *submaximal* if every dense subset of X is open.

The following is well known and shows that any dense-in-itself submaximal space (named by Hewitt as an *MI space*) is a non-$\mathcal{H}$-space.

4.1.1. Proposition. *Any submaximal space is hereditarily irresolvable.*

Proof: Suppose there is a nonempty resolvable subspace Y of a submaximal space X. We may assume Y is the disjoint union $A \sqcup B$ with A, B dense in Y. Then $X \smallsetminus A = (X \smallsetminus Y) \cup B$ is dense in X hence open by submaximality. This means that A is closed. Then $Y \subseteq \overline{A} = A$, that is, $B = \emptyset$ and therefore $Y = \emptyset$ (since $Y \subseteq \overline{B}$), a contradiction. ■

4.2. An example. Let us illustrate what happens by the following example.

Consider an infinite set X. By Zorn's Lemma, each non-trivial filter on X can be extended to a non-trivial ultrafilter (this is the Boolean Ultrafilter Theorem — in fact, BUT is a choice principle known to be weaker than the axiom of choice). Let $\mathcal{F}$ be an ultrafilter extending the filter of cofinite subsets $\mathcal{F}_0$ (the *Fréchet filter*.) Obviously

$$\tau = \mathcal{F} \cup \{\emptyset\}$$

is a topology on X.

4.2.1. Observation. *(X, τ) is a dense-in-itself T_1-space. In particular, it is not scattered.*

(Indeed, each $(X \smallsetminus \{x\})$ is already in $\mathcal{F}_0$, and hence open, and obviously no $\{x\}$ is open.)

4.2.2. Observation. *For the closure in (X, τ) we have*

$$\overline{Y} = \begin{cases} Y \ \mathit{if} \ \ Y = X \ \ \mathit{or} \ Y \notin \mathcal{F}, \\ X \ \mathit{if} \ \ Y \in \mathcal{F} \end{cases}$$

(thus, $Y \subseteq X$ is dense iff $Y \in \mathcal{F}$). Similarly,

$$\operatorname{int} Y = \begin{cases} \emptyset \ \mathit{if} \ \ Y = \emptyset \ \ \mathit{or} \ Y \notin \mathcal{F}, \\ Y \ \mathit{if} \ Y \ \in \mathcal{F}. \end{cases}$$

(This immediately follows from the property of an ultrafilter that for any $Y \subseteq X$ either $Y \in \mathcal{F}$ or $X \smallsetminus Y \in \mathcal{F}$, hence each $Y \subseteq X$ is either open or closed.)

4.2.3. Observation. *(X, τ) is hereditarily irresolvable.*

(Indeed, if $\emptyset \neq \overline{A} = \overline{B}$ and $A \cap B = \emptyset$ then, say, $A \in \mathcal{F}$ and $B \notin \mathcal{F}$ and hence $A \subseteq \overline{A} = \overline{B} = B$, a contradiction.)

4.2.4. Now let us look at the concrete phenomena relevant to our satements about sublocales. By 2.2 we have the sublocale induced by a $Y \subseteq X$ given by the formula

$$S_Y = \{\operatorname{int}((X \smallsetminus Y) \cup (U \cap Y)) \mid U \in \tau\}.$$

Since

$$(X \smallsetminus Y) \cup (U \cap Y) \supseteq (U \smallsetminus Y) \cup (U \cap Y) = U$$

we have

$$(X \smallsetminus Y) \cup (U \cap Y) = (X \smallsetminus Y) \cup U$$

and by 4.2.2,

$$S_Y = \begin{cases} \{(X \smallsetminus Y) \cup U \mid U \in \tau\} \text{ if } Y \notin \mathcal{F} \\ \{(X \smallsetminus Y) \cup U \mid U \in \tau\} \cup \{\emptyset\} \text{ if } Y \in \mathcal{F}. \end{cases}$$

To see the mechanism of the complementarity of induced sublocales in this space let us consider $Y, Z \subseteq X$ with $Y \cap Z = \emptyset$; we want to show that $S_Y \cap S_Z = \mathsf{O}$. Thus, let there be a $W \in S_Y \cap S_Z$, that is,

$$W = (X \smallsetminus Y) \cup U = Y \cup V \quad \text{for some} \quad U, V \in \mathcal{F}.$$

Then

$$(Y \cup V) \cap (X \smallsetminus Y) = ((X \smallsetminus Y) \cup U) \cap (X \smallsetminus Y) = (X \smallsetminus Y)$$

and hence $Y \smallsetminus Y \subseteq Y \cap V$, and since also $Y \subseteq Y \cap V$ we have $W = Y \cap V = X$.

A sublocale S of $\Omega(X, \tau)$ is dense if $\emptyset \in S$ (see e.g. [9, III.8]: closed sublocales of a frame L are the subsets $\uparrow a \subseteq L$ and hence the closure of T, the least closed sublocale containing T, is $\uparrow \bigwedge T$ and hence T is dense iff it contains the smallest element of L). The minimum dense sublocale of L is known to be the subset $\{a^* \mid a \in L\}$ of all the pseudocomplements in L. In our case, since $U \cap V = \emptyset$ only if some of the U, V is $\emptyset$, this set of pseudocomplements is $\{\emptyset, X\}$ and indeed a sublocale S is dense iff $\{\emptyset, X\} \subseteq S$. Finally, $\{\emptyset, X\}$ itself is not induced: if it were some of the S_Y, because of the $\emptyset$, Y would have to be in $\mathcal{F}$. But then each $(X \smallsetminus Y) \cup U$ with $U \in \mathcal{F}$, for instance with any $X \smallsetminus \{x\}$, would have to be equal to X contradicting the fact that $Y \in \mathcal{F}$ and hence is infinite.

4.2.5. Note. (1) The use of an ultrafilter was essential. Just an extension of $\mathcal{F}_0$ to a more suitable filter would not help. The question naturally arises whether one can have an example of a non-scattered hereditarily irresolvable space without using a choice principle.
(2) Theorems 2.4 and 2.11 of [4] indicate exactly when an HI space is scattered. For a space X, let $\mathcal{D}(X)$ denote the set of all dense subsets of X. By [4, 2.4, 2.11], an HI space X is scattered whenever $\mathcal{D}(Y)$ is contained in a principal ultrafilter for every nonempty closed subspace Y of X; and X is not scattered if $\mathcal{D}(X)$ is not contained in a principal ultrafilter.
(3) The minimal dense sublocale in 4.2.4 is a point in the sense of 1.4.1. It is a point of the sobrification of X, not of X itself; the sobrification is, however, not a T_D-space (no non-trivial sobrification is — see [9, VI.2.3.2]).

4.3. More examples. The space in 4.2.1 can be easily modified to provide examples of non-$\mathcal{H}$-spaces that are not submaximal. In fact, consider disjoint infinite sets X and Y and let $\mathcal{F}$ be a free ultrafilter on X and let $\mathcal{G}$ be a free ultrafilter on Y. Then let Z be the disjoint union of X and Y equipped with the topology

$$\tau = \{A \sqcup B \mid A \in \mathcal{F}, B \in \mathcal{G}\} \cup \{\emptyset\}.$$

It is easy to check that (Z, τ) is a non-submaximal, dense-in-itself, hereditarily irresolvable T_1-space (see [10, Example 2.5] for the details).

References

[1] C. E. Aull and W. J. Thron, *Separation axioms between T_0 and T_1*, Indag. Math. **24** (1963), 26-37.

[2] B. Banaschewski and A. Pultr, *Variants of openness*, Appl. Categ. Structures **2** (1994), 331-350.

[3] B. Banaschewski and A. Pultr, *Pointfree aspects of the T_D axiom of classical topology*, Quaest. Math. **33** (2010), 369-385.

[4] G. Bezhanishvili, R. Mines and P. Morandi, *Scattered, Hausdorff-reducible, and hereditarily irresolvable spaces*, Topology Appl. **132** (2003), 291-306.

[5] A.G. El'kin, *Ultrafilters and undecomposable spaces*, Moscow Univ. Math. Bull. 24 (1969), 37-40. Russian original in: *Ultrafilters and irresolvable spaces*, Vestnik Moskov. Univ. Ser. I Mat. Meh. **24** (1969), 51-56.

[6] E. Hewitt, *A problem of set-theoretic topology*, Duke Math. J. **10** (1943), 309-333.

[7] J. R. Isbell, *Atomless parts of spaces*, Math. Scand. **31** (1972), 5-32.

[8] P. T. Johnstone, *Stone Spaces*, Cambridge Univ. Press, Cambridge, 1982.

[9] J. Picado and A. Pultr, *Frames and Locales: topology without points*, Frontiers in Mathematics, vol. 28, Springer, Basel, 2012.

[10] D. Rose, K. Sizemore and B. Thurston, *Strongly irresolvable spaces*, Int. J. Math. Math. Sci. (2006), Art. ID 53653, 12 pp.

[11] H. Simmons, *Spaces with Boolean assemblies*, Colloq. Math. **43** (1980), 23-29.

[12] W. J. Thron, *Lattice-equivalence of topological spaces*, Duke Math. J. **29** (1962), 671-679.

Dharmanand Baboolal
School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal, Durban 4000, South Africa
E-mail address: `baboolald@ukzn.ac.za`

Jorge Picado
CMUC, Department of Mathematics, University of Coimbra, 3001-501 Coimbra, Portugal
E-mail address: `picado@mat.uc.pt`

Paranjothi Pillay
School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal, Durban 4000, South Africa
E-mail address: `pillaypi@ukzn.ac.za`

Aleš Pultr
Department of Applied Mathematics and CE-ITI, MFF, Charles University, Malostranské nám. 24, 11800 Praha 1, Czech Republic
E-mail address: pultr@kam.mff.cuni.cz