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RIEMANN–HILBERT PROBLEM AND MATRIX BIORTHOGONAL POLYNOMIALS

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Abstract: Recently the Riemann–Hilbert problem with jumps supported on appropriate curves in the complex plane has been presented for matrix biorthogonal polynomials, in particular non–Abelian Hermite matrix biorthogonal polynomials in the real line, understood as those whose matrix of weights is a solution of a Sylvester type Pearson equation with matrix entire function coefficients. We will explore this discussion, present some achievements and consider some new examples of weights for matrix biorthogonal polynomials.

Keywords: Riemann–Hilbert problems; matrix Pearson equations; Markov functions; matrix biorthogonal polynomials.
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1. Introduction

Back in 1949, in the seminal papers [36, 37], Krein discussed matrix extensions of real orthogonal polynomials. Afterwards, this matrix extension of the standard orthogonality was studied only sporadically until the last decade of the XX century, see [2], [33] and [1]. In[1], for a kind of discrete Sturm–Liouville operators, the authors solved the corresponding scattering problem and found a matrix version of Favard's theorem, polynomials that satisfy the three term relation

$$xP_k(x) = A_kP_{k+1}(x) + B_kP_k(x) + A^*_{k-1}P_{k-1}(x), \qquad k = 0, 1, \dots,$$

are orthogonal with respect to a positive definite matrix of measures.

Along the last decade, a number of basic results on the theory of scalar orthogonal polynomials, such as Favard's theorem [15, 16, 27, 30], quadrature

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formulae [17, 21, 29] and asymptotic properties (Markov's theorem [17], ratio [18, 19] weak [20] and zero asymptotics [28]), have been extended to the matrix scenario. The search of families of matrix orthogonal polynomials that satisfy second order differential equations with coefficients independent of n, can be found in [23, 24, 25, 26, 22]. This can be considered as a matrix extension of the classical orthogonal polynomial sequences of Hermite, Laguerre and Jacobi.

Fokas, Its and Kitaev, when discussing 2D quantum gravity, discovered that certain Riemann–Hilbert problem was solved in terms of orthogonal polynomials in the real line (OPRL), [31]. Namely, it was found that the solution of a 2×2 Riemann–Hilbert problem can be expressed in terms of orthogonal polynomials in the real line and its Cauchy transforms. Later on, Deift and Zhou combined these ideas with a non–linear steepest descent analysis in a series of papers [10, 11, 13, 14] which was the seed for a large activity in the field. Relevant results to be mentioned here are the study of strong asymptotic with applications in random matrix theory, [10, 12], the analysis of determinantal point processes [7, 8, 38, 39], applications to orthogonal Laurent polynomials [40, 41] and Painlevé equations [9, 35].

The Riemann–Hilbert problem characterization is a powerful tool that allows one to prove algebraic and analytic properties of orthogonal polynomials. The Riemann–Hilbert problem for this matrix situation and the appearance of non–Abelian discrete versions of Painlevé I (mdPI) were explored in [4], and the appearance of singularity confinement was shown in [5]. The analysis was extended further [6] for the matrix Szegő type orthogonal polynomials in the unit circle and corresponding non–Abelian versions discrete Painlevé II equations. For an alternative discussion of the use of Riemann–Hilbert problem for MOPRL see [34], were the authors focus on the algebraic aspects of the problem, obtaining difference and differential relations satisfied by the corresponding orthogonal polynomials.

In [3] we have studied a Hermite–type biorthogonal matrix polynomial system from a Riemann–Hilbert problem. In this case the matrix measure extends to an entire function on the complex plane. We have considered three types of matrix weights, W, obtained from the solution of a generalized Pearson differential or Sylvester differential equation, i.e.

$$W' = h^{\mathsf{L}}(z)W(z) + W(z)h^{\mathsf{R}}(z),$$

where h^{L} and h^{R} are polynomials of the first, second or third degrees.

In that paper we shew that for these weights the matrix solutions of the Riemann–Hilbert problem

$$Y_n^{\mathsf{L}} = \begin{bmatrix} P_n^{\mathsf{L}} & Q_n^{\mathsf{L}} \\ -C_{n-1}P_{n-1}^{\mathsf{L}} & -C_{n-1}Q_{n-1}^{\mathsf{L}} \end{bmatrix}, \quad Y_n^{\mathsf{R}} = \begin{bmatrix} P_n^{\mathsf{R}} & -C_{n-1}Q_{n-1}^{\mathsf{R}} \\ -C_{n-1}P_{n-1}^{\mathsf{R}} & Q_n^{\mathsf{R}} \end{bmatrix},$$

satisfies

$$\Big(Y_n^{\mathsf{L}} \exp\big(\int_0^z \mathcal{C}_{\mathsf{L}}'(t)\mathcal{C}_{\mathsf{L}}^{-1}(t)\,dt\big)\Big)' = M_n^{\mathsf{L}}\Big(Y_n^{\mathsf{L}} \exp\big(\int_0^z \mathcal{C}_{\mathsf{L}}'(t)\mathcal{C}_{\mathsf{L}}^{-1}(t)\,dt\big)\Big),$$

$$\Big(\exp\big(\int_0^z \mathcal{C}_{\mathsf{R}}'(t)\mathcal{C}_{\mathsf{R}}^{-1}(t)\,dt\big)Y_n^{\mathsf{R}}\Big)' = \Big(\exp\big(\int_0^z \mathcal{C}_{\mathsf{R}}'(t)\mathcal{C}_{\mathsf{R}}^{-1}(t)\,dt\big)Y_n^{\mathsf{R}}\Big)M_n^{\mathsf{R}},$$

with $\mathcal{C}_{\mathsf{L}}'(t)\mathcal{C}_{\mathsf{L}}^{-1}(t) = \begin{bmatrix} h^{\mathsf{L}} & 0_N \\ 0_N & -h^{\mathsf{R}} \end{bmatrix}$, $\mathcal{C}_{\mathsf{R}}^{-1}(t)\mathcal{C}_{\mathsf{R}}'(t) = \begin{bmatrix} h^{\mathsf{R}} & 0_N \\ 0_N & -h^{\mathsf{L}} \end{bmatrix}$, where M_n^{L} and M_n^{R} are defined by

$$(T_n^{\mathsf{L}})' = M_{n+1}^{\mathsf{L}}T_n^{\mathsf{L}} - T_n^{\mathsf{L}}M_n^{\mathsf{L}}, \qquad (T_n^{\mathsf{R}})' = T_n^{\mathsf{R}}M_{n+1}^{\mathsf{R}} - M_n^{\mathsf{R}}T_n^{\mathsf{R}},$$

and $\{T_n^{\mathsf{L}}\}$, $\{T_n^{\mathsf{R}}\}$ are sequences of transfer matrices, i.e.

$$Y_{n+1}^{\mathsf{L}} = T_n^{\mathsf{L}}Y_n^{\mathsf{L}}, \qquad Y_{n+1}^{\mathsf{R}} = Y_n^{\mathsf{R}}T_n^{\mathsf{R}}, \tag{1}$$

with

$$T_n^{\mathsf{L}}(z) = \begin{bmatrix} zI_N - \beta_n^{\mathsf{L}} & C_n^{-1} \\ -C_n & 0_N \end{bmatrix}, \qquad T_n^{\mathsf{R}}(z) = \begin{bmatrix} zI_N - \beta_n^{\mathsf{R}} & -C_n \\ C_n^{-1} & 0_N \end{bmatrix}. \tag{2}$$

In this work we extend the Riemann–Hilbert characterization for a more general class of measures, a matrix extension of the classical scalar Laguerre measures, and we find that for this class of measures appears a power logarithmic type singularities at the end point of the support of the measure.

2. Riemann–Hilbert problem for Matrix Biorthogonal Polynomials

2.1. Matrix biorthogonal polynomials.

Let

$$W = \begin{bmatrix} W^{(1,1)} & \cdots & W^{(1,N)} \\ \vdots & \ddots & \vdots \\ W^{(N,1)} & \cdots & W^{(N,N)} \end{bmatrix} \in \mathbb{C}^{N\times N}$$

be a $N \times N$ matrix of weights with support on a smooth oriented non self–intersecting curve γ in the complex plane $\mathbb{C}$, i.e. $W^{(j,k)}$ is, for each $j, k \in$

$\{1,\dots,N\}$, a complex weight with support on γ. We define the *moment of order* n associated with W as

$$W_n = \frac{1}{2\pi\,\mathrm{i}}\int_\gamma z^n W(z)\,\mathrm{d}\,z, \qquad n\in\mathbb{N}:=\{0,1,2,\dots\}.$$

We say that W is *regular* if $\det\big[W_{j+k}\big]_{j,k=0,\dots,n}\neq 0$, $n\in\mathbb{N}$. In this way, we define a *sequence of matrix monic polynomials*, $\big\{P^{\mathsf{L}}_n(z)\big\}_{n\in\mathbb{N}}$, *left orthogonal* and *right orthogonal*, $\big\{P^{\mathsf{R}}_n(z)\big\}_{n\in\mathbb{N}}$ with respect to a regular matrix measure W, by the conditions,

$$\frac{1}{2\pi\,\mathrm{i}}\int_\gamma P^{\mathsf{L}}_n(z)W(z)z^k\,\mathrm{d}\,z = \delta_{n,k}C_n^{-1}, \tag{3}$$

$$\frac{1}{2\pi\,\mathrm{i}}\int_\gamma z^k W(z)P^{\mathsf{R}}_n(z)\,\mathrm{d}\,z = \delta_{n,k}C_n^{-1}, \tag{4}$$

for $k=0,1,\dots,n$ and $n\in\mathbb{N}$, where C_n is an nonsingular matrix.

Notice that neither the matrix of weights is requested to be Hermitian nor the curve γ to be the real line, i.e., we are dealing, in principle with nonstandard orthogonality and, consequently, with biorthogonal matrix polynomials instead of orthogonal matrix polynomials.

The matrix of weights induce a sesquilinear form in the set of matrix polynomials $\mathbb{C}^{N\times N}[z]$ given by

$$\langle P,Q\rangle_W := \frac{1}{2\pi\,\mathrm{i}}\int_\gamma P(z)W(z)Q(z)\,\mathrm{d}\,z. \tag{5}$$

Moreover, we say that $\big\{P^{\mathsf{L}}_n(z)\big\}_{n\in\mathbb{N}}$ and $\big\{P^{\mathsf{R}}_n(z)\big\}_{n\in\mathbb{N}}$ are biorthogonal with respect to a matrix weight functions W if

$$\big\langle P^{\mathsf{L}}_n, P^{\mathsf{R}}_m\big\rangle_W = \delta_{n,m}C_n^{-1}, \qquad n,m\in\mathbb{N}. \tag{6}$$

As the polynomials are chosen to be monic, we can write

$$P^{\mathsf{L}}_n(z) = I_N z^n + p^1_{\mathsf{L},n}z^{n-1} + p^2_{\mathsf{L},n}z^{n-2}+\dots+p^n_{\mathsf{L},n},$$
$$P^{\mathsf{R}}_n(z) = I_N z^n + p^1_{\mathsf{R},n}z^{n-1} + p^2_{\mathsf{R},n}z^{n-2}+\dots+p^n_{\mathsf{R},n},$$

with matrix coefficients $p^k_{\mathsf{L},n}, p^k_{\mathsf{R},n}\in\mathbb{C}^{N\times N}$, $k=0,\dots,n$ and $n\in\mathbb{N}$ (imposing that $p^0_{\mathsf{L},n}=p^0_{\mathsf{R},n}=I$, $n\in\mathbb{N}$). Here $I\in\mathbb{C}^{N\times N}$ denotes the identity matrix.

We define the *sequence of second kind matrix functions* by

$$Q^{\mathsf{L}}_n(z) := \frac{1}{2\pi \mathrm{i}}\int_\gamma \frac{P^{\mathsf{L}}_n(z')}{z'-z} W(z')\,\mathrm{d}\,z', \tag{7}$$

$$Q^{\mathsf{R}}_n(z) := \frac{1}{2\pi \mathrm{i}}\int_\gamma W(z')\frac{P^{\mathsf{R}}_n(z')}{z'-z}\,\mathrm{d}\,z', \tag{8}$$

for $n \in \mathbb{N}$. From the orthogonality conditions (3) and (4) we have, for all $n \in \mathbb{N}$, the following asymptotic expansion near infinity for the sequence of functions of the second kind

$$Q^{\mathsf{L}}_n(z) = -C_n^{-1}\big(I_N z^{-n-1} + q^1_{\mathsf{L},n} z^{-n-2} + \cdots\big), \tag{9}$$

$$Q^{\mathsf{R}}_n(z) = -\big(I_N z^{-n-1} + q^1_{\mathsf{R},n} z^{-n-2} + \cdots\big)C_n^{-1}. \tag{10}$$

Assuming that the measures $W^{(j,k)}$, $j, k \in \{1, \ldots, N\}$ are Hölder continuous we obtain, by the Plemelj's formula applied to (7) and (8), the following fundamental jump identities

$$\big(Q^{\mathsf{L}}_n(z)\big)_+ - \big(Q_n(z)^{\mathsf{L}}\big)_- = P^{\mathsf{L}}_n(z)W(z), \tag{11}$$

$$\big(Q^{\mathsf{R}}_n(z)\big)_+ - \big(Q^{\mathsf{R}}_n(z)\big)_- = W(z)P^{\mathsf{R}}_n(z), \tag{12}$$

$z \in \gamma$, where, $\big(f(z)\big)_\pm = \lim\limits_{\epsilon\to 0^\pm} f(z+i\epsilon)$; here $\pm$ indicates the the positive/negative region according to the orientation of the curve γ.

2.2. Reductions: from biorthogonality to orthogonality. We consider two possible reductions for the matrix of weights, the symmetric reduction and the Hermitian reduction.

i) A matrix of weights $W(z)$ with support on γ is said to be symmetric if

$$(W(z))^\top = W(z), \qquad z \in \gamma.$$

ii) A matrix of weights $W(x)$ with support on $\mathbb{R}$ is said to be Hermitian if

$$(W(x))^\dagger = W(x), \qquad x \in \mathbb{R}.$$

These two reductions leads to orthogonal polynomials, as the two biorthogonal families are identified; i.e., for the symmetric case

$$P^{\mathsf{R}}_n(z) = \big(P^{\mathsf{L}}_n(z)\big)^\top, \qquad Q^{\mathsf{R}}_n(z) = \big(Q^{\mathsf{L}}_n(z)\big)^\top, \qquad z \in \mathbb{C},$$

and for the Hermitian case, with $\gamma = \mathbb{R}$,

$$P^{\mathsf{R}}_n(z) = \big(P^{\mathsf{L}}_n(\bar z)\big)^\dagger, \qquad Q^{\mathsf{R}}_n(z) = \big(Q^{\mathsf{L}}_n(\bar z)\big)^\dagger, \qquad z \in \mathbb{C}.$$

In both cases biorthogonality collapses into orthogonality, that for the symmetric case reads as

$$\frac{1}{2\pi \mathrm{i}}\int_\gamma P_n(z)W(z)\big(P_m(z)\big)^\top \mathrm{d}\, z=\delta_{n,m}C_n^{-1}, \qquad n,m\in\mathbb{N},$$

while for the Hermitian case can be written as follows

$$\frac{1}{2\pi \mathrm{i}}\int_{\mathbb{R}} P_n(x)W(x)\big(P_m(x)\big)^\dagger \mathrm{d}\, x=\delta_{n,m}C_n^{-1}, \qquad n,m\in\mathbb{N},$$

where $P_n=P_n^{\mathsf{L}}$.

2.3. The Riemann–Hilbert problem. Let us consider the particular case when the $N\times N$ matrix of weights with support on a smooth oriented non self–intersecting curve γ has entrywise power logarithmic type algebraic singularities at the boundary of the support of the measure, that is the entries $W^{j,k}$ of the matrix measure W can be described as

$$W^{j,k}(z)=\sum_{m\in I_{j,k}} h_m(z)(z-c)^{\alpha_m}\log^{p_m}(z)$$

where $I_{j,k}$ denotes a finite set of indexes, $\alpha_m>-1$, $p_m\in\mathbb{N}$ and $h_m(x)$ is Hölder continuous, bounded and non–vanishing on γ.

The biorthogonality can be characterized in terms of a left and right Riemann–Hilbert formulation,

Theorem 1.

i) *The matrix function*

$$Y_n^{\mathsf{L}}(z):=\begin{bmatrix} P_n^{\mathsf{L}}(z) & Q_n^{\mathsf{L}}(z)\\ -C_{n-1}P_{n-1}^{\mathsf{L}}(z) & -C_{n-1}Q_{n-1}^{\mathsf{L}}(z)\end{bmatrix},$$

is, for each $n\in\mathbb{N}$, the unique solution of the Riemann–Hilbert problem; which consists in the determination of a $2N\times 2N$ complex matrix function such that:

(RHL1): *$Y_n^{\mathsf{L}}(z)$ is holomorphic in $\mathbb{C}\setminus\gamma$;*

(RHL2): *has the following asymptotic behaviour near infinity,*

$$Y_n^{\mathsf{L}}(z)=\Big(I_N+\sum_{j=1}^{\infty}(z^{-j})Y_n^{j,\mathsf{L}}\Big)\begin{bmatrix} I_N z^n & 0_N\\ 0_N & I_N z^{-n}\end{bmatrix};$$

(RHL3): *satisfies the jump condition*

$$\big(Y_n^{\mathsf{L}}(z)\big)_+ = \big(Y_n^{\mathsf{L}}(z)\big)_- \begin{bmatrix} I_N & W(z) \\ 0_N & I_N \end{bmatrix}, \qquad z \in \gamma.$$

(RHL4): $Y_n^{\mathsf{L}}(z) = \begin{bmatrix} \mathrm{O}(1) & \mathrm{O}(s_1^{\mathsf{L}}(z)) \\ \mathrm{O}(1) & \mathrm{O}(s_2^{\mathsf{L}}(z)) \end{bmatrix}$, *as* $z \to c$,

where c denotes any of the end points of the curve γ if they exists,

$$\lim_{z\to c}(z-c)s_j^{\mathsf{L}}(z) = 0_N, \qquad j = 1, 2,$$

and the O *conditions are understood entrywise.*

ii) *The matrix function*

$$Y_n^{\mathsf{R}}(z) := \begin{bmatrix} P_n^{\mathsf{R}}(z) & -P_{n-1}^{\mathsf{R}}(z)C_{n-1} \\ Q_n^{\mathsf{R}}(z) & -Q_{n-1}^{\mathsf{R}}(z)C_{n-1} \end{bmatrix},$$

is, for each $n \in \mathbb{N}$, the unique solution of the Riemann–Hilbert problem; which consists in the determination of a $2N \times 2N$ complex matrix function such that:

(RHR1): $Y_n^{\mathsf{R}}(z)$ *is holomorphic in* $\mathbb{C} \setminus \gamma$;

(RHR2): *has the following asymptotic behaviour near infinity,*

$$Y_n^{\mathsf{R}}(z) = \begin{bmatrix} I_N z^n & 0_N \\ 0_N & I_N z^{-n} \end{bmatrix} \Big(I_N + \sum_{j=1}^{\infty}(z^{-j})Y_n^{j,\mathsf{R}}\Big);$$

(RHR3): *satisfies the jump condition*

$$\big(Y_n^{\mathsf{R}}(z)\big)_+ = \begin{bmatrix} I_N & 0_N \\ W(z) & I_N \end{bmatrix} \big(Y_n^{\mathsf{R}}(z)\big)_-, \qquad z \in \gamma.$$

(RHR4): $Y_n^{\mathsf{R}}(z) = \begin{bmatrix} \mathrm{O}(1) & \mathrm{O}(1) \\ \mathrm{O}(s_1^{\mathsf{R}}(z)) & \mathrm{O}(s_2^{\mathsf{R}}(z)) \end{bmatrix}$, *as* $z \to c$,

where c denotes any of the end points of the curve γ if they exists,

$$\lim_{z\to c}(z-c)s_j^{\mathsf{R}}(z) = 0_N, \qquad j = 1, 2,$$

and the O *conditions are understood entrywise.*

iii) *The determinant of $Y_n^{\mathsf{L}}(z)$ and $Y_n^{\mathsf{L}}(z)$ are equal to 1, for every $z \in \mathbb{C}$.*

Proof: Using the standard calculations from the scalar case it follows that the matrices Y_n^{L} and Y_n^{R} satisfy (RHL1) – (RHL3) and (RHR1) – (RHR3), respectively.

The entries $W^{j,k}$ of the matrix measure W can be described as

$$W^{j,k}(z)=\sum_{m\in I_{j,k}} h_m(z)(z-c)^{\alpha_m}\log^{p_m}(z),$$

where $I_{j,k}$ denotes a finite set of indexes, $\alpha_m>-1$, $p_m\in\mathbb{N}$ and $h_m(x)$ is Hölder continuous, bounded and non–vanishing on γ. At the boundary values of the curve γ if they exist and are denoted by c, for $z\to c$. It holds [32] that in a neighbourhood of the point c, the Cauchy transform of the function

$$\phi_m(z)=\frac{1}{2\pi i}\int_\gamma \frac{p(z')h_m(z')(z'-c)^{\alpha_m}\log^{p_m}(z')}{z'-z}dz',$$

where $p(z')$ denotes any polynomial in z', verifies

$$\lim_{z\to c}(z-c)\phi_m(z)=0,$$

and the condition (RHL4), is fulfilled for the matrix Y_n^{L} and respectively the condition (RHR4), is fulfilled for the matrix Y_n^{R}. Now let us consider

$$G(z)=Y_n^{\mathsf{L}}(z)\begin{bmatrix}0_N & I_N\\ -I_N & 0_N\end{bmatrix}Y_n^{\mathsf{R}}(z)\begin{bmatrix}0_N & -I_N\\ I_N & 0_N\end{bmatrix}.$$

It can easily be proved that G has no jump on the curve γ. In a neighborhood of the point c

$$G(z)=\begin{bmatrix}\mathrm{O}(s_1^{\mathsf{L}}(z))+\mathrm{O}(s_2^{\mathsf{R}}(z)) & \mathrm{O}(s_1^{\mathsf{L}}(z))+\mathrm{O}(s_1^{\mathsf{R}}(z))\\ \mathrm{O}(s_2^{\mathsf{L}}(z))+\mathrm{O}(s_2^{\mathsf{R}}(z)) & \mathrm{O}(s_2^{\mathsf{L}}(z))+\mathrm{O}(s_1^{\mathsf{R}}(z))\end{bmatrix},$$

so $\lim_{z\to c}(z-c)G(z)=0$, and at the point c the singularity is removable. Now using the behaviour for $z\to\infty$,

$$G(z)=\begin{bmatrix}I_Nz^n & 0_N\\ 0_N & I_Nz^{-n}\end{bmatrix}\begin{bmatrix}0_N & I_N\\ -I_N & 0_N\end{bmatrix}\begin{bmatrix}I_Nz^n & 0_N\\ 0_N & I_Nz^{-n}\end{bmatrix}\begin{bmatrix}0_N & -I_N\\ I_N & 0_N\end{bmatrix}=\begin{bmatrix}I_N & 0_N\\ 0_N & I_N\end{bmatrix},$$

and using Liouville's Theorem it holds that $G(z)=I$, the identity matrix. From this follows the unicity of the solution of each of the Riemann–Hilbert problems stated in this theorem.

Again using the standard arguments as in the scalar case we can conclude that $\det Y_n^{\mathsf{L}}(z)$ and $\det Y_n^{\mathsf{R}}(z)$ are both equal to 1. ■

We recover a representation for the inverse matrix $\left(Y_n^{\mathsf{L}}\right)^{-1}$ given by the following result

Corollary 1. *It holds that*

$$\left(Y_n^{\mathsf{L}}\right)^{-1}(z) = \begin{bmatrix} 0_N & I_N \\ -I_N & 0_N \end{bmatrix} Y_n^{\mathsf{R}}(z) \begin{bmatrix} 0_N & -I_N \\ I_N & 0_N \end{bmatrix}. \tag{13}$$

Corollary 2. *In the conditions of theorem 1 we have that for all* $n \in \mathbb{N}$,

$$Q_n^{\mathsf{L}}(z)P_{n-1}^{\mathsf{R}}(z) - P_n^{\mathsf{L}}(z)Q_{n-1}^{\mathsf{R}}(z) = C_{n-1}^{-1}, \tag{14}$$

$$P_{n-1}^{\mathsf{L}}(z)Q_n^{\mathsf{R}}(z) - Q_{n-1}^{\mathsf{L}}(z)P_n^{\mathsf{R}}(z) = C_{n-1}^{-1}, \tag{15}$$

$$Q_n^{\mathsf{L}}(z)P_n^{\mathsf{R}}(z) - P_n^{\mathsf{L}}(z)Q_n^{\mathsf{R}}(z) = 0. \tag{16}$$

Proof: As we have already proven the matrix

$$\begin{bmatrix} -Q_{n-1}^{\mathsf{R}}(z)C_{n-1} & -Q_n^{\mathsf{R}}(z) \\ P_{n-1}^{\mathsf{R}}(z)C_{n-1} & P_n^{\mathsf{R}}(z) \end{bmatrix},$$

is the inverse of $Y_n^{\mathsf{L}}(z)$, i.e.

$$Y_n^{\mathsf{L}}(z) \begin{bmatrix} -Q_{n-1}^{\mathsf{R}}(z)C_{n-1} & -Q_n^{\mathsf{R}}(z) \\ P_{n-1}^{\mathsf{R}}(z)C_{n-1} & P_n^{\mathsf{R}}(z) \end{bmatrix} = I;$$

and multiplying the two matrices we get the result. ■

2.4. Three term recurrence relation. Following the standard arguments from the Riemann–Hilbert formulation we can prove

$$Y_{n+1}^{\mathsf{L}}(z) = T_n^{\mathsf{L}}(z)Y_n^{\mathsf{L}}(z), \qquad n \in \mathbb{N},$$

where T_n^{L} is given in (2). For the right orthogonality, we similarly obtain from (1) that

$$Y_{n+1}^{\mathsf{R}}(z) = Y_n^{\mathsf{R}}(z)T_n^{\mathsf{R}}(z), \qquad n \in \mathbb{N},$$

where T_n^{R} is given in (2). Hence, we conclude that the sequence of monic polynomials $\left\{P_n^{\mathsf{L}}(z)\right\}_{n\in\mathbb{N}}$ satisfies the three term recurrence relations

$$zP_n^{\mathsf{L}}(z) = P_{n+1}^{\mathsf{L}}(z) + \beta_n^{\mathsf{L}}P_n^{\mathsf{L}}(z) + \gamma_n^{\mathsf{L}}P_{n-1}^{\mathsf{L}}(z), \qquad n \in \mathbb{N}, \tag{17}$$

with recursion coefficients $\beta_n^{\mathsf{L}} := p_{\mathsf{L},n}^1 - p_{\mathsf{L},n+1}^1$, $\gamma_n^{\mathsf{L}} := C_n^{-1}C_{n-1}$, with initial conditions, $P_{-1}^{\mathsf{L}} = 0_N$ and $P_0^{\mathsf{L}} = I_N$. We can also assert that

$$zP_n^{\mathsf{R}}(z) = P_{n+1}^{\mathsf{R}}(z) + P_n^{\mathsf{R}}(z)\beta_n^{\mathsf{R}} + P_{n-1}^{\mathsf{R}}(z)\gamma_n^{\mathsf{R}}, \qquad n \in \mathbb{N}, \tag{18}$$

where $\beta_n^{\mathsf{R}} := C_n\beta_n^{\mathsf{L}}C_n^{-1}$, $\gamma_n^{\mathsf{R}} := C_n\gamma_n^{\mathsf{L}}C_n^{-1} = C_{n-1}C_n^{-1}$.

3. Matrix weights supported on a curve γ on the complex plane that connects the point 0 to the point ∞: Laguerre weights

Motivated by different attempts that appear in the literature we try to consider some classes of weights with the aim to use the Riemann–Hilbert formulation. In this matrix case it is not so obvious which are the conditions we should to impose in order to guarantee the integrability of the matrix measure that we want to consider.

3.1. Matrix weights supported on a curve γ with one finite end point: $W(z) = z^A H(z)$.

We begin considering the weight $W(z) = z^A H(z)$ supported on a curve γ on the complex plane that connects the point 0 to the point ∞, where

i) The function z^A is defined as $z^A = e^{A \log z}$, where γ is the branch cut of the logarithmic function, and we define por $t \in \gamma$, the $t^A := (z^A)_+$, where $(z^A)_+$ is the non–tangential limit as $z \to t$, from the left side of the oriented curve γ.

ii) The constant matrix A is such that the minimum of the real part of its eigenvalues is greater than -1.

iii) The factor $H(t)$ is the restriction to the curve γ of $H(z)$, a matrix of entire functions, $z \in \mathbb{C}$ such that $H(z)$ is invertible for all $z \in \mathbb{C}$.

iv) The left logarithmic derivative $h(z) := \big(H(z)\big)^{-1}\big(H(z)\big)'$ is an entire function.

It is necessary, in order to consider the Riemann–Hilbert problem related to the weight function $W(z)$, to clarify the behaviour of this weight function $W(z)$ on a neighborhood of the point $z = 0$.

If we consider the Jordan decomposition of the matrix A, it holds that there exists an invertible matrix P such that $A = PJP^{-1}$, where $J = D + N$, D is the diagonal matrix formed whose entries are the eigenvalues of the matrix A, and N is a nilpotent matrix that commutes with the matrix D. This commutation enables us to obtain

$$z^A = z^{PJP^{-1}} = P z^J P^{-1} = P z^D z^N P^{-1},$$

where z^N is a polynomial in the variable $\log z$. The matrix z^D is a diagonal matrix whose entries are of the form $z^{\alpha_j + i\beta_j}$, where $\alpha_j + i\beta_j$ is a eigenvalue of the matrix A. Let us consider as an example the matrix

$$A = \begin{bmatrix} -\frac{1}{2} & 1 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

In this case

$$z^A = \begin{bmatrix} z^{-\frac{1}{2}} & 0 & 0 \\ 0 & z^{-\frac{1}{2}} & 0 \\ 0 & 0 & z \end{bmatrix} \begin{bmatrix} 1 & \log z & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We can see that when the matrix A is not diagonlizable, the matrix z^A, has entries of the form $z^{-\frac{1}{2}} \log z$, hence we are in presence of a power logarithmic type singularity at 0.

To assure the integrability of this kind of measure it is enough to ask that $\alpha > -1$, where is the minimum of the real part of the eigenvalues of the matrix A. So in this case the weight fulfills the hypothesis of theorem 1.

It is also valuable to comment about the factor $H(z)$ of the measure $W(z)$. In order also to have integrability of this matrix weight function we should be careful: If for example we consider $H(z) = e^{Bz}$, then it is clear that, by a similar reasoning, we should impose that the real part of the eigenvalues of the matrix B are negative; if we consider $h(z) := \big(H(z)\big)^{-1}\big(H(z)\big)'$ to be a matrix polynomial $h(z) = B_0 + B_1 z + \cdots + B_m z^m$, it should be enough in order to guarantee integrability of the measure, to impose that the real part of the eigenvalues of the matrix B_m are negative.

3.2. Matrix weights supported on a curve γ with one finite end point: $W(z) = z^\alpha H(z) G(z) z^B$.

In [23] different examples appears of Laguerre matrix weights for the matrix orthogonal polynomials on the real line. This motivates us to consider the matrix weight $W(z) = z^\alpha H(z) G(z) z^B$ supported on a curve γ on the complex plane that connects the point 0 to the point ∞. With similar considerations as in the case treated before. Nevertheless when we try to apply the general methods from the Riemann–Hilbert formulation we find a lot of difficulties, derived from the non–commutativity of the matrix product and we should impose important restrictions.

This kind of matrix weights can be treated in a more general context. Let us consider that instead of a given matrix of weights we are provided with two matrices, say $h^{\mathsf{L}}(z)$ and $h^{\mathsf{R}}(z)$, of entire functions such that the following two matrix Pearson equations are satisfied

$$z\frac{\mathrm{d}\, W^{\mathsf{L}}}{\mathrm{d}\, z} = h^{\mathsf{L}}(z)W^{\mathsf{L}}(z), \qquad z\frac{\mathrm{d}\, W^{\mathsf{R}}}{\mathrm{d}\, z} = W^{\mathsf{R}}(z)h^{\mathsf{R}}(z); \tag{19}$$

and given solutions to them we construct the corresponding matrix of weights $W = W^{\mathsf{L}}W^{\mathsf{R}}$. Moreover, this matrix of weights is also characterized by a Pearson equation.

Theorem 2 (Pearson Sylvester differential equation)**.** *Given two matrices of entire functions $h^{\mathsf{L}}(z)$ and $h^{\mathsf{R}}(z)$, any solution of the Sylvester type matrix differential equation, which we call Pearson equation for the weight,*

$$z\frac{\mathrm{d}\, W}{\mathrm{d}\, z} = h^{\mathsf{L}}(z)W(z) + W(z)h^{\mathsf{R}}(z) \tag{20}$$

is of the form $W = W^{\mathsf{L}}W^{\mathsf{R}}$ where the factor matrices W^{L} and W^{R} are solutions of (19)*, respectively.*

Proof: Given solutions W^{L} and W^{R} of (19), respectively, it follows intermediately, just using the Leibniz law for derivatives, that $W = W^{\mathsf{L}}W^{\mathsf{R}}$ fulfills (20). Moreover, given a solution W of (20) we pick a solution W^{L} of the first equation in (19), then it is easy to see that $(W^{\mathsf{L}})^{-1}W$ satisfies the second equation in (19). ■

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