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CONVEX SOBOLEV INEQUALITIES RELATED TO UNBALANCED OPTIMAL TRANSPORT

STANISLAV KONDRATYEV AND DMITRY VOROTNIKOV

ABSTRACT: We study the behaviour of various Lyapunov functionals (relative entropies) along the solutions of a family of nonlinear drift-diffusion-reaction equations coming from statistical mechanics and population dynamics. These equations can be viewed as gradient flows over the space of Radon measures equipped with the Hellinger-Kantorovich distance. The driving functionals of the gradient flows are not assumed to be geodesically convex or semi-convex. We prove new isoperimetric-type functional inequalities, allowing us to control the relative entropies by their productions, which yields the exponential decay of the relative entropies.

KEYWORDS: functional inequalities, optimal transport, reaction-diffusion, fitness-driven dispersal, entropy, exponential decay.
MATH. SUBJECT CLASSIFICATION (2010): 26D10, 35K57, 35B40, 49Q20, 58B20.

1. Introduction

The unbalanced optimal transport [36, 30, 13, 35, 14, 43] interpolates between the classical Monge-Kantorovich transport [45, 46] and the optimal information transport [41]. It equips the space of finite Radon measures with a formal Riemannian structure so that certain classes of reaction-diffusion equations and systems can be interpreted as gradient flows. This paper continues our investigation [30, 29, 31, 33, 32] of such gradient flows and associated functional inequalities, see also [12, 24, 23] for related studies.

The class of PDEs that we consider in this paper is

$$\partial_t \rho = -\operatorname{div}(\rho\nabla f) + f\rho, \qquad (x,t) \in \Omega\times(0,\infty), \tag{1.1}$$

$$\rho\frac{\partial f}{\partial \nu} = 0, \qquad (x,t) \in \partial\Omega\times(0,\infty), \tag{1.2}$$

$$\rho = \rho^0 \geq 0, \qquad (x,t) \in \Omega\times 0. \tag{1.3}$$

Here $f = f(x,\rho(x,t))$ is a nonlinear function of x and ρ which is required to have a certain structure specified below in (1.12), and $\Omega \subset \mathbb{R}^d$ is an open

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connected bounded domain admitting the relative isoperimetric inequality, cf. [40],

$$P(A;\Omega) \geq C_\Omega \min(|A|^{\frac{d-1}{d}}, |\Omega \setminus A|^{\frac{d-1}{d}}). \tag{1.4}$$

All our results remain valid if Ω is a periodic box $\mathbb{T}^d$; in this case (1.2) is omitted.

The drift-diffusion-reaction equation (1.1) appears in statistical mechanics [19]. It also describes nonlinear fitness-driven models of population dynamics, cf. [38, 15, 16, 25, 33], where it is assumed that the dispersal strategy is determined by a local intrinsic characteristic of organisms called fitness. We refer to Section 2 and to [33] for more detailed discussions.

Let $g\colon (0,\infty) \to \mathbb{R}$ and $\psi\colon [0,\infty) \to \mathbb{R}$ be fixed C^1-smooth functions, which satisfy the following assumptions:

$$g(1) = 0; \qquad g'(s) > 0 \ (s > 0), \tag{1.5}$$

$$\psi(1) = 0, \qquad \psi(s) > 0 \ (s \neq 1), \tag{1.6}$$

$$\psi \in C^2(0,+\infty),\ \psi''(s) > 0 \ (s > 0,\ s \neq 1), \tag{1.7}$$

$$\lim_{s\to\infty} \psi'(x) = \infty, \tag{1.8}$$

$$|g(s)| + s|g'(s)| \leq h(s) \quad \text{a. a. } s > 0;\ h \in L^1_{\mathrm{loc}}[0,\infty), \tag{1.9}$$

$$sg(s) \in C([0,+\infty)). \tag{1.10}$$

Let $\rho_\infty\colon \overline{\Omega} \to \mathbb{R}$ be a fixed smooth strictly positive function satisfying

$$\int_\Omega \rho_\infty \, dx = 1. \tag{1.11}$$

Define

$$f = f(x,\rho(x)) := -g\left(\frac{\rho(x)}{\rho_\infty(x)}\right). \tag{1.12}$$

Thus, the functions g and ρ_∞ determine the problem (1.1)–(1.3), and the function ψ is merely needed to define a Lyapunov functional for this problem,

$$0 \leq \mathcal{E}_\psi(\rho) := \int_\Omega \psi\left(\frac{\rho}{\rho_\infty}\right)\rho_\infty \, dx, \tag{1.13}$$

which will be referred to as the relative entropy. Obviously, $\mathcal{E}_\psi(\rho) = 0$ if and only if $\rho \equiv \rho_\infty$. Formally calculating $\partial_t \mathcal{E}_\psi(\rho_t)$ along a solution of (1.1)–(1.3) we obtain

$$\partial_t \mathcal{E}_\psi(\rho_t) = -D\mathcal{E}_\psi(\rho_t),$$

where the entropy production $D\mathcal{E}_\psi$ is defined by

$$D\mathcal{E}_\psi(\rho) := \int_\Omega g'\left(\frac{\rho}{\rho_\infty}\right)\psi''\left(\frac{\rho}{\rho_\infty}\right)\left|\nabla\left(\frac{\rho}{\rho_\infty}\right)\right|^2 \rho\, dx + \int_\Omega g\left(\frac{\rho}{\rho_\infty}\right)\psi'\left(\frac{\rho}{\rho_\infty}\right)\rho\, dx$$

Setting

$$r = \frac{\rho}{\rho_\infty},$$

we can write

$$\mathcal{E}_\psi(\rho) = \int_\Omega \psi(r)\, d\rho_\infty \tag{1.14}$$

$$D\mathcal{E}_\psi(\rho) = \int_\Omega r g(r)\psi'(r) d\rho_\infty + \int_\Omega r g'(r)\psi''(r)|\nabla r|^2 d\rho_\infty \tag{1.15}$$

Note that problem (1.1)–(1.3) can be viewed as a formal gradient flow (with respect to the unbalanced Hellinger-Kantorovich Riemannian structure) of the driving functional $D\mathcal{E}_{\psi_g}(\rho)$, where

$$\psi_g(s) := \int_1^s g(\xi)\, d\xi, \tag{1.16}$$

see Section 2 for the details. We are interested in the exponential decay of the Lyapunov functional (1.14) along the trajectories of this gradient flow. This is related to the entropy-entropy production inequalities of the form

$$\mathcal{E}_\psi(\rho) \lesssim D\mathcal{E}_\psi(\rho). \tag{1.17}$$

They can be viewed as unbalanced generalizations of the convex Sobolev inequalities [2, 3, 27], see Section 2.

The main results of the paper are convex Sobolev inequalities akin to (1.17), see Theorems 3.5 and 4.1, and existence and asymptotics of weak solutions to (1.1)–(1.3), see Theorem 3.6.

2. Background and discussion

Assume for a while that Ω is a torus or is convex, although this is not required for our main results. The gradient of a scalar functional $\mathcal{E}$ on the space of finite Radon measures over $\overline{\Omega}$ with respect to the Hellinger-Kantorovich Riemannian structure (also known as the Wasserstein-Fisher-Rao one) was calculated in [30, 35]:

$$\operatorname{grad}_{HK}\mathcal{E}(\rho) = -\operatorname{div}\left(\rho\nabla\frac{\delta\mathcal{E}}{\delta\rho}\right) + u\frac{\delta\mathcal{E}}{\delta\rho}.$$

The first term on the right-hand side is the Otto-Wasserstein gradient $\operatorname{grad}_W\mathcal{E}(\rho)$, cf. [42, 45], and the second one is the Hellinger-Fisher-Rao gradient $\operatorname{grad}_H\mathcal{E}(\rho)$, cf. [28]. It is easy to compute that $\frac{D\mathcal{E}_{\psi_g}(\rho)}{\delta\rho} = -f(x,\rho)$, hence (1.1)–(1.3) may be interpreted as a gradient flow:

$$\partial_t\rho = -\operatorname{grad}_{HK} D\mathcal{E}_{\psi_g}(\rho), \quad \rho(0) = \rho^0. \tag{2.1}$$

The production of the relative entropy $\mathcal{E}_\psi(\rho)$ along the Otto-Wasserstein gradient flow

$$\partial_t\rho = -\operatorname{grad}_W D\mathcal{E}_{\psi_g}(\rho) \tag{2.2}$$

is

$$D\mathcal{E}_\psi^W(\rho) := \int_\Omega rg'(r)\psi''(r)|\nabla r|^2 d\rho_\infty.$$

Similarly, the production of the same entropy along the Hellinger gradient flow

$$\partial_t\rho = -\operatorname{grad}_H D\mathcal{E}_{\psi_g}(\rho) \tag{2.3}$$

is

$$D\mathcal{E}_\psi^H(\rho) := \int_\Omega rg(r)\psi'(r) d\rho_\infty.$$

In the case of non-convex Ω we can abuse the terminology and still refer to (1.1)–(1.3) as to a gradient flow.

It is clear that

$$D\mathcal{E}_\psi^W(\rho) + D\mathcal{E}_\psi^H(\rho) = D\mathcal{E}_\psi(\rho).$$

Generally speaking, neither the Otto-Wasserstein nor the Fisher-Rao entropy production are able to control the relative entropy, so (1.17) is a result of an interplay between the reaction, diffusion and drift. A simple counterexample to

$$\mathcal{E}_\psi(\rho) \lesssim D\mathcal{E}_\psi^H(\rho) \tag{2.4}$$

is $\rho_\infty 1_A$ with A being a proper subset of Ω. Indeed, $D\mathcal{E}_\psi^H(\rho_\infty 1_A) = 0$ due to (1.5), (1.9) and (1.10). It is easy to construct a smooth example by mollifying this one. A trivial counterexample to

$$\mathcal{E}_\psi(\rho) \lesssim D\mathcal{E}_\psi^W(\rho) \tag{2.5}$$

is $k\rho_\infty$ where $k \neq 1$ is a non-negative constant.

Remark 2.1. Note that the two counterexamples intersect at $\rho \equiv 0$, which violates our target inequality (1.17). However, we will observe, cf. Theorems 3.5 and 4.1, that it suffices keep the total mass $\int_\Omega \rho$ bounded away from 0 to secure (1.17).

In view of (1.11), in order to obtain more interesting and instructive examples we should restrict ourselves to probability densities ρ. The sequence

$$\rho_n = \rho_\infty \frac{n}{n-1} 1_{(\frac{1}{n},1)}$$

of probability densities on $\Omega = (0,1)$ is a counterexample to (2.4). Indeed, the left-hand side of (2.4) is of order n^{-1} and the right-hand side is $\lesssim n^{-2}$.

Inequality (2.5) for $\int_\Omega \rho = 1$ deserves a more detailed discussion.

Let us start with considering $g(s) = \log s$. In this case, as first observed in the seminal paper [26], the gradient flow (2.2) is the linear Fokker-Planck equation, and the celebrated Bakry-Émery approach allows one to prove (2.5) for $\Omega = \mathbb{R}^d$ [2, 3, 27]. However, it is crucial to have concavity of $\frac{1}{\psi''(s)}$, which we never assume in this work. These instances of (2.5) are referred to as *convex Sobolev inequalities*, which inspired the title of our paper. The particular case

$$\psi(s) = \begin{cases} \frac{1}{p(p-1)}(s^p - ps + p - 1), & \text{if } 1 < p \leq 2 \\ s\log s - s + 1, & \text{if } p = 1 \end{cases} \tag{2.6}$$

implies the log-Sobolev inequality for $p = 1$, the Poincaré inequality for $p = 2$ and Beckner's inequalities [4] for $1 < p < 2$. Namely, (2.5) may be rewritten as

$$\int_\Omega r^p \, d\rho_\infty - \left(\int_\Omega r \, d\rho_\infty \right)^p \lesssim \int_\Omega r^{p-2} |\nabla r|^2 \, d\rho_\infty, \quad 1 < p \leq 2. \tag{2.7}$$

In contrast, our assumptions on ψ admit any $p > 2$ in (2.6), which yields the following "Beckner-Hellinger inequality":

$$\int_\Omega r^p\,d\rho_\infty - \left(\int_\Omega r\,d\rho_\infty\right)^p \lesssim \int_\Omega r^{p-2}|\nabla r|^2 d\rho_\infty \\ + \int_\Omega r\log\left(\frac{r}{\int_\Omega r\,d\rho_\infty}\right)\left(r^{p-1} - \left(\int_\Omega r\,d\rho_\infty\right)^{p-1}\right)d\rho_\infty, \quad p > 2. \quad (2.8)$$

Consider now the case $g(s) = \frac{s^{\alpha-1}-1}{\alpha-1}$, $\alpha > 0$, $\alpha \neq 1$. Assume for simplicity that $|\Omega| = 1$ and $\rho_\infty \equiv 1$. Then (2.2) is the porous medium equation, cf. [42]. The alleged inequality (2.5) for the relative entropy (2.6), $p \in (1,\infty)$, reads

$$\int_\Omega r^p - \left(\int_\Omega r\right)^p \lesssim \left(\int_\Omega r\right)^{1-\alpha} \int_\Omega r^{p+\alpha-3}|\nabla r|^2. \quad (2.9)$$

Setting $q := \frac{2p}{p+\alpha-1}$, $l := \frac{p+\alpha-1}{2}$, $u := r^l$, we rewrite (2.9) in the form

$$\int_\Omega u^q - \left(\int_\Omega u^{1/l}\right)^{lq} \lesssim \left(\int_\Omega u^{1/l}\right)^{l(q-2)} \int_\Omega |\nabla u|^2. \quad (2.10)$$

The inequality

$$\int_\Omega u^q - \left(\int_\Omega u^{1/l}\right)^{lq} \lesssim \left(\int_\Omega |\nabla u|^2\right)^{q/2}. \quad (2.11)$$

similar to (2.10) appears in [11], see also [10, 18]. It holds for $0 < q < 2$, $lq > 1$, that is, for $\alpha > 1$, $p > 1$. Assume for a moment that the the relative entropy, i.e., the left-hand side of (2.11), is a priori bounded. Since $ql \geq 1$, the mass $\int_\Omega u^{1/l}$ is a priori bounded. Consequently, (2.11) is weaker than (2.10) since the exponent $q/2$ is less than 1, and it is plausible that (2.10) cannot be true. Inequality (2.11) for $q = 2$ is equivalent to Beckner's inequality (2.7). As explained in [18], inequality (2.11) is wrong for $q > 2$.

In this connection, our results yield the following variant of (2.10):

$$\begin{aligned}\int_\Omega u^q - \left(\int_\Omega u^{1/l}\right)^{lq} &\lesssim \left(\int_\Omega u^{1/l}\right)^{l(q-2)} \int_\Omega |\nabla u|^2 \\ &+ \left(\int_\Omega u^{1/l}\right)^{l(q-2)} \int_\Omega u^{1/l} \left(\frac{u^{(\alpha-1)/l} - \left(\int_\Omega u^{1/l}\right)^{\alpha-1}}{\alpha-1}\right)\left(u^{(p-1)/l} - \left(\int_\Omega u^{1/l}\right)^{p-1}\right)\end{aligned} \tag{2.12}$$

for any $q > 0$, $q \neq 2$, $1 < lq < 1 + 2l$, that is, any $\alpha > 0$, $\alpha \neq 1$, $p > 1$.

The counterparts of the alleged inequalities (2.9) and (2.10) for $p = 1$ are

$$\int_\Omega r \log\left(\frac{r}{\int_\Omega r}\right) \lesssim \left(\int_\Omega r\right)^{1-\alpha} \int_\Omega r^{\alpha-2} |\nabla r|^2, \tag{2.13}$$

$$\int_\Omega u^q \log\left(\frac{u^q}{\int_\Omega u^q}\right) \lesssim \left(\int_\Omega u^q\right)^{\frac{q-2}{q}} \int_\Omega |\nabla u|^2. \tag{2.14}$$

Here $q = \frac{2}{\alpha}$. This resembles the inequality

$$\int_\Omega u^q \log\left(\frac{u^q}{\int_\Omega u^q}\right) \lesssim \left(\int_\Omega |\nabla u|^2\right)^{q/2}, \quad q < 2, \tag{2.15}$$

which was established in [10, 18]. Since $q/2 < 1$, (2.15) is weaker than (2.14), so it seems that (2.14) cannot be true. Our results imply the following variant of (2.14):

$$\begin{aligned}\int_\Omega u^q \log\left(\frac{u^q}{\int_\Omega u^q}\right) &\lesssim \left(\int_\Omega u^q\right)^{\frac{q-2}{q}} \int_\Omega |\nabla u|^2 \\ &+ \left(\int_\Omega u^q\right)^{\frac{q-2}{q}} \int_\Omega u^q \log\left(\frac{u^q}{\int_\Omega u^q}\right)\left(\frac{u^{2-q} - \left(\int_\Omega u^q\right)^{\frac{2}{q}-1}}{2-q}\right), \quad q > 0, q \neq 2.\end{aligned} \tag{2.16}$$

Remark 2.2. Inequalities (2.8), (2.12), (2.16) are obtained assuming $\int_\Omega r \, d\rho_\infty = 1$ (so that (3.4) is automatically satisfied), but hold without this normalization due to their homogeneity.

Many authors studied (2.5) or related inequalities in the particular case $\psi = \psi_g$, that is, when the driving entropy is compared to its production, cf., e.g., [42, 45, 46, 1, 9]. In this connection, the strict geodesic convexity of the driving entropy normally plays the pivotal role. In [33] (see also [30]) we studied (1.17) for $\psi = \psi_g$ without assuming neither Otto-Wasserstein nor Hellinger-Kantorovich geodesic convexity (we also never assume any similar condition in the present paper). The inequalities obtained there can be further refined [32] be means of studying gradient flows in the spherical Hellinger-Kantorovich space [34, 7], which is beyond the scope of the present paper (though it may seem strange, even non-negativity of the entropy production is uncertain for the spherical Hellinger-Kantorovich flows in the case $\psi \neq \psi_g$). The proofs in the present paper are more direct and simple than in [33] due to the "quasihomogeneous structure" (1.12).

Our last example concerns $g(s) = \frac{1}{2}\log\frac{2s^2}{1+s^2}$, which corresponds to the arctangential heat equation [6]. The relative entropy $\mathcal{E}_{\psi_g}$ generated by this g is geodesically convex neither in the Otto-Wasserstein nor in the Hellinger-Kantorovich sense, cf. [32]. Take $\psi(s) = s\log s - s + 1$. Then we infer the following inequality resembling the log-Sobolev one:

$$\int_\Omega (r\log r - r + 1)\, d\rho_\infty \lesssim \int_\Omega \frac{1}{r(1+r^2)}|\nabla r|^2 d\rho_\infty + \int_\Omega r\log r\left(\log\frac{2r^2}{1+r^2}\right) d\rho_\infty \quad (2.17)$$

provided $\int_\Omega r\, d\rho_\infty$ is bounded away from 0.

Nonlinear Fokker-Planck equations akin to (2.2) model behaviour of various stochastic systems, see [20, 44, 27, 5]. The related drift-diffusion-reaction equation (1.1) was suggested in [19]. On the other hand, equation (1.1) belongs to the class of nonlinear models (cf. [16, 25, 47, 33, 32, 38, 15]) for the spatial dynamics of populations which are tending to achieve the *ideal free distribution* [22, 21] (the distribution which happens if everybody is free to choose its location) in a heterogeneous environment. The dispersal strategy is determined by a local intrinsic characteristic of organisms called *fitness*. The fitness manifests itself as a growth rate, and simultaneously affects the dispersal as the species move along its gradient towards the most favorable environment. In (1.1), $\rho(x,t)$ is the density of organisms, and $f(x,\rho)$ is the fitness. The equilibrium $\rho \equiv \rho_\infty$ when the fitness is constantly zero corresponds to the ideal free distribution. The works [17, 8, 37, 47, 30, 29, 31, 33] perform mathematical analysis of some of such fitness-driven models. Our Theorem 3.6 indicates that the populations converge to the ideal free distribution with an exponential rate.

3. Main results

We start by introducing the weak solutions to (1.1)–(1.3), following the lines of [33, 32].

Define

$$G(s) = \int_0^s \xi g'(\xi)\, d\xi \qquad (s \ge 0),$$

where the integral exists by (1.9). Observe that

$$G'(s) = sg'(s) > 0, \quad (s > 0); \qquad G(0) = 0,$$

so that G is a nonnegative continuous increasing function on $[0,\infty)$.

Set

$$\Phi(x,u) = \rho_\infty(x) G\left(\frac{u}{\rho_\infty(x)}\right), \quad u \ge 0.$$

As in [33], we can write (1.1) in the form

$$\partial_t \rho = \Delta\Phi - \operatorname{div}(\Phi_x + \rho f_x) + \rho f, \tag{3.1}$$

where Φ stands for $\Phi(x,\rho(x,t))$.

Definition 3.1. Let $\rho^0 \in L^\infty(\Omega)$; $Q_T := \Omega \times (0,T)$. A function $\rho \in L^\infty(Q_T)$ is called a *weak solution* of (1.1)–(1.3) on $[0,T]$ if for $r = \rho/\rho_\infty$ we have $G(r(\cdot)) \in L^2(0,T;H^1(\Omega))$ and

$$\int_0^T \int_\Omega (\rho\partial_t\varphi + (-\nabla\Phi + \Phi_x + \rho f_x)\cdot\nabla\varphi + f\rho\varphi)\, dx\, dt$$
$$= \int_\Omega \rho^0(x)\varphi(x,0)\, dx \tag{3.2}$$

for any function $\varphi \in C^1(\overline{\Omega} \times [0,T])$ such that $\varphi(x,T) = 0$. A function $\rho \in L^\infty_{\text{loc}}([0,\infty); L^\infty(\Omega))$ is called a *weak solution* of (1.1)–(1.3) on $[0,\infty)$ if for any $T > 0$ it is a weak solution on $[0,T]$.

Remark 3.2. For $\rho \in L^\infty(Q_T)$ we automatically have $G(r) \in L^\infty(Q_T)$, so the condition $G(r(\cdot)) \in L^2(0,T;H^1(\Omega))$ is equivalent to $rg'(r)\nabla r \in L^2(Q_T)$. Here $r = \rho/\rho_\infty$.

Formally, the integrand $rg'(r)\psi''(r)|\nabla r|^2$ vanishes if $r = 0$. Otherwise it can be written as

$$rg'(r)\psi''(r)|\nabla r|^2 = \frac{1}{r}\frac{\psi''(r)}{g'(r)}|rg'(r)\nabla r|^2 = \frac{1}{r}\frac{\psi''(r)}{g'(r)}|\nabla G(r)|^2.$$

This motivates the following extension of the entropy production suitable for weak solutions.

Definition 3.3. If $\rho \in L^\infty(\Omega)$ and $G(r) \in H^1(\Omega)$, then the *entropy production* is defined by

$$\begin{aligned} D\mathcal{E}_\psi(\rho) &= \int_\Omega rg(r)\psi'(r)d\rho_\infty + \int_{[r>0]} rg'(r)\psi''(r)|\nabla r|^2 d\rho_\infty \\ &\equiv \int_\Omega rg(r)\psi'(r)d\rho_\infty + \int_{[r>0]} \frac{1}{r}\frac{\psi''(r)}{g'(r)}|\nabla G(r)|^2 d\rho_\infty. \end{aligned} \tag{3.3}$$

Remark 3.4. Observe that although the integrand with the gradient in (3.3) is a nonnegative measurable function on Ω, the integral, and hence the entropy production, may be infinite.

The following entropy-entropy production inequality applicable to weak solutions is based on an isoperimetric-type inequality established in Section 4.

Theorem 3.5 (Entropy-entropy production inequality)**.** *Suppose that g and ψ satisfy* (1.5)–(1.10)*. Let $U \subset L^\infty_+(\Omega)$ be a set of functions such that for any $\rho \in U$ and $r = \rho/\rho_\infty$, we have $G(r) \in H^1(\Omega)$ and*

$$\inf_{\rho \in U} \|\rho\|_{L^1(\Omega)} > 0, \tag{3.4}$$

$$\sup\{\mathcal{E}_\psi(\rho) \colon \rho \in U\} < \infty. \tag{3.5}$$

Then there exists C_U such that

$$\mathcal{E}_\psi(\rho) \le C_U D\mathcal{E}_\psi(\rho) \quad (\rho \in U). \tag{3.6}$$

Proof: The idea is to use the isoperimetric-type inequality provided by Theorem 4.1 (see Section 4). Since we are dealing with a less regular setting at the moment, we argue by approximation.

Take $\rho \in U$ and as usual, put $r = \rho/\rho_\infty$. Arguing as in [33, proof of Theorem 1.7], we see that there exists a sequence of functions $G_n \in C(\overline{\Omega}) \cap C^\infty(\Omega)$ taking values in $(0, a)$, where $a < G(\infty)$, such that

$$G_n \to G(r(\cdot)) \qquad \text{in } H^1 \text{ and a. e. in } \Omega.$$

Set $r_n(x) = G^{-1}(G_n(x))$ and $\rho_n(x) = r_n(x)\rho_\infty(x)$, so that $G_n(x) = G(r_n(x))$. Clearly, r_n and ρ_n are positive and reasonably smooth, the sequences $\{r_n\}$ and $\{\rho_n\}$ are bounded in $L^\infty(Q_T)$ (specifically, the former is bounded by $G^{-1}(a)$), and by the continuity of G^{-1} we have

$$r_n \to r, \quad \rho_n \to \rho \text{ a. e. in } \Omega.$$

In particular, this implies that ρ_n converges to ρ in $L^1(\Omega)$. Further, by the Lebesgue Dominated Convergence we have

$$\mathcal{E}_\psi(\rho_n) \to \mathcal{E}_\psi(\rho). \tag{3.7}$$

Thus, if we denote the infimum in (3.4) by d_U and the supremum in (3.5) by E_U, there is no loss of generality in assuming that $\|\rho_n\|_{L^1(\Omega)} \geq d_U/2$ and $\mathcal{E}_\psi(\rho_n) \leq 2E_U$. It follows from Theorem 4.1 that there exist C and σ both depending on d_U and E_U (but not on the approximation nor on ρ itself) such that

$$\mathcal{E}_\psi(\rho_n) \leq C\left(\int_\Omega r_n g(r_n)\psi'(r_n)\, d\rho_\infty + \int_{[r\geq\sigma]} r_n g'(r_n)\psi''(r_n)|\nabla r_n|^2\, d\rho_\infty\right). \tag{3.8}$$

By the Lebesgue Dominated Convergence we have

$$\int_\Omega r_n g(r_n)\psi'(r_n)\, d\rho_\infty \to \int_\Omega r g(r)\psi'(r)\, d\rho_\infty. \tag{3.9}$$

Further, we have

$$\int_{[r_n\geq\sigma]} r_n g'(r_n)\psi''(r_n)|\nabla r_n|^2\, d\rho_\infty = \int_\Omega 1_{[r_n\geq\sigma]}\frac{\psi''(r_n)}{r_n g'(r_n)}|\nabla G_n|^2\, d\rho_\infty.$$

On one hand, $\nabla G_n \to \nabla G$ in $L^2(\Omega)$. On the other hand, the functions

$$h_n = 1_{[r_n\geq\sigma]}\frac{\psi''(r_n)}{r_n g'(r_n)}$$

are uniformly bounded in $L^\infty(\Omega)$, and since we obviously have

$$\limsup_{n\to\infty} 1_{[r_n\geq\sigma]} \leq 1_{[r\geq\sigma]} \qquad \text{a. e. in } \Omega,$$

we also have

$$\limsup_{n\to\infty} h_n(x) \leq 1_{[r\geq\sigma]} \frac{\psi''(r)}{rg'(r)} \qquad \text{a. e. in } \Omega.$$

Using Reverse Fatou's Lemma for products (Lemma A.1 in the Appendix), we obtain

$$\begin{aligned}\limsup_{n\to\infty} \int_{[r_n\geq\sigma]} r_n g'(r_n)\psi''(r_n)|\nabla r_n|^2\, d\rho_\infty &= \limsup_{n\to\infty} \int_\Omega h_n|\nabla G_n|^2\, d\rho_\infty \\ &\leq \int_\Omega 1_{[r\geq\sigma]} \frac{\psi''(r)}{rg'(r)} |\nabla G|^2\, d\rho_\infty \\ &\leq \int_{[r>0]} rg'(r)\psi''(r)|\nabla r|^2\, d\rho_\infty.\end{aligned}$$

Combining this with (3.7) and (3.9), we see that we can pass to the limit in (3.8) and obtain (3.6) with $C_U = C$. ■

Theorem 3.6 (Existence and asymptotics of weak solutions)**.** *Assume* (1.5)–(1.10)*. Then for any $\rho^0 \in L^\infty_+(\Omega)$ there exists a nonnegative weak solution $\rho \in L^\infty(\Omega\times(0,\infty))$ of problem* (1.1)–(1.3) *which enjoys the following properties:*

(1) *ρ satisfies the entropy dissipation inequality in the sense of measures: for any smooth nonnegative compactly supported function $\chi\colon (0,T) \to \mathbb{R}$ we have*

$$-\int_0^T \chi'(t)\mathcal{E}_\psi(\rho)\, dt \leq \int_0^T \chi(t) D\mathcal{E}_\psi(\rho)\, dt; \tag{3.10}$$

(2) *the initial entropy satisfies*

$$\operatorname*{ess\,sup}_{t>0} \mathcal{E}_\psi(\rho(t)) \leq \mathcal{E}_\psi(\rho^0); \tag{3.11}$$

(3) *ρ satisfies the lower L^1-bound*

$$\|\rho(t)\|_{L^1(\Omega)} \geq \|\min(\rho^0,\rho_\infty)\|_{L^1(\Omega)} \quad \textit{a. a. } t>0; \tag{3.12}$$

(4) *ρ exponentially converges to ρ_∞ in the sense of entropy:*

$$\mathcal{E}_\psi(\rho(t)) \leq \mathcal{E}_\psi(\rho^0)e^{-\gamma_\psi t} \quad \textit{a. a. } t>0, \tag{3.13}$$

where $\gamma_\psi > 0$ *can be chosen uniformly over initial data satisfying*

$$\|\min(\rho^0, \rho_\infty)\|_{L^1(\Omega)} \geq c, \quad \mathcal{E}_\psi(\rho^0) \leq C \tag{3.14}$$

with some $c, C > 0$*;*

(5) *for any* $p \in [2, +\infty)$*,*

$$\|\rho(t) - \rho_\infty\|_{L^p(\Omega)} \leq e^{-\gamma_p t}\left(1 + \frac{\sup\ \rho_\infty}{\inf\ \rho_\infty}\right)\|\rho^0 - \rho_\infty\|_{L^p(\Omega)} \quad a.\ a.\ t > 0, \tag{3.15}$$

where $\gamma_p > 0$ *can be chosen uniformly over initial data satisfying*

$$\|\min(\rho^0, \rho_\infty)\|_{L^1(\Omega)} \geq c, \quad \|\rho^0\|^p_{L^p(\Omega)} \leq C. \tag{3.16}$$

Proof: For the proof of existence, the approximating procedure used in [33] is still applicable in the current setting. As a matter of fact, the existence result in [33] requires that $|f(x,\xi)|$ is either large or does not depend on x when ξ is near 0 or near $+\infty$. A similar requirement was imposed for large ξ. However, these assumptions are only needed in order to ensure that any $u \in L^\infty_+(\Omega)$ can be bounded from above by a function $u_c \colon \Omega \to \mathbb{R}$ satisfying $f(x, u_c(x)) \equiv cst$ and that u can be bounded from below by another such function provided that u is uniformly bounded away from 0. This is still the case in the current setting. Indeed, assume for simplicity that u is continuous on $\overline{\Omega}$. Set $c = \max_\Omega g(u/\rho_\infty)$ and put $u_c = \rho_\infty g^{-1}(c)$, then clearly $f(x, u_c(x)) = -g(u_c(x)/\rho_\infty) = -c$; moreover, it follows from the monotonicity of g that $u \leq u_c$, as required. The existence of a lower bound is proved in a similar way, cf. [33, Remark 3.4].

Inequality (3.11) is proved in the same way as the analogous inequality in [33].

We prove that the solution constructed as in [33] satisfies (3.10). To this end it suffices to check that this inequality is preserved under the passage to the limit. Specifically, assume that smooth enough approximate solutions $\{\rho_n\}$ are uniformly bounded in $L^\infty(Q_T)$ and converge to ρ a. e. in Q_T, while

$$G_n := G(r_n) \to G(r) \qquad \text{weakly in } L^2(\Omega).$$

By the Lebesgue Dominated Convergence we have

$$\mathcal{E}_\psi(\rho_n) \to \mathcal{E}_\psi(\rho), \tag{3.17}$$

$$\int_\Omega r_n g(r_n)\psi'(r_n)\,d\rho_\infty \to \int_\Omega r g(r)\psi'(r)\,d\rho_\infty. \tag{3.18}$$

Arguing as in [33, proof of Theorem 3.9] and, in particular, taking into account that $\nabla G = 0$ a. e. on the set $\{(x,t) \in Q_T : r = 0\}$ and $\nabla G_n = 0$ a. e. on the set $\{(x,t) \in Q_T : r_n = 0\}$, we conclude that for any $\delta > 0$ we have

$$\begin{aligned}
&\iint\limits_{\{(x,t)\in Q_T:\ r>0\}} \frac{\chi(t)\psi''(r)}{\max(r,\delta)g'(r)}|\nabla G|^2\,d\rho_\infty\,dt \\
&\qquad \le \liminf_{n\to\infty} \iint\limits_{\{(x,t)\in Q_T:\ r_n>0\}} \frac{\chi(t)\psi''(r_n)}{\max(r_n,\delta)g'(r_n)}|\nabla G_n|^2\,d\rho_\infty\,dt \\
&\qquad\qquad \le \liminf_{n\to\infty} \iint\limits_{\{(x,t)\in Q_T:\ r_n>0\}} \frac{\chi(t)\psi''(r_n)}{r_n g'(r_n)}|\nabla G_n|^2\,d\rho_\infty\,dt,
\end{aligned}$$

so sending $\delta \to \infty$ and applying Beppo Levy's theorem, we obtain

$$\begin{aligned}
&\iint\limits_{\{(x,t)\in Q_T:\ r>0\}} \frac{\chi(t)\psi''(r)}{r g'(r)}|\nabla G|^2\,d\rho_\infty\,dt \\
&\qquad\qquad \le \liminf_{n\to\infty} \iint\limits_{\{(x,t)\in Q_T:\ r_n>0\}} \frac{\chi(t)\psi''(r_n)}{r_n g'(r_n)}|\nabla G_n|^2\,d\rho_\infty\,dt
\end{aligned}$$

or, equivalently,

$$\begin{aligned}
&\iint\limits_{\{(x,t)\in Q_T:\ r>0\}} \chi(t) r g'(r)\psi''(r)|\nabla r|^2\,d\rho_\infty\,dt \\
&\qquad\qquad \le \liminf_{n\to\infty} \iint\limits_{\{(x,t)\in Q_T:\ r_n>0\}} \chi(t) r_n g_n'(r)\psi''(r_n)|\nabla r_n|^2\,d\rho_\infty\,dt.
\end{aligned}$$

Combining this with (3.17) and (3.18), we obtain (3.10).

We now prove the exponential convergence of the solution to the steady state. Let ρ be a weak solution of (1.1)–(1.3) with the initial data satisfying (3.14). Let $U \subset L^\infty_+$ be the set of functions such that for any $u \in U$, we have $G(u/\rho_\infty) \in H^1(\Omega)$ and $\|u\|_{L^1(\Omega)} \geq c$, $\mathcal{E}_\psi(u) \leq C$ with the same c and C as in (3.14). By Theorem 3.5 we have the entropy-entropy production inequality (3.6) for U. It follows from the bounds (3.11) and (3.12) that $\rho(t) \in U$ for a. a. $t > 0$. Combining the entropy dissipation and entropy-entropy production inequalities, we get

$$\partial_t \mathcal{E}_\psi(\rho(t)) \leq -C_U^{-1} \mathcal{E}_\psi(\rho(t))$$

in the sense of measures. Set $\gamma_\psi = C_U^{-1}$ and $\phi(t) = \mathcal{E}_\psi(\rho(t))e^{\gamma_\psi t}$. It is easy to check that that $\partial_t \phi(t) \leq 0$ in the sense of measures, whence ϕ a. e. coincides with a nonincreasing function. Moreover,

$$\operatorname*{ess\,sup}_{t>0} \phi(t) = \operatorname*{ess\,lim\,sup}_{t\to 0} \phi(t) = \operatorname*{ess\,lim\,sup}_{t\to 0} \mathcal{E}_\psi(\rho(t))e^{\gamma_\psi t} \leq \mathcal{E}_\psi(\rho^0)$$

by virtue of (3.11), so $\phi(t) \leq \mathcal{E}_\psi(\rho^0)$ for a. a. $t > 0$, which implies (3.13).

We will now use (3.13) with $\psi(s) = |s-1|^p$, which is a C^2-function for $p \geq 2$, and satisfies the assumptions (1.6)–(1.8). We immediately get

$$\begin{aligned}
\|\rho(t) - \rho_\infty\|_{L^p(\Omega)} &\leq (\sup\ \rho_\infty)^{(p-1)/p} [\mathcal{E}_\psi(\rho(t))]^{1/p} \\
&\leq (\sup\ \rho_\infty)^{(p-1)/p} [\mathcal{E}_\psi(\rho^0)]^{1/p} e^{-\gamma_\psi t/p} \\
&\leq \left(\frac{\sup\ \rho_\infty}{\inf\ \rho_\infty}\right)^{(p-1)/p} \|\rho^0 - \rho_\infty\|_{L^p(\Omega)} e^{-\gamma_p t} \\
&\leq \left(1 + \frac{\sup\ \rho_\infty}{\inf\ \rho_\infty}\right) \|\rho^0 - \rho_\infty\|_{L^p(\Omega)} e^{-\gamma_p t},
\end{aligned} \tag{3.19}$$

where $\gamma_p = \gamma_\psi/p$. Uniform boundedness of $\|\rho^0\|_{L^p}^p$ implies a bound on $\mathcal{E}_\psi(\rho^0)$. ■

4. Inequality

In this section we prove a refined version of our unbalanced convex Sobolev inequality in the smooth case.

Theorem 4.1. *Assume* (1.5)–(1.10)*. Let* $U \in C_+^\infty(\Omega)$ *be such that*

$$\inf\left\{\|\rho\|_{L^1(\Omega)} : \rho \in U\right\} > 0,$$
$$\sup\{\mathcal{E}_\psi(\rho) : \rho \in U\} < \infty.$$

Then there exist constants (independent of ρ*)* $C > 0$*,* $0 < \alpha < \beta < \infty$*, such that*

$$\mathcal{E}_\psi(\rho) \le C\left(\int_\Omega rg(r)\psi'(r)\, d\rho_\infty + \int_{[\alpha<r<\beta]} rg'(r)\psi''(r)|\nabla r|^2\, d\rho_\infty \right) \quad (\rho \in U). \quad (4.1)$$

The proof of Theorem 4.1 is based on the next two lemmas.

Lemma 4.2. *Fix* $0 < \alpha < \beta < 1$*. Then*

$$\left|[\alpha < r < \beta]\right| \int_{[\alpha<r<\beta]} rg'(r)\psi''(r)|\nabla r|^2\, d\rho_\infty \ge C_{\alpha\beta} \min\left(\left|[r \le \alpha]\right|^{2(d-1)/d}, \left|[r \ge \beta]\right|^{2(d-1)/d}\right) \quad (4.2)$$

Proof: If the minimum on the right-hand side vanishes, there is nothing to prove. Otherwise the set $[\alpha < r < \beta]$ has nonzero measure. In what follows, we use some facts from geometric measure theory, which can be found in [39]. The relative perimeter of a Lebesgue measurable set A of locally finite perimeter with respect to Ω is $P(A;\Omega) = |\mu_A|(\Omega)$, where $\mu_A := \nabla 1_A$ is the Gauss-Green measure associated with A. The support of μ_A is contained in the topological boundary of A.

We have:

$$\int_{[\alpha<r<\beta]} rg'(r)\psi''(r)|\nabla r|^2\, d\rho_\infty$$
$$\ge \inf_\Omega \rho_\infty \min_{s\in[\alpha,\beta]} (sg'(s)\psi''(s)) \int_{[\alpha<r<\beta]} |\nabla r|^2\, dx$$
$$\ge \frac{\inf_\Omega \rho_\infty \min_{s\in[\alpha,\beta]}(sg'(s)\psi''(s))}{\left|[\alpha < r < \beta]\right|} \left(\int_{[\alpha<r<\beta]} |\nabla r|\, dx \right)^2 \quad (4.3)$$

The last integral is the variation of r over $[\alpha < r < \beta]$, which can be computed using the coarea formula:

$$\begin{aligned}\int_{[\alpha<r<\beta]} |\nabla r|\,dx &= \int_{-\infty}^{\infty} P([r<t];[\alpha<r<\beta])\,dt \\ &= \int_{\alpha}^{\beta} P([r<t];[\alpha<r<\beta])\,dt \\ &= \int_{\alpha}^{\beta} P([r<t];\Omega)\,dt, \end{aligned} \tag{4.4}$$

where we first use the observation that the support of the Gauss–Green measure associated with $[r < t]$ is disjoint with $[\alpha < r < \beta]$ whenever $t \le \alpha$ or $t \ge \beta$, and then we notice that if $\alpha < t < \beta$, then the part of the support of the Gauss–Green measure of $[r < t]$ lying in Ω is contained in $[\alpha < r < \beta]$.

Invoking the relative isoperimetric inequality (1.4), we estimate

$$P([r<t];\Omega) \ge C_\Omega \min\left(\left|[r<t]\right|^{(d-1)/d}, \left|\Omega \setminus [r<t]\right|^{(d-1)/d}\right)$$

and since for $t \in (\alpha, \beta)$ we have

$$[r \le \alpha] \subset [r<t] \subset [r<\beta] = \Omega \setminus [r \ge \beta]$$

we see that

$$P([r<t];\Omega) \ge C_\Omega \min\left(\left|[r\le\alpha]\right|^{(d-1)/d}, \left|[r\ge\beta]\right|^{(d-1)/d}\right)$$

Combining this estimate with (4.3) and (4.4), we obtain (4.2). ■

Lemma 4.3. *Given $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that*

$$\psi(s) \le C_\varepsilon s g(s) \psi'(s) \quad (s \ge \varepsilon). \tag{4.5}$$

Proof: Applying L'Hôpital's rule for $\liminf$, and remembering that g is an increasing function, we obtain

$$\begin{aligned}\liminf_{s\to\infty} \frac{sg(s)\psi'(s)}{\psi(s)} &\ge \liminf_{s\to\infty}\left(g(s) + sg'(s) + \frac{sg(s)\psi''(s)}{\psi'(s)}\right) \\ &\ge \lim_{s\to\infty} g(s) > 0, \end{aligned} \tag{4.6}$$

$$\liminf_{s\to 1}\frac{sg(s)\psi'(s)}{\psi(s)}=\liminf_{s\to 1}\frac{g(s)\psi'(s)}{\psi(s)}$$
$$\geq \liminf_{s\to 1}\left(g'(s)+\frac{g(s)\psi''(s)}{\psi'(s)}\right)\geq g'(1)>0. \quad (4.7)$$

In (4.6) and (4.7) we have used the fact that for $s\neq 1$, the signs of $g(s)$ and $\psi'(s)$ coincide, while $\psi''(s)>0$. Obviously, (4.6) and (4.7) imply (4.5). ■

Proof of Theorem 4.1: We claim that there exists $\beta>0$ such that

$$\delta := \inf_{\rho\in U}\left|[r\geq\beta]\right|>0 \quad (4.8)$$

Indeed, it follows from (1.8) (L'Hôpital's rule) that

$$\lim_{s\to\infty}\frac{\psi(s)}{s}=\infty.$$

As the entropy $\mathcal{E}_\psi$ is bounded on U, by de la Vallée Poussin's theorem the set U is uniformly integrable. Put

$$m=\frac{1}{2|\Omega|}\inf_{\rho\in U}\|\rho\|_{L^1(\Omega)};$$

for any $\rho\in U$ we have

$$2|\Omega|m\leq\|\rho\|_{L^1(\Omega)}=\int_{[\rho<m]}\rho\,dx+\int_{[\rho\geq m]}\rho\,dx\leq|\Omega|m+\omega_U\left(\left|[\rho\geq m]\right|\right),$$

where ω_U is the modulus of integrability of U. Hence

$$\omega_U\left(\left|[\rho\geq m]\right|\right)\geq|\Omega|m,$$

which clearly implies a lower bound on $\left|[\rho\geq m]\right|$ and a fortiori on $\left|[r\geq\beta]\right|$ with $\beta=\frac{m}{\sup\rho_\infty}$.

Clearly, there is no loss in generality in assuming $\beta<1$ in (4.8).

In what follows we fix α and β such that $0<\alpha<\beta<1$ and β satisfies (4.8). Denote

$$\sigma := \left|[r\leq\alpha]\right|,$$
$$\tau := \left|[\alpha<r<\beta]\right|$$

and also

$$D_{\alpha\beta}\mathcal{E}_\psi(\rho) := \int_\Omega rg(r)\psi'(r)\,d\rho_\infty + \int_{[\alpha<r<\beta]} rg'(r)\psi''(r)|\nabla r|^2\,d\rho_\infty.$$

Assume for now that $\sigma > 0$. Using Lemma 4.2, we have

$$\begin{aligned} D_{\alpha\beta}\mathcal{E}_\psi(\rho) &\geq \int_{[\alpha<r<\beta]} rg(r)\psi'(r)\,d\rho_\infty + \int_{[\alpha<r<\beta]} rg'(r)\psi''(r)|\nabla r|\,d\rho_\infty \\ &\geq \left(\min_{s\in[\alpha,\beta]} sg(s)\psi'(s)\right)\tau + C_{\alpha\beta}\frac{1}{\tau}\min\left(\sigma^{2(d-1)/d}, \left|[r\geq\beta]\right|^{2(d-1)/d}\right). \end{aligned}$$

Taking into account (4.8), we can write

$$D_{\alpha\beta}\mathcal{E}_\psi(\rho) \geq \frac{c}{2}\left(\tau + \frac{\min(\sigma^{2(d-1)/d}, \delta^{2(d-1)/d})}{\tau}\right)$$

with c independent of ρ. Estimating

$$\tau + \frac{\min(\sigma^{2(d-1)/d}, \delta^{2(d-1)/d})}{\tau} \geq 2\min(\sigma^{(d-1)/d}, \delta^{(d-1)/d}),$$

we obtain

$$D_{\alpha\beta}\mathcal{E}_\psi(\rho) \geq c\min(\sigma^{(d-1)/d}, \delta^{(d-1)/d}). \tag{4.9}$$

If $\sigma = 0$, this estimate trivially holds with any c. Since σ is a priori bounded from above by $|\Omega|$, (4.9) implies that

$$\begin{aligned} \sigma \leq C\min\left(\frac{\sigma}{|\Omega|^{1/d}}, \frac{\delta^{(d-1)/d}\sigma}{|\Omega|}\right) \\ \leq C\min(\sigma^{(d-1)/d}, \delta^{(d-1)/d}) \leq CD_{\alpha\beta}\mathcal{E}_\psi(\rho). \end{aligned} \tag{4.10}$$

Evoking Lemma 4.3, we obtain

$$\begin{aligned} \mathcal{E}_\psi(\rho) &= \int_{[r>\alpha]} \psi(r)\,d\rho_\infty + \int_{[r\leq\alpha]} \psi(r)\,d\rho_\infty \\ &\leq C_\alpha \int_{[r>\alpha]} r\psi'(r)g(r)\,d\rho_\infty + \psi(0)\int_{[r\leq\alpha]} d\rho_\infty \\ &\leq C_\alpha D_{\alpha\beta}\mathcal{E}_\psi(\rho) + C_0\left|[r\leq\alpha]\right| \\ &\leq CD_{\alpha\beta}\mathcal{E}_\psi(\rho) + C\sigma. \end{aligned}$$

Using (4.10) to estimate σ by $D_{\alpha\beta}\mathcal{E}_\psi$, we obtain (4.1) ∎

Appendix A. Reverse Fatou's Lemma for products

Lemma A.1. *Let (S,Σ,μ) be a measure space. Suppose that $\{f_n\}$ is bounded in $L^\infty(S,\mu)$ and $\{g_n\}$ converges to a nonnegative limit g in $L^1(S,\mu)$. Then*

$$\limsup_{n\to\infty}\int_S f_n g_n\,d\mu \le \int_S \left(\limsup_{n\to\infty} f_n\right) g\,d\mu. \tag{A.1}$$

Proof: As we have $|f_n g| \le (\sup_n \|f_n\|)g$, we can use Reverse Fatou's Lemma obtaining

$$\begin{aligned}\limsup_{n\to\infty}\int_S f_n g\,d\mu &\le \int_S \left(\limsup_{n\to\infty} f_n g\right) d\mu \\ &= \int_S \left(\limsup_{n\to\infty} f_n\right) g\,d\mu. \end{aligned}\tag{A.2}$$

Further, it is clear that

$$\lim_{n\to\infty}\int_S f_n(g_n-g)\,d\mu = 0. \tag{A.3}$$

Using (A.2) and (A.3) we obtain

$$\begin{aligned}\limsup_{n\to\infty}\int_S f_n g_n &= \limsup_{n\to\infty}\left(\int_S f_n g\,d\mu + \int_S f_n(g_n-g)\,d\mu\right) \\ &= \limsup_{n\to\infty}\int_S f_n g\,d\mu + \lim_{n\to\infty}\int_S f_n(g_n-g)\,d\mu \\ &\le \int_S \left(\limsup_{n\to\infty} f_n\right) g\,d\mu,\end{aligned}$$

as claimed. ■

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Stanislav Kondratyev
CMUC, Department of Mathematics, University of Coimbra,
3001-501 Coimbra, Portugal
E-mail address: kondratyev@mat.uc.pt

Dmitry Vorotnikov
CMUC, Department of Mathematics, University of Coimbra,
3001-501 Coimbra, Portugal
E-mail address: mitvorot@mat.uc.pt