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INTRINSIC SCHREIER SPLIT EXTENSIONS

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ABSTRACT: In the context of regular unital categories we introduce an intrinsic version of the notion of a *Schreier split epimorphism*, originally considered for monoids.

We show that such split epimorphisms satisfy the same homological properties as Schreier split epimorphisms of monoids do. This gives rise to new examples of $\mathscr{S}$-protomodular categories, and allows us to better understand the homological behaviour of monoids from a categorical perspective.

KEYWORDS: Fibration of points; jointly extremal-epimorphic pair; regular, unital, protomodular category; monoid; Jónsson–Tarski variety.
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1. Introduction

Schreier extensions of monoids, introduced in [23], have been studied by Patchkoria in [21, 20] in connection with the cohomology of monoids with coefficients in semimodules. Indeed, Patchkoria's second cohomology monoids can be described in terms of Schreier extensions. Moreover, Schreier split extensions correspond actually to monoid actions [22, 17], where an action of a monoid B on a monoid X is a monoid homomorphism from B to the monoid $End(X)$ of endomorphisms of X. These split extensions turned out to have the classical homological properties of split extensions of groups, such as the Split Short Five Lemma (see [9, 10] for more details on these properties).

In order to better understand this phenomenon of a distinguished class of (split) extensions of monoids behaving as (split) extensions of groups, Schreier extensions of monoids have been studied from a categorical point of view. The category of groups is protomodular [2], while the category of monoids is not. This led to the study of the notion of protomodularity relativised with respect

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to a suitable class $\mathscr{S}$ of split epimorphisms, giving rise to the notion of an $\mathscr{S}$-protomodular category [11], having the category of monoids with Schreier split epimorphisms as a key example. Later, in [16] it was shown that every Jónsson–Tarski variety of universal algebras [15] is an $\mathscr{S}$-protomodular category with respect to (a suitable generalisation of) the class of Schreier split epimorphisms. These categories satisfy relative versions of the basic properties of protomodular categories.

However, this categorical description of the homological properties of Schreier extensions of monoids is not entirely satisfactory. The definition of a Schreier (split) extension is not categorical, because it depends crucially on the element-wise approach involving a *Schreier retraction*, which is just a set-theoretical map (rather than a morphism of monoids). Moreover, for the same category, there may be several different classes of split epimorphisms that give rise to a structure of an $\mathscr{S}$-protomodular category: in the case of monoids, some of such different classes have been identified (see [9] and [12] for a description of these examples). For these reasons, the notion of an $\mathscr{S}$-protomodular category is able to capture only some of the (very strong) homological properties of Schreier split epimorphisms. Indeed, this notion covers other situations that are not so well-behaved.

The aim of the present paper is to give a characterisation of Schreier split epimorphisms in completely categorical terms, without using elements. In order to do that, we use *imaginary morphisms*—in the sense of Bourn and Janelidze, see [6, 8, 7, 24]—for the categorical Schreier retractions. The advantage of obtaining this characterisation is two-fold. On one hand, this approach may allow a sharper categorical interpretation of the homological properties of Schreier extensions than the one obtained through the notion of an $\mathscr{S}$-protomodular category. On the other hand, our notion of *intrinsic Schreier split extension*, being categorical, can be considered in a wider context than that of Jónsson–Tarski varieties, namely in regular unital [3] categories (under some additional assumptions). This may allow us to develop a meaningful cohomology theory for regular unital categories, which on one hand extends the well-established cohomology theory for semi-abelian categories [13], and on the other hand interprets categorically Patchkoria's cohomology of monoids. This is material for future work.

2. Schreier split extensions of monoids

In this section we recall from [9, 10] the main definitions and properties concerning Schreier split extensions.

A split epimorphism of monoids f with chosen section s and kernel K

$$\begin{tikzcd} K \arrow[r, tail, "k"'] & (X, \cdot, 1) \arrow[r, shift right, two heads, "f"'] & Y \arrow[l, shift right, "s"'] \end{tikzcd} \tag{1}$$

is called a **Schreier split epimorphism** if, for every $x \in X$, there exists a unique element $a \in K$ such that $x = k(a) \cdot sf(x)$. Equivalently, if there exists a unique function $q\colon X \dashrightarrow K$ such that $x = kq(x) \cdot sf(x)$ for all $x \in X$. We emphasise the fact that q is just a function (not necessarily a morphism of monoids) by using an arrow of type $\dashrightarrow$.

The uniqueness property may be replaced [10, Proposition 2.4] by an extra condition on q: the couple (f, s) is a Schreier split epimorphism if and only if

(S1) $x = kq(x) \cdot sf(x)$, for all $x \in X$;
(S2) $q(k(a) \cdot s(y)) = a$, for all $a \in K$, $y \in Y$.

Remark 2.1. Recall from [9] that Schreier split epimorphisms are also called **right homogenous** split epimorphisms. A split epimorphism as in (1) is called **left homogenous** if, for every $x \in X$, there exists a unique element $a \in K$ such that $x = sf(x) \cdot k(a)$.

Proposition 2.2. [9, Proposition 2.1.5] *Given a Schreier split extension as in* (1)*, the following hold:*

(S3) $qk = 1_K$*;*
(S4) $qs = 0$*;*
(S5) $q(1) = 1$*;*
(S6) $kq(s(y) \cdot k(a)) \cdot s(y) = s(y) \cdot k(a)$*, for all* $a \in K$*,* $y \in Y$*.*

We say that a split epimorphism (with fixed section s) is a **strongly split epimorphism** [4] (see also [18], where the same notion was considered, in the regular context, under the name of *regular point*) if its kernel k and section s form a jointly extremal-epimorphic pair (k, s). It is **stably strong** [19] if every pullback of it along any morphism is a strongly split epimorphism (with the section induced by s).

Lemma 2.3. [9, Lemma 2.1.6] *Any Schreier split epimorphism is strongly split.*

It is easy to see that every strongly split epimorphism (f, s) is such that f is the cokernel of its kernel, hence it gives rise to a split extension. The split extension (1) is then called a **Schreier split extension** and the map q is called the associated **(Schreier) retraction**. It is indeed a retraction, by **(S3)** above.

Actually any Schreier split epimorphism is stably strong, since Schreier split epimorphisms are stable under pullbacks:

Proposition 2.4. [9, Proposition 2.3.4] *Schreier split epimorphisms are stable under pullbacks along arbitrary morphisms.*

Corollary 2.5. *Any Schreier split epimorphism is stably strong.*

Some examples of Schreier split extensions are given by direct products:

Proposition 2.6. [9, Proposition 2.2.1] *A split extension underlying a product of monoids*

$$\begin{tikzcd} X \arrow[r, tail, "\langle 1_X, 0\rangle"'] & X \times Y \arrow[r, two heads, shift right, "\pi_Y"'] & Y \arrow[l, shift right, "\langle 0, 1_Y\rangle"'] \end{tikzcd}$$

is always a Schreier split extension.

Corollary 2.7. *Any terminal split extension and identity split extension*

$$\begin{tikzcd} X \arrow[r, tail, "1_X"'] & X \arrow[r, two heads, shift right, "!_X"'] & 0 \arrow[l, shift right, "0_X"'] \end{tikzcd} \qquad \textit{and} \qquad \begin{tikzcd} 0 \arrow[r, tail, "0_X"'] & X \arrow[r, two heads, shift right, "1_X"'] & X \arrow[l, shift right, "1_X"'] \end{tikzcd}$$

is a Schreier split extension.

Several other examples of Schreier split extensions are considered in [9]. A useful property of such split extensions is the following:

Proposition 2.8. [10, Lemma 4.1] *Any morphism between two Schreier split extensions*

$$\begin{tikzcd} K \arrow[r, tail, shift right, "k"'] \arrow[d, "\overline{g}"'] & X \arrow[l, dashed, shift right, "q"'] \arrow[r, two heads, shift right, "f"'] \arrow[d, "g"] & Y \arrow[l, shift right, "s"'] \arrow[d, "h"] \\ K' \arrow[r, tail, shift right, "k'"'] & X' \arrow[l, dashed, shift right, "q'"'] \arrow[r, two heads, shift right, "f'"'] & Y' \arrow[l, shift right, "s'"'] \end{tikzcd}$$

is compatible with respect to their retractions, i.e., $\overline{g}q = q'g$.

3. $\mathscr{S}$-protomodular categories

In this section we recall the definition of an $\mathscr{S}$-protomodular category, with respect to a class $\mathscr{S}$ of points in a category with a zero object.

Let $\mathbb{C}$ be a finitely complete category. A **point** in $\mathbb{C}$ is a split epimorphism f with a chosen section s:

$$X \underset{f}{\overset{s}{\leftrightarrows}} Y, \qquad\qquad fs = 1_Y.$$

We say that a point is **(stably) strong** when it is a (stably) strongly split epimorphism. Consequently, its kernel k and section s form a jointly extremal-epimorphic pair (k, s), and (f, s) is part of a split extension.

We denote by $\mathsf{Pt}(\mathbb{C})$ the category of points in $\mathbb{C}$, whose morphisms are pairs of morphisms which form commutative squares with both the split epimorphisms and their sections. The functor $\mathrm{cod}\colon \mathsf{Pt}(\mathbb{C}) \to \mathbb{C}$ associates with every split epimorphism its codomain. It is a fibration, usually called the **fibration of points**. For each object Y of $\mathbb{C}$, we denote by $\mathsf{Pt}_Y(\mathbb{C})$ the fibre of this fibration, whose objects are the points with codomain Y.

Let $\mathscr{S}$ be a class of points in $\mathbb{C}$ which is stable under pullbacks along any morphism. If we look at it as a full subcategory $\mathscr{S}$-$\mathsf{Pt}(\mathbb{C})$ of $\mathsf{Pt}(\mathbb{C})$, then it gives rise to a subfibration $\mathscr{S}$-cod of the fibration of points.

Definition 3.1. [9, Definition 8.1.1] Let $\mathbb{C}$ be a pointed finitely complete category, and $\mathscr{S}$ a pullback-stable class of points. We say that $\mathbb{C}$ is **$\mathscr{S}$-protomodular** when:

(1) every point in $\mathscr{S}$-$\mathsf{Pt}(\mathbb{C})$ is a strong point;
(2) $\mathscr{S}$-$\mathsf{Pt}(\mathbb{C})$ is closed under finite limits in $\mathsf{Pt}(\mathbb{C})$.

Example 3.2. [9] The category Mon of monoids is $\mathscr{S}$-protomodular with respect to the class $\mathscr{S}$ of Schreier split epimorphism.

Example 3.3. [16] We recall that a variety of universal algebras is called a **Jónsson–Tarski** variety [15] when its theory contains a unique constant 0 and a binary operation $+$ such that $x + 0 = x = 0 + x$. So an algebra is a unitary magma, possibly equipped with additional operations.

Every Jónsson–Tarski variety is an $\mathscr{S}$-protomodular category with respect to the class of Schreier split epimorphisms. Indeed, the definition of a Schreier split epimorphism makes sense also in this wider context, and it gives rise to a whole family of examples of $\mathscr{S}$-protomodular categories.

4. Imaginary morphisms

The technique of *imaginary morphisms* stems from the work of Bourn and Janelidze [6, 8, 7]; it was further explored in [24] by the second and third authors of the present article. Here we use imaginary morphisms in order to capture certain characteristic properties of Jónsson–Tarski varieties and to define an intrinsic version of the concept of a Schreier retraction.

Here we assume that $\mathbb{C}$ is a regular category with enough projectives and that we can choose projective covers functorially. We write $\varepsilon_X \colon P(X) \twoheadrightarrow X$ for the chosen projective cover of some object X in $\mathbb{C}$: ε_X is a regular epimorphism and $P(X)$ is a projective object, which means that for any morphism $z \colon P(X) \to Z$ and any regular epimorphism $f \colon Y \twoheadrightarrow Z$, there exists a morphism $y \colon P(X) \to Y$ in $\mathbb{C}$ such that $fy = z$. In what follows, it will be convenient for us to let P be part of a comonad (P, δ, ε); we say that $\mathbb{C}$ is equipped with **functorial (comonadic) projective covers**. Note that for any morphism $f \colon X \to Y$ in $\mathbb{C}$

$$f\varepsilon_X = \varepsilon_Y P(f) \tag{2}$$

and

$$P^2(f)\delta_X = \delta_Y P(f), \tag{3}$$

where $P^2 = PP$. Also

$$\varepsilon_{P(X)}\delta_X = 1_{P(X)} = P(\varepsilon_X)\delta_X \tag{4}$$

and

$$P(\delta_X)\delta_X = \delta_{P(X)}\delta_X, \tag{5}$$

for all objects X in $\mathbb{C}$.

Example 4.1. If $\mathbb{V}$ is a variety of universal algebras, then we may consider the free algebra comonad (P, δ, ε). For any algebra X, we have

$$\begin{array}{rcl} \varepsilon_X \colon P(X) & \twoheadrightarrow & X \\ {}[x] & \mapsto & x \end{array} \quad \text{and} \quad \begin{array}{rcl} \delta_X \colon P(X) & \hookrightarrow & P^2(X), \\ {}[x] & \mapsto & [[x]] \end{array}$$

where $[x]$ denotes the one letter word x, which are the generators of $P(X)$. In this case, any function $\overline{f} \colon X \dashrightarrow Y$ between algebras X and Y extends uniquely to a morphism

$$\begin{array}{rcl} f \colon P(X) & \to & Y \\ {}[x] & \mapsto & \overline{f}(x) \end{array}$$

in $\mathbb{V}$.

This example motivates the following definition:

imaginary composition	corresponding morphism in $\mathbb{C}$
$\begin{tikzcd} X \arrow[r, dashed, "\overline{f}"] \arrow[rr, dashed, bend right, "g\overline{f}"'] & Y \arrow[r, "g"] & Z \end{tikzcd}$	$P(X) \xrightarrow{f} Y \xrightarrow{g} Z$
$\begin{tikzcd} W \arrow[r, "h"] \arrow[rr, dashed, bend right, "\overline{f}h"'] & X \arrow[r, dashed, "\overline{f}"] & Y \end{tikzcd}$	$P(W) \xrightarrow{P(h)} P(X) \xrightarrow{f} Y$

FIGURE 1. Imaginary compositions

Definition 4.2. A morphism $f\colon P(X) \to Y$ is called an **imaginary morphism from X to Y**; we write $\overline{f}\colon X \dashrightarrow Y$.

Example 4.3. In a variety of universal algebras $\mathbb{V}$ equipped with the free algebra comonad (Example 4.1), each function from an algebra X to an algebra Y may be considered as an imaginary morphism $X \dashrightarrow Y$ in $\mathbb{V}$.

An imaginary morphism $X \dashrightarrow Y$ is not actually a morphism $X \to Y$ in $\mathbb{C}$. Rather, it is a morphism in $\mathbb{C}$ with domain $P(X)$. Any real morphism $f\colon X \to Y$ may be considered as an imaginary morphism $\overline{f}\colon X \dashrightarrow Y$, namely the composite $f\varepsilon_X$. In particular, $1_Y\colon Y \to Y$, considered as an imaginary morphism $Y \dashrightarrow Y$, is $\varepsilon_Y\colon P(Y) \to Y$.

Imaginary morphisms do not compose, however they do compose with real ones. Let $\overline{f}\colon X \dashrightarrow Y$ be an imaginary morphism and let $g\colon Y \to Z$ and $h\colon W \to X$ be morphisms in $\mathbb{C}$. Then they compose as in Figure 1.

Lemma 4.4. *In a regular category with functorial projective covers, a morphism $f\colon X \to Y$ is a regular epimorphism if and only if it admits an imaginary splitting $\overline{s}\colon Y \dashrightarrow X$. This means that $f\overline{s} = \overline{1_Y}\colon Y \dashrightarrow Y$ or, equivalently, that $s\colon P(Y) \to X$ satisfies $fs = \varepsilon_Y$.*

Proof: This is an immediate consequence of the definitions. ■

A pointed and regular category with binary coproducts is **unital** [3, 1] when, for all objects A, B, the comparison morphism

$$r_{A,B} = \begin{pmatrix} 1_A & 0 \\ 0 & 1_B \end{pmatrix}\colon A + B \twoheadrightarrow A \times B$$

is a regular epimorphism. If the category has functorial projective covers, then, by Lemma 4.4, this is equivalent to saying that $r_{A,B}$ admits an imaginary splitting

$$\overline{t_{A,B}}\colon A \times B \dashrightarrow A + B,$$

i.e., there exists a morphism $t_{A,B}\colon P(A\times B)\to A+B$ such that

$$r_{A,B}t_{A,B}=\varepsilon_{A\times B}. \tag{6}$$

Example 4.5. A variety of universal algebras $\mathbb{V}$ is unital if and only if it is a Jónsson–Tarski variety [1]. In this case, for any pair of algebras (A,B) in $\mathbb{V}$, we make the following canonical choices of imaginary splittings for $r_{A,B}$: the **direct imaginary splitting** t^d

$$[(a,b)]\mapsto \underline{a}+\overline{b}$$

which sends a generator $[(a,b)]\in P(A\times B)$ to the sum of $\underline{a}=\iota_A(a)$ with $\overline{b}=\iota_B(b)$ in $A+B$, and the **twisted imaginary splitting** t^w

$$[(a,b)]\mapsto \overline{b}+\underline{a}$$

which does the same, but in the opposite order. Note that each of those choices determines a natural transformation

$$t\colon P((\cdot)\times(\cdot))\Rightarrow(\cdot)+(\cdot)$$

such that $rt=\varepsilon_{(\cdot)\times(\cdot)}$, where $r\colon(\cdot)+(\cdot)\Rightarrow(\cdot)\times(\cdot)$ and

$$\varepsilon_{(\cdot)\times(\cdot)}\colon P((\cdot)\times(\cdot))\Rightarrow(\cdot)\times(\cdot).$$

Definition 4.6. In a pointed regular (unital) category $\mathbb{C}$ with binary coproducts and functorial projective covers, a **natural imaginary splitting (of the comparison from sum to product)** is a natural transformation

$$t\colon P((\cdot)\times(\cdot))\Rightarrow(\cdot)+(\cdot)$$

such that $rt=\varepsilon_{(\cdot)\times(\cdot)}$.

Note that, by Lemma 4.4, the existence of a natural imaginary splitting in a pointed regular category with binary coproducts and functorial projective covers implies that this category is unital. Any Jónsson–Tarski variety comes equipped with a direct and a twisted natural imaginary splitting t^d and t^w as in Example 4.5. On the other hand, outside the varietal context there seems to be no reasonable way to characterise these cases categorically.

Remark 4.7. Any natural imaginary splitting $t\colon P((\cdot)\times(\cdot))\Rightarrow(\cdot)+(\cdot)$ has the following properties:

1. $t_{A,0}$ is isomorphic to ε_A

$$\begin{tikzcd}
P(A) \arrow[r, two heads, "\varepsilon_A"] \arrow[d, "\cong"] & A \arrow[r, equal, "1_A"] \arrow[d, "\cong"] & A \arrow[d, "\cong"] \\
P(A\times 0) \arrow[r, "t_{A,0}"] \arrow[rr, two heads, bend right, "\varepsilon_{A\times 0}"'] & A+0 \arrow[r, two heads, "r_{A,0}"] & A\times 0,
\end{tikzcd}$$

   for all objects A in $\mathbb{C}$;
2. the naturality of t gives the commutative diagram

$$\begin{tikzcd}
P(A\times B) \arrow[r, "t_{A,B}"] \arrow[d, "P(u\times v)"'] & A+B \arrow[d, "u+v"] \\
P(C\times D) \arrow[r, "t_{C,D}"'] & C+D
\end{tikzcd} \tag{7}$$

   for all $u\colon A\to C$, $v\colon B\to D$ in $\mathbb{C}$;
3. from (6), we deduce

$$(1_A\ \ 0)t_{A,B} = \pi_A\varepsilon_{A\times B} \overset{(2)}{=} \varepsilon_A P(\pi_A) \tag{8}$$

   and

$$(0\ \ 1_B)t_{A,B} = \pi_B\varepsilon_{A\times B} \overset{(2)}{=} \varepsilon_B P(\pi_B) \tag{9}$$

   for all objects A and B in $\mathbb{C}$;
4. using properties 1. and 2. above, we obtain the (regular epimorphism, monomorphism) factorisations

$$\begin{tikzcd}
P(A) \arrow[r, "P(\langle 1_A{,}0\rangle)"] \arrow[dr, two heads, "\varepsilon_A"'] & P(A\times B) \arrow[r, "t_{A,B}"] & A+B \\
& A \arrow[ur, tail, "\iota_A"'] &
\end{tikzcd} \tag{10}$$

   and

$$\begin{tikzcd}
P(B) \arrow[r, "P(\langle 0{,}1_B\rangle)"] \arrow[dr, two heads, "\varepsilon_B"'] & P(A\times B) \arrow[r, "t_{A,B}"] & A+B, \\
& B \arrow[ur, tail, "\iota_B"'] &
\end{tikzcd}$$

   for all objects A and B in $\mathbb{C}$.

5. Intrinsic Schreier split extensions

In this section we describe a categorical approach towards Schreier extensions. Here $\mathbb{C}$ will denote a regular unital category with binary coproducts, functorial projective covers and a natural imaginary splitting t.

Definition 5.1. A split epimorphism f with chosen section s and kernel K

$$\begin{tikzcd} K \arrow[r, tail, "k"'] & X \arrow[r, shift right, two heads, "f"'] & Y, \arrow[l, shift right, "s"'] \end{tikzcd} \tag{11}$$

is called an **intrinsic Schreier split epimorphism** (with respect to t) if there exists an imaginary morphism $q\colon X \dashrightarrow K$ (i.e. a morphism $q\colon P(X) \to K$), called the **imaginary (Schreier) retraction**, such that

(iS1) $\lfloor kq\ \ sf\varepsilon_X \rceil t_{P(X),P(X)} P(\langle 1_{P(X)}, 1_{P(X)}\rangle)\delta_X = \varepsilon_X$, i.e., the diagram

$$\begin{tikzcd} P^2(X) \arrow[r, "P(\langle 1_{P(X)},1_{P(X)}\rangle)"] & P(P(X)\times P(X)) \arrow[r, "t_{P(X),P(X)}"] & P(X)+P(X) \arrow[d, "\lfloor kq\ sf\varepsilon_X \rceil"] \\ P(X) \arrow[u, tail, "\delta_X"] \arrow[rr, two heads, "\varepsilon_X"'] & & X \end{tikzcd}$$

commutes;

(iS2) $qP(\lfloor k\ \ s\rceil)P(t_{K,Y})\delta_{K\times Y} = \pi_K\varepsilon_{K\times Y}$, i.e., the diagram

$$\begin{tikzcd} P^2(K\times Y) \arrow[r, "P(t_{K,Y})"] & P(K+Y) \arrow[r, "P(\lfloor k\ s\rceil)"] & P(X) \arrow[d, "q"] \\ P(K\times Y) \arrow[u, tail, "\delta_{K\times Y}"] \arrow[r, two heads, "\varepsilon_{K\times Y}"'] & K\times Y \arrow[r, two heads, "\pi_K"'] & K \end{tikzcd}$$

commutes.

Proposition 5.2. *If the point (f, s) in* (11) *admits an imaginary morphism $q\colon X \dashrightarrow K$ satisfying* **(iS1)**, *then it is a strong point.*

Proof: From **(iS1)** we see that $\lfloor kq\ \ sf\varepsilon_X\rceil\colon\ P(X)+P(X) \twoheadrightarrow X$ is a regular epimorphism. It easily follows that also $\lfloor k\ \ s\rceil\colon K+Y \twoheadrightarrow X$ is a regular epimorphism, thus (k, s) is a jointly extremal-epimorphic pair. ■

We then call the point and split extension in (11) an **intrinsic Schreier point** and an **intrinsic Schreier split extension**, respectively. The properties **(iS1)** and **(iS2)** are the respective translations of **(S1)** and **(S2)** to the "imaginary" context.

Proposition 5.3. *For an intrinsic Schreier split extension* (11)*, the imaginary retraction* $q\colon X \dashrightarrow K$ *is unique.*

Proof: Suppose that there exist two imaginary retractions $q, q'\colon P(X) \to K$ such that **(iS1)** and **(iS2)** hold. From **(iS1)** applied to q we get

$$\begin{aligned}
&\varepsilon_X = (kq\ \ sf\varepsilon_X)t_{P(X),P(X)}P(\langle 1_{P(X)}, 1_{P(X)}\rangle)\delta_X\\
\Leftrightarrow\ &\varepsilon_X = (k\ \ s)(q + f\varepsilon_X)t_{P(X),P(X)}P(\langle 1_{P(X)}, 1_{P(X)}\rangle)\delta_X\\
\overset{(7)}{\Leftrightarrow}\ &\varepsilon_X = (k\ \ s)t_{K,Y}P(q \times (f\varepsilon_X))P(\langle 1_{P(X)}, 1_{P(X)}\rangle)\delta_X\\
\Leftrightarrow\ &\varepsilon_X = (k\ \ s)t_{K,Y}P(\langle q, f\varepsilon_X\rangle)\delta_X.
\end{aligned} \tag{12}$$

Applying **(iS1)** to q' we obtain a similar equality, namely

$$(k\ \ s)t_{K,Y}P(\langle q, f\varepsilon_X\rangle)\delta_X = (k\ \ s)t_{K,Y}P(\langle q', f\varepsilon_X\rangle)\delta_X.$$

We use this equality and **(iS2)** applied to q to obtain $q = q'$. Indeed

$$\begin{aligned}
&qP((k\ \ s))P(t_{K,Y})P^2(\langle q, f\varepsilon_X\rangle)P(\delta_X)\delta_X\\
&\qquad = qP((k\ \ s))P(t_{K,Y})P^2(\langle q', f\varepsilon_X\rangle)P(\delta_X)\delta_X\\
\overset{(5)}{\Leftrightarrow}\quad &qP((k\ \ s))P(t_{K,Y})P^2(\langle q, f\varepsilon_X\rangle)\delta_{P(X)}\delta_X\\
&\qquad = qP((k\ \ s))P(t_{K,Y})P^2(\langle q', f\varepsilon_X\rangle)\delta_{P(X)}\delta_X\\
\overset{(3)}{\Leftrightarrow}\quad &qP((k\ \ s))P(t_{K,Y})\delta_{K\times Y}P(\langle q, f\varepsilon_X\rangle)\delta_X\\
&\qquad = qP((k\ \ s))P(t_{K,Y})\delta_{K\times Y}P(\langle q', f\varepsilon_X\rangle)\delta_X\\
\overset{\textbf{(iS2)}}{\Leftrightarrow}\quad &\pi_K\varepsilon_{K\times Y}P(\langle q, f\varepsilon_X\rangle)\delta_X = \pi_K\varepsilon_{K\times Y}P(\langle q', f\varepsilon_X\rangle)\delta_X\\
\overset{(2)}{\Leftrightarrow}\quad &\pi_K\langle q, f\varepsilon_X\rangle\varepsilon_{P(X)}\delta_X = \pi_K\langle q', f\varepsilon_X\rangle\varepsilon_{P(X)}\delta_X\\
\overset{(4)}{\Leftrightarrow}\quad &q = q'.
\end{aligned}$$

■

The next results give the intrinsic versions of those recalled in Proposition 2.2.

Proposition 5.4. *Let $\mathbb{C}$ be a regular unital category with binary coproducts, functorial projective covers and a natural imaginary splitting t. If* (11) *is an intrinsic Schreier split extension with imaginary retraction q, then:*

(iS3) $qP(k) = \varepsilon_K$;
(iS4) $qP(s) = 0$;
(iS5) $q0_{P(X)} = 0_K$;

(iS6) $\lfloor s\ \ k\rceil t_{Y,K} = \lfloor kqP(\lfloor s\ \ k\rceil)P(t_{Y,K})\ \ s\pi_Y\varepsilon_{Y\times K}\varepsilon_{P(Y\times K)}\rceil t_{P^2(Y\times K),P^2(Y\times K)}$
$\circ\, P(\langle 1_{P^2(Y\times K)}, 1_{P^2(Y\times K)}\rangle)\delta_{P(Y\times K)}\delta_{Y\times K}.$

Proof: If we compose each side of **(iS2)** with $P(\langle 1_K, 0\rangle)$, use (3) and (2), we obtain

$$\begin{aligned} & qP(\lfloor k\ \ s\rceil)P(t_{K,Y})P^2(\langle 1_K,0\rangle)\delta_K = \pi_K\langle 1_K,0\rangle\varepsilon_K \\ \overset{(10)}{\Leftrightarrow}\quad & qP(\lfloor k\ \ s\rceil)P(\iota_K)P(\varepsilon_K)\delta_K = \varepsilon_K \\ \overset{(4)}{\Leftrightarrow}\quad & qP(k) = \varepsilon_K; \end{aligned}$$

this proves **(iS3)**. Similarly, we prove **(iS4)** by composing each side of **(iS2)** with $P(\langle 0, 1_Y\rangle)$; **(iS5)** is obvious.

Next, we prove a stronger equality from which **(iS6)** easily follows (by precomposing with $\delta_{Y\times K}$ and using (4)):

$$\begin{aligned} \lfloor s\ \ k\rceil t_{Y,K}\varepsilon_{P(Y\times K)} &\overset{(2)}{=} \varepsilon_X P(\lfloor s\ \ k\rceil t_{Y,K}) \\ &\overset{\textbf{(iS1)}}{=} \lfloor kq\ \ sf\varepsilon_X\rceil t_{P(X),P(X)}P(\langle 1_{P(X)},1_{P(X)}\rangle)\delta_X P(\lfloor s\ \ k\rceil t_{Y,K}) \\ &\overset{(3)}{=} \lfloor kq\ \ sf\varepsilon_X\rceil t_{P(X),P(X)}P(\langle 1_{P(X)},1_{P(X)}\rangle)P^2(\lfloor s\ \ k\rceil t_{Y,K})\delta_{P(Y\times K)} \\ &= \lfloor kq\ \ sf\varepsilon_X\rceil t_{P(X),P(X)}P(\langle 1_{P(X)},1_{P(X)}\rangle P(\lfloor s\ \ k\rceil t_{Y,K}))\delta_{P(Y\times K)} \\ &= \lfloor kq\ \ sf\varepsilon_X\rceil t_{P(X),P(X)}P(P(\lfloor s\ \ k\rceil t_{Y,K})\times P(\lfloor s\ \ k\rceil t_{Y,K})) \\ &\qquad \circ\, P(\langle 1_{P^2(Y\times K)},1_{P^2(Y\times K)}\rangle)\delta_{P(Y\times K)} \\ &\overset{(7)}{=} \lfloor kqP(\lfloor s\ \ k\rceil t_{Y,K})\ \ sf\varepsilon_X P(\lfloor s\ \ k\rceil t_{Y,K})\rceil t_{P^2(Y\times K),P^2(Y\times K)} \\ &\qquad \circ\, P(\langle 1_{P^2(Y\times K)},1_{P^2(Y\times K)}\rangle)\delta_{P(Y\times K)}. \end{aligned}$$

To finish, we use

$$\begin{aligned} sf\varepsilon_X P(\lfloor s\ \ k\rceil t_{Y,K}) &\overset{(2)}{=} sf\lfloor s\ \ k\rceil t_{Y,K}\varepsilon_{P(Y\times K)} \\ &= s\lfloor 1_Y\ \ 0\rceil t_{Y,K}\varepsilon_{P(Y\times K)} \\ &\overset{(8)}{=} s\pi_Y\varepsilon_{Y\times K}\varepsilon_{P(Y\times K)}. \end{aligned}$$

■

Any binary product gives an example of an intrinsic Schreier split extension, like for Schreier split extensions for monoids (see Proposition 2.6).

Proposition 5.5. *Any split extension given by a binary product is a Schreier split extension.*

Proof: Given any split extension underlying a binary product in $\mathbb{C}$

$$\begin{tikzcd} X \arrow[r, tail, "\langle 1_X,0\rangle"'] & X\times Y \arrow[r, shift right, two heads, "\pi_Y"'] & Y \arrow[l, shift right, "\langle 0,1_Y\rangle"'] \end{tikzcd}$$

we define $q=\pi_X\varepsilon_{X\times Y}\colon P(X\times Y)\to X$ for its imaginary retraction. To prove **(iS1)**, we use

$$\begin{aligned}
&\big(\langle 1_X,0\rangle\pi_X\varepsilon_{X\times Y}\ \ \langle 0,1_Y\rangle\pi_Y\varepsilon_{X\times Y}\big)t_{P(X\times Y),P(X\times Y)}\\
&\qquad\circ P(\langle 1_{P(X\times Y)},1_{P(X\times Y)}\rangle)\delta_{X\times Y}\\
&=\big(\langle \pi_X,0\rangle\ \ \langle 0,\pi_Y\rangle\big)(\varepsilon_{X\times Y}+\varepsilon_{X\times Y})t_{P(X\times Y),P(X\times Y)}\\
&\qquad\circ P(\langle 1_{P(X\times Y)},1_{P(X\times Y)}\rangle)\delta_{X\times Y}\\
&\overset{(7)}{=}(\pi_X\times\pi_Y)\big(\begin{smallmatrix}1_{X\times Y} & 0\\ 0 & 1_{X\times Y}\end{smallmatrix}\big)t_{X\times Y,X\times Y}P(\varepsilon_{X\times Y}\times\varepsilon_{X\times Y})\\
&\qquad\circ P(\langle 1_{P(X\times Y)},1_{P(X\times Y)}\rangle)\delta_{X\times Y}\\
&\overset{(6)}{=}(\pi_X\times\pi_Y)\varepsilon_{X\times Y\times X\times Y}P(\langle\varepsilon_{X\times Y},\varepsilon_{X\times Y}\rangle)\delta_{X\times Y}\\
&\overset{(2)}{=}\varepsilon_{X\times Y}P(\pi_X\times\pi_Y)P(\langle\varepsilon_{X\times Y},\varepsilon_{X\times Y}\rangle)\delta_{X\times Y}\\
&=\varepsilon_{X\times Y}P(\varepsilon_{X\times Y})\delta_{X\times Y}\\
&\overset{(4)}{=}\varepsilon_{X\times Y}.
\end{aligned}$$

The proof of **(iS2)** is quite straightforward:

$$\pi_X\varepsilon_{X\times Y}P\big(\big(\begin{smallmatrix}1_X & 0\\ 0 & 1_Y\end{smallmatrix}\big)\big)P(t_{X,Y})\delta_{X\times Y}\overset{(6)}{=}\pi_X\varepsilon_{X\times Y}P(\varepsilon_{X\times Y})\delta_{X\times Y}\overset{(4)}{=}\pi_X\varepsilon_{X\times Y}.\quad\blacksquare$$

Corollary 5.6. *For any object X,*

$$\begin{tikzcd} X \arrow[r, tail, "1_X"'] & X \arrow[r, shift right, two heads, "!_X"'] & 0 \arrow[l, shift right, "0_X"'] \end{tikzcd}\qquad \textit{and}\qquad \begin{tikzcd} 0 \arrow[r, tail, "0_X"'] & X \arrow[r, shift right, two heads, "1_X"'] & X \arrow[l, shift right, "1_X"'] \end{tikzcd}$$

are intrinsic Schreier split extensions.

Proposition 5.7. *Any morphism in* $\mathsf{Pt}(\mathbb{C})$ *between intrinsic Schreier split extensions*

$$\begin{tikzcd} K \arrow[r, tail, shift right, "k"'] \arrow[d, "\overline{g}"'] & X \arrow[l, dashed, shift right, "q"'] \arrow[r, shift right, two heads, "f"'] \arrow[d, "g"] & Y \arrow[l, shift right, "s"'] \arrow[d, "h"] \\ K' \arrow[r, tail, shift right, "k'"'] & X' \arrow[l, dashed, shift right, "q'"'] \arrow[r, shift right, two heads, "f'"'] & Y' \arrow[l, shift right, "s'"'] \end{tikzcd}$$

is compatible with respect to their imaginary retractions, i.e., $\overline{g}q = q'P(g)$.

Proof: We start by using (12) for the bottom intrinsic Schreier extension

$$
\begin{aligned}
& (k' \;\; s')t_{K',Y'}P(\langle q', f'\varepsilon_{X'}\rangle)\delta_{X'}P(g) = \varepsilon_{X'}P(g) \\
\overset{(3),(2)}{\Leftrightarrow}\;\; & (k' \;\; s')t_{K',Y'}P(\langle q', f'\varepsilon_{X'}\rangle)P^2(g)\delta_X = g\varepsilon_X \\
\overset{(12)}{\Leftrightarrow}\;\; & (k' \;\; s')t_{K',Y'}P(\langle q'P(g), f'\varepsilon_{X'}P(g)\rangle)\delta_X = g(k \;\; s)t_{K,Y}P(\langle q, f\varepsilon_X\rangle)\delta_X \\
\overset{(2)}{\Leftrightarrow}\;\; & (k' \;\; s')t_{K',Y'}P(\langle q'P(g), f'g\varepsilon_X\rangle)\delta_X = (gk \;\; gs)t_{K,Y}P(\langle q, f\varepsilon_X\rangle)\delta_X \\
\Leftrightarrow\;\; & (k' \;\; s')t_{K',Y'}P(\langle q'P(g), f'g\varepsilon_X\rangle)\delta_X = (k' \;\; s')t_{K',Y'}P(\langle \overline{g}q, f'g\varepsilon_X\rangle)\delta_X,
\end{aligned}
$$

by using the commutativity of the diagram and (7).

We may now proceed as in the second part of the proof of Proposition 5.3 to conclude that $q'P(g) = \overline{g}q$. ■

Proposition 5.8. *If the point* (f, s) *in* (11) *admits an imaginary morphism* $q\colon X \dashrightarrow K$ *satisfying* **(iS1)**, *then it is a stably strong point.*

Proof: We already know that (f, s) is strong by Proposition 5.2. Now to see that (f, s) is a stably strong point, we take its pullback along an arbitrary morphism g

$$
\begin{tikzcd}
K \arrow[r, equal] \arrow[d, tail, "\langle 0,k\rangle"'] & K \arrow[d, tail, "k"] \\
Z \times_Y X \arrow[r, "\pi_X"] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[d, shift right, two heads, "\pi_Z"'] & X \arrow[d, shift right, two heads, "f"'] \\
Z \arrow[r, "g"'] \arrow[u, shift right, "\langle 1_Z, sg\rangle"'] & Y. \arrow[u, shift right, "s"']
\end{tikzcd}
\tag{13}
$$

To prove that $(\pi_Z, \langle 1_Z, sg\rangle)$ is strong it suffices to show the existence of an imaginary morphism $q'\colon Z \times_Y X \dashrightarrow K$ satisfying **(iS1)**. We define $q' = qP(\pi_X)\colon P(Z \times_Y X) \to K$ and check that

$$
\begin{aligned}
\varepsilon_{Z\times_Y X} = {} & (\langle 0, k\rangle qP(\pi_X) \;\; \langle 1_Z, sg\rangle \pi_Z \varepsilon_{Z\times_Y X}) \\
& \circ t_{P(Z\times_Y X),P(Z\times_Y X)}P(\langle 1_{P(Z\times_Y X)}, 1_{P(Z\times_Y X)}\rangle)\delta_{Z\times_Y X}.
\end{aligned}
$$

Indeed, this equality follows from

$$\begin{aligned}
&\pi_Z(\langle 0,k\rangle qP(\pi_X)\ \ \langle 1_Z, sg\rangle \pi_Z\varepsilon_{Z\times_Y X})\\
&\qquad \circ t_{P(Z\times_Y X),P(Z\times_Y X)}P(\langle 1_{P(Z\times_Y X)},1_{P(Z\times_Y X)}\rangle)\delta_{Z\times_Y X}\\
&= (0\ \ \pi_Z\varepsilon_{Z\times_Y X})t_{P(Z\times_Y X),P(Z\times_Y X)}P(\langle 1_{P(Z\times_Y X)},1_{P(Z\times_Y X)}\rangle)\delta_{Z\times_Y X}\\
&= \pi_Z\varepsilon_{Z\times_Y X}(0\ \ 1_{P(Z\times_Y X)})t_{P(Z\times_Y X),P(Z\times_Y X)}P(\langle 1_{P(Z\times_Y X)},1_{P(Z\times_Y X)}\rangle)\delta_{Z\times_Y X}\\
&\overset{(9)}{=} \pi_Z\varepsilon_{Z\times_Y X}\varepsilon_{P(Z\times_Y X)}P(\pi_2)P(\langle 1_{P(Z\times_Y X)},1_{P(Z\times_Y X)}\rangle)\delta_{Z\times_Y X}\\
&= \pi_Z\varepsilon_{Z\times_Y X}\varepsilon_{P(Z\times_Y X)}\delta_{Z\times_Y X} \overset{(4)}{=} \pi_Z\varepsilon_{Z\times_Y X}
\end{aligned}$$

and

$$\begin{aligned}
&\pi_X(\langle 0,k\rangle qP(\pi_X)\ \ \langle 1_Z, sg\rangle \pi_Z\varepsilon_{Z\times_Y X})\\
&\qquad \circ t_{P(Z\times_Y X),P(Z\times_Y X)}P(\langle 1_{P(Z\times_Y X)},1_{P(Z\times_Y X)}\rangle)\delta_{Z\times_Y X}\\
&= (kqP(\pi_X)\ \ sg\pi_Z\varepsilon_{Z\times_Y X})\\
&\qquad \circ t_{P(Z\times_Y X),P(Z\times_Y X)}P(\langle 1_{P(Z\times_Y X)},1_{P(Z\times_Y X)}\rangle)\delta_{Z\times_Y X}\\
&= (kqP(\pi_X)\ \ sf\pi_X\varepsilon_{Z\times_Y X})\\
&\qquad \circ t_{P(Z\times_Y X),P(Z\times_Y X)}P(\langle 1_{P(Z\times_Y X)},1_{P(Z\times_Y X)}\rangle)\delta_{Z\times_Y X}\\
&\overset{(2)}{=} (kq\ \ sf\varepsilon_X)(P(\pi_X)+P(\pi_X))\\
&\qquad \circ t_{P(Z\times_Y X),P(Z\times_Y X)}P(\langle 1_{P(Z\times_Y X)},1_{P(Z\times_Y X)}\rangle)\delta_{Z\times_Y X}\\
&\overset{(7)}{=} (kq\ \ sf\varepsilon_X)t_{P(X),P(X)}P(P(\pi_X)\times P(\pi_X))P(\langle 1_{P(Z\times_Y X)},1_{P(Z\times_Y X)}\rangle)\delta_{Z\times_Y X}\\
&= (kq\ \ sf\varepsilon_X)t_{P(X),P(X)}P(\langle P(\pi_X),P(\pi_X)\rangle)\delta_{Z\times_Y X}\\
&= (kq\ \ sf\varepsilon_X)t_{P(X),P(X)}P(\langle 1_{P(X)},1_{P(X)}\rangle)P^2(\pi_X)\delta_{Z\times_Y X}\\
&\overset{(3)}{=} (kq\ \ sf\varepsilon_X)t_{P(X),P(X)}P(\langle 1_{P(X)},1_{P(X)}\rangle)\delta_X P(\pi_X)\\
&\overset{(\mathbf{iS1})}{=} \varepsilon_X P(\pi_X) \overset{(2)}{=} \pi_X\varepsilon_{Z\times_Y X}.
\end{aligned}$$

■

We observe that the proof of the previous proposition actually tells us that the points (f,s) satisfying **(iS1)** for a certain imaginary retraction q are stable under pullbacks along any morphism.

Recall from [19] that an object Y is said to be a **protomodular object** when all points over it are stably strong. Consequently, a finitely complete category is protomodular if and only if all of its objects are protomodular. It was shown there that the protomodular objects in Mon are precisely the groups.

The next result gives a partial version of Corollary 3.1.7 from [9] which says that a monoid Y is a group if and only if all points over Y are Schreier points.

Corollary 5.9. *If all points over Y are intrinsic Schreier points, then Y is a protomodular object.*

Proof: All points over Y are stably strong by Proposition 5.8. ■

The converse implication is false in general, as we will show in Section 7.

We prove now that, if we apply our intrinsic definition to the category Mon of monoids, we regain the original definition of a Schreier split epimorphism (= right homogeneous split epimorphism). Also that of a left homogeneous split epimorphism (see Remark 2.1) fits the picture.

Theorem 5.10. *In the case of monoids, the intrinsic Schreier split epimorphisms with respect to the direct imaginary splitting t^d are precisely the Schreier split epimorphisms. Similarly, the intrinsic Schreier split epimorphisms with respect to the twisted imaginary splitting t^w are the left homogeneous split epimorphisms.*

Proof: Let (1) be an intrinsic Schreier split extension of monoids with respect to t^d. Then there exists an imaginary retraction $q\colon P(X) \to K$ such that **(iS1)** and **(iS2)** hold, where $P(X)$ is the free monoid on X. We define the function

$$\begin{tikzcd} X \arrow[r, dashed] \arrow[rr, dashed, bend left, "q'"] & P(X) \arrow[r] & K \\ x \arrow[r, mapsto] & {[x]} \arrow[r, mapsto] & q([x]) \end{tikzcd}$$

to be the Schreier retraction in Mon. From **(iS1)** we prove **(S1)**: for all $x \in X$,

$$\begin{aligned} & (kq\ \ sf\varepsilon_X)t^d_{P(X),P(X)}P(\langle 1_{P(X)}, 1_{P(X)}\rangle)\delta_X([x]) = \varepsilon_X([x]) \\ \Leftrightarrow\quad & (kq\ \ sf\varepsilon_X)t^d_{P(X),P(X)}P(\langle 1_{P(X)}, 1_{P(X)}\rangle)([[x]]) = x \\ \Leftrightarrow\quad & (kq\ \ sf\varepsilon_X)t^d_{P(X),P(X)}([([x],[x])]) = x \\ \Leftrightarrow\quad & (kq\ \ sf\varepsilon_X)(\underline{[x]} \cdot \overline{[x]}) = x \\ \Leftrightarrow\quad & kq([x]) \cdot sf(x) = x \\ \Leftrightarrow\quad & kq'(x) \cdot sf(x) = x. \end{aligned}$$

Note that here the Jónsson–Tarski operation is written as a product. Similarly, from **(iS2)** we prove **(S2)**: for all $a \in K$, $y \in Y$,

$$\begin{aligned} & qP([k\ \ s])P(t^d_{K,Y})\delta_{K\times Y}([(a,y)]) = \pi_K \varepsilon_{K\times Y}([(a,y)]) \\ \Leftrightarrow\quad & qP([k\ \ s])P(t^d_{K,Y})([[(a,y)]]) = \pi_K(a,y) \\ \Leftrightarrow\quad & qP([k\ \ s])([\underline{a}\cdot\overline{y}]) = a \\ \Leftrightarrow\quad & q([k(a)\cdot s(y)]) = a \\ \Leftrightarrow\quad & q'(k(a)\cdot s(y)) = a. \end{aligned}$$

Conversely, suppose that (1) is a Schreier split extension of monoids with Schreier retraction q'. We define the morphism of monoids $q\colon P(X) \to K$ by

$$q([x_1]\cdot\ \cdots\ \cdot[x_n]) = q'(x_1)\cdot\ \cdots\ \cdot q'(x_n).$$

So, on the generators we have $q([x]) = q'(x)$, $\forall x \in X$, as above. Then (1) is an intrinsic Schreier extension with respect to t^d. Indeed, to prove **(iS1)** and **(iS2)** it suffices to check these equalities for the generators $[x]$, for all $x \in X$, and $[(a,y)]$, for all $a \in K$ and $y \in Y$, respectively. They follow immediately from **(S1)** and **(S2)** and the fact that $q([x]) = q'(x)$ for all $x \in X$, as for the previous implication.

The proof for left homogeneous split epimorphisms is similar: replace t^d by t^w. ∎

Remark 5.11. It is not difficult to extend this result to Jónsson–Tarski varieties. The main issue is to define $q(\theta([x_1],\dots,[x_n]))$, where θ is an n-ary operation; we can let it be equal to $\theta(q'(x_1),\dots,q'(x_n))$.

6. Stability properties

The aim of this section is to show that any regular unital category $\mathbb{C}$ with binary coproducts, functorial projective covers and a natural imaginary splitting t is an $\mathscr{S}$-protomodular category with respect to the class $\mathscr{S}$ of intrinsic Schreier split epimorphisms. First of all, we show that the class $\mathscr{S}$ is stable under pullbacks, which gives a subfibration of the fibration of points.

Proposition 6.1. *Intrinsic Schreier split extensions are stable under arbitrary pullbacks.*

Proof: Consider an intrinsic Schreier split extension and an arbitrary pullback as in (13). We know from Proposition 5.8 that $q' = qP(\pi_X)\colon P(Z\times_Y X) \to K$

satisfies **(iS1)** for the split epimorphism $(\pi_Z, \langle 1_Z, sg\rangle)$. To prove **(iS2)** we calculate

$$\begin{aligned}
& qP(\pi_X)P((\langle 0,k\rangle\ \langle 1_Z, sg\rangle))P(t_{K,Z})\delta_{K\times Z}\\
= {} & qP((k\ \ sg))P(t_{K,Z})\delta_{K\times Z}\\
= {} & qP((k\ \ s))P(1_K + g)P(t_{K,Z})\delta_{K\times Z}\\
\overset{(7)}{=} {} & qP((k\ \ s))P(t_{K,Y})P^2(1_K \times g))\delta_{K\times Z}\\
\overset{(3)}{=} {} & qP((k\ \ s))P(t_{K,Y})\delta_{K\times Y}P(1_K \times g)\\
\overset{(\mathbf{iS2})}{=} {} & \pi_K\varepsilon_{K\times Y}P(1_K \times g)\\
\overset{(2)}{=} {} & \pi_K(1_K \times g)\varepsilon_{K\times Z} = \pi_K\varepsilon_{K\times Z}.
\end{aligned}$$

■

We have already explained that every intrinsic Schreier split epimorphism is a (stably) strong point (see Proposition 5.8). What remains to be shown—see Definition 3.1—is that the full subcategory $\mathscr{S}$-$\mathsf{Pt}(\mathbb{C})$ of intrinsic Schreier points is closed in $\mathsf{Pt}(\mathbb{C})$ under finite limits.

Proposition 6.2. *Intrinsic Schreier split extensions are stable under products.*

Proof: Let

$$K \xrightarrow{k} X \underset{f}{\overset{s}{\rightleftarrows}} Y \qquad \text{and} \qquad K' \xrightarrow{k'} X' \underset{f'}{\overset{s'}{\rightleftarrows}} Y'$$

be intrinsic Schreier split extensions. Suppose that the imaginary retractions are $q\colon P(X) \to K$ and $q'\colon\ P(X') \to K'$, respectively. We want to prove that

$$K \times K' \xrightarrow{k\times k'} X \times X' \underset{f\times f'}{\overset{s\times s'}{\rightleftarrows}} Y \times Y'$$

is an intrinsic Schreier split extension. As imaginary retraction we use the imaginary morphism $\rho = (q \times q')\langle P(\pi_X), P(\pi_{X'})\rangle\colon P(X \times X') \to K \times K'$. For **(iS1)** we must prove that

$$\begin{aligned}
\varepsilon_{X\times X'} = ((k \times k')\rho\ \ (s \times s')(f \times f')\varepsilon_{X\times X'})&\\
\circ\, t_{P(X\times X'),P(X\times X')}P(\langle 1_{P(X\times X')}, 1_{P(X\times X')}\rangle)\delta_{X\times X'}&.
\end{aligned}$$

We have

$$
\begin{aligned}
&\pi_X[(k \times k')\rho \;\; (s \times s')(f \times f')\varepsilon_{X\times X'}] \\
&\qquad \circ t_{P(X\times X'),P(X\times X')}P(\langle 1_{P(X\times X')}, 1_{P(X\times X')}\rangle)\delta_{X\times X'} \\
&\overset{(2)}{=} [kqP(\pi_X) \;\; sf\varepsilon_X P(\pi_X)]t_{P(X\times X'),P(X\times X')}P(\langle 1_{P(X\times X')}, 1_{P(X\times X')}\rangle)\delta_{X\times X'} \\
&= [kq \;\; sf\varepsilon_X](P(\pi_X) + P(\pi_X)) \\
&\qquad \circ t_{P(X\times X'),P(X\times X')}P(\langle 1_{P(X\times X')}, 1_{P(X\times X')}\rangle)\delta_{X\times X'} \\
&\overset{(7)}{=} [kq \;\; sf\varepsilon_X]t_{P(X),P(X)}P(P(\pi_X) \times P(\pi_X))P(\langle 1_{P(X\times X')}, 1_{P(X\times X')}\rangle)\delta_{X\times X'} \\
&= [kq \;\; sf\varepsilon_X]t_{P(X),P(X)}P(\langle P(\pi_X), P(\pi_X)\rangle)\delta_{X\times X'} \\
&= [kq \;\; sf\varepsilon_X]t_{P(X),P(X)}P(\langle 1_{P(X)}, 1_{P(X)}\rangle)P^2(\pi_X)\delta_{X\times X'} \\
&\overset{(3)}{=} [kq \;\; sf\varepsilon_X]t_{P(X),P(X)}P(\langle 1_{P(X)}, 1_{P(X)}\rangle)\delta_X P(\pi_X) \\
&\overset{\textbf{(iS1)}}{=} \varepsilon_X P(\pi_X) \overset{(2)}{=} \pi_X \varepsilon_{X\times X'}.
\end{aligned}
$$

The proof that

$$
\begin{aligned}
\pi_{X'}\varepsilon_{X\times X'} = \pi_{X'}[(k \times k')\rho \;\; (s \times s')(f \times f'\varepsilon_{X\times X'})] \\
\circ t_{P(X\times X'),P(X\times X')}P(\langle 1_{P(X\times X')}, 1_{P(X\times X')}\rangle)\delta_{X\times X'}
\end{aligned}
$$

is similar. For **(iS2)** we must show that

$$\rho P([k \times k' \;\; s \times s'])P(t_{K\times K',Y\times Y'})\delta_{K\times K'\times Y\times Y'} = \pi_{K\times K'}\varepsilon_{K\times K'\times Y\times Y'}.$$

We have

$$
\begin{aligned}
&\pi_K \rho P([k \times k' \;\; s \times s'])P(t_{K\times K',Y\times Y'})\delta_{K\times K'\times Y\times Y'} \\
&= qP(\pi_X)P([k \times k' \;\; s \times s'])P(t_{K\times K',Y\times Y'})\delta_{K\times K'\times Y\times Y'} \\
&= qP([k \;\; s](\pi_K + \pi_Y)t_{K\times K',Y\times Y'})\delta_{K\times K'\times Y\times Y'} \\
&\overset{(7)}{=} qP([k \;\; s]t_{K,Y}P(\pi_K \times \pi_Y))\delta_{K\times K'\times Y\times Y'} \\
&= qP([k \;\; s])P(t_{K,Y})P^2(\pi_K \times \pi_Y)\delta_{K\times K'\times Y\times Y'} \\
&\overset{(3)}{=} qP([k \;\; s])P(t_{K,Y})\delta_{K\times Y}P(\pi_K \times \pi_Y) \\
&\overset{\textbf{(iS2)}}{=} \pi_K \varepsilon_{K\times Y}P(\pi_K \times \pi_Y) \\
&\overset{(2)}{=} \pi_K(\pi_K \times \pi_Y)\varepsilon_{K\times K'\times Y\times Y'} = \pi_K \pi_{K\times K'}\varepsilon_{K\times K'\times Y\times Y'}.
\end{aligned}
$$

The proof that

$$\pi_{K'}\rho P([k \times k' \quad s \times s'])P(t_{K\times K',Y\times Y'})\delta_{K\times K'\times Y\times Y'} = \pi_{K'}\pi_{K\times K'}\varepsilon_{K\times K'\times Y\times Y'}$$

is similar. ■

Proposition 6.3. *Intrinsic Schreier split extensions in* $\mathbb{C}$ *are closed under equalisers in* $\mathsf{Pt}(\mathbb{C})$.

Proof: In the following diagram, the middle and right vertical extensions are intrinsic Schreier split extensions. Consider the morphisms (g, u) and (h, v) between them and the induced morphisms $\overline{g}$ and $\overline{h}$. Let e be the equaliser of (g, h), w be the equaliser of (u, v), φ and σ be the induced morphisms and L the kernel of φ

$$\begin{tikzcd}
L \arrow[r, tail, "\overline{e}"] \arrow[d, tail, "l"'] & K \arrow[r, shift left, "\overline{g}"] \arrow[r, shift right, "\overline{h}"'] \arrow[d, tail, "k"'] & K' \arrow[d, tail, "k'"'] \\
E \arrow[r, tail, "e"] \arrow[d, shift right, two heads, "\varphi"'] & X \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] \arrow[d, shift right, two heads, "f"'] & X' \arrow[d, shift right, two heads, "f'"'] \\
W \arrow[r, tail, "w"'] \arrow[u, shift right, "\sigma"'] & Y \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] \arrow[u, shift right, "s"'] & Y'. \arrow[u, shift right, "s'"']
\end{tikzcd}$$

Since limits commute with limits, we know that the induced morphism $\overline{e}$ is the equaliser of $(\overline{g}, \overline{h})$. Moreover, (e, w) is also the equaliser of $((g, u), (h, v))$ in $\mathsf{Pt}(\mathbb{C})$.

We want to prove that the left vertical extension above is an intrinsic Schreier split extension. Suppose $q\colon P(X) \to K$ and $q'\colon P(X') \to K'$ are the imaginary retractions for (f, s) and (f', s'), respectively. The morphism $\gamma\colon P(E) \to L$,

$$\begin{tikzcd}
L \arrow[r, "\overline{e}"] & K \arrow[r, shift left, "\overline{g}"] \arrow[r, shift right, "\overline{h}"'] & K' \\
P(E) \arrow[u, dotted, "\gamma"] \arrow[r, "P(e)"'] & P(X) \arrow[u, "q"'] &
\end{tikzcd}$$

which is induced by the universal property of the equaliser $\overline{e}$ because

$$\overline{g}qP(e) \overset{5.7}{=} q'P(g)P(e) = q'P(ge) = q'P(he) = \cdots = \overline{h}qP(e),$$

gives the imaginary retraction with respect to (φ, σ). We prove **(iS1)** by using that e is a monomorphism:

$$
\begin{aligned}
& e(l\gamma \ \ \sigma\varphi\varepsilon_E)t_{P(E),P(E)}P(\langle 1_{P(E)}, 1_{P(E)}\rangle)\delta_E \\
&= (el\gamma \ \ e\sigma\varphi\varepsilon_E)t_{P(E),P(E)}P(\langle 1_{P(E)}, 1_{P(E)}\rangle)\delta_E \\
&\overset{(2)}{=} (kqP(e) \ \ sf\varepsilon_X P(e))t_{P(E),P(E)}P(\langle 1_{P(E)}, 1_{P(E)}\rangle)\delta_E \\
&= (kq \ \ sf\varepsilon_X)(P(e)+P(e))t_{P(E),P(E)}P(\langle 1_{P(E)}, 1_{P(E)}\rangle)\delta_E \\
&\overset{(7)}{=} (kq \ \ sf\varepsilon_X)t_{P(X),P(X)}P(P(e)\times P(e))P(\langle 1_{P(E)}, 1_{P(E)}\rangle)\delta_E \\
&= (kq \ \ sf\varepsilon_X)t_{P(X),P(X)}P(\langle P(e), P(e)\rangle)\delta_E \\
&= (kq \ \ sf\varepsilon_X)t_{P(X),P(X)}P(\langle 1_{P(X)}, 1_{P(X)}\rangle)P^2(e)\delta_E \\
&\overset{(3)}{=} (kq \ \ sf\varepsilon_X)t_{P(X),P(X)}P(\langle 1_{P(X)}, 1_{P(X)}\rangle)\delta_X P(e) \\
&\overset{\textbf{(iS1)}}{=} \varepsilon_X P(e) \overset{(2)}{=} e\varepsilon_E.
\end{aligned}
$$

To prove **(iS2)** we use that $\overline{e}$ is a monomorphism

$$
\begin{aligned}
\overline{e}\gamma P((l \ \ \sigma))P(t_{L,W})\delta_{L\times W} &= qP(e)P((l \ \ \sigma))P(t_{L,W})\delta_{L\times W} \\
&= qP((el \ \ e\sigma)t_{L,W})\delta_{L\times W} \\
&= qP((k\overline{e} \ \ sw)t_{L,W})\delta_{L\times W} \\
&= qP((k \ \ s)(\overline{e}+w)t_{L,W})\delta_{L\times W} \\
&\overset{(7)}{=} qP((k \ \ s)t_{K,Y}P(\overline{e}\times w))\delta_{L\times W} \\
&= qP((k \ \ s))P(t_{K,Y})P^2(\overline{e}\times w)\delta_{L\times W} \\
&\overset{(3)}{=} qP((k \ \ s))P(t_{K,Y})\delta_{K\times Y}P(\overline{e}\times w) \\
&\overset{\textbf{(iS2)}}{=} \pi_K\varepsilon_{K\times Y}P(\overline{e}\times w) \\
&\overset{(2)}{=} \pi_K(\overline{e}\times w)\varepsilon_{L\times W} \\
&= \overline{e}\pi_L\varepsilon_{L\times W}.
\end{aligned}
$$

■

Corollary 6.4. *Intrinsic Schreier split extensions in $\mathbb{C}$ are closed under finite limits in $\mathsf{Pt}(\mathbb{C})$.*

Proof: By Corollary 5.6, the terminal object of $\mathsf{Pt}(\mathbb{C})$, the point $0 \leftrightarrows 0$, is an intrinsic Schreier point. Then the result follows by Propositions 6.2 and 6.3. ■

Corollary 6.5. *Any regular unital category with binary coproducts, functorial projective covers and a natural imaginary splitting is an $\mathscr{S}$-protomodular category with respect to the class $\mathscr{S}$ of intrinsic Schreier split epimorphisms.*

It is easy to see that the closedness of $\mathscr{S}$-$\mathsf{Pt}(\mathbb{C})$ in $\mathsf{Pt}(\mathbb{C})$ under finite limits implies that, for every object Y, the fibre $\mathscr{S}$-$\mathsf{Pt}_Y(\mathbb{C})$ is closed in $\mathsf{Pt}_Y(\mathbb{C})$ under finite limits.

7. (Intrinsic) Schreier special objects

In this section we investigate $\mathscr{S}$-special objects and the link between them and protomodular objects.

Definition 7.1. An object X of an $\mathscr{S}$-protomodular category is said to be **$\mathscr{S}$-special** when the point

$$X \xrightarrow[\langle 1_X,0\rangle]{} X \times X \underset{\pi_2}{\overset{\langle 1_X,1_X\rangle}{\leftrightarrows}} X \tag{14}$$

(or, equivalently, the point $(\pi_1, \langle 1_X, 1_X\rangle)$) belongs to the class $\mathscr{S}$.

Proposition 7.2. [11, Proposition 6.2] *Given an $\mathscr{S}$-protomodular category $\mathbb{C}$, the full subcategory of $\mathscr{S}$-special objects is a protomodular category, called the **protomodular core** of $\mathbb{C}$ relative to the class $\mathscr{S}$.*

If $\mathbb{C}$ is the category Mon of monoids, and $\mathscr{S}$ is the class of Schreier split epimorphisms, then the protomodular core is the category Gp of groups ([11], Proposition 6.4). On the other hand, in [19] we showed that a monoid is a group if and only if it is a protomodular object. Thus one could be led to think that the notions of $\mathscr{S}$-special object and protomodular object are always equivalent, like in the case of monoids. This is false: actually, neither condition is implied by the other.

Theorem 7.3. *In Jónsson–Tarski varieties, the concepts of a protomodular object and of an $\mathscr{S}$-special object (for $\mathscr{S}$ the class of Schreier split epimorphisms) are independent.*

We start with a counterexample: a variety where not all $\mathscr{S}$-special objects are protomodular.

Example 7.4. Let Mag be the Jónsson–Tarski variety of unitary magmas (whose theory contains a unique constant 0 and a binary operation $+$ such that $0+x =$

$x = x + 0$). We show that the unitary magma $Y = C_2 = (\{0, 1\}, +)$, the cyclic group of order 2, is an $\mathscr{S}$-special object which is not protomodular in Mag.

It is easy to see that Y is a special Schreier object: the map q in

$$\begin{tikzcd} Y \arrow[r, tail, shift right, "\langle 1_Y, 0\rangle"'] & Y \times Y \arrow[l, dashed, shift right, "q"'] \arrow[r, two heads, shift right, "\pi_2"'] & Y \arrow[l, shift right, "\langle 1_Y, 1_Y\rangle"'] \end{tikzcd}$$

which sends $(0, 0)$ and $(1, 1)$ to 0, and $(1, 0)$ and $(0, 1)$ to 1, is a Schreier retraction.

Next we show that Y is not a protomodular object, by giving an example of a split epimorphism over Y which is not a strong point. In Mag we consider the point (f, s) and its kernel

$$\begin{tikzcd} \{0\} \arrow[r, hook, "k"'] & X \arrow[r, two heads, shift right, "f"'] & Y, \arrow[l, shift right, "s"'] \end{tikzcd}$$

where $X = (\{0, a, b\}, +)$ is defined by

$+$	0	a	b
0	0	a	b
a	a	0	0
b	b	0	0

and

$$\begin{cases} f(0) = 0 \\ f(a) = 1 = f(b), \end{cases} \qquad\qquad \begin{cases} s(0) = 0 \\ s(1) = a. \end{cases}$$

Note that X is not associative since, for instance, $(a + a) + b = 0 + b = b$ and $a + (a + b) = a + 0 = a$.

The point (f, s) is not strong, because there exists a monomorphism s which is not an isomorphism, yet makes the diagram

$$\begin{tikzcd} & Y \arrow[d, tail, "s"] \arrow[dr, equal] & \\ \{0\} \arrow[ur, hook] \arrow[r, hook, "k"'] & X & Y \arrow[l, "s"] \end{tikzcd}$$

commute.

Proposition 7.5. *Let $\mathbb{V}$ be a Jónsson–Tarski variety. An algebra in $\mathbb{V}$ is special with respect to the class of Schreier split epimorphisms if and only if it has a right loop structure.*

Proof: We recall that a **right loop** $(X, +, -, 0)$ is a set X with two binary operations $+$ and $-$ and a unique constant 0 such that the following axioms are satisfied:

$$x + 0 = 0 + x = x, \qquad (x - y) + y = x, \qquad (x + y) - y = x.$$

Now, given an object X in $\mathbb{V}$, suppose that the split epimorphism (14) is a Schreier split epimorphism. Then there exists a map $q\colon X \times X \dashrightarrow X$ such that, for all x, $y \in X$, we have

$$(x, y) = (q(x, y), 0) + (y, y),$$

from which we deduce $x = q(x, y) + y$. Let us then define

$$x - y = q(x, y).$$

Clearly we have $(x - y) + y = q(x, y) + y = x$. Moreover, **(S2)** tell us that

$$q(x + y, y) = x,$$

or, in other words,

$$(x + y) - y = x.$$

Conversely, given a right loop $(X, +, -, 0)$, we can define $q\colon X \times X \dashrightarrow X$ by putting $q(x, y) = x - y$. It is then immediate to check that **(S1)** and **(S2)** are satisfied. ■

Remark 7.6. In any Jónsson–Tarski variety, a similar proof shows that an object X is special with respect to the class of left homogeneous split epimorphisms if and only if it has a **left loop** structure:

$$x + 0 = 0 + x = x, \qquad x + (-x + y) = y, \qquad -x + (x + y) = y.$$

As consequence of these observations, we are able to complete the proof of Theorem 7.3 with Example 7.7.

We recall from [5] that a pointed variety is protomodular if and only if its theory contains only one constant 0, and there exists a positive integer n, as well as n binary operations $\alpha_1, \ldots, \alpha_n$ and an $(n + 1)$-ary operation β such that

$$\begin{cases} \alpha_i(x, x) = 0 & \forall i \in \{1, \ldots, n\}, \\ \beta(\alpha_1(x, y), \ldots, \alpha_n(x, y), y) = x. \end{cases}$$

In many of the known examples of pointed protomodular varieties, such as Gp (or any variety which contains the right loop operations), the characterisation above works for $n = 1$.

Example 7.7. Let HSLat be the variety of **Heyting semilattices**, namely the variety defined by a unique constant $\top$ and two binary operations $\wedge$, $\to$ such that $(\wedge, \top)$ gives the structure of a semilattice with a top element, and the following axioms are satisfied:

$$\begin{aligned} (x \to x) &= \top \\ x \wedge (x \to y) &= x \wedge y \\ y \wedge (x \to y) &= y \\ x \to (y \wedge z) &= (x \to y) \wedge (x \to z). \end{aligned}$$

In [14] it was shown that HSLat is a protomodular variety. Accordingly, all objects in HSLat are protomodular. It was also shown that HSLat is protomodular for $n = 2$, but not for $n = 1$. This allows us to conclude that not all objects are $\mathscr{S}$-special. Indeed, HSLat is a Jónsson–Tarski variety, so by Proposition 7.5 all $\mathscr{S}$-special objects are right loops. If all objects were $\mathscr{S}$-special, then HSLat would be a protomodular variety with $n = 1$, which is false.

This argument can be used on any pointed protomodular variety for which the smallest number n that makes the characterisation recalled above work is strictly larger than 1. Hence our notion of intrinsic Schreier split epimorphism gives a categorical method for distinguishing general pointed protomodular varieties from varieties of right Ω-loops. Indeed, in every protomodular variety $\mathbb{V}$ all objects are protomodular, while every object in $\mathbb{V}$ is $\mathscr{S}$-special with respect to the class $\mathscr{S}$ of (intrinsic) Schreier split epimorphisms if and only if $\mathbb{V}$ is a variety of right Ω-loops.

Proof of Theorem 7.3: Combine Example 7.4 with Example 7.7. ■

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