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SOME EMBEDDINGS OF MORREY SPACES WITH CRITICAL SMOOTHNESS

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Abstract: We study embeddings of Besov-Morrey spaces $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)$ and of Triebel-Lizorkin-Morrey spaces $\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)$ in the limiting cases when the smoothness s equals $s_0 = d\max(1/u - p/u, 0)$ or $s_\infty = d/u$, which is related to the embeddings in $L_1^{\rm loc}(\mathbb{R}^d)$ or in $L_\infty(\mathbb{R}^d)$, respectively. When $s = s_0$ we characterise the embeddings in $L_1^{\rm loc}(\mathbb{R}^d)$ and when $s = s_\infty$ we obtain embeddings into Orlicz-Morrey spaces of exponential type and into generalised Morrey spaces.

Keywords: Besov-Morrey spaces, Triebel-Lizorkin-Morrey spaces, limiting embeddings, generalised Morrey spaces, Orlicz-Morrey spaces.

AMS Subject Classification (2010): 46E35.

1. Introduction

In recent years function spaces built upon Morrey spaces $\mathcal{M}_{u,p}(\mathbb{R}^d)$, $0 < p \le u < \infty$, attracted some attention. They include in particular Besov-Morrey spaces $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)$ and Triebel-Lizorkin-Morrey spaces $\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)$, $0 < p \le u < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$. The attention paid to the spaces was motivated first of all by possible applications. The Besov-Morrey spaces $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)$ were introduced by Kozono and Yamazaki in [7] and used by them and later on by Mazzucato [10] in the study of Navier-Stokes equations. In [28] Tang and Xu introduced the corresponding Triebel-Lizorkin-Morrey spaces $\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)$, thanks to establishing the Morrey version of the Fefferman-Stein vector-valued inequality. Some properties of these spaces including their atomic decompositions and wavelet characterisations were later described in the papers by Sawano [18, 19], Sawano and Tanaka [20, 21] and

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Rosenthal [17]. Also embedding properties of these spaces were investigated in a series of papers [4, 5].

In our two recent papers [2] and [3] we studied under which conditions the spaces $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)$ and $\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)$ contain only regular distributions, i.e., when

$$\mathcal{N}^s_{u,p,q}(\mathbb{R}^d) \subset L_1^{\rm loc}(\mathbb{R}^d), \quad \mathcal{E}^s_{u,p,q}(\mathbb{R}^d) \subset L_1^{\rm loc}(\mathbb{R}^d) \tag{1.1}$$

and when they consist of bounded functions

$$\mathcal{N}^s_{u,p,q}(\mathbb{R}^d) \subset L_\infty(\mathbb{R}^d), \quad \mathcal{E}^s_{u,p,q}(\mathbb{R}^d) \subset L_\infty(\mathbb{R}^d). \tag{1.2}$$

The embeddings (1.1) hold if the smoothness s is related to $s_o = \frac{p}{u} d \max(\frac{1}{p} - 1, 0)$. Similarly the embeddings (1.2) are valid if s is related to $s_\infty = \frac{d}{u}$. Therefore we called s_o and s_∞ critical smoothnesses. In particular, the spaces contain a non-regular distribution if $s < s_o$ and consist of locally integrable functions if $s > s_o$. The behaviour in the critical case $s = s_o$ is a delicate question. The analogous situation occurs in the case of boundedness of the functions.

This paper is the direct continuation of the above mentioned recent articles. First in Section 3 we prove that the embeddings (1.1) with smoothness $s = s_o$ hold if and only if the spaces $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)$ and $\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)$ are embedded into some Morrey spaces $\mathcal{M}_{v,r}(\mathbb{R}^d)$ with properly defined indices v and r, cf. Theorem 3.2 and Theorem 3.4.

In Section 4 we investigate unboundedness properties of the functions belonging to smoothness spaces of Morrey type with $s = s_\infty$. First we prove that the spaces $\mathcal{N}^{d/u}_{u,p,q}(\mathbb{R}^d)$ and $\mathcal{E}^{d/u}_{u,p,q}(\mathbb{R}^d)$ can be embedded into Orlicz-Morrey spaces of exponential type, cf. Theorem 4.5 and Corollary 4.6. The idea that exponential Young functions can control the unboundedness of functions belonging to the Sobolev spaces of critical smoothness goes back to Trudinger [34], cf. also Strichartz [27]. There are several definitions of Orlicz-Morrey spaces. Here we follow the approach proposed by Nakai [13]. Afterwards we embed the above spaces of smoothness $\frac{d}{u}$ into the so-called generalised Morrey spaces, cf. Theorem 4.10 and Remark 4.11. The generalised Morrey spaces have been investigated by several authors. Our results here are close to that ones in papers by Sawano and Wadade [22], Nakamura, Noi and Sawano [15] and by Eridani, Gunawan, Nakai and Sawano [1]. However in contrast to the above papers we study also the Besov-Morrey spaces. Moreover in the case of Sobolev-Morrey spaces we are able to prove the embeddings for a larger set of parameters.

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Again the embedding (5.3) would thus lead to $\varrho \le v$ in view of [4, Thm. 3.3], i.e., to a contradiction by our choice of ϱ. Here we also used the identification $F^{s}_{\infty,q}(\mathbb{R}^d) = F^{s,\frac{1}{p}}_{p,q}(\mathbb{R}^d)$, see [24, Props. 3.4 and 3.5] and [25, Rem. 10]. In a completely parallel way one can show that an embedding $B^{s,\tau}_{p,q}(\mathbb{R}^d) \hookrightarrow \mathcal{M}_{v,r}(\mathbb{R}^d)$ is never possible when $\tau \ge \frac{1}{p}$.

Note that the limiting case $v = \infty$ is covered by [37, Prop. 5.4], [36, Prop. 4.1], in view of $\mathcal{M}_{\infty,r}(\mathbb{R}^d) = L_\infty(\mathbb{R}^d)$. In particular, if $0 < p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$ and $\tau > 0$, then

$$F^{s,\tau}_{p,q}(\mathbb{R}^d) \hookrightarrow L_\infty(\mathbb{R}^d) \quad \text{if, and only if,} \quad s > d\left(\frac{1}{p} - \tau\right).$$

The result for $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ looks alike.

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In the very short last section, somewhat supplementary, we formulate some outcome concerning embeddings into spaces with smoothness 0, in particular to $\mathrm{bmo}(\mathbb{R}^d)$. In Section 2 we collect the definition and basic facts concerning the smoothness Morrey spaces that are needed in the next sections.

2. Smoothness spaces of Morrey type

First we fix some notation. By $\mathbb{N}$ we denote the *set of natural numbers*, by $\mathbb{N}_0$ the set $\mathbb{N} \cup \{0\}$, and by $\mathbb{Z}^d$ the *set of all lattice points in* $\mathbb{R}^d$ *having integer components*. Let $\mathbb{N}_0^d$, where $d \in \mathbb{N}$, be the set of all multi-indices, $\alpha = (\alpha_1, \ldots, \alpha_d)$ with $\alpha_j \in \mathbb{N}_0$ and $|\alpha| := \sum_{j=1}^d \alpha_j$. If $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ and $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$, then we put $x^\alpha := x_1^{\alpha_1} \cdots x_d^{\alpha_d}$. For $a \in \mathbb{R}$, let $\lfloor a \rfloor := \max\{k \in \mathbb{Z} : k \le a\}$ and $a_+ := \max(a, 0)$. If $0 < u \le \infty$, the number u' is given by $\frac{1}{u'} = (1 - \frac{1}{u})_+$, with the convention that $1/\infty = 0$. All unimportant positive constants will be denoted by C, occasionally the same letter C is used to denote different constants in the same chain of inequalities. By the notation $A \lesssim B$, we mean that there exists a positive constant c such that $A \le c\, B$, whereas the symbol $A \sim B$ stands for $A \lesssim B \lesssim A$. We denote by $B(x, r) := \{y \in \mathbb{R}^d : |x - y| < r\}$ the ball centred at $x \in \mathbb{R}^d$ with radius $r > 0$ and $|\cdot|$ denotes the Lebesgue measure when applied to measurable subsets of $\mathbb{R}^d$. Let $\mathcal{Q}$ be the collection of all *dyadic cubes* in $\mathbb{R}^d$, namely, $\mathcal{Q} := \{Q_{j,k} := 2^{-j}([0,1)^d + k) : \ j \in \mathbb{Z},\ k \in \mathbb{Z}^d\}$.

Given two (quasi-)Banach spaces X and Y, we write $X \hookrightarrow Y$ if $X \subset Y$ and the natural embedding of X into Y is continuous.

We introduce smoothness spaces of Morrey type. We recall first the definition of Morrey spaces.

Definition 2.1. Let $0 < p \le u < \infty$. The *Morrey space* $\mathcal{M}_{u,p}(\mathbb{R}^d)$ is the set of all locally p-integrable functions $f \in L_p^{\mathrm{loc}}(\mathbb{R}^d)$ such that

$$\|f \mid \mathcal{M}_{u,p}(\mathbb{R}^d)\| := \sup_{Q \in \mathcal{Q}} |Q|^{\frac{1}{u} - \frac{1}{p}} \left(\int_Q |f(y)|^p \, \mathrm{d}y \right)^{1/p} < \infty \,. \tag{2.1}$$

Remark 2.2. The spaces $\mathcal{M}_{u,p}(\mathbb{R}^d)$ are quasi-Banach spaces (Banach spaces for $p \ge 1$), that can be equivalently defined if the supremum in (2.1) is taken over all balls $B(x, r)$ or over all cubes $Q(x, r)$, with center $x \in \mathbb{R}^d$ and side length $r > 0$, and sides parallel to the axis of coordinates.
These spaces originated from Morrey's study on PDE (see [12]) and are part of the wider class of Morrey-Campanato spaces; cf. [16]. They can be

considered as a complement to L_p spaces, since $\mathcal{M}_{p,p}(\mathbb{R}^d) = L_p(\mathbb{R}^d)$ with $p \in (0,\infty)$, extended by $\mathcal{M}_{\infty,\infty}(\mathbb{R}^d) = L_\infty(\mathbb{R}^d)$. In a parallel way one can define the spaces $\mathcal{M}_{\infty,p}(\mathbb{R}^d)$, $p \in (0,\infty)$, but using the Lebesgue differentiation theorem, one arrives at $\mathcal{M}_{\infty,p}(\mathbb{R}^d) = L_\infty(\mathbb{R}^d)$. Moreover, $\mathcal{M}_{u,p}(\mathbb{R}^d) = \{0\}$ for $u < p$, and for $0 < p_2 \le p_1 \le u < \infty$,

$$L_u(\mathbb{R}^d) = \mathcal{M}_{u,u}(\mathbb{R}^d) \hookrightarrow \mathcal{M}_{u,p_1}(\mathbb{R}^d) \hookrightarrow \mathcal{M}_{u,p_2}(\mathbb{R}^d).$$

Let $\mathcal{S}(\mathbb{R}^d)$ be the set of all *Schwartz functions* on $\mathbb{R}^d$, endowed with the usual topology, and denote by $\mathcal{S}'(\mathbb{R}^d)$ its *topological dual*, namely, the space of all bounded linear functionals on $\mathcal{S}(\mathbb{R}^d)$ endowed with the weak $*$-topology. Let $\varphi_0 = \varphi \in \mathcal{S}(\mathbb{R}^d)$ be such that

$$\operatorname{supp}\varphi \subset \{x \in \mathbb{R}^d : |x| \le 2\} \qquad \text{and} \qquad \varphi(x) = 1 \quad \text{if} \quad |x| \le 1, \tag{2.2}$$

and for each $j \in \mathbb{N}$ let $\varphi_j(x) := \varphi(2^{-j}x) - \varphi(2^{-j+1}x)$. Then $\{\varphi_j\}_{j=0}^\infty$ forms a smooth dyadic resolution of unity.

Definition 2.3. Let $0 < p \le u < \infty$ or $p = u = \infty$. Let $q \in (0,\infty]$, $s \in \mathbb{R}$ and $\{\varphi_j\}_{j=0}^\infty$ a smooth dyadic resolution of unity.

(i) The *Besov-Morrey space* $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)$ is defined to be the set of all distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\big\|f \mid \mathcal{N}^s_{u,p,q}(\mathbb{R}^d)\big\| := \bigg(\sum_{j=0}^\infty 2^{jsq}\big\|\mathcal{F}^{-1}(\varphi_j \mathcal{F} f)|\mathcal{M}_{u,p}(\mathbb{R}^d)\big\|^q\bigg)^{1/q} < \infty$$

with the usual modification made in case of $q = \infty$.

(ii) Let $u \in (0,\infty)$. The *Triebel-Lizorkin-Morrey space* $\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)$ is defined to be the set of all distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\big\|f \mid \mathcal{E}^s_{u,p,q}(\mathbb{R}^d)\big\| := \bigg\|\bigg(\sum_{j=0}^\infty 2^{jsq}|\mathcal{F}^{-1}(\varphi_j \mathcal{F} f)(\cdot)|^q\bigg)^{1/q} \mid \mathcal{M}_{u,p}(\mathbb{R}^d)\bigg\| < \infty$$

with the usual modification made in case of $q = \infty$.

Remark 2.4. Occasionally we adopt the usual custom to write $\mathcal{A}^s_{u,p,q}(\mathbb{R}^d)$ instead of $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)$ or $\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)$, when both scales are meant simultaneously in some context. The spaces $\mathcal{A}^s_{u,p,q}(\mathbb{R}^d)$ are independent of the particular dyadic partition of unity $\{\varphi_j\}_{j=0}^\infty$ appearing in their definitions. They are quasi-Banach spaces (Banach spaces for $p, q \ge 1$), and $\mathcal{S}(\mathbb{R}^d) \hookrightarrow \mathcal{A}^s_{u,p,q}(\mathbb{R}^d) \hookrightarrow$

To prove necessity let us take $s < \frac{d}{u}$. If $\mathcal{N}^s_{u,p,\infty}(\mathbb{R}^d) \hookrightarrow \operatorname{bmo}(\mathbb{R}^d)$, then $\mathcal{E}^s_{u,p,q}(\mathbb{R}^d) \hookrightarrow \operatorname{bmo}(\mathbb{R}^d)$. This contradicts (5.1).

Step 3. Part (iii) in case of $p < u$ can be found in [4,5], whereas the classical results for $p = u$ are well-known. ∎

Remark 5.2. A partial forerunner of (i) can be found in [23, Prop. 1.11] dealing with the sufficiency part; see also [25]. In some sense the embeddings into $C(\mathbb{R}^d)$ and $\operatorname{bmo}(\mathbb{R}^d)$ can be understood as limiting cases of Corollary 3.5 when $v \to \infty$, whereas the embedding into $L_r(\mathbb{R}^d)$ refers to the situation of $r = v = q$ in Corollary 3.5.

Remark 5.3. Based on arguments on the known properties of Triebel-Lizorkin type spaces one can easily strengthen Remark 3.6, in particular (3.12) with (3.11), as follows. Let $0 < p < \infty$, $0 < q \le \infty$, $1 \le r \le v < \infty$, $s \in \mathbb{R}$, $\tau \ge 0$. Then

$$F^{s,\tau}_{p,q}(\mathbb{R}^d) \hookrightarrow \mathcal{M}_{v,r}(\mathbb{R}^d) \tag{5.3}$$

if, and only if,

$$r \le v(1-p\tau), \quad \text{and} \quad \begin{cases} s > d(\frac{1}{p} - \tau - \frac{1}{v}) \ge 0, & \text{or} \\ s = d(\frac{1}{p} - \tau - \frac{1}{v}) > 0, & \text{or} \\ s = d(\frac{1}{p} - \tau - \frac{1}{v}) = 0, & \text{and} \quad q \le 2. \end{cases}$$

The case $0 \le \tau < \frac{1}{p}$ is covered by Remark 3.6 together with the usual monotonicity for spaces $F^{s,\tau}_{p,q}(\mathbb{R}^d)$ and $\mathcal{M}_{v,r}(\mathbb{R}^d)$, see also the forerunner [5, Thm. 3.1]. Thus it remains to disprove any embedding of type (5.3) whenever $\tau \ge \frac{1}{p}$. Assume first $\tau > \frac{1}{p}$ or $\tau = \frac{1}{p}$ and $q = \infty$. Thus the coincidence $F^{s,\tau}_{p,q}(\mathbb{R}^d) = B^{s+d(\tau-\frac{1}{p})}_{\infty,\infty}(\mathbb{R}^d)$, cf. [35], together with (5.3) would imply

$$B^{s+d(\tau-\frac{1}{p})}_{\infty,\infty}(\mathbb{R}^d) = F^{s,\tau}_{p,q}(\mathbb{R}^d) \hookrightarrow \mathcal{M}_{v,r}(\mathbb{R}^d) \hookrightarrow \mathcal{N}^0_{v,r,\infty}(\mathbb{R}^d),$$

and hence $v = \infty$ in view of [4, Thm. 3.3]. But this contradicts our general assumption. Otherwise, if $\tau = \frac{1}{p}$ and $q < \infty$, then we may use [38, Prop. 2.6] which states, that $F^{s,\tau}_{p,q}(\mathbb{R}^d) \hookrightarrow B^{s+d(\tau-\frac{1}{p})}_{\infty,\infty}(\mathbb{R}^d)$. We choose a number ϱ such that $v < \varrho < \infty$, and apply an embedding proved by Marschall, cf. [9], to obtain

$$B^{s+\frac{d}{\varrho}}_{\varrho,\infty}(\mathbb{R}^d) \hookrightarrow F^s_{\infty,q}(\mathbb{R}^d) = F^{s,\frac{1}{p}}_{p,q}(\mathbb{R}^d) \hookrightarrow \mathcal{M}_{v,r}(\mathbb{R}^d) \hookrightarrow \mathcal{N}^0_{v,r,\infty}(\mathbb{R}^d).$$

and

$$\mathcal{E}^s_{u,p,q}(\mathbb{R}^d) \hookrightarrow C(\mathbb{R}^d) \quad \Longleftrightarrow \quad \begin{cases} s > \frac{d}{u}, & \text{or} \\ s = \frac{d}{u} & \text{and} \quad u = p \le 1. \end{cases}$$

Here $C(\mathbb{R}^d)$ can be replaced by $L_\infty(\mathbb{R}^d)$.

(ii) *Then*

$$\mathcal{A}^s_{u,p,q}(\mathbb{R}^d) \hookrightarrow \mathrm{bmo}(\mathbb{R}^d) \qquad \text{if, and only if,} \qquad s \ge \frac{d}{u}. \tag{5.1}$$

(iii) *Let $1 \le r < \infty$. If $p < u$, then $\mathcal{A}^s_{u,p,q}(\mathbb{R}^d)$ is never embedded into $L_r(\mathbb{R}^d)$. If $p = u$, then*

$$\mathcal{N}^s_{u,u,q}(\mathbb{R}^d) \hookrightarrow L_r(\mathbb{R}^d) \iff r \ge u \quad \text{and} \quad \begin{cases} s > \frac{d}{u} - \frac{d}{r}, & \text{or} \\ s = \frac{d}{u} - \frac{d}{r} & \text{and} \quad q \le r, \end{cases}$$

and

$$\mathcal{E}^s_{u,u,q}(\mathbb{R}^d) \hookrightarrow L_r(\mathbb{R}^d) \iff r \ge u \quad \text{and} \quad \begin{cases} s \ge \frac{d}{u} - \frac{d}{r} & \text{and} \quad s > 0, \ \text{or} \\ s = \frac{d}{u} - \frac{d}{r} = 0 & \text{and} \quad q \le 2. \end{cases}$$

Proof: *Step 1.* Part (i) is well-known, we refer to [4, Prop. 5.5] and [5, Prop. 3.8] for the Morrey situation when $p < u$, while the classical setting $p = u$ can be found in [31, Theorem 11.4].

Step 2. We prove (ii). In case of $\mathcal{A}^s_{u,p,q} = \mathcal{E}^s_{u,p,q}$, this follows from the analogous statement for $F^{s,\tau}_{p,q}(\mathbb{R}^d)$ spaces and the coincidence $\mathcal{E}^s_{u,p,q}(\mathbb{R}^d) = F^{s,\tau}_{p,q}(\mathbb{R}^d)$ if $\tau = \frac{1}{p} - \frac{1}{u}$, cf. [37, Props. 5.13, 5.14] and (3.8). The similar statement for $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)$ spaces

$$\mathcal{N}^s_{u,p,q}(\mathbb{R}^d) \hookrightarrow \mathrm{bmo}(\mathbb{R}^d) \qquad \text{if, and only if,} \qquad s \ge \frac{d}{u} \tag{5.2}$$

can be proved analogously. Let $\tau = \frac{1}{p} - \frac{1}{u}$. Then, in view of (3.9) and the subsequent remark,

$$\mathcal{N}^s_{u,p,\infty}(\mathbb{R}^d) = B^{s,\tau}_{p,\infty}(\mathbb{R}^d) \hookrightarrow \mathrm{bmo}(\mathbb{R}^d) = B^{0,\frac{1}{2}}_{2,2}(\mathbb{R}^d)$$

if $\frac{1}{p} - \frac{1}{u} - \frac{1}{2} \ne 0$, cf. [37, Prop. 5.10, Theorem 2.5]. If $\frac{1}{p} - \frac{1}{u} - \frac{1}{2} = 0$, then we can choose r such that $u < r < p$ and $\frac{1}{r} - \frac{1}{u} - \frac{1}{2} > 0$. Hence

$$\mathcal{N}^s_{u,p,\infty}(\mathbb{R}^d) \hookrightarrow \mathcal{N}^s_{u,r,\infty}(\mathbb{R}^d) \hookrightarrow \mathrm{bmo}(\mathbb{R}^d).$$

This proves the sufficiency of the conditions.

$\mathcal{S}'(\mathbb{R}^d)$. Moreover, for $u = p$ we re-obtain the usual Besov and Triebel-Lizorkin spaces $\mathcal{A}^s_{p,p,q}(\mathbb{R}^d) = A^s_{p,q}(\mathbb{R}^d)$. Besov-Morrey spaces were introduced by Kozono and Yamazaki in [7]. They studied semi-linear heat equations and Navier-Stokes equations with initial data belonging to Besov-Morrey spaces. The investigations were continued by Mazzucato [10], where one can find the atomic decomposition of some spaces. The Triebel-Lizorkin-Morrey spaces were later introduced by Tang and Xu [28], we follow their approach. The ideas were further developed by Sawano and Tanaka [18–21]. Closely related, alternative approaches can be found in the monographs [32, 33, 38] or in the survey papers by Sickel [24, 25].

We list some elementary embeddings within this scale of spaces. It holds

$$\mathcal{A}^{s+\varepsilon}_{u,p,r}(\mathbb{R}^d) \hookrightarrow \mathcal{A}^s_{u,p,q}(\mathbb{R}^d), \quad \text{if} \quad \varepsilon > 0, \quad r, q \in (0, \infty],$$

and

$$\mathcal{A}^s_{u,p,q_1}(\mathbb{R}^d) \hookrightarrow \mathcal{A}^s_{u,p,q_2}(\mathbb{R}^d), \qquad \text{if} \quad q_1 \le q_2. \tag{2.3}$$

Sawano proved in [18] that, for $s \in \mathbb{R}$ and $0 < p < u < \infty$,

$$\mathcal{N}^s_{u,p,\min(p,q)}(\mathbb{R}^d) \hookrightarrow \mathcal{E}^s_{u,p,q}(\mathbb{R}^d) \hookrightarrow \mathcal{N}^s_{u,p,\infty}(\mathbb{R}^d), \tag{2.4}$$

where, for the latter embedding, $r = \infty$ cannot be improved – unlike in case of $u = p$. More precisely, $\mathcal{E}^s_{u,p,q}(\mathbb{R}^d) \hookrightarrow \mathcal{N}^s_{u,p,r}(\mathbb{R}^d)$ if, and only if, $r = \infty$ or $u = p$ and $r \ge \max(p, q)$. On the other hand, Mazzucato has shown in [10, Prop. 4.1] that

$$\mathcal{E}^0_{u,p,2}(\mathbb{R}^d) = \mathcal{M}_{u,p}(\mathbb{R}^d), \quad 1 < p \le u < \infty, \tag{2.5}$$

in particular,

$$\mathcal{E}^0_{p,p,2}(\mathbb{R}^d) = L_p(\mathbb{R}^d), \quad 1 < p < \infty.$$

This is nothing else than the well-known classical coincidence $F^0_{p,2}(\mathbb{R}^d) = L_p(\mathbb{R}^d)$, $1 < p < \infty$, cf. [30, Thm. 2.5.6]. Further embedding results for the above scales of function spaces on $\mathbb{R}^d$ can be found in [3–5, 36].

The atomic decompositions. An important tool in our later considerations is the characterisation of the Besov-Morrey and Triebel-Lizorkin-Morrey spaces by means of atomic decompositions. We follow [17]; see also [20].

Definition 2.5. Let $0 < p \le u < \infty$, $q \in (0, \infty]$ and $s \in \mathbb{R}$. Let $K \in \mathbb{N}_0$ and $N \in \{-1\} \cup \mathbb{N}_0$. A collection of L_∞-functions $a_{jm} : \mathbb{R}^d \to \mathbb{C}$, $j \in \mathbb{N}_0$, $m \in \mathbb{Z}^d$, is a family of (K, N)-atoms if, for some $c_1 > 1$ and $c_2 > 0$, it holds

(i) $\operatorname{supp} a_{jm} \subset c_1 Q_{jm}, \quad j \in \mathbb{N}_0,\ m \in \mathbb{Z}^d$,

(ii) there exist all (classical derivatives) $\mathrm{D}^\alpha a_{jm}$ for $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \le K$ and

$$\| \mathrm{D}^\alpha a_{jm} | L_\infty(\mathbb{R}^d) \| \le c_2 2^{j|\alpha|}, \quad j \in \mathbb{N}_0,\ m \in \mathbb{Z}^d, \tag{2.6}$$

(iii) if $\gamma \in \mathbb{N}_0^d$ with $|\gamma| \le N$, then

$$\int_{\mathbb{R}^d} x^\gamma a_{jm}(x)\,\mathrm{d}x = 0, \quad j \in \mathbb{N}_0, \quad m \in \mathbb{Z}^d.$$

$N = -1$ means that no moment condition is required.

For $0 < u < \infty$, $j \in \mathbb{N}_0$ and $m \in \mathbb{Z}^d$ we denote by $\chi_{j,m}$ the characteristic function of the cube $Q_{j,m}$ and by $\chi_{j,m}^{(u)} := 2^{jd/u}\chi_{j,m}$ the u-normalised characteristic function of the same cube, i.e., such that $\|\chi_{j,m}^{(u)} \mid \mathcal{M}_{u,p}(\mathbb{R}^d)\| = 1$.

Definition 2.6. Let $0 < p \le u < \infty$, $q \in (0,\infty]$ and $s \in \mathbb{R}$.

(i) The *sequence space* $e^s_{u,p,q}(\mathbb{R}^d)$ is defined to be the set of all sequences $\lambda := \{\lambda_{j,m}\}_{j\in\mathbb{N}_0, m\in\mathbb{Z}^d} \subset \mathbb{C}$ such that

$$\|\lambda \mid e^s_{u,p,q}(\mathbb{R}^d)\| := \Big\| \Big(\sum_{j=0}^\infty 2^{jq(s-\frac{d}{u})} \sum_{m\in\mathbb{Z}^d} |\lambda_{j,m}|^q \chi_{j,m}^{(u)q} \Big)^{1/q} | \mathcal{M}_{u,p}(\mathbb{R}^d) \Big\| < \infty$$

with the usual modification in case of $q = \infty$.

(ii) The *sequence space* $n^s_{u,p,q}(\mathbb{R}^d)$ is defined to be the set of all sequences $\lambda := \{\lambda_{j,m}\}_{j\in\mathbb{N}_0, m\in\mathbb{Z}^d} \subset \mathbb{C}$ such that

$$\|\lambda \mid n^s_{u,p,q}(\mathbb{R}^d)\| := \Big(\sum_{j=0}^\infty 2^{jq(s-\frac{d}{u})} \Big\| \sum_{m\in\mathbb{Z}^d} |\lambda_{j,m}| \chi_{j,m}^{(u)} | \mathcal{M}_{u,p}(\mathbb{R}^d) \Big\|^q \Big)^{1/q} < \infty$$

with the usual modification in case of $q = \infty$.

Remark 2.7. It was proved in [4] that

$$\|\lambda \mid n^s_{u,p,q}(\mathbb{R}^d)\| \sim \|\lambda | n^s_{u,p,q}\|^* := $$

$$= \Bigg(\sum_{j=0}^\infty 2^{qj(s-\frac{d}{u})} \sup_{\nu:\nu\le j; k\in\mathbb{Z}^d} 2^{qd(j-\nu)(\frac{1}{u}-\frac{1}{p})} \Big(\sum_{m: Q_{j,m}\subset Q_{\nu,k}} |\lambda_{j,m}|^p \Big)^{\frac{q}{p}} \Bigg)^{1/q} < \infty.$$

For $p, q \in (0,\infty]$, let

$$\sigma_p := d\left(\frac{1}{p} - 1\right)_+ \quad \text{and} \quad \sigma_{p,q} := d\left(\frac{1}{\min(p,q)} - 1\right)_+. \tag{2.7}$$

Remark 4.13. (i) As in Remark 4.11, we can state the following: Let $0 < p \le u < \infty$, $0 < q \le \infty$, and $1 \le r < \infty$. Then

$$\mathcal{E}^{\frac{d}{u}}_{u,p,q}(\mathbb{R}^d) \hookrightarrow \mathcal{M}^{\varphi_r}_r(\mathbb{R}^d) \qquad \text{and} \qquad \mathcal{E}^{\frac{d}{u}}_{u,p,q}(\mathbb{R}^d) \hookrightarrow L\mathcal{M}^{\varphi}_r(\mathbb{R}^d)$$

with $\varphi_r = \varphi_{r,\infty}$ given by (4.23) and $\varphi(t) = |\log(t)|^{-1}$.

(ii) In the particular case of $p > 1$, $q = 2$ and $r = 1$, the result in the above corollary coincides with Theorem 5.1 of [22].

(iii) In the particular case of $p = 1$, $q = 2$ and $r = 1$, the result in the above corollary is comparable with Proposition 1.7 of [1].

5. Further embeddings into spaces with smoothness $s = 0$

In the preceding subsection we dealt with (limiting) embeddings of spaces $\mathcal{A}^s_{u,p,q}$ when $s = \frac{d}{u}$, into spaces of Orlicz-Morrey type or generalised Morrey type. In Corollary 3.5 the parallel setting was studied for embeddings into Morrey spaces $\mathcal{M}_{v,q}(\mathbb{R}^d)$ when $s = \frac{d}{u} - \frac{d}{v}$. For convenience we briefly recall the forerunners, that is, when $\mathcal{A}^s_{u,p,q}(\mathbb{R}^d)$ is embedded into classical Lebesgue spaces $L_r(\mathbb{R}^d)$, and into the space $C(\mathbb{R}^d)$ of bounded, uniformly continuous functions. We also consider the target space $\operatorname{bmo}(\mathbb{R}^d)$, i.e., the local (non-homogeneous) space of functions of bounded mean oscillation, consisting of all locally integrable functions $f \in L_1^{\mathrm{loc}}(\mathbb{R}^d)$ satisfying that

$$\|f\|_{\operatorname{bmo}(\mathbb{R}^d)} := \sup_{|Q|\le 1} \frac{1}{|Q|} \int_Q |f(x) - f_Q|\,\mathrm{d}x + \sup_{|Q|>1} \frac{1}{|Q|} \int_Q |f(x)|\,\mathrm{d}x < \infty,$$

where Q appearing in the above definition runs over all cubes in $\mathbb{R}^d$, and f_Q denotes the mean value of f with respect to Q, namely, $f_Q := \frac{1}{|Q|}\int_Q f(x)\,\mathrm{d}x$.

Most of these results have been obtained in different papers before, we recall it for completeness and to simplify the comparison with our new findings presented above.

Corollary 5.1. *Let $0 < p \le u < \infty$, $q \in (0,\infty]$ and $s \in \mathbb{R}$.*

(i) *Then*

$$\mathcal{N}^s_{u,p,q}(\mathbb{R}^d) \hookrightarrow C(\mathbb{R}^d) \quad \Longleftrightarrow \quad \begin{cases} s > \frac{d}{u}, & \text{or} \\ s = \frac{d}{u} & \text{and} \quad q \le 1, \end{cases}$$

Step 3. Let $1 \le r < p$. Using Hölder's inequality we obtain

$$\left(\int_Q |f(x)|^r \,\mathrm{d}x\right)^{\frac{1}{r}} \le |Q|^{\frac{1}{r}-\frac{1}{p}} \left(\int_Q |f(x)|^p \,\mathrm{d}x\right)^{\frac{1}{p}} \le |Q|^{\frac{1}{r}-\frac{1}{u}} \|f|\mathcal{M}_{u,p}(\mathbb{R}^d)\|$$
$$\le c|Q|^{\frac{1}{r}-\frac{1}{u}} \|f|\mathcal{N}^{\frac{d}{u}}_{u,p,q}(\mathbb{R}^d)\|, \tag{4.22}$$

for all dyadic cubes Q and all $f \in \mathcal{N}^{\frac{d}{u}}_{u,p,q}(\mathbb{R}^d)$, where in the last step we used the fact that

$$\mathcal{N}^{\frac{d}{u}}_{u,p,\infty}(\mathbb{R}^d) \hookrightarrow \mathcal{N}^0_{u,p,1}(\mathbb{R}^d) \hookrightarrow \mathcal{E}^0_{u,p,2}(\mathbb{R}^d) = \mathcal{M}_{u,p}(\mathbb{R}^d).$$

Again the final outcome in this case follows from Step 1 and (4.22). ∎

Remark 4.11. (i) In terms of embeddings in generalised Morrey spaces, what has been proved could be stated as follows. Let $0 < p \le u < \infty$, $0 < q \le \infty$, and $1 \le r < \infty$. Then

$$\mathcal{N}^{\frac{d}{u}}_{u,p,q}(\mathbb{R}^d) \hookrightarrow \mathcal{M}^{\varphi_{r,q}}_r(\mathbb{R}^d)$$

where

$$\varphi_{r,q}(t) = \begin{cases} \left(\log(t^{-1})\right)^{-\frac{1}{q}} & \text{if} \quad 0 < t < e^{-\frac{u}{d}\max(1,\frac{r}{p})}, \\ t^{\frac{d}{u}\min(1,\frac{p}{r})} & \text{if} \quad t \ge e^{-\frac{u}{d}\max(1,\frac{r}{p})}. \end{cases} \tag{4.23}$$

(ii) If we would consider local generalised Morrey spaces $L\mathcal{M}^{\varphi}_p(\mathbb{R}^d)$, where the supremum taken in the definition of the norm is restricted to cubes with volume less or equal than 1 (cf. [33, page 7]), then we can state that

$$\mathcal{N}^{\frac{d}{u}}_{u,p,q}(\mathbb{R}^d) \hookrightarrow L\mathcal{M}^{\varphi}_r(\mathbb{R}^d), \qquad \varphi(t) = |\log(t)|^{-\frac{1}{q}}, \quad 1 \le r < \infty.$$

Corollary 4.12. *Let $0 < p \le u < \infty$ and $0 < q \le \infty$. If $1 \le r < \infty$, then there exists a positive constant c such that*

$$\left(\frac{1}{|Q|}\int_Q |f(x)|^r \,\mathrm{d}x\right)^{\frac{1}{r}} \le c\,(1+|Q|)^{-\frac{\min(1,\frac{p}{r})}{u}} \log\left(e+|Q|^{-1}\right) \|f|\mathcal{E}^{\frac{d}{u}}_{u,p,q}(\mathbb{R}^d)\|$$

holds for all dyadic cubes Q and all $f \in \mathcal{E}^{\frac{d}{u}}_{u,p,q}(\mathbb{R}^d)$.

Proof: The outcome is a direct consequence of Theorem 4.10 taking into account the embedding

$$\mathcal{E}^{\frac{d}{u}}_{u,p,q}(\mathbb{R}^d) \hookrightarrow \mathcal{N}^{\frac{d}{u}}_{u,p,\infty}(\mathbb{R}^d).$$

∎

According to [17, Thms. 2.30 and 2.36] (see also [20, Thm. 4.12]), we have the following atomic decomposition characterisation of $\mathcal{A}^s_{u,p,q}(\mathbb{R}^d)$, where we adopt the same custom to write $a^s_{u,p,q}(\mathbb{R}^d)$ instead of $n^s_{u,p,q}(\mathbb{R}^d)$ or $e^s_{u,p,q}(\mathbb{R}^d)$, for convenience.

Proposition 2.8. *Let $0 < p \le u < \infty$, $q \in (0,\infty]$ and $s \in \mathbb{R}$. Let*

$$K \ge \max(\lfloor s+1 \rfloor, 0)$$

and

$$N \ge \max(\lfloor \sigma_{p,q} - s \rfloor, -1)\ (\mathcal{E}\text{−case}) \quad \text{or} \quad N \ge \max(\lfloor \sigma_p - s \rfloor, -1)\ (\mathcal{N}\text{−case}).$$

Then for each $f \in \mathcal{A}^s_{u,p,q}(\mathbb{R}^d)$, there exist a family $\{a_{jm}\}_{j\in\mathbb{N}_0,m\in\mathbb{Z}^d}$ of (K,N)-atoms and a sequence $\lambda = \{\lambda_{jm}\}_{j\in\mathbb{N}_0,m\in\mathbb{Z}^d} \in a^s_{u,p,q}(\mathbb{R}^d)$ such that

$$f = \sum_{j=0}^{\infty}\sum_{m\in\mathbb{Z}^d} \lambda_{jm}\, a_{jm} \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^d)$$

and

$$\|\lambda \mid a^s_{u,p,q}(\mathbb{R}^d)\| \le C\, \|f \mid \mathcal{A}^s_{u,p,q}(\mathbb{R}^d)\|,$$

where C is a positive constant independent of λ and f.

Conversely, there exists a positive constant C such that for all families $\{a_{jm}\}_{j\in\mathbb{N}_0,m\in\mathbb{Z}^d}$ of (K,N)-atoms and $\lambda = \{\lambda_{jm}\}_{j\in\mathbb{N}_0,m\in\mathbb{Z}^d} \in a^s_{u,p,q}(\mathbb{R}^d)$,

$$\Big\| \sum_{j=0}^{\infty}\sum_{m\in\mathbb{Z}^d} \lambda_{jm}\, a_{jm} \mid \mathcal{A}^s_{u,p,q}(\mathbb{R}^d) \Big\| \le C\, \|\lambda \mid a^s_{u,p,q}(\mathbb{R}^d)\|.$$

3. Embeddings with smoothness $\frac{p}{u}\sigma_p$

We return to the remarkable coincidence (2.5) and consider the limiting situation when $p = 1$. Recall that in case of $p = u = 1$ it is well-known that $F^0_{1,2}(\mathbb{R}^d) \hookrightarrow L_1(\mathbb{R}^d)$ properly embedded, cf. [26]. Now we concentrate on the Morrey situation when $p = 1 < u$ and can prove some partial counterpart of (2.5).

Proposition 3.1. *Let $1 < u < \infty$. Then*

$$\mathcal{E}^0_{u,1,2}(\mathbb{R}^d) \hookrightarrow \mathcal{M}_{u,1}(\mathbb{R}^d).$$

Proof: *Step 1.* We prove that there exists a positive constant $C > 0$ such that

$$\|f|\mathcal{M}_{u,1}(\mathbb{R}^d)\| \le C\, \|f|\mathcal{E}^0_{u,1,2}(\mathbb{R}^d)\|. \tag{3.1}$$

Let $f \in \mathcal{E}^0_{u,1,2}(\mathbb{R}^d)$. By the atomic decomposition theorem with atoms satisfying the conditions from Definition 2.5 with $N > \lfloor d(u-1) \rfloor$ and $c_1 = 3$, we have

$$f = \sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^d} \lambda_{jm} a_{jm}$$

with

$$\Big\| \Big(\sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^d} |\lambda_{jm}|^2 \, \chi_{jm}(\cdot) \Big)^{\frac{1}{2}} \big| \mathcal{M}_{u,1}(\mathbb{R}^d) \Big\| \le c \| f | \mathcal{E}^0_{u,1,2}(\mathbb{R}^d) \|.$$

Let $Q = Q_{\nu,k}$ be a dyadic cube and decompose f as follows

$$f = \sum_{j=0}^{\nu} \sum_{m \in \mathbb{Z}^d} \lambda_{jm} a_{jm} + \sum_{j=\nu+1}^{\infty} \sum_{m \in \mathbb{Z}^d} \lambda_{jm} a_{jm} = f_1 + f_2. \tag{3.2}$$

Remark that $f_i \in \mathcal{E}^0_{u,1,2}(\mathbb{R}^d)$ and $\|f_i | \mathcal{E}^0_{u,1,2}(\mathbb{R}^d)\| \le c \|f | \mathcal{E}^0_{u,1,2}(\mathbb{R}^d)\|$, $i = 1, 2$.

Step 2. We deal first with f_1. Recall $Q = Q_{\nu,k}$. Let $\mu_{j,\nu} = \{m \in \mathbb{Z}^d : Q \cap \operatorname{supp} a_{j,m} \neq \emptyset\}$ and note that $\mu_{j,\nu} \le c$ if $j \le \nu$. Then we have

$$\begin{aligned}
&|Q|^{\frac{1}{u}-1} \int_Q |f_1(x)| \, \mathrm{d}x \le |Q|^{\frac{1}{u}-1} \sum_{j=0}^{\nu} \sum_{m \in \mu_{j,\nu}} |\lambda_{jm}| \int_Q |a_{jm}(x)| \, \mathrm{d}x \\
&\le c |Q|^{\frac{1}{u}} \sum_{j=0}^{\nu} \sum_{m \in \mu_{j,\nu}} |\lambda_{jm}| = c |Q|^{\frac{1}{u}} \sum_{j=0}^{\nu} \sum_{m \in \mu_{j,\nu}} |Q_{jm}|^{-1} \int_{Q_{jm}} \Big(|\lambda_{jm}|^2 \chi_{jm}(x) \Big)^{\frac{1}{2}} \, \mathrm{d}x \\
&\le c \sum_{j=0}^{\nu} 2^{(j-\nu)\frac{d}{u}} \sum_{m \in \mu_{j,\nu}} |Q_{jm}|^{\frac{1}{u}-1} \int_{Q_{jm}} \Big(\sum_{\ell=1}^{\infty} \sum_{n \in \mathbb{Z}^d} |\lambda_{\ell n}|^2 \chi_{\ell n}(x) \Big)^{\frac{1}{2}} \, \mathrm{d}x \\
&\le c \Big(\sum_{j=0}^{\nu} 2^{(j-\nu)\frac{d}{u}} \Big) \sup_{0 \le j \le \nu} \sup_{m \in \mu_{j,\nu}} \Big\{ |Q_{jm}|^{\frac{1}{u}-1} \int_{Q_{jm}} \Big(\sum_{\ell=1}^{\infty} \sum_{n \in \mathbb{Z}^d} |\lambda_{\ell n}|^2 \chi_{\ell,n}(x) \Big)^{\frac{1}{2}} \, \mathrm{d}x \Big\} \\
&\le c \Big\| \Big(\sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^d} |\lambda_{jm}|^2 \, \chi_{j,m}(\cdot) \Big)^{\frac{1}{2}} \big| \mathcal{M}_{u,1}(\mathbb{R}^d) \Big\| \le c \| f | \mathcal{E}^0_{u,1,2}(\mathbb{R}^d) \|.
\end{aligned} \tag{3.3}$$

Step 3. Now we deal with f_2 for what we rely on the results in [6]. By checking the proof in [6, Thm. 4.3], we can see that $f_2 \in \mathcal{E}^0_{u,1,2}(\mathbb{R}^d)$ can be

holds for all dyadic cubes Q and all $f \in \mathcal{N}^{\frac{d}{u}}_{u,p,q}(\mathbb{R}^d)$. The Hölder inequality and (4.18) yield

$$\begin{aligned}
\Big(\int_Q |f(x)|^r \, \mathrm{d}x \Big)^{\frac{1}{r}} &\le |Q|^{\frac{1}{r}-\frac{1}{r_0}} \Big(\int_Q |f(x)|^{r_0} \, \mathrm{d}x \Big)^{\frac{1}{r_0}} \\
&\le c \, |Q|^{\frac{1}{r}-\frac{1}{v_0}} \, v_0^{\frac{1}{q'}} \, \|f | \mathcal{N}^{\frac{d}{u}}_{u,p,q}(\mathbb{R}^d)\|
\end{aligned} \tag{4.19}$$

for all dyadic cubes Q and all $f \in \mathcal{N}^{\frac{d}{u}}_{u,p,q}(\mathbb{R}^d)$.

For convenience we deal with the case $q \ge 1$, the other case is even easier. Assume first that the cubes are small, that is, they satisfy $|Q| < e^{-u \max(1, \frac{r}{p})}$. Then $v_0 = \log(|Q|^{-1})$ and $r_0 = \frac{p}{u} v_0$ satisfy the above assumptions. Hence (4.19) leads to

$$\Big(\frac{1}{|Q|} \int_Q |f(x)|^r \, \mathrm{d}x \Big)^{\frac{1}{r}} \le c \, \big(\log \big(e + |Q|^{-1} \big) \big)^{\frac{1}{q'}} \, \|f | \mathcal{N}^{\frac{d}{u}}_{u,p,q}(\mathbb{R}^d)\|, \tag{4.20}$$

for any small enough cube Q with $|Q| < e^{-u \max(1, \frac{r}{p})}$, and all $f \in \mathcal{N}^{\frac{d}{u}}_{u,p,q}(\mathbb{R}^d)$. It remains to deal with the bigger cubes.

Step 2. Let $r \ge p$. Elementary embeddings and [4, Thm. 3.3] yield

$$\mathcal{N}^{\frac{d}{u}}_{u,p,q}(\mathbb{R}^d) \hookrightarrow \mathcal{N}^0_{\frac{ur}{p},r,1}(\mathbb{R}^d) \hookrightarrow \mathcal{E}^0_{\frac{ur}{p},r,2}(\mathbb{R}^d) = \mathcal{M}_{\frac{ur}{p},r}(\mathbb{R}^d), \quad \text{if} \quad r > 1,$$

and

$$\mathcal{N}^{\frac{d}{u}}_{u,p,q}(\mathbb{R}^d) \hookrightarrow \mathcal{N}^0_{\frac{u}{p},1,1}(\mathbb{R}^d) \hookrightarrow \mathcal{M}_{\frac{u}{p},1}(\mathbb{R}^d), \quad \text{if} \quad r = 1,$$

where the last embedding is due to Theorem 3.4. Therefore, for $r \ge 1$ and $r \ge p$, we have

$$\Big(\frac{1}{|Q|} \int_Q |f(x)|^r \, \mathrm{d}x \Big)^{\frac{1}{r}} \le c \, |Q|^{-\frac{p}{ur}} \|f | \mathcal{N}^{\frac{d}{u}}_{u,p,q}(\mathbb{R}^d)\|, \tag{4.21}$$

for all dyadic cubes Q and all $f \in \mathcal{N}^{\frac{d}{u}}_{u,p,q}(\mathbb{R}^d)$. Together with Step 1 this completes the argument in case of $r \ge p$.

Corollary 4.9. *Let $0 < p \le u < \infty$ and $0 < q \le \infty$.*

(i) *If $1 < q \le \infty$ and $r \ge p + q'$. Then there is a positive constant C such that the inequality*

$$\left(\frac{1}{|Q|}\int_Q |f(x)|^r \,\mathrm{d}x\right)^{\frac{1}{r}} \le c\,(1+|Q|)^{-\frac{p}{ru}}\left(\log\left(e+|Q|^{-1}\right)\right)^{\frac{1}{q'}}\|f|\mathcal{N}^{\frac{d}{u}}_{u,p,q}(\mathbb{R}^d)\| \tag{4.16}$$

holds for all dyadic cubes Q and all $f \in \mathcal{N}^{\frac{d}{u}}_{u,p,q}(\mathbb{R}^d)$.

(ii) *If $0 < q \le \infty$ and $r \ge p + 1$. Then there is a positive constant C such that the inequality*

$$\left(\frac{1}{|Q|}\int_Q |f(x)|^r \,\mathrm{d}x\right)^{\frac{1}{r}} \le c\,(1+|Q|)^{-\frac{p}{ru}}\left(\log\left(e+|Q|^{-1}\right)\right)\|f|\mathcal{E}^{\frac{d}{u}}_{u,p,q}(\mathbb{R}^d)\| \tag{4.17}$$

holds for all dyadic cubes Q and all $f \in \mathcal{E}^{\frac{d}{u}}_{u,p,q}(\mathbb{R}^d)$.

The inequalities (4.16) and (4.17) can be extended to the smallest values of r and q. Recall that q' is defined by $q' = \frac{q}{q-1}$ if $1 < q < \infty$ and $q' = \infty$ if $0 < q \le 1$, where the usual convention $1/\infty = 0$ is assumed.

Theorem 4.10. *Let $0 < p \le u < \infty$ and $0 < q \le \infty$. If $1 \le r < \infty$, then there exists a positive constant c such that*

$$\left(\frac{1}{|Q|}\int_Q |f(x)|^r \,\mathrm{d}x\right)^{\frac{1}{r}} \le c\,(1+|Q|)^{-\frac{\min(1,\frac{p}{r})}{u}}\left(\log\left(e+|Q|^{-1}\right)\right)^{\frac{1}{q'}}\|f|\mathcal{N}^{\frac{d}{u}}_{u,p,q}(\mathbb{R}^d)\|$$

holds for all dyadic cubes Q and all $f \in \mathcal{N}^{\frac{d}{u}}_{u,p,q}(\mathbb{R}^d)$.

Proof: *Step 1.* Given $r \ge 1$, let r_0 be such that $r_0 > \max(p, r)$. By Lemma 4.1, there exists a positive constant c, not depending on r_0 and v_0, with $v_0 = \frac{ur_0}{p}$, such that the inequality

$$|Q|^{\frac{1}{v_0}-\frac{1}{r_0}}\left(\int_Q |f(x)|^{r_0}\,\mathrm{d}x\right)^{\frac{1}{r_0}} \le c\,v_0^{\frac{1}{q'}}\|f|\mathcal{N}^{\frac{d}{u}}_{u,p,q}(\mathbb{R}^d)\| \tag{4.18}$$

decomposed in terms of non-smooth atoms b_{jm} supported in cubes cQ_{jm} with side lengths less than $2^{-\nu}$, that is,

$$f_2 = \sum_{j=\nu+1}^{\infty}\sum_{m\in\mathbb{Z}^d}\lambda_{jm}a_{jm} = \sum_{j=\nu+1}^{\infty}\sum_{m\in\mathbb{Z}^d}t_{jm}b_{jm}$$

where

$$\|t|m_{u,1}\| := \sup_{P\in\mathcal{Q}}|P|^{\frac{1}{u}-1}\Big(\sum_{Q_{\ell k}\subset P}|t_{\ell k}|\Big) \le c\|f_2|\mathcal{E}^0_{u,1,2}(\mathbb{R}^d)\|$$

and $t = \{t_{jm}\}_{j\in\mathbb{N}_0, m\in\mathbb{Z}^d}$.

We remark that the non-smooth atoms from [6] differ from the smooth ones of Definition 2.5 in condition (2.6); in particular, we have

$$\|b_{jm}|L_u(\mathbb{R}^d)\| \le |Q_{jm}|^{\frac{1}{u}-1}, \quad j\in\mathbb{N}_0,\ m\in\mathbb{Z}^d.$$

Then, using Hölder's inequality and the properties of non-smooth atoms we obtain

$$\begin{aligned}
|Q|^{\frac{1}{u}-1}\int_Q |f_2(x)|\,\mathrm{d}x &\le |Q|^{\frac{1}{u}-1}\sum_{j=\nu+1}^{\infty}\sum_{m\in\mathbb{Z}^d}|t_{jm}|\int_{3Q_{jm}\cap Q}|b_{jm}(x)|\,\mathrm{d}x\\
&\le c|Q|^{\frac{1}{u}-1}\sum_{j=\nu+1}^{\infty}\sum_{m\in\mathbb{Z}^d:Q_{jm}\subset cQ}|t_{jm}||Q_{jm}|^{1-\frac{1}{u}}\|b_{jm}|L_u(\mathbb{R}^d)\|\\
&\le c|Q|^{\frac{1}{u}-1}\sum_{j=\nu+1}^{\infty}\sum_{m\in\mathbb{Z}^d:Q_{jm}\subset c'Q}|t_{jm}| \le c\|(t_{jm})|m_{u,1}\|\\
&\le c\|f|\mathcal{E}^0_{u,1,2}(\mathbb{R}^d)\|. \qquad (3.4)
\end{aligned}$$

The desired outcome is then a consequence of (3.2), (3.3) and (3.4). ∎

We equip the spaces $L_1^{\mathrm{loc}}(\mathbb{R}^d)$ with the metric

$$d(f,g) = \sum_{n=1}^{\infty}\frac{1}{2^n}\frac{\|f-g|L_1(\widetilde{Q}_n)\|}{1+\|f-g|L_1(\widetilde{Q}_n)\|},$$

where $\widetilde{Q}_n = [-n,n]^d$. The space $L_1^{\mathrm{loc}}(\mathbb{R}^d)$ with this metric is a complete locally convex metric space, i.e., a Fréchet space, cf. [11, page 40]. One

can easily see that a Morrey space $\mathcal{M}_{u,p}(\mathbb{R}^d)$ is continuously embedded into $L_1^{\rm loc}(\mathbb{R}^d)$ if $p \geq 1$. Indeed we have

$$d(f,g) \leq C \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|\widetilde{Q}_n|^{1-\frac{1}{u}}}{1+\|f-g|L_1(\widetilde{Q}_n)\|} \|f-g|\mathcal{M}_{u,p}(\mathbb{R}^d)\| \leq$$
$$\leq C \sum_{n=1}^{\infty} \frac{|\widetilde{Q}_n|^{1-\frac{1}{u}}}{2^n} \|f-g|\mathcal{M}_{u,p}(\mathbb{R}^d)\| \leq C\|f-g|\mathcal{M}_{u,p}(\mathbb{R}^d)\|.$$

Theorem 3.2. *Let $0 < p \leq u < \infty$, $0 < q \leq \infty$, and $s = \frac{p}{u}\sigma_p$. The following assertions are equivalent:*

(i) $\mathcal{E}^s_{u,p,q}(\mathbb{R}^d) \hookrightarrow L_1^{\rm loc}(\mathbb{R}^d)$,
(ii) $\mathcal{E}^s_{u,p,q}(\mathbb{R}^d) \hookrightarrow \mathcal{M}_{\frac{u}{\min(p,1)},\max(p,1)}(\mathbb{R}^d)$,
(iii) *either* $p \geq 1$ *and* $q \leq 2$, *or* $0 < p < 1$.

Proof: Note that the implication (ii) $\Rightarrow$ (i) is already shown, and the part (i) $\Rightarrow$ (iii) is covered by [3, Thm. 3.4].

Step 1. Consider first the case $p = 1$. The implication (iii) $\Rightarrow$ (ii) is a consequence of Proposition 3.1 and an elementary embedding,

$$\mathcal{E}^0_{u,1,q}(\mathbb{R}^d) \hookrightarrow \mathcal{E}^0_{u,1,2}(\mathbb{R}^d) \hookrightarrow \mathcal{M}_{u,1}(\mathbb{R}^d) \quad \text{if} \quad 0 < q \leq 2.$$

Step 2. Let $0 < p < 1$. Then $s = \frac{p}{u}\sigma_p = \frac{d}{u}(1-p)$. The general properties of embeddings between Triebel-Lizorkin-Morrey spaces, cf. [5, Thm. 3.1], and the first step imply

$$\mathcal{E}^s_{u,p,\infty}(\mathbb{R}^d) \hookrightarrow \mathcal{E}^0_{\frac{u}{p},1,1}(\mathbb{R}^d) \hookrightarrow \mathcal{M}_{\frac{u}{p},1}(\mathbb{R}^d),$$

which shows that (iii) $\Rightarrow$ (ii), and there is nothing more to be proved in this case.

Step 3. Now assume $p > 1$. The implication (iii) $\Rightarrow$ (ii) follows from

$$\mathcal{E}^0_{u,p,q}(\mathbb{R}^d) \hookrightarrow \mathcal{E}^0_{u,p,2}(\mathbb{R}^d) = \mathcal{M}_{u,p}(\mathbb{R}^d) \quad \text{if} \quad 0 < q \leq 2.$$

■

Remark 3.3. The above theorem improves the statements of [3, Thm. 3.4] and extends Theorem 3.3.2(i) and Corollary 3.3.1 in [26] from classical Triebel-Lizorkin spaces to Triebel-Lizorkin-Morrey spaces.

holds for all dyadic cubes Q and all $f \in \mathcal{M}_{\frac{u}{p},\Phi_{p,q}}(\mathbb{R}^d)$.

Proof: It should be clear that it is sufficient to consider the case $r = p + j_0 q$ for some $j_0 \in \mathbb{N}$. Note that this refers to $r = p(j_0)$ in the notation of Lemma 4.4.

Let Q be a dyadic cube with side length 2^{-j} and $|Q| = 2^{-jd}$, $j \in \mathbb{Z}$. Assume first $j \leq 0$. Then by Lemma 4.4 we get

$$\left(\frac{1}{|Q|}\int_Q |f(x)|^r \,\mathrm{d}x\right)^{\frac{1}{r}} \leq C\, j_0^{1/q} |Q|^{-\frac{p}{ur}} \|f|\mathcal{M}_{\frac{u}{p},\Phi_{p,q}}(\mathbb{R}^d)\|$$
$$\leq C(1+|Q|)^{-\frac{p}{ur}} \|f|\mathcal{M}_{\frac{u}{p},\Phi_{p,q}}(\mathbb{R}^d)\|. \tag{4.13}$$

Next we assume that $j \geq j_0$. Then Hölder's inequality and Lemma 4.4 imply

$$\left(\frac{1}{|Q|}\int_Q |f(x)|^r \,\mathrm{d}x\right)^{\frac{1}{r}} \leq \left(\frac{1}{|Q|}\int_Q |f(x)|^{p(j)} \,\mathrm{d}x\right)^{\frac{1}{p(j)}}$$
$$\leq C\, j^{1/q} |Q|^{-\frac{p}{up(j)}} \|f|\mathcal{M}_{\frac{u}{p},\Phi_{p,q}}(\mathbb{R}^d)\|$$
$$\leq C\left(\log(e+|Q|^{-1})\right)^{\frac{1}{q}} \|f|\mathcal{M}_{\frac{u}{p},\Phi_{p,q}}(\mathbb{R}^d)\| \tag{4.14}$$

since

$$\log(e+|Q|^{-1}) \sim j \quad \text{and} \quad |Q|^{-\frac{p}{up(j)}} = 2^{\frac{jdp}{u(p+jq)}} \leq 2^{\frac{dp}{uq}}.$$

At the end we consider the cubes with $0 < j < j_0$. We have

$$2^{-\frac{j_0 d}{v(j_0)}} \sup_{Q:\, |Q|=2^{-jd},\, 0<j<j_0} \left(\frac{1}{|Q|}\int_Q |f(x)|^r \,\mathrm{d}x\right)^{\frac{1}{r}}$$
$$\leq \sup_Q |Q|^{\frac{1}{v(j_0)}} \left(\frac{1}{|Q|}\int_Q |f(x)|^r \,\mathrm{d}x\right)^{\frac{1}{r}} \leq C\, j_0^{\frac{1}{q}} \|f|\mathcal{M}_{\frac{u}{p},\Phi_{p,q}}(\mathbb{R}^d)\|,$$

by Lemma 4.4, recall $r = p(j_0)$. Thus

$$\left(\frac{1}{|Q|}\int_Q |f(x)|^r \,\mathrm{d}x\right)^{\frac{1}{r}} \leq C\left(\log(e+|Q|^{-1})\right)^{\frac{1}{q}} \|f|\mathcal{M}_{\frac{u}{p},\Phi_{p,q}}(\mathbb{R}^d)\|. \tag{4.15}$$

Consequently, the inequalities (4.13)-(4.15) prove the proposition. ■

The next corollary is an immediate consequence of Theorem 4.5, Corollary 4.6 and Proposition 4.8.

Remark 4.7. According to [22, Cor. 1.5] it holds

$$\mathcal{E}^{\frac{d}{u}}_{u,p,2}(\mathbb{R}^d) \hookrightarrow \mathcal{M}_{\frac{u}{p},\Phi_p}(\mathbb{R}^d),$$

where $1 < p \le u < \infty$ and

$$\Phi_p(t) := \sum_{j=j_p}^{\infty} \frac{t^j}{j!}, \quad t \ge 0,$$

with $j_p := \min\{j \in \mathbb{N} : j \ge p\}$. Since there exists a constant $c > 0$ such that

$$\Phi_{p,1}(t) \ge c\Phi_p(t) \quad \text{for all} \quad t \ge 0,$$

it turns out that

$$\mathcal{M}_{\frac{u}{p},\Phi_{p,1}}(\mathbb{R}^d) \hookrightarrow \mathcal{M}_{\frac{u}{p},\Phi_p}(\mathbb{R}^d).$$

Hence Corollary 4.6 does not only extend [22, Cor. 1.5] from $q = 2$ to any $1 < q \le \infty$, it improves it.

4.2. Embeddings in generalised Morrey spaces. Now we turn to the generalised Morrey spaces. The spaces were extensively studied, cf. Nakai [14], Nakamura, Noi and Sawano [15], Sawano and Wadade [22] and the references given there. Let $0 < r < \infty$ and let $\varphi : [0,\infty) \to [0,\infty)$ be a suitable function. For a locally r-integrable function f we put

$$\|f|\mathcal{M}^{\varphi}_{r}\| := \sup_{Q\in\mathcal{Q}} \varphi(|Q|)\left(|Q|^{-1}\int_Q |f(y)|^r \,\mathrm{d}y\right)^{1/r}. \tag{4.12}$$

The space $\mathcal{M}^{\varphi}_{r}(\mathbb{R}^d)$ is the set of all measurable functions f for which the quasi-norm (4.12) is finite. If $\varphi(t) = t^{1/u}$, $0 < r \le u$, then the definition coincides with the definition of the Morrey space $\mathcal{M}_{u,r}(\mathbb{R}^d)$. Since we will work with the given examples of functions φ we avoid the discussions which functions φ define the reasonable spaces, and we refer the interested reader to the above mentioned papers.

We start with a proposition that somehow compares the Orlicz-Morrey spaces with generalised Morrey spaces we will use.

Proposition 4.8. *Let $0 < p \le u < \infty$, $1 \le q < \infty$ and $r \ge p+q$. Then there is a positive constant C depending on u, p, q and r such that the following inequality*

$$\left(\frac{1}{|Q|}\int_Q |f(x)|^r \,\mathrm{d}x\right)^{\frac{1}{r}} \le c\,(1+|Q|)^{-\frac{p}{ru}}\left(\log\left(e+|Q|^{-1}\right)\right)^{\frac{1}{q}} \|f|\mathcal{M}_{\frac{u}{p},\Phi_{p,q}}(\mathbb{R}^d)\|$$

Theorem 3.4. *Let $0 < p \le u < \infty$, $0 < q \le \infty$, and $s = \frac{p}{u}\sigma_p$. The following assertions are equivalent:*

(i) $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d) \hookrightarrow L_1^{\mathrm{loc}}(\mathbb{R}^d)$,

(ii) $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d) \hookrightarrow \mathcal{M}_{\frac{u}{\min(p,1)},\max(p,1)}(\mathbb{R}^d)$,

(iii) $0 < q \le \min\big(\max(p,1),2\big)$.

Proof: The case $p = u$ is well known, cf. [26, Thm. 3.3.2, Cor. 3.3.1], so we can restrict ourselves to the case $p < u$.

Step 1. We prove that (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i). The second implication has been already shown so it remains to prove the first one.

Let $0 < p \le 1$. For $0 < q \le 1$, by general properties of embeddings, in particular [4, Thm. 3.2], and Theorem 3.2, we have

$$\mathcal{N}^s_{u,p,q}(\mathbb{R}^d) \hookrightarrow \mathcal{N}^0_{\frac{u}{p},1,1}(\mathbb{R}^d) \hookrightarrow \mathcal{E}^0_{\frac{u}{p},1,1}(\mathbb{R}^d) \hookrightarrow \mathcal{M}_{\frac{u}{p},1}(\mathbb{R}^d)$$

which proves the implication (iii) $\Rightarrow$ (ii).

Consider the case $p \ge 1$. If $0 < q \le \min(p,2)$, then elementary embeddings, (2.5) and Theorem 3.2 yield

$$\mathcal{N}^0_{u,p,q}(\mathbb{R}^d) \hookrightarrow \mathcal{E}^0_{u,p,2}(\mathbb{R}^d) \hookrightarrow \mathcal{M}_{u,p}(\mathbb{R}^d),$$

which shows that (iii) $\Rightarrow$ (ii).

Step 2. We prove that the condition (i) implies (iii). For $0 < p < 1$ the implication was proved in [3], cf. Step 2 of the proof of Theorem 3.4. But the same argument works for $p = 1$. Also the case $2 \le p < \infty$ is covered by Theorem 3.4 in [3].

It remains to consider the case $1 < p < 2$. We assume that embedding (i) holds for some $q > p$. We choose a smooth function $\tilde{a}$ such that

$$\operatorname{supp}\tilde{a} \subset \left[0,\frac{1}{2}\right]^d, \qquad 0 \le \tilde{a} \le 1 \qquad \text{and} \qquad \left|\frac{\partial\tilde{a}}{\partial x_i}\right| \le 1, \quad i = 1,\dots,d, \tag{3.5}$$

and put

$$a(x) = \tilde{a}(x) - \tilde{a}(-x). \tag{3.6}$$

Then a is an atom satisfying the first moment condition and supported in $[-\frac{1}{2},\frac{1}{2}]^d$. Moreover we consider the family of atoms $a_{j,m}$, $j = 0,1,\dots$ and $m \in \mathbb{Z}^d$ that are the dilations and translations of a, i.e.,

$$a_{j,m}(x) = a(2^j x - m - (1/2,\dots,1/2)).$$

The function $a_{j,m}$ is an atom supported in $Q_{j,m}$.

Let us fix $n \in \mathbb{N}$. For any cube $Q_{j,m} \subset Q_{0,0}$, $1 \le j \le n$, $m \in \mathbb{Z}^d$, we define a function

$$b_{j,m}(x) = 2^{j-n} \sum_{k:Q_{n,k}\subset Q_{j,m}} a_{n,k}(x).$$

The function $b_{j,m}$ is an atom satisfying the first moment condition supported in $Q_{j,m}$. We define a smooth function f_n by the following finite sum

$$f_n(x) = \sum_{j=1}^{n} j^{-\frac{1}{p}} \sum_{m:Q_{j,m}\subset Q_{0,0}} b_{j,m}(x).$$

The functions f_n belong to $\mathcal{N}^0_{u,p,q}(\mathbb{R}^d)$ and their norms are uniformly bounded since

$$\begin{aligned}
\|f_n|\mathcal{N}^0_{u,p,q}(\mathbb{R}^d)\| &\le \left(\sum_{j=1}^{n} 2^{-jq\frac{d}{u}} \sup_{\nu:\nu\le j, k\in\mathbb{Z}^d} 2^{dq(j-\nu)(\frac{1}{u}-\frac{1}{p})} \Big(\sum_{Q_{j,m}\subset Q_{\nu,k}\subset Q_{0,0}} j^{-1}\Big)^{\frac{q}{p}}\right)^{\frac{1}{q}} \\
&\le \left(\sum_{j=1}^{n} j^{-\frac{q}{p}} 2^{-jq\frac{d}{u}} \sup_{\nu:0\le\nu\le j, k\in\mathbb{Z}^d} 2^{dq\frac{(j-\nu)}{u}}\right)^{\frac{1}{q}} \\
&\le \left(\sum_{j=1}^{\infty} j^{-\frac{q}{p}}\right)^{\frac{1}{q}} = C < \infty,
\end{aligned}$$

recall $q > p$. On the other hand, for $\nu \in \mathbb{N}$,

$$\begin{aligned}
\|f_n|L_1(\widetilde{Q}_\nu)\| = \|f_n|L_1(\widetilde{Q}_1)\| &= \int_{\widetilde{Q}_1} \left|\sum_{j=1}^{n} j^{-\frac{1}{p}} \sum_{Q_{j,m}\subset Q_{0,0}} b_{j,m}(x)\right| \mathrm{d}x \\
&= \int_{\widetilde{Q}_1} \left|\sum_{j=1}^{n} j^{-\frac{1}{p}} 2^{j-n} \sum_{Q_{j,m}\subset Q_{0,0}} \sum_{Q_{n,k}\subset Q_{j,m}} a_{n,k}(x)\right| \mathrm{d}x \\
&= \int_{\widetilde{Q}_1} \left|\sum_{j=1}^{n} j^{-\frac{1}{p}} 2^{j-n} n \sum_{Q_{n,k}\subset Q_{0,0}} a_{n,k}(x)\right| \mathrm{d}x \\
&= \int_{\widetilde{Q}_1} \left(\sum_{j=1}^{n} j^{-\frac{1}{p}} 2^{j-n} n\right) \left|\sum_{Q_{n,k}\subset Q_{0,0}} a_{n,k}(x)\right| \mathrm{d}x
\end{aligned}$$

Taking the infimum over ε we get

$$j^{-\frac{p}{q}-j}|Q|^{\frac{p(j)}{v(j)}-1} \int_Q |f(x)|^{p+jq}\,\mathrm{d}x \le cj^{-\kappa}\|f\|^{p+jq}_{(r,\Phi);Q}$$

and afterwards taking the supremum over all dyadic cubes gives

$$j^{-1/q}\|f|\mathcal{M}_{v(j),p(j)}(\mathbb{R}^d)\| \le c^{1/p(j)} j^{-\kappa/p(j)}\|f|\mathcal{M}_{r,\Phi}(\mathbb{R}^d)\|.$$

Now we take the supremum over all j. The expression on the right-hand side is of the size $j^{1/j}$ so it can be bounded by a positive constant. This yields

$$\sup_{j\ge1} j^{-1/q}\|f|\mathcal{M}_{v(j),p(j)}(\mathbb{R}^d)\| \le c\|f|\mathcal{M}_{r,\Phi}(\mathbb{R}^d)\|.$$

■

Theorem 4.5. *Let $0 < p \le u < \infty$ and $1 < q \le \infty$. Then*

$$\mathcal{N}^{\frac{d}{u}}_{u,p,q}(\mathbb{R}^d) \hookrightarrow \mathcal{M}_{\frac{u}{p},\Phi_{p,q'}}(\mathbb{R}^d),$$

where q' is the conjugate exponent of q.

Proof: For each $j \in \mathbb{N}$, by Lemma 4.1 with $r = p(j) = p + qj$ and $v(j) = \frac{u}{p}p(j)$,

$$\|f|\mathcal{M}_{v(j),p(j)}(\mathbb{R}^d)\| \le c\{v(j)\}^{1/q'}\|f|\mathcal{N}^{\frac{d}{u}}_{u,p,q}(\mathbb{R}^d)\|,$$

where c is a positive constant independent of $v(j)$ and $p(j)$, and thus of j. Since $\{v(j)\}^{1/q'} \sim j^{1/q'}$, using also Lemma 4.4, we get

$$\|f|\mathcal{M}_{\frac{u}{p},\Phi_{p,q'}}(\mathbb{R}^d)\| \sim \sup_{j\ge1} j^{-1/q'}\,\|f|\mathcal{M}_{v(j),p(j)}(\mathbb{R}^d)\| \le c\|f|\mathcal{N}^{\frac{d}{u}}_{u,p,q}(\mathbb{R}^d)\|.$$

■

Corollary 4.6. *Let $0 < p \le u < \infty$ and $0 < q \le \infty$. Then*

$$\mathcal{E}^{\frac{d}{u}}_{u,p,q}(\mathbb{R}^d) \hookrightarrow \mathcal{M}_{\frac{u}{p},\Phi_{p,1}}(\mathbb{R}^d).$$

Proof: The result follows from the above theorem and elementary embeddings:

$$\mathcal{E}^{\frac{d}{u}}_{u,p,\infty}(\mathbb{R}^d) \hookrightarrow \mathcal{N}^{\frac{d}{u}}_{u,p,\infty}(\mathbb{R}^d) \hookrightarrow \mathcal{M}_{\frac{u}{p},\Phi_{p,1}}(\mathbb{R}^d).$$

■

Therefore,

$$\int_Q \Phi_{p,q}\Big(\frac{|f(x)|}{\lambda}\Big)\,\mathrm{d}x \sim \sum_{j=0}^{\infty} j^{\kappa-\frac{p}{q}-j}\lambda^{-(p+jq)}\int_Q |f(x)|^{p+jq}\,\mathrm{d}x. \tag{4.11}$$

Step 2. We prove the left-hand side inequality in (4.10). Assume that

$$\sup_{j\geq 1} j^{-1/q}\,\|f|\mathcal{M}_{v(j),p(j)}(\mathbb{R}^d)\| \leq \lambda.$$

Then, for any dyadic cube Q and any $j\in\mathbb{N}$, it holds

$$\lambda^{-(p+qj)} j^{-\frac{p+qj}{q}}|Q|^{\frac{p}{u}-1}\int_Q |f(x)|^{p(j)}\,\mathrm{d}x \leq 1.$$

For this λ, inserting the above inequality in (4.11) entails

$$\int_Q \Phi_{p,q}\Big(\frac{|f(x)|}{\lambda}\Big)\,\mathrm{d}x \leq c\sum_{j=0}^{\infty} j^{\kappa}|Q|^{1-\frac{p}{u}}.$$

By choosing $\kappa<-1$, we conclude that

$$\sup_{Q\in\mathcal{Q}}|Q|^{\frac{p}{u}-1}\int_Q \Phi_{p,q}\Big(\frac{|f(x)|}{\lambda}\Big)\,\mathrm{d}x \leq c.$$

Step 3. It remains to show the right-hand side inequality in (4.10). Let now

$$\|f|\mathcal{M}_{r,\Phi}(\mathbb{R}^d)\| := \sup_{Q\in\mathcal{Q}}\|f\|_{(r,\Phi);Q}\leq 1$$

and let $\varepsilon>0$. Then for any dyadic cube Q there is λ_Q, $0<\lambda_Q\leq \|f\|_{(r,\Phi);Q}+\varepsilon$, such that

$$|Q|^{\frac{p}{u}-1}\int_Q \Phi_{p,q}\Big(\frac{|f(x)|}{\lambda_Q}\Big)\,\mathrm{d}x \leq 1.$$

Then, by (4.11),

$$\sum_{j=0}^{\infty} j^{\kappa-\frac{p}{q}-j}\lambda_Q^{-(p+jq)}|Q|^{\frac{p(j)}{v(j)}-1}\int_Q |f(x)|^{p+jq}\,\mathrm{d}x \leq c$$

for all dyadic cubes Q and for some positive constant c independent of Q. Hence, for any $j\in\mathbb{N}$ and any dyadic cube Q, it holds

$$j^{-\frac{p}{q}-j}|Q|^{\frac{p(j)}{v(j)}-1}\int_Q |f(x)|^{p+jq}\,\mathrm{d}x \leq cj^{-\kappa}\lambda_Q^{p+jq} \leq cj^{-\kappa}(\|f\|_{(r,\Phi);Q}+\varepsilon)^{p+jq}.$$

$$= \sum_{j=1}^{n} j^{-\frac{1}{p}}2^{j-n}n\sum_{Q_{n,k}\subset Q_{0,0}}\int_{\widetilde{Q}_1}|a_{n,k}(x)|\,\mathrm{d}x$$

$$= cn2^{-n}\sum_{j=1}^{n} j^{-\frac{1}{p}}2^{j} \geq cn^{1-\frac{1}{p}}.$$

Thus $\|f_n|L_1(\widetilde{Q}_\nu)\| = \|f_n|L_1(\widetilde{Q}_1)\|\to\infty$ if $n\to\infty$ since $1<p$. The local base at zero in $L_1^{\mathrm{loc}}(\mathbb{R}^d)$ is given by the sets $V_{k,\varepsilon}=\{f:\|f|L_1(\widetilde{Q}_\nu)\|<\varepsilon,\quad \nu=1,\dots,k\}$. So the sequence f_n is not bounded in $L_1^{\mathrm{loc}}(\mathbb{R}^d)$. This contradicts the continuity of the embedding (i). ∎

Corollary 3.5. *Let $0<p\leq u<v<\infty$, $1\leq q\leq v$ and $s=\frac{d}{u}-\frac{d}{v}$. Then the following assertions are equivalent:*

(i) $\mathcal{E}^s_{u,p,\infty}(\mathbb{R}^d)\hookrightarrow\mathcal{M}_{v,q}(\mathbb{R}^d)$,
(ii) $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)\hookrightarrow\mathcal{M}_{v,q}(\mathbb{R}^d)$,
(iii) $q\leq v\frac{p}{u}$.

Proof: *Step 1.* First we prove the sufficiency of the condition **(iii)**. If $p=\frac{u}{v}$, then $\frac{p}{u}\sigma_p=\frac{d}{u}-\frac{d}{v}=s$. So **(i)** follows from Theorem 3.2. If $p>\frac{u}{v}$, then $s=\frac{d}{u}-\frac{d}{v}>0$ and $q_{\max}=\frac{pv}{u}>1$. In that case the embedding is a consequence of Sobolev embeddings [5, Thm. 3.2], the Paley-Littlewood formula (2.5) and the embeddings between Morrey spaces.

Similarly, the embedding **(ii)** in the case $p=\frac{u}{v}$ follows from Theorem 3.4 since $q=1$. To prove **(ii)** for $p>\frac{u}{v}$ and $1<q\leq p\frac{v}{u}$ one can use the Franke-Jawerth embeddings for smoothness Morrey spaces, cf. [5, Thm. 4.3]. Indeed, we have

$$\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)\hookrightarrow\mathcal{E}^0_{v,q,2}(\mathbb{R}^d)=\mathcal{M}_{v,q}(\mathbb{R}^d).$$

If $p>\frac{u}{v}$ and $q=1$, then

$$\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)\hookrightarrow\mathcal{N}^s_{u,\frac{u}{v},q}(\mathbb{R}^d)\hookrightarrow\mathcal{M}_{v,1}(\mathbb{R}^d),$$

the statement coincides with our previous statement, Theorem 3.4. We refer to [4] for a proof of the first embedding.

Step 2. Now we prove the necessity of the conditions.

First we assume that

$$\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)\hookrightarrow\mathcal{M}_{v,q}(\mathbb{R}^d). \tag{3.7}$$

The last embeddings implies that the space $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)$ consists of locally integrable functions, so $s \geq \frac{p}{u}\sigma_p$, cf. Theorem 3.3. in [3]. This implies $p \geq \frac{u}{v}$. If $1 < q$, then by (2.4) and (2.5) we get

$$\mathcal{N}^s_{u,p,q}(\mathbb{R}^d) \hookrightarrow \mathcal{M}_{v,q}(\mathbb{R}^d) = \mathcal{E}^0_{v,q,2}(\mathbb{R}^d) \hookrightarrow \mathcal{N}^0_{v,q,\infty}(\mathbb{R}^d).$$

But Theorem 3.3 of [4], implies $q \leq \frac{pv}{u}$. If $q = 1$, then the condition $q \leq \frac{pv}{u}$ is satisfied automatically since we have already proved that $p \geq \frac{u}{v}$.

Using the same argument as above we can prove that $q \leq \frac{pv}{u}$ if

$$\mathcal{E}^s_{u,p,\infty}(\mathbb{R}^d) \hookrightarrow \mathcal{M}_{v,q}(\mathbb{R}^d).$$

■

Remark 3.6. The embeddings **(i)** and **(ii)** of the last corollary holds also for $q < 1$ if $v\frac{p}{u} \geq 1$ since then we increase the target space whereas the source space is unchanged or smaller.

Remark 3.7. Another class of generalisations of Besov and Triebel-Lizorkin spaces are Besov-type spaces $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ and Triebel-Lizorkin-type spaces $F^{s,\tau}_{p,q}(\mathbb{R}^d)$, $0 < p, q \leq \infty$ (with $p < \infty$ in case of $F^{s,\tau}_{p,q}$), $\tau \geq 0$, $s \in \mathbb{R}$ introduced in [38]. The spaces are strictly related to $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)$ and $\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)$ spaces if $0 \leq \tau < \frac{1}{p}$. Namely

$$F^{s,\tau}_{p,q}(\mathbb{R}^d) = \mathcal{E}^s_{u,p,q}(\mathbb{R}^d) \qquad \text{if} \qquad 0 \leq \tau = \frac{1}{p} - \frac{1}{u} < \frac{1}{p}, \tag{3.8}$$

and

$$\mathcal{N}^s_{u,p,q}(\mathbb{R}^d) \hookrightarrow B^{s,\tau}_{p,q}(\mathbb{R}^d) \qquad \text{if} \qquad 0 \leq \tau = \frac{1}{p} - \frac{1}{u} < \frac{1}{p}. \tag{3.9}$$

The Besov-Morrey and Besov-type spaces coincide only if $\tau = 0$ or $q = \infty$. In contrast to the spaces $\mathcal{A}^s_{u,p,q}(\mathbb{R}^d)$ these scales are embedded into each other like their classical counterparts, that is,

$$B^{s,\tau}_{p,\min(p,q)}(\mathbb{R}^d) \hookrightarrow F^{s,\tau}_{p,q}(\mathbb{R}^d) \hookrightarrow B^{s,\tau}_{p,\max(p,q)}(\mathbb{R}^d), \tag{3.10}$$

whenever $0 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$, $\tau \geq 0$. Using these relations we can easily transfer our results to the new class of function spaces if $0 \leq \tau < \frac{1}{p}$. In particular, if $0 \leq \tau < \frac{1}{p}$ and $s = (1 - \tau p)\sigma_p$, then the following three conditions are equivalent

(i) $F^{s,\tau}_{p,q}(\mathbb{R}^d) \hookrightarrow L^{\mathrm{loc}}_1(\mathbb{R}^d)$,

We consider also the following expression

$$\|f|\mathcal{M}_{r,\Phi}(\mathbb{R}^d)\|^* := \inf\left\{\lambda > 0 : \sup_{Q\in\mathcal{Q}} |Q|^{\frac{1}{r}-1} \int_Q \Phi\Big(\frac{|f(x)|}{\lambda}\Big)\,\mathrm{d}x \leq 1\right\}.$$

Since $\sup_Q \inf_\lambda \leq \inf_\lambda \sup_Q$ we have $\|f|\mathcal{M}_{r,\Phi}(\mathbb{R}^d)\| \leq \|f|\mathcal{M}_{r,\Phi}(\mathbb{R}^d)\|^*$. In the next lemma we show that if the Young function Φ is of exponential type, then both expressions are equivalent and the space $\mathcal{M}_{r,\Phi}(\mathbb{R}^d)$ can be characterised by extrapolation.

Lemma 4.4. *Let $0 < p \leq u < \infty$, $0 < q < \infty$, and $\Phi_{p,q}(t) := t^p \exp(t^q)$. Then f belongs to the Orlicz-Morrey space $\mathcal{M}_{\frac{u}{p},\Phi_{p,q}}(\mathbb{R}^d)$ if, and only if,*

$$\sup_{j\geq 1} j^{-1/q}\,\|f|\mathcal{M}_{v(j),p(j)}(\mathbb{R}^d)\| < \infty,$$

where $p(j) = p + jq$ and $v(j) = \frac{u}{p}p(j)$. Moreover,

$$\|f|\mathcal{M}_{\frac{u}{p},\Phi_{p,q}}(\mathbb{R}^d)\| \sim \|f|\mathcal{M}_{\frac{u}{p},\Phi_{p,q}}(\mathbb{R}^d)\|^* \sim \sup_{j\geq 1} j^{-1/q}\,\|f|\mathcal{M}_{v(j),p(j)}(\mathbb{R}^d)\|.$$

Proof: *Step 1.* It is sufficient to prove that there are constants $c, C > 0$ such that for any measurable function f we have

$$c\|f|\mathcal{M}_{\frac{u}{p},\Phi_{p,q}}(\mathbb{R}^d)\|^* \leq \sup_{j\geq 1} j^{-1/q}\,\|f|\mathcal{M}_{v(j),p(j)}(\mathbb{R}^d)\| \leq C\|f|\mathcal{M}_{\frac{u}{p},\Phi_{p,q}}(\mathbb{R}^d)\|. \tag{4.10}$$

Consider a dyadic cube Q and $\lambda > 0$. Using the Taylor expansion of $\Phi_{p,q}$ and the Stirling's formula, we have

$$\begin{aligned}\int_Q \Phi_{p,q}\Big(\frac{|f(x)|}{\lambda}\Big)\,\mathrm{d}x &= \sum_{j=0}^{\infty} \frac{1}{j!} \int_Q \frac{|f(x)|^{p+jq}}{\lambda^{p+jq}}\,\mathrm{d}x \\ &= \sum_{j=0}^{\infty} j^{-j} e^j (2\pi j)^{-1/2} \lambda^{-(p+jq)} \int_Q |f(x)|^{p+jq}\,\mathrm{d}x.\end{aligned}$$

Let $\kappa \in \mathbb{R}$ (be at our disposal) and let

$$\lambda_j := (2\pi)^{\frac{1}{2(p+qj)}} e^{-\frac{j}{p+qj}} j^{(\kappa - \frac{p}{q} + \frac{1}{2})/(p+qj)}, \quad j \in \mathbb{N}.$$

It can be easily seen that the sequence $(\lambda_j)_j$ converges to $e^{-1/q}$, thus there are positive constants c_0, c_1 such that $0 < c_0 < \lambda_j < c_1 < \infty$ for any $j \in \mathbb{N}$.

with obvious changes if $q=\infty$; similarly,

$$\|f|\mathcal{M}_{v,r}(\mathbb{R}^d)\| \le C\sum_{k=0}^{\infty} 2^{kd(\frac{1}{u}-\frac{1}{v})}\|\mathcal{F}^{-1}\varphi_k\mathcal{F}f|\mathcal{M}_{u,p}(\mathbb{R}^d)\| \le C\|f|\mathcal{N}^{\frac{d}{u}}_{u,p,q}(\mathbb{R}^d)\|$$

if $1\le r$ and $0<q\le 1$.

Now let $p<r<1$ and $q=\infty$. We have

$$\begin{aligned}\|f|\mathcal{M}_{v,r}(\mathbb{R}^d)\|^r &\le \sum_{k=0}^{\infty}\|\mathcal{F}^{-1}\varphi_k\mathcal{F}f|\mathcal{M}_{v,r}(\mathbb{R}^d)\|^r\\ &\le C\sum_{k=0}^{\infty}2^{kdr(\frac{1}{u}-\frac{1}{v})}\|\mathcal{F}^{-1}\varphi_k\mathcal{F}f|\mathcal{M}_{u,p}(\mathbb{R}^d)\|^r\\ &\le C\Big(\sup_{k\in\mathbb{N}_0}2^{k\frac{d}{u}}\|\mathcal{F}^{-1}\varphi_k\mathcal{F}f|\mathcal{M}_{u,p}(\mathbb{R}^d)\|\Big)^r\ \sum_{k=0}^{\infty}2^{-kd\frac{r}{v}}.\end{aligned}$$

But

$$\Big(\sum_{k=0}^{\infty}2^{-kd\frac{r}{v}}\Big)^{\frac{1}{r}} \le C_{u,p}<\infty,$$

so

$$\|f|\mathcal{M}_{v,r}(\mathbb{R}^d)\| \le C\|f|\mathcal{N}^{\frac{d}{u}}_{u,p,\infty}(\mathbb{R}^d)\|.$$

This concludes the argument. ∎

4.1. Embeddings in Orlicz-Morrey spaces. The Orlicz-Morrey spaces considered below were introduced by Nakai [13]. They are a generalisation of both Morrey and Orlicz spaces.

Definition 4.3. Let $\Phi:[0,\infty)\to[0,\infty)$ be a Young function, i.e., a continuous convex function with $\Phi(0)=0$ and $\lim_{t\to\infty}\Phi(t)=\infty$. For $1\le r<\infty$ and a cube Q we put

$$\|f\|_{(r,\Phi);Q} := \inf\left\{\lambda>0 : |Q|^{\frac{1}{r}-1}\int_Q \Phi\Big(\frac{|f(x)|}{\lambda}\Big)\,\mathrm{d}x\le 1\right\}.$$

The Orlicz-Morrey space $\mathcal{M}_{r,\Phi}(\mathbb{R}^d)$ is the set of all measurable functions f such that

$$\|f|\mathcal{M}_{r,\Phi}(\mathbb{R}^d)\| := \sup_{Q\in\mathcal{Q}}\|f\|_{(r,\Phi);Q}<\infty.$$

(ii) $F^{s,\tau}_{p,q}(\mathbb{R}^d)\hookrightarrow\mathcal{M}_{v,\max(p,1)}(\mathbb{R}^d)$, where $v=\frac{p}{(1-\tau p)\min(p,1)}$,

(iii) either $p\ge 1$ and $q\le 2$, or $0<p<1$.

The above statement improves Theorem 3.8 in [3] and corrects some misprint concerning non-embeddings of $F^{s,\tau}_{p,q}(\mathbb{R}^d)$ spaces stated in formula (3.31) there.

Moreover we have the following counterpart of Corollary 3.5. If

$$0\le\tau<\frac{1}{p},\quad \frac{p}{1-p\tau}<v<\infty,\quad s=d\left(\frac{1}{p}-\tau\right)-\frac{d}{v},\quad 1\le q\le v, \tag{3.11}$$

then

$$F^{s,\tau}_{p,\infty}(\mathbb{R}^d)\hookrightarrow\mathcal{M}_{v,q}(\mathbb{R}^d)\quad\Longleftrightarrow\quad q\le v(1-p\tau). \tag{3.12}$$

This is a direct consequence of (3.8) and Corollary 3.5. In case of spaces $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ we have a partial counterpart only at the moment: assume (3.11) and $q\le p$. Then

$$B^{s,\tau}_{p,q}(\mathbb{R}^d)\hookrightarrow\mathcal{M}_{v,q}(\mathbb{R}^d), \tag{3.13}$$

as by assumption, $q\le p<v(1-p\tau)$; thus (3.12) together with (3.10) conclude the argument. In case of $q>p$ the situation is not yet complete: while the embedding (3.9) together with Corollary 3.5 always lead to $q\le v(1-p\tau)$ whenever $B^{s,\tau}_{p,q}(\mathbb{R}^d)\hookrightarrow\mathcal{M}_{v,q}(\mathbb{R}^d)$, the sufficiency is not clear in all cases: assume, in addition to (3.11) that $p<q\le\frac{v}{\tau v+1}$. Then using some Franke-Jawerth embedding, cf. [36, Thm. 3.10],

$$B^{s,\tau}_{p,q}(\mathbb{R}^d)\hookrightarrow F^{\sigma,\tau}_{q,\infty}(\mathbb{R}^d),\quad \sigma=s-\frac{d}{p}+\frac{d}{q}<s,$$

and, by (3.12) again,

$$F^{\sigma,\tau}_{q,\infty}(\mathbb{R}^d)\hookrightarrow\mathcal{M}_{v,q}(\mathbb{R}^d)\quad\text{if}\quad q\le v(1-q\tau),$$

which is satisfied by our additional assumption on q, we get (3.13).

4. Embeddings with smoothness $\frac{d}{u}$

In this section we are interested in embeddings of spaces with smoothness $s=\frac{d}{u}$. This is once more the borderline smoothness since if the smoothness is strictly bigger than $\frac{d}{u}$, then the space consists of bounded functions. We describe the properties of the functions belonging to the spaces with smoothness $s=\frac{d}{u}$ in terms of exponential Orlicz-Morrey spaces and some generalised Morrey spaces.

Our approach is based on the extrapolation argument that in the case of Besov and Triebel-Lizorkin spaces was elaborated by Triebel in [29]. We

follow the general idea of his work. The extrapolation inequalities we need are formulated in the following lemma.

Lemma 4.1. *Let $0 < p \le u < \infty$, $0 < q \le \infty$, $p < r < \infty$ and $v = \frac{ur}{p}$.*

(i) *If $r \ge 1$, then there is a constant $C > 0$ depending on d, p, u and q but independent of v, r such that for any $f \in \mathcal{N}^{\frac{d}{u}}_{u,p,q}(\mathbb{R}^d)$ the following inequalities hold*

$$\|f|\mathcal{M}_{v,r}(\mathbb{R}^d)\| \le Cv^{1-\frac{1}{q}}\|f|\mathcal{N}^{\frac{d}{u}}_{u,p,q}(\mathbb{R}^d)\| \qquad \text{if} \quad q \ge 1, \tag{4.1}$$

$$\|f|\mathcal{M}_{v,r}(\mathbb{R}^d)\| \le C\|f|\mathcal{N}^{\frac{d}{u}}_{u,p,q}(\mathbb{R}^d)\| \qquad \text{if} \quad q \le 1. \tag{4.2}$$

(ii) *If $p < r < 1$, then there is a constant $C > 0$ depending on d, p, u but independent of v and r such that for any $f \in \mathcal{N}^{\frac{d}{u}}_{u,p,\infty}(\mathbb{R}^d)$ the following inequality holds*

$$\|f|\mathcal{M}_{v,r}(\mathbb{R}^d)\| \le C\|f|\mathcal{N}^{\frac{d}{u}}_{u,p,\infty}(\mathbb{R}^d)\|. \tag{4.3}$$

Remark 4.2. The constant C is independence of v in that sense that it depends on u and p but takes the same value as far as the the quotient $\frac{u}{p}$ is constant.

Proof: First we prove that

$$|\psi(x)| \le C\|\psi|\mathcal{M}_{u,p}(\mathbb{R}^d)\| \tag{4.4}$$

if $\operatorname{supp}\mathcal{F}\psi \subset B(0,2)$ and $0 < p \le u < \infty$. The proof is standard and based on the Plancherel-Polya-Nikol'skii inequality

$$\sup_{y\in\mathbb{R}^d} \frac{|\psi(x-y)|}{(1+|y|)^{d/\delta}} \le C\Big(M(|\psi|^\delta)(x)\Big)^{\frac{1}{\delta}}, \qquad x \in \mathbb{R}^d,\ \delta > 0, \tag{4.5}$$

cf. [30, Theorem 1.3.1]. Here M stands for the Hardy-Littlewood maximal operator, as usual.

We repeat the argument for completeness, cf. also [15]. One can easily prove that if $|x-y| \le 2$, then

$$|\psi(x)| \le c\sup_{z\in\mathbb{R}^d} \frac{|\psi(z)|}{(1+|z-y|)^{d/\delta}} \le C\Big(M(|\psi|^\delta)(y)\Big)^{\frac{1}{\delta}}. \tag{4.6}$$

So, if $\delta < p$, then the boundedness of the maximal function in Morrey spaces gives us

$$\begin{aligned}
|\psi(x)| &\le c\bigg(|B(x,2)|^{-1}\int_{B(x,2)}\Big(M(|\psi|^\delta)(y)\Big)^{\frac{p}{\delta}}\,\mathrm{d}y\bigg)^{1/p}\\
&\le C\,\Big\|\Big(M(|\psi|^\delta)\Big)^{\frac{1}{\delta}}|\mathcal{M}_{u,p}(\mathbb{R}^d)\Big\| = C\,\big\|M(|\psi|^\delta)|\mathcal{M}_{u/\delta,p/\delta}(\mathbb{R}^d)\big\|^{\frac{1}{\delta}}\\
&\le C\,\big\||\psi|^\delta|\mathcal{M}_{u/\delta,p/\delta}(\mathbb{R}^d)\big\|^{\frac{1}{\delta}} = C\|\psi|\mathcal{M}_{u,p}(\mathbb{R}^d)\|, \qquad x \in \mathbb{R}^d.
\end{aligned}$$

Thus, if $p < r$, then for any cube Q we have

$$\Big(\int_Q |\psi(x)|^r\,\mathrm{d}x\Big)^{\frac{1}{r}} \le \sup_{x\in Q}|\psi(x)|^{1-\frac{p}{r}}\Big(\int_Q|\psi(x)|^p\,\mathrm{d}x\Big)^{\frac{1}{r}} \le C|Q|^{\frac{1}{r}-\frac{p}{ru}}\|\psi|\mathcal{M}_{u,p}(\mathbb{R}^d)\|.$$

So

$$\|\psi|\mathcal{M}_{v,r}(\mathbb{R}^d)\| \le C\|\psi|\mathcal{M}_{u,p}(\mathbb{R}^d)\|. \tag{4.7}$$

The last inequality implies

$$\|\mathcal{F}^{-1}\big(D_{2^j}\varphi_j\mathcal{F}f\big)|\mathcal{M}_{v,r}(\mathbb{R}^d)\| \le C\|\mathcal{F}^{-1}\big(D_{2^j}(\varphi_j\mathcal{F}f)\big)|\mathcal{M}_{u,p}(\mathbb{R}^d)\| \tag{4.8}$$

since $\operatorname{supp} D_{2^j}(\varphi_j\mathcal{F}f) \subset B(0,2)$, where $D_\delta g(x) = g(\delta x)$. Now the formula for Fourier dilations and the relation between dilations and the Morrey norms give us

$$\|\mathcal{F}^{-1}\big(\varphi_j\mathcal{F}f\big)|\mathcal{M}_{v,r}(\mathbb{R}^d)\| \le C2^{jd(\frac{1}{u}-\frac{1}{v})}\|\mathcal{F}^{-1}\big(\varphi_j\mathcal{F}f\big)|\mathcal{M}_{u,p}(\mathbb{R}^d)\|. \tag{4.9}$$

In consequence, if $1 \le r$ and $1 < q \le \infty$ we have

$$\begin{aligned}
\|f|\mathcal{M}_{v,r}(\mathbb{R}^d)\| &\le \sum_{k=0}^\infty \|\mathcal{F}^{-1}\varphi_k\mathcal{F}f|\mathcal{M}_{v,r}(\mathbb{R}^d)\|\\
&\le C\sum_{k=0}^\infty 2^{kd(\frac{1}{u}-\frac{1}{v})}\|\mathcal{F}^{-1}\varphi_k\mathcal{F}f|\mathcal{M}_{u,p}(\mathbb{R}^d)\|\\
&\le C\Big(\sum_{k=0}^\infty 2^{-kd\frac{q'}{v}}\Big)^{\frac{1}{q'}}\Big(\sum_{k=0}^\infty 2^{k\frac{d}{u}q}\|\mathcal{F}^{-1}\varphi_k\mathcal{F}f|\mathcal{M}_{u,p}(\mathbb{R}^d)\|^q\Big)^{\frac{1}{q}}\\
&\le Cv^{1-\frac{1}{q}}\|f|\mathcal{N}^{\frac{d}{u}}_{u,p,q}(\mathbb{R}^d)\|,
\end{aligned}$$