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COMPACTLY GENERATED SPACES AND QUASI-SPACES IN TOPOLOGY

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Abstract: The notions of compactness and Hausdorff separation for generalized enriched categories allow us, as classically done for the category Top of topological spaces and continuous functions, to study *compactly generated spaces* and *quasi-spaces* in this setting. Moreover, for a class $\mathcal{C}$ of objects we generalize the notion of *$\mathcal{C}$-generated spaces*, from which we derive, for instance, a general concept of *Alexandroff spaces*. Furthermore, as done for Top, we also study, in our level of generality, the relationship between compactly generated spaces and quasi-spaces.

Keywords: $(\mathbb{T},\mathsf{V})$-categories, compact and Hausdorff space, compactly generated space, Alexandroff space, cartesian closedness, quasi-space.

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Introduction

Amongst the classical approaches to the inconvenience of non-cartesian closedness of Top – restricting ourselves to a subcategory thereof or including it in a supercategory – in this paper we turn our attention to the subcategory of *compactly generated spaces* and the supercategory of *quasi-topological spaces*.

Compactly generated spaces were widely studied in the second half of the past century, having as early references the works of Kelley [Kel55] and Gale [Gal50], who indicates Hurewicz as the first to define the notion. The main motivation for its investigation was the search for a category of spaces convenient for homotopy theory and algebraic topology [Ste67, Vog71]. The full subcategory of Top of compactly generated spaces, that we will denote by Top_c, where $\mathcal{C}$ stands for the class of compact Hausdorff topological spaces, is well known to be cartesian closed and coreflective in Top; it is actually the *coreflective hull* of $\mathcal{C}$ in Top, that is, the smallest coreflective subcategory of Top that contains $\mathcal{C}$, from where we conclude that compactly generated topological spaces coincide with colimits of compact Haudorff topological spaces. This coreflection is also shown in [Mac71]. In order to establish our results, we directly follow the approach of Escardó, Lawson and Simpson [ELS04], which is centered in the concept of a generating class of objects, in this case, the compact Hausdorff ones.

Concerning quasi-topological spaces, as indicated by Spanier [Spa63], they were introduced with the intention of constructing internal function spaces in the resulting category QsTop of quasi-topological spaces and quasi-continuous maps. The latter supercategory of Top, that is convenient for homotopy theory, was suitably used in the results of Booth [Boo73] and Day [Day81]. Although carrying a size *illegitimacy* demonstrated by Herrlich and Rajagopalan [HR83], the category of quasi-topological spaces provided a useful tool in current works as the ones by Dadarlat and Meyer [DM12] and Browne [Bro17] on *E-theory of C^*-algebras*, which share a common topic with a past work of Dubuc and Porta [DP80]; we also refer to Petrakis' PhD thesis [Pet15] that highlights the relation between quasi-topological spaces and Bishop spaces. Moreover, the category QsTop also serves as a paradigm for the results of Escardó and Xu [XE13], and Dubuc and Español [Dub79, DE06] who presented a much more general context of quasi-topologies using the notion of *Grothendieck topologies*.

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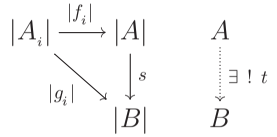
The main goal of the present work is to carry these two concepts – compactly generated spaces and quasi-spaces – from Top to $(\mathbb{T},\mathsf{V})$-Cat. The $(\mathbb{T},\mathsf{V})$-*setting* [CT03, CH03] (see also [HST14]) has been broadly studied and applied to problems in topology [Hof07, CH04a, Hof11, CHR20], for it provides a unified effective way of studying classical categories from Analysis and Topology such as categories of ordered, metric, topological, and approach spaces.

This setting also allows for a generalization of compact and Hausdorff spaces, which is fundamental for our discussion. We recall this generalization in the first section, as well as some essential facts about topological functors, and necessary background of $(\mathbb{T},\mathsf{V})$-spaces and $(\mathbb{T},\mathsf{V})$-continuous maps. In Section 2 we develop the concept of $\mathcal{C}$-generated $(\mathbb{T},\mathsf{V})$-spaces generalizing the one established for Top in [ELS04]; in particular, we get the notions of *compactly generated* $(\mathbb{T},\mathsf{V})$-*spaces* and *Alexandroff* $(\mathbb{T},\mathsf{V})$-*spaces*. We finish with Section 4 presenting the quasi-spaces in our general context and studying a relationship between the two main notions, extending the case of Top due to Day [Day68], whose work we follow closely. Examples are presented throughout the paper.

1. Prerequisites

1.1. A comment on topological functors. Concerning topological functors, we recall some facts that can be found in the literature. Let A and X be categories and $|\text{-}|\colon \mathsf{A}\to\mathsf{X}$ be a functor. A sink $(f_i\colon A_i\to A)_{i\in I}$ of morphisms of A is $|\text{-}|$-*final* if, for every sink $(g_i\colon A_i\to B)_{i\in I}$ in A and every morphism $s\colon |A|\to|B|$ of X with $s\cdot|f_i|=|g_i|$, there exists a unique $t\colon A\to B$ in A with $|t|=s$ and, for all $i\in I$, $t\cdot f_i=g_i$.



The dual concept is that of a $|\text{-}|$-*initial source*. The functor $|\text{-}|$ is *topological* if every $|\text{-}|$-*structured sink*, that is, a sink of the form $(f_i\colon |A_i|\to X)_{i\in I}$ in X, with $A_i\in\mathsf{A}$, admits a $|\text{-}|$-*final lifting*, or, equivalently, if every $|\text{-}|$-*structured source* $(f_i\colon X\to|A_i|)_{i\in I}$ in X admits a $|\text{-}|$-*initial lifting* [AHS90, Theorem 21.9].

Observe that, as usual, we do not make any assumption of smallness on the entity I of indexes. However, in our case, we will be dealing with a *fibre-small* (the fibre of each object in the codomain is a set) forgetful functor, hence we will employ the fact that each $|\text{-}|$-final lifting of a sink $(g_i\colon |A_i| \to X)_{i\in I}$ is actually the $|\text{-}|$-final lifting of a sink $(g_j\colon |A_j| \to X)_{j\in J}$, with J a set contained in I [AHS90, Proposition 21.34].

1.2. $(\mathbb{T},\mathsf{V})$-spaces and $(\mathbb{T},\mathsf{V})$-continuous maps. We introduce briefly the $(\mathbb{T},\mathsf{V})$-Cat setting for our purposes, and refer the reader to the references [CH03, CT03, HST14] for details.

Let $\mathsf{V} = (\mathsf{V},\otimes,k)$ be a unital commutative quantale (see, for instance, [HST14, II-1.10]), which is also a *Heyting algebra*, so that the operation infimum $\wedge$ has a right adjoint. Consider the order-enriched category V-Rel: objects are sets and morphisms are V-relations (or V-matrices) $r\colon X \nrightarrow Y$, which are V-valued maps $r\colon X\times Y \to \mathsf{V}$; the *relational composition* of $r\colon X\nrightarrow Y$ and $s\colon Y\nrightarrow Z$ is given by: for each $(x,z)\in X\times Z$, $s\cdot r(x,z) = \bigvee_{y\in Y}(r(x,y)\otimes s(y,z))$, and the order between V-relations is defined *componentwise*. There exists an *involution* given by transposition: for each $r\colon X\nrightarrow Y$, $r^\circ\colon Y\nrightarrow X$ is given by, for each $(y,x)\in Y\times X$, $r^\circ(y,x) = r(x,y)$. Denoting the bottom element of the complete lattice V by $\bot$, each map $f\colon X\to Y$ can be seen as a V-relation $f\colon X\nrightarrow Y$: for $(x,y)\in X\times Y$,

$$f(x,y) = \begin{cases} k, & \text{if } f(x) = y \\ \bot, & \text{otherwise.}\end{cases}$$

Let $\mathbb{T} = (T,m,e)\colon \mathsf{Set}\to\mathsf{Set}$ be a monad satisfying the *Beck-Chevalley condition*, (BC) for short, that is, T preserves weak pullbacks and the naturality squares of m are weak pullbacks [CHJ14]. We fix a lax extension of $\mathbb{T}$ to V-Rel, again denoted by $\mathbb{T}$, so that $T\colon \mathsf{V}\text{-}\mathsf{Rel}\to\mathsf{V}\text{-}\mathsf{Rel}$ is a lax functor and the natural transformations m and e become oplax: for each V-relation $r\colon X\nrightarrow Y$,

$$\begin{array}{ccccc} X & \xrightarrow{e_X} & TX & \xleftarrow{m_X} & T^2X \\ {\scriptstyle r}\downarrow & \leq & \downarrow{\scriptstyle \hat{T}r} & \geq & \downarrow{\scriptstyle \hat{T}^2r} \\ Y & \xrightarrow[e_Y]{} & TY. & \xleftarrow[m_Y]{} & T^2Y \end{array}$$

We assume that this lax extension is *flat*, that is, for each set X, $T1_X = 1_{TX}$, and, moreover, that it commutes with involution, i.e., for each $r\colon X\nrightarrow Y$

map $f\colon (X,(Q(C,X))_{C\in\mathcal{C}}) \to (Y,(Q_b(C,Y))_{C\in\mathcal{C}})$ induces a continuous $f\colon (X,a_Q)\to(Y,b)$, hence a quasi-continuous $f\colon (X,(Q_{a_Q}(C,X))_{C\in\mathcal{C}}) \to (Y,(Q_b(C,Y))_{C\in\mathcal{C}})$.

$$\begin{array}{ccc} (X,Q(C,X)) & \xrightarrow{1_X} & (X,Q_{a_Q}(C,X)) \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle f} \\ & & (Y,Q_b(C,Y)) \end{array}$$

■

We can draw the following explanatory diagram, where $\mathcal{C}$ stands for the class of compact Hausdorff $(\mathbb{T},\mathsf{V})$-spaces, as fixed in the beginning of Subsection 4.1.

$$\begin{array}{ccc} (\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}_{\mathcal{C}} & \overset{\top}{\hookrightarrow} & (\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat} \\ \cong\downarrow & & \downarrow \text{non-full} \\ \mathcal{C}\text{-}(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat} & \underset{\top}{\hookrightarrow} & \mathsf{Qs}(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat} \end{array}$$

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in general is not full. However, the situation is different when restricting ourselves to compactly generated spaces.

Let (X,a) be a compactly generated space; by definition, a map $f\colon (X,a)\to(Y,b)$, for (Y,b) in $(\mathbb{T},\mathsf{V})$-Cat, is continuous if, and only if, for all continuous maps $\alpha\colon C\to Y$, with $C\in\mathcal{C}$, $f\cdot\alpha\colon C\to Y$ is continuous. Considering the quasi-spaces associated with (X,a) and (Y,b), then $f\colon X\to Y$ is continuous if, and only if, $f\colon X\to Y$ is quasi-continuous, that is,

$$(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}(X,Y)=\mathsf{Qs}(X,Y). \tag{5.i}$$

Conversely, if (X,a) is a $(\mathbb{T},\mathsf{V})$-space such that, for each $(\mathbb{T},\mathsf{V})$-space (Y,b), its associated quasi-space satisfies (5.i), then, by the same reasoning, (X,a) is compactly generated.

We identify the elements of $(\mathbb{T},\mathsf{V})$-Cat_c with their associated quasi-spaces, and denote the resulting subcategory of $\mathsf{Qs}(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$ by $\mathcal{C}$-$(\mathbb{T},\mathsf{V})$-Cat.

Proposition $\mathcal{C}$-$(\mathbb{T},\mathsf{V})$-Cat *is fully reflective in* $\mathsf{Qs}(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$.

Proof. For $X,Y\in\mathcal{C}$-$(\mathbb{T},\mathsf{V})$-Cat, by (5.i),

$$\mathcal{C}\text{-}(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}(X,Y)=(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}(X,Y)=\mathsf{Qs}(X,Y),$$

hence it is a full subcategory.

For each $(X,(Q(C,X))_{C\in\mathcal{C}})$ in $\mathsf{Qs}(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$, define (X,a_Q), with a_Q the $(\mathbb{T},\mathsf{V})$-structure from the $|\text{-}|$-final lifting in $(\mathbb{T},\mathsf{V})$-Cat of the sink $(\alpha\colon(C,c)\to X)_{C\in\mathcal{C},\alpha\in Q(C,X)}$. Consider the quasi-space $(X,(Q_{a_Q}(C,X))_{C\in\mathcal{C}})$ associated with (X,a_Q).

Let (Y,b) be a $(\mathbb{T},\mathsf{V})$-space and $f\colon X\to Y$ be a map. If $f\colon(X,a_Q)\to(Y,b)$ is continuous, then

$$f\colon(X,(Q_{a_Q}(C,X))_{C\in\mathcal{C}})\to(Y,(Q_b(C,Y))_{C\in\mathcal{C}})$$

is quasi-continuous. Conversely, if f is quasi-continuous with respect to the associated quasi-structures, then, for each $C\in\mathcal{C}$ and each continuous map $\alpha\colon C\to(X,a_Q)$, the composite $f\cdot\alpha\colon C\to(Y,b)$ is continuous. Hence $f\colon(X,a_Q)\to(Y,b)$ is continuous, by definition of a_Q. Therefore, for every $(\mathbb{T},\mathsf{V})$-space (Y,b), (X,a_Q) satisfies (5.i), that is, $(X,(Q_{a_Q}(C,X))_{C\in\mathcal{C}})$ belongs to $\mathcal{C}$-$(\mathbb{T},\mathsf{V})$-Cat.

The identity map $1_X\colon(X,(Q(C,X))_{C\in\mathcal{C}})\to(X,(Q_{a_Q}(C,X))_{C\in\mathcal{C}})$ is quasi-continuous, since each $\alpha\in Q(C,X)$, for $C\in\mathcal{C}$, is a continuous map $\alpha\colon C\to(X,a_Q)$. For each $(Y,(Q_b(C,Y))_{C\in\mathcal{C}})$ in $\mathcal{C}$-$(\mathbb{T},\mathsf{V})$-Cat, each quasi-continuous

in V-Rel, $T(r^\circ)=(Tr)^\circ$. Then we have a lax monad on V-Rel in the sense of [CH04b] and $(\mathbb{T},\mathsf{V})$-Cat is defined as its category of Eilenberg-Moore lax algebras. Hence objects are pairs (X,a), where X is a set and $a\colon TX\nrightarrow X$ is a *reflexive* and *transitive* V-relation, so that the diagram

$$\begin{array}{ccccc} X & \xrightarrow{e_X} & TX & \xleftarrow{Ta} & T^2X \\ & \searrow^{1_X} & \leq\ \downarrow a & \leq & \downarrow m_X \\ & & X & \xleftarrow[a]{} & TX \end{array}$$

is lax commutative; such pairs are called $(\mathbb{T},\mathsf{V})$-*categories* or $(\mathbb{T},\mathsf{V})$-*spaces*; and a morphism from (X,a) to (Y,b) is a map $f\colon X\to Y$ such that the square

$$\begin{array}{ccc} TX & \xrightarrow{Tf} & TY \\ {\scriptstyle a}\downarrow & \leq & \downarrow{\scriptstyle b} \\ X & \xrightarrow[f]{} & Y \end{array}$$

is lax commutative; such a map is called a $(\mathbb{T},\mathsf{V})$-*functor* or a $(\mathbb{T},\mathsf{V})$-*continuous* map. When this diagram is strictly commutative, what is equivalent to $a=f^\circ\cdot b\cdot Tf$, f is said to be *fully faithful*. The V-relation $a\colon TX\nrightarrow X$ of the $(\mathbb{T},\mathsf{V})$-space (X,a) is referred usually as the $(\mathbb{T},\mathsf{V})$-*structure* of X. For a $(\mathbb{T},\mathsf{V})$-space (X,a), each subset $A\subseteq X$ can be endowed with a *subspace* $(\mathbb{T},\mathsf{V})$-*structure* $a_A=i_A^\circ\cdot a\cdot Ti_A\colon TA\nrightarrow A$, where $i_A\colon A\hookrightarrow X$ is the inclusion map, which becomes a fully faithful map $i_A\colon(A,a_A)\to(X,a)$.

The forgetful functor $|\text{-}|\colon(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}\to\mathsf{Set}$ is topological [CH03, CT03] and fibre-small: for each set X, a $(\mathbb{T},\mathsf{V})$-structure a on X is an element of V-$\mathsf{Rel}(TX,X)=\mathsf{Set}(TX\times X,\mathsf{V})$.

We also need to assume the fairly restrictive condition that each constant map between $(\mathbb{T},\mathsf{V})$-spaces is continuous, what implies, as we show below, that, in particular, the quantale V is *integral*, i.e., $k=\top$ is the top element of V.

Lemma *The following statements are equivalent.*
(i) Any constant map $y_0\colon(X,a)\to(Y,b)$ *between* $(\mathbb{T},\mathsf{V})$*-spaces is continuous.*
(ii) If $(\mathsf{1},c)$ *is a* $(\mathbb{T},\mathsf{V})$*-space, where* $\mathsf{1}$ *denotes a singular set* $\{*\}$*, then, for each* $\mathfrak{z}\in T\mathsf{1}$*,* $c(\mathfrak{z},*)=\top$.
(iii) $k=\top$ *and* $T1=1$.

Proof. (i)⇔(ii) Let c be a $(\mathbb{T},\mathsf{V})$-structure on $\mathbf{1}$. The identity map $1_{_1}\colon (1,\top)\to(\mathbf{1},c)$ is constant, hence it is continuous, so $\top\leq c$. Conversely, for the constant map $y_{_0}\colon (X,a)\to(Y,b)$, consider the factorization

$$(X,a)\xrightarrow{\ y_{_0}\ }(\mathbf{1},b_{_1})\xrightarrow{\ i_{_1}\ }(Y,b),$$

where $\mathbf{1}=\{y_{_0}\}\subseteq Y$ and $b_{_1}$ is the subspace $(\mathbb{T},\mathsf{V})$-structure. By hypothesis, $b_{_1}$ is constantly equal to $\top$, what implies that $y_{_0}\colon (X,a)\to(\mathbf{1},b_{_1})$ is continuous, whence the composite $y_{_0}\colon (X,a)\to(Y,b)$ is continuous.
(ii)⇔(iii) Since T is flat, the *discrete* $(\mathbb{T},\mathsf{V})$*-structure* on $\mathbf{1}$ is given by $e_{_1}^{\circ}$, so we have the *discrete* $(\mathbb{T},\mathsf{V})$*-space* $(\mathbf{1},e_{_1}^{\circ})$ [HST14, III-3.2]. Assuming (ii), for each $\mathfrak{z}\in T\mathbf{1}$, $e_{_1}^{\circ}(\mathfrak{z},*)=\top$. In particular, for $\mathfrak{z}=e_{_1}(*)$,

$$k=e_{_1}^{\circ}(e_{_1}(*),*)=\top.$$

Then, for each $\mathfrak{z}\in T\mathbf{1}$,

$$e_{_1}^{\circ}(\mathfrak{z},*)=\top=k\iff\mathfrak{z}=e_{_1}(*),$$

whence $T\mathbf{1}=\mathbf{1}$. On the other hand, for V integral and $T\mathbf{1}=1$, one readily checks condition (ii). ∎

Under those conditions, $(\mathbb{T},\mathsf{V})$-Cat is a topological category in the classical sense of [Her74], that is, there exist $|\text{-}|$-initial $(\mathbb{T},\mathsf{V})$-structures, with the forgetful functor $|\text{-}|\colon(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}\to\mathsf{Set}$ being fibre-small, and on a singleton set there exists exactly one $(\mathbb{T},\mathsf{V})$-structure. To give examples we will consider V as the following integral and *completely distributive* quantales (see, for instance, [HST14, II-1.11]): $\mathsf{2}=(\{\bot,\top\},\wedge,\top)$, $\mathsf{P}_{_+}=([0,\infty]^{\mathrm{op}},+,0)$, $\mathsf{P}_{_{\max}}=([0,\infty]^{\mathrm{op}},\max,0)$, and $[0,1]_{_\odot}=([0,1],\odot,1)$, where $\odot$ is the *Lukasiewicz tensor* given by $u\odot v=\max(0,u+v-1)$, for each $u,v\in[0,1]$; in addition to $\mathbb{T}$ as:
• the identity monad $\mathbb{I}=(\mathrm{Id},\mathbf{1},\mathbf{1})$ on Set extended to the identity monad on V-Rel, and
• the ultrafilter monad $\mathbb{U}$ with the Barr extension to V-Rel [HST14, IV-2.4.5].

In the categories of the following table, all constant maps are $(\mathbb{T},\mathsf{V})$-continuous.

$\mathbb{T}$ \ V	$\mathsf{2}$	$\mathsf{P}_{_+}$	$\mathsf{P}_{_{\max}}$	$[0,1]_{_\odot}$
$\mathbb{I}$	Ord	Met	UltMet	$\mathsf{B}_{_1}$Met
$\mathbb{U}$	Top	App	NA-App	$(\mathbb{U},[0,1]_{_\odot})$-Cat

(1.i)

Then $\gamma\cdot h\in Q(B,Z)$ and $\langle\gamma\cdot h,\alpha\rangle\in Q(B,Z\times X)$, whence $f\cdot\langle\gamma\cdot h,\alpha\rangle\in Q(B,Y)$. The result follows from the equalities: for each $b\in B$,

$$\begin{aligned}\mathrm{ev}\cdot\left\langle\overline{f}\cdot\gamma\cdot h,\alpha\right\rangle(b) &= \overline{f}\cdot\gamma\cdot h(b)(\alpha(b))\\ &= \overline{f}(\gamma\cdot h(b))(\alpha(b))\\ &= f(\gamma\cdot h(b),\alpha(b))\\ &= f\cdot\langle\gamma\cdot h,\alpha\rangle(b).\end{aligned}$$

We have proved the following:

Theorem $\mathsf{Qs}(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$ *is cartesian closed.*

Examples (1) Let us begin with V-Cat. C is a compact Hausdorff V-space if, and only if, C is discrete, that is, of the form $(C,1_{_C})$; this way, a quasi-V-space is a set X with, for each $(C,1_{_C})\in\mathcal{C}$, a set of maps $Q(C,X)$ satisfying conditions (QS1), (QS2), and (QS3). Moreover, a quasi-space $(X,(Q_{_a}(C,X))_{C\in\mathcal{C}})$ associated with a V-space (X,a) necessarily satisfies

$$Q_{_a}(C,X)=\{\text{maps }\alpha\colon C\to X\}=\mathsf{Set}(C,X).$$

Therefore, associated quasi-spaces coincide with indiscrete quasi-spaces (items **(III)** and **(V)** of Subsection 4.1). In particular, this happens for the quantales $\mathsf{2}$, $\mathsf{P}_{_+}$, $\mathsf{P}_{_{\max}}$, and $[0,1]_{_\odot}$. The respective categories of quasi-spaces coincide:

$$\mathsf{QsOrd}=\mathsf{QsMet}=\mathsf{QsUltMet}=\mathsf{QsB}_{_1}\mathsf{Met}.$$

(2) For Top we recover Spanier's category of quasi-topological spaces [Spa63]. For App, NA-App, and $(\mathbb{U},[0,1]_{_\odot})$-$\mathsf{Cat}$, similar to the previous item, our observation in Section 1 that

$$(\mathbb{U},[0,1]_{_\odot})\text{-}\mathsf{Cat}_{\mathrm{CompHaus}}\cong\mathsf{NA}\text{-}\mathsf{App}_{\mathrm{CompHaus}}\cong\mathsf{App}_{\mathrm{CompHaus}}\cong\mathsf{Top}_{\mathrm{CompHaus}}\cong\mathsf{Set}^{\mathbb{U}}$$

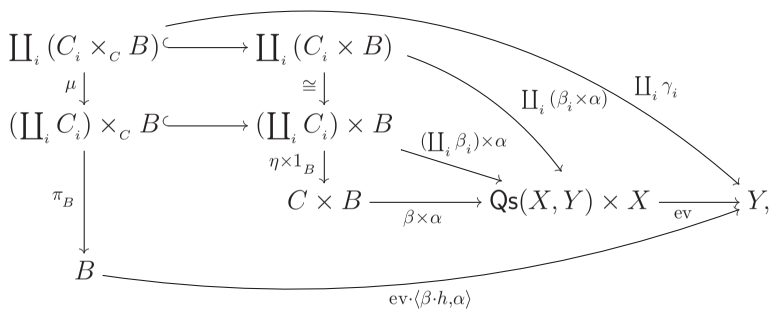
allow us to conclude that

$$\mathsf{Qs}(\mathbb{U},[0,1]_{_\odot})\text{-}\mathsf{Cat}=\mathsf{QsNA}\text{-}\mathsf{App}=\mathsf{QsApp}=\mathsf{QsTop}.$$

5. Relationship between quasi-spaces and compactly generated spaces

This relationship is studied for Top in [Day68]. We have commented in item **(I)** of Subsection 4.1 that the inclusion

$$(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}\hookrightarrow\mathsf{Qs}(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}$$



where μ is the surjective $(\mathbb{T}, \mathsf{V})$-continuous map

$$\begin{array}{rccc} \mu\colon & \coprod_i (C_i \times_C B) & \longrightarrow & (\coprod_i C_i) \times_C B \\ & ((c_i, b), i) & \longmapsto & ((c_i, i), b). \end{array}$$

We observe that we also use distributivity of $(\mathbb{T}, \mathsf{V})$-Cat. The map $\mathrm{ev} \cdot \langle \beta \cdot h, \alpha \rangle$ is then covered by the family of maps $(\gamma_i)_{i \in I}$, where, for each $(c_i, b) \in C_i \times_C B$,

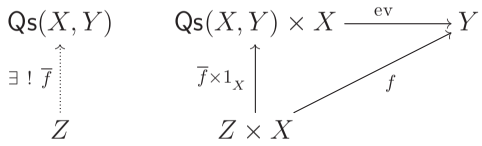
$$\gamma_i(c_i, b) = \beta_i(\eta_i(c_i))(\alpha(b)) = \beta_i(\eta_i \cdot \pi_{C_i}(c_i, b))(\alpha \cdot \pi^i_B(c_i, b)),$$

so $\gamma_i = \mathrm{ev} \cdot \left\langle \beta_i \cdot \eta_i \cdot \pi_{C_i}, \alpha \cdot \pi^i_B \right\rangle$ is admissible, since β_i is admissible, $\eta_i \cdot \pi_{C_i} \colon C_i \times_C B \to C$ is continuous, and α admissible implies $\alpha \cdot \pi^i_B$ admissible.

Choosing $h = 1_C$ in the definition of the quasi-structure, the map

$$\mathrm{ev}\colon \mathsf{Qs}(X,Y) \times X \to Y$$

proves to be quasi-continuous. Furthermore, for each quasi-continuous map $f\colon Z \times X \to Y$, for $Z \in \mathsf{Qs}(\mathbb{T}, \mathsf{V})$-$\mathsf{Cat}$, there exists a unique Set-map $\overline{f}\colon Z \to \mathsf{Qs}(X,Y)$, the transpose of f, such that $\mathrm{ev} \cdot (\overline{f} \times 1_X) = f$. We verify next that $\overline{f}$ is quasi-continuous.



Let $\gamma \in Q(C, Z)$, with $C \in \mathcal{C}$; we wish to prove that $\overline{f} \cdot \gamma \in Q(C, \mathsf{Qs}(X,Y))$. For that, let $h\colon B \to C$ be a continuous map, for $B \in \mathcal{C}$, and $\alpha \in Q(B, X)$.

- Ord is the category of pre-ordered spaces and monotone maps;
- Met is the category of Lawvere's generalized metric spaces and non-expansive maps [Law73];
- UltMet is the full subcategory of Met of ultrametric spaces [HST14, III-Exercise 2.B];
- $\mathsf{B_1Met}$ is the category of *bounded-by-1* metric spaces and non-expansive maps (see [CHR20]);
- Top is the usual category of topological spaces and continuous functions;
- App is the category of Lowen's approach spaces and contractive maps [Low97];
- $\mathsf{NA\text{-}App}$ is the full subcategory of App of non-Archimedean approach spaces [CVO17].

1.3. Injective and exponentiable $(\mathbb{T}, \mathsf{V})$-spaces. Details on the following concepts can be found in [CT03, CH09, Hof11].

For each $(X, a), (Y, b)$ in $(\mathbb{T}, \mathsf{V})$-Cat, there exists an induced (pre-)order on the set $(\mathbb{T}, \mathsf{V})$-$\mathsf{Cat}((X, a), (Y, b))$ of $(\mathbb{T}, \mathsf{V})$-continuous maps from (X, a) to (Y, b) given by, for each $f, g\colon (X, a) \to (Y, b)$,

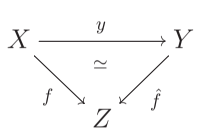
$$f \leq g \iff \forall x \in X,\ k \leq b(e_Y(f(x)), g(x)). \tag{1.ii}$$

Denoting by $f \simeq g$ when $f \leq g$ and $g \leq f$, the $(\mathbb{T}, \mathsf{V})$-space (Y, b) is *separated* if, for every $(\mathbb{T}, \mathsf{V})$-space (X, a) and every $f, g\colon (X, a) \to (Y, b)$, $f \simeq g$ implies $f = g$, that is, for every (X, a), the order (1.ii) is anti-symmetric. Furthermore, the space (Y, b) is separated precisely when the following order on Y is anti-symmetric: for each $y, y' \in Y$,

$$y \leq y' \iff k \leq b(e_Y(y), y'), \tag{1.iii}$$

i.e., when $y \leq y'$ in the order (1.ii) for $y, y'\colon 1 \to Y$ the morphisms induced by y and y', respectively. The full subcategory of $(\mathbb{T}, \mathsf{V})$-Cat of separated spaces, which is denoted by $(\mathbb{T}, \mathsf{V})$-$\mathsf{Cat}_{\mathrm{sep}}$, is closed under mono-sources.

A space (Z, c) is *injective* if, for each fully faithful map $y\colon (X, a) \to (Y, b)$ and $(\mathbb{T}, \mathsf{V})$-continuous map $f\colon (X, a) \to (Z, c)$, there exists a $(\mathbb{T}, \mathsf{V})$-continuous map $\hat{f}\colon (Y, b) \to (Z, c)$, called an *extension of f along y*, such that $\hat{f} \cdot y \simeq f$.



(1.iv)

Observe that when considering separated $(\mathbb{T},\mathsf{V})$-spaces, injectivity assumes its usual notion. In [Hof11], injective $(\mathbb{T},\mathsf{V})$-spaces are characterized as the ones satisfying a *cocompleteness* condition.

As usual, a $(\mathbb{T},\mathsf{V})$-space (X,a) is said to be *exponentiable* in $(\mathbb{T},\mathsf{V})$-Cat if the functor $-\times(X,a)\colon(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}\to(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}$ has a right adjoint. In order to recall conditions under which injective $(\mathbb{T},\mathsf{V})$-spaces are exponentiable, which are established in [CHR20], we restrict ourselves to the case when the extension of T to V-Rel is fully determined by a $\mathbb{T}$-algebra structure $\xi\colon T\mathsf{V}\to\mathsf{V}$, so we are in the setting of *strict topological theories* [Hof07]; such extensions are characterized in [CT14] as the *algebraic extensions*. In particular, the following diagrams are commutative:

$$\begin{array}{ccccc}\mathsf{V} & \xrightarrow{e_{\mathsf{V}}} & T\mathsf{V} & \xleftarrow{T\xi} & T^2\mathsf{V}\\ & {\scriptstyle 1_{\mathsf{V}}}\searrow & \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle m_{\mathsf{V}}}\\ & & \mathsf{V} & \xleftarrow[\xi]{} & T\mathsf{V}\end{array}\qquad\qquad \begin{array}{ccccc}T(\mathsf{V}\times\mathsf{V}) & \xrightarrow{T(\otimes)} & T\mathsf{V} & \xleftarrow{T(k)} & T1\\ {\scriptstyle\langle\xi\cdot T\pi_1,\xi\cdot T\pi_2\rangle}\downarrow & & \downarrow{\scriptstyle\xi} & & \downarrow{\scriptstyle !}\\ \mathsf{V}\times\mathsf{V} & & \xrightarrow[\otimes]{} \mathsf{V} & \xleftarrow[k]{} & 1,\end{array}\qquad (1.v)$$

where π_1 and π_2 are the product projections $\mathsf{V}\times\mathsf{V}\to\mathsf{V}$, and $\langle\xi\cdot T\pi_1,\xi\cdot T\pi_2\rangle$ is the unique map rendering the diagram below commutative.

$$\begin{array}{ccccc} & & T(\mathsf{V}\times\mathsf{V}) & & \\ & {\scriptstyle\xi\cdot T\pi_1}\swarrow & \downarrow{\scriptstyle\langle\xi\cdot T\pi_1,\xi\cdot T\pi_2\rangle} & \searrow{\scriptstyle\xi\cdot T\pi_2} & \\ \mathsf{V} & \xleftarrow[\pi_1]{} & \mathsf{V}\times\mathsf{V} & \xrightarrow[\pi_2]{} & \mathsf{V}\end{array}$$

The extension of T to V-Rel is given by, for each $r\colon X\nrightarrow Y$, $\mathfrak{x}\in TX$, $\mathfrak{y}\in TY$,

$$Tr(\mathfrak{x},\mathfrak{y})=\bigvee\{\xi\cdot T\vec{r}(\mathfrak{w})\mid \mathfrak{w}\in T(X\times Y),\ T\pi_X(\mathfrak{w})=\mathfrak{x},\ T\pi_Y(\mathfrak{w})=\mathfrak{y}\},\tag{1.vi}$$

where π_X and π_Y are the product projections from $X\times Y$ to X and Y, respectively [Hof07, Definition 3.4], and we adopt the notation from [CT14]: $T\vec{r}\colon T(X\times Y)\to T\mathsf{V}$ is the image of the map $\vec{r}\colon X\times Y\to\mathsf{V}$ by the functor T, while Tr continues to have the meaning so far used, that is, it is the image of the V-relation $r\colon X\nrightarrow Y$ by the extension of the functor T to V-Rel.

The initial object is then given by $\emptyset$ endowed with the quasi-structure:

$$Q(C,\emptyset)=\begin{cases}\emptyset,\ \text{if } C\neq\emptyset\\ \{1_\emptyset\},\ \text{otherwise.}\end{cases}$$

As for coequalizers of quasi-continuous maps $f,g\colon X\to Y$, consider in Y the smallest equivalence relation containing the pairs $(f(x),g(x))$, for $x\in X$, and endow $\tilde{Y}=Y/\!\sim$ with the quotient quasi-structure with respect to the projection map $p_Y\colon Y\to\tilde{Y}$.

4.3. $\mathsf{Qs}(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$ is cartesian closed. In general $(\mathbb{T},\mathsf{V})$-spaces and $(\mathbb{T},\mathsf{V})$-continuous maps do not form a cartesian closed category. Hence this property is desirable for a supercategory of $(\mathbb{T},\mathsf{V})$-Cat.

The natural candidate for an exponential of quasi-spaces X and Y is $\mathsf{Qs}(X,Y)$. Consider the evalutation map $\mathrm{ev}\colon\mathsf{Qs}(X,Y)\times X\to Y$, $(f,x)\mapsto f(x)$. First we wish to define a quasi-structure on $\mathsf{Qs}(X,Y)$ such that ev is a quasi-continuous map, that is, for each $\gamma\in Q(C,\mathsf{Qs}(X,Y)\times X)$, with $C\in\mathcal{C}$, $\mathrm{ev}\cdot\gamma\in Q(C,Y)$. Hence, for each $\beta\in Q(C,\mathsf{Qs}(X,Y))$ and $\alpha\in Q(C,X)$, the composite $\mathrm{ev}\cdot\langle\beta,\alpha\rangle$ must belong to $Q(C,Y)$.

Under this intuition and keeping in mind conditions (QS1), (QS2), and (QS3), define, for each $C\in\mathcal{C}$, $\beta\in Q(C,\mathsf{Qs}(X,Y))$ if for each $(\mathbb{T},\mathsf{V})$-continuous map $h\colon B\to C$, for $B\in\mathcal{C}$, and each $\alpha\in Q(B,X)$, the map $\mathrm{ev}\cdot\langle\beta\cdot h,\alpha\rangle : B\to Y$ belongs to $Q(B,Y)$.

The latter data indeed define a quasi-structure and to verify, for instance, (QS3), take a map $\beta\colon C\to\mathsf{Qs}(X,Y)$ covered by a family of admissible maps $(\beta_i)_{i\in I}$ as in (4.i). For a continuous map $h\colon B\to C$, with $B\in\mathcal{C}$, and $\alpha\in Q(B,X)$, form the following pullbacks

$$\begin{array}{ccc}(\coprod_i C_i)\times_C B & \xrightarrow{\pi_B} & B\\ {\scriptstyle\pi_{\coprod_i C_i}}\downarrow & & \downarrow{\scriptstyle h}\\ \coprod_i C_i & \xrightarrow[\eta]{} & C\end{array}\qquad\qquad \begin{array}{ccc}C_i\times_C B & \xrightarrow{\pi_B^i} & B\\ {\scriptstyle\pi_{C_i}}\downarrow & & \downarrow{\scriptstyle h}\\ C_i & \xrightarrow[\eta_i]{} & C,\end{array}$$

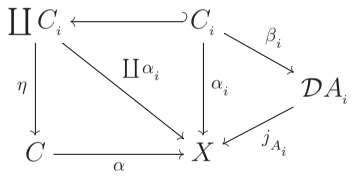
where, for each $i\in I$, $\eta_i=\eta\cdot j_i$, with $j_i\colon C_i\hookrightarrow\coprod_i C_i$ the coproduct inclusion, and consider the commutative diagram

described in the previous Proposition. In particular, the product of a family $((X_i, (Q(C, X_i))_{C\in\mathcal{C}}))_{i\in I}$ of quasi-spaces is given by the set $\prod_{i\in I} X_i$ endowed with the quasi-structure: for each $C \in \mathcal{C}$,

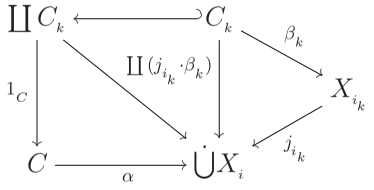
$$\alpha \in Q(C, \prod_{i\in I} X_i) \iff \forall i \in I,\ \pi_i \cdot \alpha \in Q(C, X_i),$$

where the π_i's are product projections. One can see that for the empty family, the product is given by the singleton $\mathsf{1}$ endowed with the quasi-structure: for each $C \in \mathcal{C}$, $Q(C, \mathsf{1}) = \{!_C : C \to \mathsf{1}\}$, which was described in Proposition 4.1. As for equalizers of quasi-continuous maps $f, g : X \to Y$, endow the set $E = \{x \in X \mid f(x) = g(x)\} \subseteq X$ with the subspace quasi-structure.

Colimits. For a diagram $\mathcal{D} : \mathsf{A} \to \mathsf{Qs}(\mathbb{T}, \mathsf{V})\text{-}\mathsf{Cat}$, we form the colimit in Set of $|\text{-}| \cdot \mathcal{D} : \mathsf{A} \to \mathsf{Set}$, denoted by $(j_A : |\mathcal{D}A| \to X)_{A\in \mathrm{Obj}\mathsf{A}}$, and then we take the $|\text{-}|$-final lifting of this sink. The quasi-structure on X is given by: for each $C \in \mathcal{C}$, $\alpha \in Q(C, X)$ if, and only if, α is covered by a family $(\alpha_i)_{i\in I}$ such that each α_i factorizes through a colimit inclusion $j_{A_i} : \mathcal{D}A_i \to X$ and an admissible map $\beta_i \in Q(C_i, \mathcal{D}A_i)$.



In particular cases, we can reduce this quasi-structure (see [Day68]). The coproduct of a family $((X_i, (Q(C, X_i))_{C\in\mathcal{C}}))_{i\in I}$ of quasi-spaces is given by the disjoint union $\dot{\bigcup} X_i$ endowed with the quasi-structure: for $C \in \mathcal{C}$, $\alpha \in Q(C, \dot{\bigcup} X_i)$ if, and only if, α is covered by a family $(j_{i_k} \cdot \beta_k)_{k\in K}$, K a finite set, with $\eta = 1_C$, $j_{i_k} : X_{i_k} \to \dot{\bigcup} X_i$ the coproduct inclusion, and $\beta_k \in Q(C_k, X_{i_k})$.



In this context, V can be endowed with a $(\mathbb{T}, \mathsf{V})$-structure $\hom_\xi : T\mathsf{V} \nrightarrow \mathsf{V}$ given by the composite

$$T\mathsf{V} \xrightarrow{\ \xi\ } \mathsf{V} \xrightarrow{\ \mathrm{hom}\ } \mathsf{V},$$

where $\hom : \mathsf{V} \times \mathsf{V} \to \mathsf{V}$ is the left adjoint of the tensor operator: for each $u, v, w \in \mathsf{V}$,

$$u \otimes v \le w \iff u \le \hom(v, w),$$

and it is given by, for each $u, v \in \mathsf{V}$,

$$\hom(u, v) = \bigvee \{w \in \mathsf{V} \mid w \otimes u \le v\}. \tag{1.vii}$$

In $\mathsf{Top} \cong (\mathbb{U}, 2)\text{-}\mathsf{Cat}$, with $\xi = e_2^\circ : U2 \to 2$ [Hof07], $(\mathsf{V}, \hom_\xi)$ is the usual *Sierpiński space* $\mathbb{S} = (\{\bot, \top\}, \{\emptyset, \{\bot\}, \{\bot, \top\}\})$, and by analogy we call $(\mathsf{V}, \hom_\xi)$ the *Sierpiński* $(\mathbb{T}, \mathsf{V})$*-space*. For $\mathsf{V}\text{-}\mathsf{Cat}$ ($\mathbb{T} = \mathbb{I}$), $\xi = 1_\mathsf{V} : \mathsf{V} \to \mathsf{V}$ [Hof07], hence $\hom_\xi = \hom$, and the Sierpiński V-space has easy descriptions for our examples of quantales in Table (1.i): for $\mathsf{V} = \mathsf{2}$, it is given by the ordered set $(\{\bot, \top\}, \le)$ with $\bot < \top$; for $\mathsf{V} = \mathsf{P}_+$, $\mathsf{V} = \mathsf{P}_{\max}$, and $\mathsf{V} = [0, 1]_\odot$ it is given, respectively, by $([0, \infty], \ominus)$, $([0, \infty], \oslash)$, and $([0, 1], \circledast)$, where, for each $u, v \in [0, \infty]$,

$$v \ominus u = \begin{cases} v - u, & \text{if } v \ge u \\ 0, & \text{otherwise} \end{cases} \qquad \& \qquad u \oslash v = \begin{cases} v, & \text{if } u < v \\ 0, & \text{otherwise,} \end{cases} \tag{1.viii}$$

and for each $u', v' \in [0, 1]$, $u' \circledast v' = \min(1, 1 - u + v)$. Moreover, since we are assuming $T\mathsf{1} = \mathsf{1}$, by [HT10, Lemma 4.18] (see also the comment after the proof of [Hof11, Theorem 2.9]), $(\mathsf{V}, \hom_\xi)$ is an injective $(\mathbb{T}, \mathsf{V})$-space and, consequently, so is the binary product $(\mathsf{V} \times \mathsf{V}, \hom_\xi \times \hom_\xi)$, where, for each $\mathfrak{q} \in T(\mathsf{V} \times \mathsf{V})$, $(u, v) \in \mathsf{V} \times \mathsf{V}$,

$$(\hom_\xi \times \hom_\xi)(\mathfrak{q}, (u, v)) = \hom_\xi(T\pi_1(\mathfrak{q}), u) \wedge \hom_\xi(T\pi_2(\mathfrak{q}), v).$$

Since $\xi : T\mathsf{V} \to \mathsf{V}$ is a $\mathbb{T}$-algebra, for $(\mathsf{V}, \hom_\xi)$ the order (1.iii) gives exactly the order of V, which is anti-symmetric, since V is a quantale. Hence $(\mathsf{V}, \hom_\xi)$ and, consequently, $(\mathsf{V} \times \mathsf{V}, \hom_\xi \times \hom_\xi)$ are separated $(\mathbb{T}, \mathsf{V})$-spaces, whence the equivalence $\simeq$ in diagram (1.iv) is an equality when considering extensions with codomain $\mathsf{V} \times \mathsf{V}$; this fact will be used in Subsection 3.2.

For V-relations $r : X \nrightarrow X'$ and $s : Y \nrightarrow Y'$, consider the V-relation $r \oslash s : X \times Y \nrightarrow X' \times Y'$, given by, for $(x, y) \in X \times Y$, $(x', y') \in X' \times Y'$,

$r \oslash s((x,y),(x',y')) = r(x,x') \wedge s(y,y')$. Assume that the diagram

$$\begin{array}{ccc} T(\mathsf{V}\times\mathsf{V}) & \xrightarrow{T(\wedge)} & TV \\ {\scriptstyle \langle \xi\cdot T\pi_1, \xi\cdot T\pi_2\rangle}\downarrow & \leq & \downarrow{\scriptstyle \xi} \\ \mathsf{V}\times\mathsf{V} & \xrightarrow[\wedge]{} & \mathsf{V} \end{array}$$

is lax commutative, what is true for the examples in Table (1.i) [Hof07, Hof14]. Thus, by [CHR20, 7.4], for all V-relations $r\colon X \to X'$ and $s\colon Y \to Y'$, the following diagram is lax commutative,

$$\begin{array}{ccc} T(X\times Y) & \xrightarrow{\mathrm{can}_{X,Y}} & TX\times TY \\ {\scriptstyle T(r\oslash s)}\downarrow & \geq & \downarrow{\scriptstyle (Tr)\oslash(Ts)} \\ T(X'\times Y') & \xrightarrow[\mathrm{can}_{X',Y'}]{} & TX'\times TY' \end{array} \tag{1.ix}$$

where, under the previous notation, $\mathrm{can}_{X,Y} = \langle T\pi_X, T\pi_Y\rangle : T(X\times Y) \to TX\times TY$. Then, by [CHR20, 3.1], a $(\mathbb{T},\mathsf{V})$-space (X,a) is exponentiable provided that, for each $\mathfrak{X}\in TTX$, $x\in X$, $u,v\in\mathsf{V}$,

$$\bigvee_{\mathfrak{x}\in TX} (Ta(\mathfrak{X},\mathfrak{x})\wedge u)\otimes(a(\mathfrak{x},x)\wedge v) \geq a(m_X(\mathfrak{X}),x)\wedge(u\otimes v). \tag{1.x}$$

Now, within the framework of strict topological theories, the tensor product of V induces a tensor product between $(\mathbb{T},\mathsf{V})$-spaces: for each $(X,a),(Y,b)$, $(X,a)\otimes(Y,b) = (X\times Y, c)$, where, for each $\mathfrak{w}\in T(X\times Y)$, $(x,y)\in X\times Y$, $c(\mathfrak{w},(x,y)) = a(T\pi_X(\mathfrak{w}),x)\otimes b(T\pi_Y(\mathfrak{w}),y)$ (see [Hof07, Lemma 6.1]). Consider the maps

$$\mathsf{V}\otimes\mathsf{V} \xrightarrow{\otimes} \mathsf{V} \quad \& \quad X \xrightarrow{(-,u)} X\otimes\mathsf{V}, \tag{1.xi}$$

and define, for a V-relation $r\colon X \nrightarrow Y$ and an element $u\in\mathsf{V}$, the V-relation $r\otimes u\colon X\nrightarrow Y$ given by, for each $(x,y)\in X\times Y$, $(r\otimes u)(x,y) = r(x,y)\otimes u$. Finally, consider the condition:

$$\forall u,v,w\in\mathsf{V},\ w\wedge(u\otimes v) = \{u'\otimes v' \mid u'\leq u,\ v'\leq v,\ u'\otimes v'\leq w\}, \tag{1.xii}$$

which is equivalent to exponentiability of every injective V-space in V-Cat [HR13, Theorem 5.3]. Then, by [CHR20, 5.8], we have

so the map $g\cdot\alpha$ is covered by an admissible map:

$$\begin{array}{ccc} C' & & \\ {\scriptstyle f'}\downarrow & \searrow^{g\cdot f\cdot\alpha'} & \\ C & \xrightarrow[g\cdot\alpha]{} & Z, \end{array}$$

whence $g\cdot\alpha\in Q(C,Z)$, and so $g\in\mathsf{Qs}(Y,Z)$.

The constructions above lead us to the following result.

Proposition *The forgetful functor* $|\text{-}|\colon \mathsf{Qs}(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}\to\mathsf{Set}$ *is topological.*

Proof. Let $(f_j\colon X\to|(X_j,(Q(C,X_j))_{C\in\mathcal{C}})|)_{j\in J}$ be a source in Set. To construct a $|\text{-}|$-initial lifting

$$(f_j\colon (X,(Q(C,X))_{C\in\mathcal{C}})\to(X_j,(Q(C,X_j))_{C\in\mathcal{C}}))_{j\in J}$$

for this source, define, for each $C\in\mathcal{C}$,

$$\alpha\in Q(C,X) \iff \forall j\in J,\ f_j\cdot\alpha\in Q(C,X_j).$$

Properties (QS1) and (QS2) are immediately satisfied. To check (QS3), observe that if a map $\alpha\colon C\to X$ is covered by admissible maps $\alpha_i\colon C_i\to X$, $i\in I$ finite, then, for each $j\in J$, $f_j\cdot\alpha$ is covered by the family of maps $(\beta_{ji} = f_j\cdot\alpha_i)_{i\in I}$, which are admissible by definition of $Q(C_i,X)$.

$$\begin{array}{ccccc} \coprod_{i\in I} C_i & & \coprod_{i\in I}\alpha_i & & \coprod_{i\in I}\beta_{ji} \\ {\scriptstyle\eta}\downarrow & \searrow & & & \\ C & \xrightarrow[\alpha]{} & X & \xrightarrow[f_j]{} & X_j \end{array}$$

Hence, for all $j\in J$, $f_j\cdot\alpha\in Q(C,X_j)$, so $\alpha\in Q(C,X)$.

It is straightforward to verify that the latter lifting is indeed $|\text{-}|$-initial and uniqueness follows from amnesticity of $|\text{-}|$ [AHS90, Proposition 21.5]. ∎

Among other properties, the latter result implies completeness and cocompleteness of $\mathsf{Qs}(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$ [AHS90, Proposition 13.15]; following this reference we describe limits and colimits.

Limits. Let A be a small category and $\mathcal{D}\colon\mathsf{A}\to\mathsf{Qs}(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}$ be a diagram. First construct the limit in Set of $|\text{-}|\cdot\mathcal{D}\colon\mathsf{A}\to\mathsf{Set}$, that we denote by $(p_A\colon X\to|\mathcal{D}A|)_{A\in\mathrm{Obj}\mathsf{A}}$, and then take the $|\text{-}|$-initial lifting of this source,

Furthermore, for a quasi-space $(X, (Q(C,X))_{C\in\mathcal{C}})$ and a surjective map $f\colon X \to Y$, we can define a *quotient* quasi-structure by: for each $C \in \mathcal{C}$, $\alpha \in Q(C,Y)$ if there exist a surjective map $f'\colon C' \to C$ in $\mathcal{C}$, and a map $\alpha' \in Q(C',X)$ such that the square below is commutative.

$$\begin{array}{ccc} C' & \xrightarrow{f'} & C \\ {\scriptstyle\alpha'}\downarrow & & \downarrow{\scriptstyle\alpha} \\ X & \xrightarrow[f]{} & Y \end{array} \tag{4.iii}$$

One can check that the latter structure satisfies (QS1) and (QS3). To verify (QS2), take $\alpha \in Q(C,Y)$ and $h\colon B \to C$ be a continuous map, with $B, C \in \mathcal{C}$. By definition, there exist a surjective map $f'\colon C' \to C$ and a map $\alpha' \in Q(C',X)$, for $C' \in \mathcal{C}$, as in (4.iii). Take the pullback of f' along h as in the diagram below.

$$\begin{array}{ccccc} B \times_C C' & & \xrightarrow{\pi_1} & & B \\ {\scriptstyle\pi_2}\downarrow & \lrcorner & & & \downarrow{\scriptstyle h} \\ C' & & \xrightarrow[f']{} & & C \\ {\scriptstyle\alpha'}\downarrow & & & & \downarrow{\scriptstyle\alpha} \\ X & & \xrightarrow[f]{} & & Y \end{array}$$

Since f' is a surjective Set-map, π_1 is surjective too. Also, by our assumptions, $B \times_C C' \in \mathcal{C}$, and because $\alpha' \in Q(C',X)$, we have $\alpha' \cdot \pi_2 \in Q(B \times_C C', X)$.

When Y is endowed with the quotient quasi-structure with respect to the surjection $f\colon X \to Y$, the map f becomes not only quasi-continuous, but also a $|\text{-}|$-final morphism: if $g\colon Y \to Z$ is a map such that $g \cdot f\colon X \to Z \in \mathsf{Qs}(X,Z)$, for $(Z, (Q(C,Z))_{C\in\mathcal{C}}) \in \mathsf{Qs}(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}$, then, for each $C \in \mathcal{C}$ and $\alpha \in Q(C,Y)$, there exist a surjection $f'\colon C' \to C$, and a map $\alpha' \in Q(C',X)$ commuting the square in (4.iii), hence

$$g \cdot \alpha \cdot f' = g \cdot f \cdot \alpha' \in Q(C',Z),$$

Theorem *Suppose that:*
- *for all* V*-relations* $r\colon X \nrightarrow X'$ *and* $s \nrightarrow Y \to Y'$*, diagram* (1.ix) *is commutative;*
- *for every injective* $(\mathbb{T},\mathsf{V})$*-space* (X,a) *and every* $u \in \mathsf{V}$*, the maps* $\otimes$ *and* $(-,u)$ *in* (1.xi) *are* $(\mathbb{T},\mathsf{V})$*-continuous, and* $T(a \otimes u) = Ta \otimes u$*; and*
- (1.xii) *holds.*

Then every injective $(\mathbb{T},\mathsf{V})$*-space is exponentiable in* $(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}$.

Remark As examples, in the categories of Table (1.i) every injective $(\mathbb{T},\mathsf{V})$-space, and in particular the Sierpiński $(\mathbb{T},\mathsf{V})$-space $(\mathsf{V}, \hom_\xi)$, is exponentiable.

1.4. Compact and Hausdorff $(\mathbb{T},\mathsf{V})$-spaces. As discussed in [HST14, Notes on Chapter V], the work of Manes [Man74] can be considered a predecessor of the notions of compactness and Hausdorff separation in categories of V-relational algebras. We follow here the definitions outlined in [HST14, V-1.1.1]: a $(\mathbb{T},\mathsf{V})$-space (X,a) is *compact* if $1_{TX} \leq a^\circ \cdot a$, or componentwise, if for each $\mathfrak{x} \in TX$,

$$k \leq \bigvee_{x\in X} a(\mathfrak{x},x) \otimes a(\mathfrak{x},x);$$

and it is *Hausdorff* if $a \cdot a^\circ \leq 1_X$, that is, for each $x, y \in X$, $\mathfrak{x} \in TX$,

$$(\bot < a(\mathfrak{x},x) \otimes a(\mathfrak{x},y) \implies x = y) \quad \& \quad a(\mathfrak{x},x) \otimes a(\mathfrak{x},x) \leq k.$$

Under our assumption that V is integral, the second condition for Hausdorff separation holds trivially. Observe that, for $(\mathbb{U},2)\text{-}\mathsf{Cat} \cong \mathsf{Top}$, we recover the fact that a topological space (X,τ) is compact and Hausdorff if, and only if, each ultrafilter has a unique convergence point.

Under the condition that V is *lean*, that is, for each $u, v \in \mathsf{V}$,

$$(u \vee v = \top \text{ and } u \otimes v = \bot) \implies (u = \top \text{ or } v = \top),$$

by [HST14, V-1.2.1], a $(\mathbb{T},\mathsf{V})$-space (X,a) is compact and Hausdorff precisely when (X,a) is a $\mathbb{T}$-algebra. Condition (1.x) is satisfied by any $\mathbb{T}$-algebra, whence compact Hausdorff $(\mathbb{T},\mathsf{V})$-spaces are exponentiable. Moreover, under those conditions – V integral and lean, and a flat lax extension – since

$$(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}_{\mathrm{CompHaus}} \cong \mathsf{Set}^{\mathbb{T}}, \tag{1.xiii}$$

by [HST14, V-1.2.3], limits of compact Hausdorff $(\mathbb{T},\mathsf{V})$-spaces are compact and Hausdorff. Finally, by [HST14, V-1.1.6(2)], assuming that the functor T

preserves finite coproducts, we have that finite coproducts of compact Haudorff $(\mathbb{T},\mathsf{V})$-spaces are compact and Hausdorff. Observe that the examples of Table (1.i) satisfy the conditions discussed above (see [Bör87]), so the latter facts, which we summarize below, hold for those categories:
(I) each compact Hausdorff $(\mathbb{T},\mathsf{V})$-space is exponentiable in $(\mathbb{T},\mathsf{V})$-Cat;
(II) the binary product of compact Haudorff $(\mathbb{T},\mathsf{V})$-spaces is compact and Hausdorff;
(III) finite coproducts of compact Haudorff $(\mathbb{T},\mathsf{V})$-spaces are compact and Hausdorff.

Examples [HST14, V-1.1.4] (1) For V-Cat with V integral and lean, by (1.xiii), $\mathsf{V}\text{-}\mathsf{Cat}_{\text{CompHaus}} \cong \mathsf{Set}^{\mathbb{I}}$, hence a V-space (X,a) is compact and Hausdorff if, and only if, it is *discrete*, that is, $a = 1_X$.
(2) For $\mathsf{Top} \cong (\mathbb{U},2)$-$\mathsf{Cat}$, we already observed that compactness and Hausdorff separation coincide with the classical notions: each ultrafilter converges to a unique point.
(3) For $\mathsf{App} \cong (\mathbb{U},\mathsf{P}_{+})$-$\mathsf{Cat}$ and NA-$\mathsf{App} \cong (\mathbb{U},\mathsf{P}_{\max})$-$\mathsf{Cat}$, a space (X,a) is compact exactly when it is *0-compact*, that is, for each $\mathfrak{x} \in UX$, $\inf\{a(\mathfrak{x},x) \mid x \in X\} = 0$; it is Hausdorff exactly when, for each $x,y \in X$, $\mathfrak{x} \in UX$,

$$(a(\mathfrak{x},x) < \infty \ \ \& \ \ a(\mathfrak{x},y) < \infty) \implies (x = y),$$

or, equivalently, if the *pseudotopological modification* of (X,a) is Hausdorff. As observed in [HST14, V-1.2.2(1)], by (1.xiii), we have:

$$(\mathbb{U},[0,1]_{\odot})\text{-}\mathsf{Cat}_{\text{CompHaus}} \cong \mathsf{NA}\text{-}\mathsf{App}_{\text{CompHaus}} \cong \mathsf{App}_{\text{CompHaus}} \cong \mathsf{Top}_{\text{CompHaus}} \cong \mathsf{Set}^{\mathbb{U}}.$$

2. $\mathcal{C}$-generated $(\mathbb{T},\mathsf{V})$-spaces

2.1. The category $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}_{\mathcal{C}}$. This section should be compared with [ELS04, Section 3], whose work we follow directly. From now on, in order to keep the text lighter, we sometimes drop the prefix $(\mathbb{T},\mathsf{V})$ when referring to $(\mathbb{T},\mathsf{V})$-spaces, $(\mathbb{T},\mathsf{V})$-continuity, or $(\mathbb{T},\mathsf{V})$-structures; we also drop the prefix $|\text{-}|$ when referring to $|\text{-}|$-initial and $|\text{-}|$-final structures, which are taken with respect to the forgetful functor $|\text{-}|\colon (\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat} \to \mathsf{Set}$.

We fix a class $\mathcal{C} \subseteq (\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$ of objects, containing at least one non-empty element. Although the class $\mathcal{C}$ is arbitrary, the reader can keep it in mind as being the class of compact Hausdorff spaces.

Definition The elements of $\mathcal{C}$ are called *generating spaces*. For a space (X,a), a continuous map from a generating space to (X,a) is called a *probe*

We observe that the indiscrete quasi-space is associated with the indiscrete space $(X,\top)$, because, for each $C \in \mathcal{C}$, every map $f\colon C \to (X,\top)$ is continuous. The same does not happen in general with the discrete quasi-space, as one verifies, for example in Top, that there exist continuous maps from compact Hausdorff spaces to discrete spaces that are not constant.

With the definitions in item **(V)** we prove the following:

Proposition *The forgetful functor* $|\text{-}|\colon \mathsf{Qs}(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat} \to \mathsf{Set}$ *has left and right adjoints, and it is represented by the singleton quasi-space.*

Proof: For the left adjoint, define $D\colon \mathsf{Set} \to \mathsf{Qs}(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$ assigning to each set X the discrete quasi-space DX, and on maps D is the identity. As observed before, for each quasi-space $(Y,(Q(C,Y))_{C\in\mathcal{C}})$, each Set-map $f\colon X \to Y$ is a quasi-continuous map $f\colon DX \to (Y,(Q(C,Y))_{C\in\mathcal{C}})$, whence

$$\mathsf{Qs}(DX,(Y,(Q(C,Y))_{C\in\mathcal{C}})) = \mathsf{Set}(X,Y) = \mathsf{Set}(X,|(Y,(Q(C,Y))_{C\in\mathcal{C}})|).$$

For the right adjoint, define $I\colon \mathsf{Set} \to \mathsf{Qs}(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$ assigning to each set X the indiscrete quasi-space IX, and leaving the morphisms unchanged; for each quasi-space $(Y,(Q(C,Y))_{C\in\mathcal{C}})$, a map $f\colon X \to Y$ is a quasi-continuous map $f\colon (X,(Q(C,X))_{C\in\mathcal{C}}) \to IY$, so

$$\mathsf{Set}(|(X,(Q(C,X))_{C\in\mathcal{C}})|,Y) = \mathsf{Set}(X,Y) = \mathsf{Qs}((X,(Q(C,X))_{C\in\mathcal{C}}),IY).$$

Since $|\text{-}|$ has a left adjoint D, it is represented by $D1$, which coincides with $I1$, and is given by the singleton $1 = \{*\}$ endowed with the quasi-structure defined by

$$Q(C,1) = \{!_C\colon C \to 1\},$$

for each $C \in \mathcal{C}$. ∎

4.2. $\mathsf{Qs}(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$ is topological over Set. Given a quasi-space

$$(X,(Q(C,X))_{C\in\mathcal{C}})$$

and a subset $A \subseteq X$, we can consider the *subspace quasi-structure* on A, which is given by, for each $C \in \mathcal{C}$,

$$\alpha \in Q(C,A) \iff i_A \cdot \alpha \in Q(C,X), \tag{4.ii}$$

where $i_A\colon A \hookrightarrow X$ is the inclusion map. When A is endowed with this structure, i_A becomes a quasi-continuous map which is also $|\text{-}|$-initial, with $|\text{-}|\colon \mathsf{Qs}(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat} \to \mathsf{Set}$ the forgetful functor.

Lemma 5.5]. Let us study the case of V-Cat. As seen in the Examples of Section 1, the compact Hausdorff V-spaces coincide with the discrete V-spaces, that is, spaces of the form $(C, 1_C)$, $C \in \mathsf{Set}$. Moreover, for each V-space (X, a), any map $\alpha\colon (C, 1_C) \to (X, a)$ is V-continuous, whence the associated quasi-V-structure on X is given by, for each $C \in \mathsf{Set}$,

$$Q_a(C, X) = \{\text{maps } \alpha\colon C \to X\}.$$

Therefore, if X is not a singleton, the quasi-V-structure on X defined by

$$Q'(C, X) = \{\text{constant maps } \alpha\colon C \to X\},$$

for each $C \in \mathsf{Set}$, is not associated with any V-space (X, a) in V-Cat.

(IV) *Admissible maps of the associated quasi-space, quasi-continuous maps, and $(\mathbb{T}, \mathsf{V})$-continuous maps coincide when considering compact Hausdorff $(\mathbb{T}, \mathsf{V})$-spaces.* Consider $(C, c) \in \mathcal{C} \subseteq (\mathbb{T}, \mathsf{V})$-$\mathsf{Cat}$ and its associated quasi-structure $(Q_c(B, C))_{B\in\mathcal{C}} = (\mathcal{C}(B, C))_{B\in\mathcal{C}}$. Let $(X, (Q(B, X))_{B\in\mathcal{C}})$ be a quasi-space, and $\alpha\colon C \to X$ be a map. If α is quasi-continuous, then $\alpha \cdot 1_C = \alpha \in Q(C, X)$, since $1_C \in Q_c(C, C)$. Conversely, if α is an admissible map from C to X, then, for each $B \in \mathcal{C}$ and $\beta \in Q_c(B, C) = \mathcal{C}(B, C)$, $\alpha \cdot \beta \in Q(B, X)$, by (QS2), so α is quasi-continuous. Hence

$$Q(C, X) = \mathsf{Qs}(C, X).$$

In particular, if $X = (D, d) \in \mathcal{C}$, quasi-continuous maps between the associated quasi-spaces C and D coincide with the admissible maps $Q_d(C, D)$, which are all continuous maps from C to D by definition, that is,

$$Q_d(C, D) = \mathsf{Qs}(C, D) = (\mathbb{T}, \mathsf{V})\text{-}\mathsf{Cat}(C, D).$$

(V) *Discrete and indiscrete quasi-structures.* For each set X we have a *discrete* quasi-structure given by, for each $C \in \mathcal{C}$,

$$Q_{\text{dis}}(C, X) = \{\text{constant maps from } C \text{ to } X\}.$$

Denoting $(X, (Q_{\text{dis}}(C, X))_{C\in\mathcal{C}})$ by DX, we see that, for each quasi-space $(Y, (Q(C, Y))_{C\in\mathcal{C}})$, each map $f\colon X \to Y$ is a quasi-continuous map $f\colon DX \to (Y, (Q(C, Y))_{C\in\mathcal{C}})$. Analogously, we can endow the set X with an *indiscrete* quasi-structure: for each $C \in \mathcal{C}$,

$$Q_{\text{ind}}(C, X) = \mathsf{Set}(C, X).$$

Denoting $(X, (Q_{\text{ind}}(C, X))_{C\in\mathcal{C}})$ by IX, for each quasi-space $(Y, (Q(C, Y))_{C\in\mathcal{C}})$, each map $f\colon Y \to X$ is a quasi-continuous map $f\colon (Y, (Q(C, Y))_{C\in\mathcal{C}}) \to IX$.

over (X, a), or simply a *probe*. The *$\mathcal{C}$-generated structure* a^c on X is the final structure with respect to all probes over (X, a). A space (X, a) is *$\mathcal{C}$-generated* if $a = a^c$. The full subcategory of $(\mathbb{T}, \mathsf{V})$-Cat of $\mathcal{C}$-generated spaces is denoted by $(\mathbb{T}, \mathsf{V})$-$\mathsf{Cat}_{\mathcal{C}}$.

By definition, for a space (X, a), a map $t\colon (X, a^c) \to (Y, b)$ is continuous if, and only if, for each probe $p\colon C \to (X, a)$, the composite $t \cdot p$ is continuous. Hence, for each space (X, a), the identity map $1_X\colon (X, a^c) \to (X, a)$ is continuous, that is, $a^c \leq a$. Moreover, each generating space is $\mathcal{C}$-generated, for if (D, d) in $\mathcal{C}$, then $1_D\colon D \to D$ is a probe, hence, by definition of final structure, it is a continuous map $1_D\colon (D, d) \to (D, d^c)$, thus $d \leq d^c$, and since $d^c \leq d$, we have $d = d^c$.

Lemma *For a $\mathcal{C}$-generated space (X, a), a map $f\colon (X, a) \to (Y, b)$ is continuous if, and only if, $f\colon (X, a) \to (Y, b^c)$ is so.*

Proof. For sufficiency, we can factorize $f\colon (X, a) \to (Y, b)$ as

$$(X, a) \xrightarrow{\ f\ } (Y, b^c) \xrightarrow{\ 1_Y\ } (Y, b),$$

which is continuous, since $1_Y\colon (Y, b^c) \to (Y, b)$ is continuous. For necessity, for each probe $p\colon C \to (X, a)$, the composite $f \cdot p\colon C \to (Y, b)$ is continuous, hence a probe, then $f \cdot p\colon C \to (Y, b^c)$ is continuous, and we conclude that $f\colon (X, a^c) = (X, a) \to (Y, b^c)$ is continuous. ∎

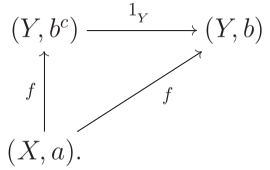
Remark *For each space (X, a), (X, a^c) is $\mathcal{C}$-generated:* each probe $p\colon C \to (X, a)$ is a probe over (X, a^c), and, consequently, it is a probe over $(X, (a^c)^c)$. Hence $1_X\colon (X, a^c) \to (X, (a^c)^c)$ is a continuous map, that is, $a^c \leq (a^c)^c$, and since $(a^c)^c \leq a^c$, we conclude $(a^c)^c = a^c$.

The next result is also proven for the particular case of Top in [Mac71].

Theorem $(\mathbb{T}, \mathsf{V})$-$\mathsf{Cat}_{\mathcal{C}}$ *is coreflective in* $(\mathbb{T}, \mathsf{V})$-Cat.

Proof. For each (Y, b) in $(\mathbb{T}, \mathsf{V})$-Cat, (Y, b^c) belongs to $(\mathbb{T}, \mathsf{V})$-$\mathsf{Cat}_{\mathcal{C}}$. Moreover, each continuous map $f\colon (X, a) \to (Y, b)$, with (X, a) in $(\mathbb{T}, \mathsf{V})$-$\mathsf{Cat}_{\mathcal{C}}$, factorizes

through the identity $1_Y\colon (Y,b^c)\to(Y,b)$:



So the coreflector $G_{\mathcal{C}}$ takes (X,a) to (X,a^c) and $f\colon(X,a)\to(Y,b)$ to $f\colon(X,a^c)\to(Y,b^c)$; each coreflection is given by an identity.

$$(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}_{\mathcal{C}} \underset{\top}{\overset{G_{\mathcal{C}}}{\hookleftarrow}} (\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat} \qquad \blacksquare$$

As a corollary, we obtain that $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}_{\mathcal{C}}$ is complete and cocomplete, since $(\mathbb{T},\mathsf{V})$-Cat is so. Next we characterize the $\mathcal{C}$-generated spaces in terms of colimits. To do so, recall from the first section that each $|$-$|$-final lifting of a sink is actually the $|$-$|$-final lifting of a small sink. We use also the fact that each constant map is continuous.

Proposition *(1) $\mathcal{C}$-generated spaces are closed under the formation of coproducts and coequalizers, hence closed under colimits.*
(2) A space is $\mathcal{C}$-generated if, and only if, it is a coequalizer of a coproduct of generating spaces.

Proof. (1) This follows from the fact that the inclusion functor is a left adjoint, so it preserves colimits.
(2) By the first assertion, a coequalizer of a coproduct of generating spaces is $\mathcal{C}$-generated, since generating spaces are $\mathcal{C}$-generated. Now let (X,a) be a $\mathcal{C}$-generated space, hence $a=a^c$ is the final structure with respect to a sink of continuous maps $(p_i\colon(X_i,a_i)\to(X,a))_{i\in I}$, with (X_i,a_i) in $\mathcal{C}$, where I is a set. Take the coproduct $(\dot{\bigcup}_{i\in I}X_i,a_I)$ in $(\mathbb{T},\mathsf{V})$-Cat. From its universal property we get a continuous map $t\colon(\dot{\bigcup}_{i\in I}X_i,a_I)\to(X,a)$ such that, for each

by a family of admissible maps $(\alpha_i)_{i\in I}$ as in (4.i). Observe that $\alpha\cdot\eta=\coprod_{i\in I}\alpha_i\in Q_a(\coprod_{i\in I}C_i,X)$ and then, Axiom of Choice granted, we conclude

$$\begin{array}{rcll} c &=& c\cdot T\eta\cdot(T\eta)^\circ & (T\eta \text{ is a surjective map})\\ &\leq& c\cdot T\eta\cdot b^\circ\cdot b\cdot(T\eta)^\circ & ((\amalg_i C_i,b) \text{ is compact})\\ &\leq& c\cdot c^\circ\cdot\eta\cdot b\cdot(T\eta)^\circ & (\eta \text{ is } (\mathbb{T},\mathsf{V})\text{-continuous})\\ &\leq& \eta\cdot b\cdot(T\eta)^\circ & ((C,c) \text{ is Hausdorff})\\ &\leq& \eta\cdot(\alpha\cdot\eta)^\circ\cdot a\cdot T(\alpha\cdot\eta)\cdot(T\eta)^\circ & (\alpha\cdot\eta \text{ is } (\mathbb{T},\mathsf{V})\text{-continuous})\\ &=& \eta\cdot\eta^\circ\cdot\alpha^\circ\cdot a\cdot T\alpha\cdot T\eta\cdot(T\eta)^\circ & \\ &\leq& \alpha^\circ\cdot a\cdot T\alpha & (\eta \text{ and } T\eta \text{ are maps}). \end{array}$$

Observe that, by the same argument, one can conclude the following lemma (see also [HST14, V-4.3.1]).

Lemma *Every surjective $(\mathbb{T},\mathsf{V})$-continuous map from a compact $(\mathbb{T},\mathsf{V})$-space to a Hausdorff $(\mathbb{T},\mathsf{V})$-space is a proper $(\mathbb{T},\mathsf{V})$-continuous map, which is also a quotient map.*

For details on proper $(\mathbb{T},\mathsf{V})$-continuous maps see [CH04a] and also [HST14, V-3]. The quasi-space $(X,(Q_a(C,X))_{C\in\mathcal{C}})$ is said to be *associated* with the space (X,a). Moreover, each continuous map $f\colon(X,a)\to(Y,b)$ is quasi-continuous with respect to the associated quasi-spaces: for each $C\in\mathcal{C}$ and admissible $\alpha\colon C\to X$ (that is, α continuous), $f\cdot\alpha\colon C\to Y$ is continuous, hence admissible by definition. However, in general this inclusion of $(\mathbb{T},\mathsf{V})$-Cat into $\mathsf{Qs}(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$ is not full.

(II) *Different $(\mathbb{T},\mathsf{V})$-structures on the set X may originate the same quasi-space*: take (X,a) and the corresponding $\mathcal{C}$-generated space (X,a^c) defined in Section 2. By Lemma 2.1, for each $C\in\mathcal{C}$, a map $\alpha\colon C\to(X,a)$ is continuous if, and only if, $\alpha\colon C\to(X,a^c)$ is continuous. Actually, the $\mathcal{C}$-generated structure on X is the least one inducing the same associated quasi-space, for if $\overline{a}$ is such a structure, then, in particular, each probe over (X,a) is a continuous map $\alpha_i\colon(X_i,a_i)\to(X,\overline{a})$, what implies that the identity $1_X\colon(X,a^c)\to(X,\overline{a})$ is continuous, hence $a^c\leq\overline{a}$.

(III) *In general, there exist quasi-spaces in $\mathsf{Qs}(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$ wich are not associated with any space in $(\mathbb{T},\mathsf{V})$-Cat.* For Top, this is shown in [Spa63,

there exists a surjective continuous map $\eta\colon \coprod_{i\in I} C_i \to C$, where $\coprod_{i\in I} C_i$ denotes the coproduct of the family $(C_i)_{i\in I}$ in $(\mathbb{T},\mathsf{V})$-Cat, such that the triangle below is commutative.

$$\begin{array}{ccc} \coprod_{i\in I} C_i & & \coprod_{i\in I} \alpha_i \\ \eta\downarrow & \searrow & \\ C & \xrightarrow{\alpha} & X \end{array} \tag{4.i}$$

In particular, every map α is covered by itself.
(2) A *quasi-*$(\mathbb{T},\mathsf{V})$*-space*, or simply a *quasi-space*, is a set X together with, for each $C\in\mathcal{C}$, a set $Q(C,X)$ of functions from C to X, whose elements are called *admissible maps*, satisfying the conditions:
(QS1) for all $C\in\mathcal{C}$, $Q(C,X)$ contains all constant maps,
(QS2) for all $C_1, C_2 \in \mathcal{C}$, if $h\colon C_1 \to C_2$ is a $(\mathbb{T},\mathsf{V})$-continuous map and $\alpha\in Q(C_2,X)$, then $\alpha\cdot h\in Q(C_1,X)$, and
(QS3) for all $C\in\mathcal{C}$, $\alpha\in Q(C,X)$ if, and only if, α is covered by a family of admissible maps.
We denote a quasi-space by $(X,(Q(C,X))_{C\in\mathcal{C}})$, or simply by X when the corresponding *quasi-*$(\mathbb{T},\mathsf{V})$*-structure* $(Q(C,X))_{C\in\mathcal{C}}$ is clear from the context. A map $f\colon X\to Y$ between quasi-spaces is said to be a *quasi-*$(\mathbb{T},\mathsf{V})$*-continuous* map, or simply a *quasi-continuous* map, if, for all $C\in\mathcal{C}$ and $\alpha\in Q(C,X)$, $f\cdot\alpha\in Q(C,Y)$; we denote the set of quasi-continuous maps from X to Y by $\mathsf{Qs}(X,Y)$.

Identity maps and composition of quasi-continuous maps are quasi-continuous, so we have a category $\mathsf{Qs}(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$.

Remark Although carrying a size illegitimacy – its collection of objects do not form a class – proved by Herrlich and Rajagopalan [HR83], we still call $\mathsf{Qs}(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$ a category.

In the following items we discuss some properties of quasi-spaces.

(I) *Each space* (X,a) *is a quasi-space*: for each $C\in\mathcal{C}$, define

$$Q_a(C,X) = \{\alpha\colon (C,c)\to(X,a) \mid \alpha \text{ is continuous}\}.$$

Thus $Q_a(C,X)$ satisfies (QS1) and (QS2). Let us verify (QS3). If $\alpha\in Q_a(C,X)$, then it is trivially covered by itself. Consider $\alpha\colon C\to X$ covered

$i\in I$, $t\cdot\iota_i = p_i$, where ι_i is the canonical inclusion of X_i into the coproduct.

$$\begin{array}{ccc} (X_i,a_i) & \xrightarrow{\iota_i} & (\dot{\bigcup} X_i, a_I) \\ & \searrow^{p_i} & \downarrow t \\ & & (X,a) \end{array}$$

Let us prove that t is a final surjection, hence a regular epimorphism in $(\mathbb{T},\mathsf{V})$-Cat. We can assume that all points of X are covered by probes: for $x_0\in X$, consider a constant map $x_0\colon C_0\to(X,a)$, for a non-empty element C_0 of $\mathcal{C}$, which exists by our assumptions; adding those maps to our set of probes does not affect its finality neither its smallness, so we consider, without loss of generality, that those constant probes are already indexed by I. Hence, for each $x\in X$, there exists $i\in I$ such that $x = p_i(x_i) = t(\iota_i(x_i))$, and t is surjective. Next consider a map $s\colon (X,a)\to(Y,b)$ such that $s\cdot t$ is a continuous map; this is equivalent to, for each $i\in I$, the map $s\cdot t\cdot\iota_i$ being continuous, hence, for each $i\in I$, $s\cdot p_i$ is continuous, what implies that s is a continuous map, since the structure a is final with respect to the sink $(p_i)_{i\in I}$. ∎

For a complete account on regular epimorphisms in $(\mathbb{T},\mathsf{V})$-Cat see [Hof05]. As a corollary, we can conclude that $(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}_{\mathcal{C}}$ is the *coreflective hull* of $\mathcal{C}$ in $(\mathbb{T},\mathsf{V})$-Cat. Hence the matter of cartesian closedness fits the goals of [Nel78], where the author also established the conditions used below. However, following the lines of [ELS04], a direct approach to the question is given.

2.2. The category $\mathcal{C}$-Map. We start by the following:

Definition A map $f\colon (X,a)\to(Y,b)$ is *$\mathcal{C}$-continuous* if the composite $f\cdot p\colon C\to(Y,b)$ is continuous, for every probe $p\colon C\to(X,a)$.

Notice that, for spaces $(X,a),(Y,b)$ and a map $f\colon X\to Y$, the following assertions are equivalent:
(i) $f\colon (X,a)\to(Y,b)$ is $\mathcal{C}$-continuous;
(ii) $f\colon (X,a^c)\to(Y,b)$ is continuous;
(iii) $f\colon (X,a^c)\to(Y,b^c)$ is continuous.
Continuity obviously implies $\mathcal{C}$-continuity, and from (ii) we see that for maps defined on $\mathcal{C}$-generated spaces the converse is also true.

Lemma *(1) $(\mathbb{T},\mathsf{V})$-spaces and $\mathcal{C}$-continuous maps form a category, denoted by $\mathcal{C}$-*Map*.*
*(2) The identity map $1_X\colon (X,a^c)\to(X,a)$ is an isomorphism in $\mathcal{C}$-*Map*.*
(3) The assignment that sends a space (X,a) to (X,a^c), and a $\mathcal{C}$-continuous map to itself, is an equivalence of categories $F_{\mathcal{C}}\colon \mathcal{C}\text{-}\mathsf{Map}\to(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}_{\mathcal{C}}$.

Proof. (1) Identity maps and composition of $\mathcal{C}$-continuous maps are readily seen to be $\mathcal{C}$-continuous.
(2) The identity map $1_X\colon (X,a^c)\to(X,a)$ is continuous, hence it is $\mathcal{C}$-continuous, and $1_X\colon (X,a^c)\to(X,a^c)$ is continuous, thus $1_X\colon (X,a)\to(X,a^c)$ is also $\mathcal{C}$-continuous.
(3) $(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}_{\mathcal{C}}$ is a full subcategory of $\mathcal{C}$-Map and the inclusion $(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}_{\mathcal{C}}\hookrightarrow\mathcal{C}$-Map is essentially surjective: for each space (X,a), by item (2), $(X,a^c)\cong(X,a)$ in $\mathcal{C}$-Map. ∎

2.3. Cartesian closedness of $\mathcal{C}$-Map and $(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}_{\mathcal{C}}$. Firstly we prove the following result:

Lemma *For $(\mathbb{T},\mathsf{V})$-spaces $(X,a),(Y,b),(Z,c)$, if $f\colon (X\times Y,a\times b)\to(Z,c)$ is a $\mathcal{C}$-continuous map, then, for each $x\in X$, the map $f_x\colon (Y,b)\to(Z,c)$, given by $f_x(y)=f(x,y)$ for each $y\in Y$, is $\mathcal{C}$-continuous.*

Proof. Every constant map $x\colon (Y,b)\to(X,a)$ is continuous. Then

$$\langle x,1_Y\rangle : (Y,b)\to(X\times Y,a\times b)$$

is a continuous map, hence it is $\mathcal{C}$-continuous, and so is the composite $f_x = f\cdot\langle x,1_Y\rangle$. ∎

This provides, for each $\mathcal{C}$-continuous map $f\colon X\times Y\to Z$, a map $\overline{f}\colon X\to\mathcal{C}\text{-}\mathsf{Map}(Y,Z)$ given by, for each $x\in X$, $\overline{f}(x)=f_x$; as usual we call $\overline{f}$ the *transpose* of f. We wish to endow $\mathcal{C}\text{-}\mathsf{Map}(Y,Z)$ with a $(\mathbb{T},\mathsf{V})$-structure d such that f is $\mathcal{C}$-continuous if, and only if, $\overline{f}$ is so. In order to do that, we assume the condition:

(EP) *each element of $\mathcal{C}$ is exponentiable in $(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}$ and the product of two elements of $\mathcal{C}$ is a $\mathcal{C}$-generated space.*

The class $\mathcal{C}$ is referred as being *productive* [ELS04, Definition 3.5] (see also [Nel78, Day72]).

Consider the spaces $(Y,b),(Z,c)$ and the sink $(q_j\colon (Y_j,b_j)\to(Y,b))_{j\in J}$ of all probes over (Y,b). Since each generating space (Y_j,b_j) is exponentiable, we

of $(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}$. A criterion for exponentiability in $(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}$ is proved in [CHR20] (see (1.x)).

In Ord, $\mathcal{C}=\mathsf{Ord}_{\mathcal{C}}=\mathsf{Ord}$. For Top, exponentiable spaces are characterized as the *core-compact* spaces (see [EH01] for a complete account on the matter). In [Hof14, Definition 1.8] a space (X,a) is said to be *core-compact* if $a\cdot Ta = a\cdot m_X$, condition that implies (X,a) to be $\otimes$-exponentiable [Hof07, Hof14], what in Top is equivalent to be (cartesian) exponentiable, since $\otimes=\wedge$ in the quantale $\mathsf{2}$. Exponentiably generated spaces in Top, which are quotients of disjoint sums of core-compact spaces, are then called *core-compactly generated* [ELS04]. For an account on exponentiable metric spaces and exponentiable approach spaces see [CH06] and [HS15], respectively.

3.4. Injectively generated $(\mathbb{T},\mathsf{V})$-spaces. Under the conditions of Theorem 1.3, hence in particular for the categories of Table (1.i), one can consider the class $\mathcal{C}$ of injective spaces. Then each element of $\mathcal{C}$ is exponentiable and binary products of injective spaces are again injective, hence condition **(EP)** holds for $\mathcal{C}$. Proposition 2.1 and Theorem 2.3 imply that coequalizers of coproducts of injective spaces form a cartesian closed subcategory of $(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}$.

The injective spaces in the usual sense, that is, with "=" instead of "$\simeq$" in diagram (1.iv), also form a class satisfying condition **(EP)**. In particular, in Top quotients of disjoint sums of retracts of powers D^I, I a set and $D=(\{0,1,2\},\{\emptyset,\{0,1,2\},\{0,1\}\})$ [AHS90, Examples 9.3(4)], form a cartesian closed subcategory.

4. Quasi-$(\mathbb{T},\mathsf{V})$-spaces

4.1. The category of quasi-spaces and quasi-continuous maps. Following the presentation of [Day68], we generalize the work of [Spa63] from Top to $(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}$. Throughout the rest of the paper, let $\mathcal{C}$ denote the full subcategory of $(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}$ of compact Hausdorff spaces. We also assume the necessary conditions on $\mathbb{T}$ and V so that constant maps are continuous in $(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}$, and $\mathcal{C}$ is closed under finite coproducts, binary products, and equalizers.

Definition (1) For a map $\alpha\colon C\to X$ and a finite family $(\alpha_i)_{i\in I}$ of maps $\alpha_i\colon C_i\to X$, with $C,C_i\in\mathcal{C}$, one says that α *is covered by the family* $(\alpha_i)_{i\in I}$ if

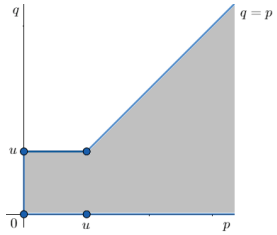
satisfies (BC) and $U1 = 1$, the diagram

$$\begin{array}{ccc} U([0,\infty]\times[0,\infty]) & \xrightarrow{U\pi_2} & U[0,\infty] \\ {\scriptstyle U\pi_1}\downarrow & & \downarrow \\ U[0,\infty] & \longrightarrow & 1 \end{array}$$

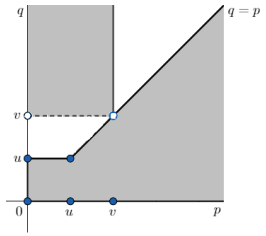
is a weak pullback, hence there exists $\mathfrak{w} \in U([0,\infty]\times[0,\infty])$ such that $U\pi_i(\mathfrak{w}) = \mathfrak{v}_i$, $i = 1, 2$. Now $\xi \cdot U\vec{\oslash}(\mathfrak{w}) = \inf\{u \in [0,\infty] \mid D_u \in \mathfrak{w}\}$, where

$$D_u = \{(p,q) \in [0,\infty]\times[0,\infty] \mid p \geq q \text{ or } q \leq u\},$$

which is depicted in the graphic below.



Consider an element $u < v$. Since $[0,v] \in \mathfrak{v}_1$ and $]v,\infty] \in \mathfrak{v}_2$, we have $[0,v]\times]v,\infty] \in \mathfrak{w}$, and because $[0,v]\times]v,\infty] \subseteq ([0,\infty]\times[0,\infty])\setminus D_u$, then $D_u \notin \mathfrak{w}$. Hence, if $D_u \in \mathfrak{w}$, then $v \leq u$, so that $\xi(\mathfrak{v}_1) \oslash \xi(\mathfrak{v}_2) = 0 < v \leq \xi \cdot U\vec{\oslash}(\mathfrak{w})$.



3.3. Exponentiably generated $(\mathbb{T},\mathsf{V})$-spaces. Consider the class $\mathcal{C}$ of exponentiable spaces. One readily checks that property **(EP)** of Subsection 2.3 holds. By Proposition 2.1 and Theorem 2.3, we conclude that coequalizers of coproducts of exponentiable spaces form a cartesian closed subcategory

have an exponential (Z^{Y_j}, d_j) in $(\mathbb{T},\mathsf{V})$-Cat, which is given by the set

$$Z^{Y_j} = \{h\colon (Y_j,b_j) \to (Z,c) \mid h \text{ is a } (\mathbb{T},\mathsf{V})\text{-continuous map}\}$$

endowed with the following $(\mathbb{T},\mathsf{V})$-structure: for each $\mathfrak{p} \in T(Z^{Y_j})$, $h \in Z^{Y_j}$,

$$d_j(\mathfrak{p},h) = \bigvee\{v \in \mathsf{V} \mid (\forall \mathfrak{q} \in (T\pi_Z)^{-1}(\mathfrak{p}))\ (\forall y_j \in Y_j)\ b_j(T\pi_X(\mathfrak{q}), y_j) \wedge v \leq \leq c(T\mathrm{ev}(\mathfrak{q}), h(x))\},$$

where π_X and π_Z are the product projections from $X \times Z$ into X and Z, respectively [CHT03]. Consequently, each probe $q_j\colon (Y_j,b_j) \to (Y,b)$ induces a map

$$\begin{array}{rcl} t_{q_j}\colon \mathcal{C}\text{-Map}(Y,Z) & \longrightarrow & (Z^{Y_j}, d_j) \\ g & \longmapsto & g \cdot q_j, \end{array}$$

which is well-defined: if g is $\mathcal{C}$-continuous, then $g \cdot q_j$ is continuous. We endow $\mathcal{C}$-Map(Y,Z) with the initial structure d with respect to the source $(t_{q_j}\colon \mathcal{C}\text{-Map}(Y,Z) \to (Z^{Y_j},d_j))_{j\in J}$, so that $d = \bigwedge_{j\in J} t^{\circ}_{q_j} \cdot d_j \cdot Tt_{q_j}$ is such that a map $h\colon (W,l) \to (\mathcal{C}\text{-Map}(Y,Z),d)$, with (W,l) in $(\mathbb{T},\mathsf{V})$-Cat, is continuous precisely when, for each $j \in J$, $t_{q_j} \cdot h\colon (W,l) \to (Z^{Y_j},d_j)$ is continuous.

Proposition *For $(\mathbb{T},\mathsf{V})$-spaces $(X,a),(Y,b),(Z,c)$, a map $f\colon (X\times Y, a\times b) \to (Z,c)$ is $\mathcal{C}$-continuous if, and only if, $\overline{f}\colon (X,a) \to (\mathcal{C}\text{-Map}(Y,Z),d)$ is $\mathcal{C}$-continuous.*

Proof. Suppose that $f\colon X\times Y \to Z$ is $\mathcal{C}$-continuous. To prove that $\overline{f}\colon X \to \mathcal{C}\text{-Map}(Y,Z)$ is $\mathcal{C}$-continuous, take a probe $p\colon C \to (X,a)$ and consider the composite $\overline{f}\cdot p\colon C \to \mathcal{C}\text{-Map}(Y,Z)$, which we wish to verify to be a continuous map. By definition of d, it suffices to prove that, for each probe $q_j\colon (Y_j,b_j) \to (Y,b)$, $t_{q_j} \cdot \overline{f} \cdot p\colon C \to Z^{Y_j}$ is a continuous map. We have a natural bijection

$$(\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}(C, Z^{Y_j}) \cong (\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}(C\times Y_j, Z), \tag{2.i}$$

and we calculate, for each $c \in C$, $y_j \in Y_j$,

$$(t_{q_j} \cdot \overline{f} \cdot p(c))(y_j) = \overline{f} \cdot p(c)(q_j(y_j)) = f(p(c), q_j(y_j)) = f \cdot (p\times q_j)(c,y_j),$$

hence $t_{q_j} \cdot \overline{f} \cdot p$ corresponds to the map $f\cdot(p\times q_j)\colon C\times Y_j \to Z$ by the bijection (2.i), which is continuous, since $p\times q_j$ is continuous, $C\times Y_j$ is $\mathcal{C}$-generated, and, by hypothesis, f is $\mathcal{C}$-continuous.

Now let $\overline{f}\colon (X,a) \to (\mathcal{C}\text{-Map}(Y,Z),d)$ be $\mathcal{C}$-continuous. To prove that $f\colon X\times Y \to Z$ is $\mathcal{C}$-continuous, take a probe $r\colon C \to (X\times Y, a\times b)$ and consider the composite $f\cdot r\colon C\to Z$. Composing with the product projections $\pi_{_X}$ and $\pi_{_Y}$, we get the probes $r_{_X} = \pi_{_X}\cdot r\colon C\to X$ and $r_{_Y} = \pi_{_Y}\cdot r\colon C\to Y$. By hypothesis, $\overline{f}\cdot r_{_X}\colon C\to \mathcal{C}\text{-Map}(Y,Z)$ is a continuous map, what implies, by definition of $(\mathcal{C}\text{-Map}(Y,Z),d)$, that $t_{r_{_Y}}\cdot \overline{f}\cdot r_{_X}\colon C\to Z^C$ is a continuous map. Then, for each $c\in C$, we have

$$f\cdot r(c) = f\cdot\langle r_{_X}, r_{_Y}\rangle(c) = \overline{f}(r_{_X}(c))(r_{_Y}(c)) = (t_{r_{_Y}}\cdot\overline{f}\cdot r_{_X}(c))(c),$$

and we conclude that $f\cdot r$ is a continuous map. ∎

Corollary $\mathcal{C}$-Map *is a cartesian closed category.*

Proof. For spaces $(Y,b),(Z,c)$, the evaluation map

$$\begin{array}{rcl} \mathrm{ev}_{_{Y,Z}}\colon (\mathcal{C}\text{-Map}(Y,Z)\times Y, d\times b) & \longrightarrow & (Z,c) \\ (f,y) & \longmapsto & f(y), \end{array}$$

is $\mathcal{C}$-continuous, since its transpose $\overline{\mathrm{ev}}_{_{Y,Z}}\colon \mathcal{C}\text{-Map}(Y,Z)\to\mathcal{C}\text{-Map}(Y,Z)$ is an identity map, hence ($\mathcal{C}$-)continuous. Moreover, for each $\mathcal{C}$-continuous map $f\colon (X\times Y, a\times b)\to(Z,c)$, the $\mathcal{C}$-continuous map $\overline{f}\colon (X,a)\to(\mathcal{C}\text{-Map}(Y,Z),d)$ is the unique one such that $\mathrm{ev}_{_{Y,Z}}\cdot(\overline{f}\times 1_{_Y}) = f$.

$$\begin{array}{ccc} \mathcal{C}\text{-Map}(Y,Z) & \qquad & \mathcal{C}\text{-Map}(Y,Z)\times Y \xrightarrow{\ \mathrm{ev}_{_{Y,Z}}\ } Z \\ \exists\,!\ \overline{f}\ \uparrow & & \overline{f}\times 1_{_Y}\uparrow \quad \nearrow_{f} \\ X & & X\times Y \end{array}$$

∎

By Lemma 2.2(3) and the previous corollary, we conclude:

Theorem $(\mathbb{T},\mathsf{V})$-Cat_c *is a cartesian closed category.*

The exponential of objects $(X,a),(Y,b)$ in $(\mathbb{T},\mathsf{V})$-Cat_c is given by

$$\begin{aligned} \mathcal{C}(\mathcal{C}\text{-Map}((X,a),(Y,b)),d) &= (\mathcal{C}\text{-Map}((X,a),(Y,b)),d^c) \\ &= ((\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat}((X,a),(Y,b)),d^c). \end{aligned}$$

3. Examples of $\mathcal{C}$-generated $(\mathbb{T},\mathsf{V})$-spaces

3.1. Compactly generated $(\mathbb{T},\mathsf{V})$-spaces. Let $\mathcal{C}\subseteq(\mathbb{T},\mathsf{V})$-$\mathsf{Cat}$ be the class of compact Hausdorff spaces; $\mathcal{C}$-generated spaces are called, as usual, *compactly generated*. As discussed in Subsection 1.4, $\mathcal{C}$ satisfies condition

and, since $\xi(\mathfrak{v}_{_2})-\epsilon<\xi(\mathfrak{v}_{_2})$, then $[\xi(\mathfrak{v}_{_2})-\epsilon,1]\in\mathfrak{v}_{_2}$. Hence $[0,\xi(\mathfrak{v}_{_2})-\epsilon-t+1[\times[\xi(\mathfrak{v}_{_2})-\epsilon,1]\in\mathfrak{w}$, but one can see that $[0,\xi(\mathfrak{v}_{_2})-\epsilon-t+1[\times[\xi(\mathfrak{v}_{_2})-\epsilon,1]\subseteq R_{_t}$, a contradiction.

Thus, for each $\mathfrak{w}\in U([0,1]\times[0,1])$, $\xi\cdot U\vec{\circledast}(\mathfrak{w}) = \xi(\mathfrak{v}_{_1})\circledast\xi(\mathfrak{v}_{_2})$ and the diagram is commutative. We have proved in the first item that the ultrafilter functor U satisfies inequality (3.iii) for every set X. Therefore we conclude that Alexandroff $(\mathbb{U},[0,1]_{\odot})$-spaces form a cartesian closed subcategory of $(\mathbb{U},[0,1]_{\odot})$-$\mathsf{Cat}$. Furthermore, Alexandroff $(\mathbb{U},[0,1]_{\odot})$-spaces are precisely the image by the functor $A^{\circ}\colon \mathsf{B}_{_1}\mathsf{Met}\to(\mathbb{U},[0,1]_{\odot})$-$\mathsf{Cat}$ of the Alexandroff bounded-by-1 metric spaces. Explicitly, if (X,d) is an Alexandroff bounded-by-1 metric space, then $(X,d_{_\#})$ is an Alexandroff $(\mathbb{U},[0,1]_{\odot})$-space, where

$$d_{_\#}(\mathfrak{x},x) = \inf_{A\in\mathfrak{x}}\sup_{x'\in A} d(x',x);$$

if (X,a) is an Alexandroff $(\mathbb{U},[0,1]_{\odot})$-space, then $(X,a\cdot e_{_X})$ is an Alexandroff bounded-by-1 metric space, and, moreover, $a = d_{_\#}$, with $d = a\cdot e_{_X}$, so that, for each $\mathfrak{x}\in UX$, $x\in X$,

$$a(\mathfrak{x},x) = \inf_{A\in\mathfrak{x}}\sup_{x'\in A} a(e_{_X}(x'),x).$$

(3) For the category NA-App we cannot apply Corollary 3.2 nor Lemma 2, since the diagram

$$\begin{array}{ccc} U([0,\infty]\times[0,\infty]) & \xrightarrow{\ U\vec{\otimes}\ } & U[0,\infty] \\ {\scriptstyle\langle\xi\cdot U\pi_{_1},\xi\cdot U\pi_{_2}\rangle}\downarrow & & \downarrow{\scriptstyle\xi} \\ [0,\infty]\times[0,\infty] & \xrightarrow[\ \vec{\otimes}\ ]{} & [0,\infty] \end{array}$$

is not commutative, where ξ is defined as in the first item. To prove this fact, let $0<v<\infty$ and consider the filters on $[0,\infty]$ defined by

$$\mathfrak{f}_{_1} = \{A\subseteq[0,\infty] \mid \exists\ u<v;]u,v]\subseteq A \text{ or } [0,v]\subseteq A\}$$

and

$$\mathfrak{f}_{_2} = \{A\subseteq[0,\infty] \mid \exists\ u>v, [0,u]\subseteq A \text{ or }]v,u]\subseteq A\}.$$

There exist ultrafilters $\mathfrak{v}_{_1}\supseteq\mathfrak{f}_{_1}$ and $\mathfrak{v}_{_2}\supseteq\mathfrak{f}_{_2}$; moreover, one can see that $\xi(\mathfrak{v}_{_1}) = \xi(\mathfrak{v}_{_2}) = v$, whence $\xi(\mathfrak{v}_{_1})\mathbin{\text{⍟}}\xi(\mathfrak{v}_{_2}) = 0$. Since the ultrafilter monad

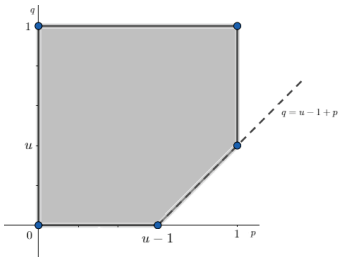
Consider the diagram

$$\begin{array}{ccc} U([0,1]\times[0,1]) & \xrightarrow{U\vec{\circledast}} & U[0,1] \\ {\scriptstyle \langle \xi\cdot U\pi_1, \xi\cdot U\pi_2\rangle}\downarrow & & \downarrow{\scriptstyle \xi} \\ [0,1]\times[0,1] & \xrightarrow[\vec{\circledast}]{} & [0,1]. \end{array}$$

We know that $\xi\cdot U\vec{\circledast} \leq \vec{\circledast}\cdot\langle \xi\cdot U\pi_1, \xi\cdot U\pi_2\rangle$. Let $\mathfrak{w}\in U([0,1]\times[0,1])$ and suppose we have $\xi\cdot U\vec{\circledast}(\mathfrak{w}) < \xi(\mathfrak{v}_1)\circledast\xi(\mathfrak{v}_2)$, where $\mathfrak{v}_i = U\pi_i(\mathfrak{w})$, $i=1,2$. Hence there exists $t\in[0,1]$ such that $\xi\cdot U\vec{\circledast}(\mathfrak{w}) < t < \xi(\mathfrak{v}_1)\circledast\xi(\mathfrak{v}_2)$. Analogously to the first item, one calculates

$$\xi\cdot U\vec{\circledast}(\mathfrak{w}) = \sup\{u\in[0,1] \mid R_u\in\mathfrak{w}\},$$

where $R_u = \{(p,q)\in[0,1]\times[0,1] \mid u-1+p\leq q\}$, which is depicted as the gray area of the graphic below.



Hence $R_t\notin\mathfrak{w}$. Let us study the cases below.

$\boxed{\xi(\mathfrak{v}_1)=0}$ Since $0<t<1$, there exists $\epsilon>0$ such that $\epsilon<1-t$. Hence

$$\epsilon = \xi(\mathfrak{v}_1)+\epsilon > \xi(\mathfrak{v}_1) \;\Rightarrow\; [\epsilon,1]\notin\mathfrak{v}_1 \;\Rightarrow\; [0,\epsilon[\in\mathfrak{v}_1 \;\Rightarrow\; [0,\epsilon[\times[0,1]\in\mathfrak{w},$$

but $[0,\epsilon[\times[0,1]\subseteq R_t$, what implies $R_t\in\mathfrak{w}$, a contradiction.

$\boxed{\xi(\mathfrak{v}_1)>0}$ If $\xi(\mathfrak{v}_1)<1-t$, then we can choose $\epsilon>0$ with $\xi(\mathfrak{v}_1)<\epsilon<1-t$. Hence $[0,\epsilon[\times[0,1]\in\mathfrak{w}$ and $[0,\epsilon[\times[0,1]\subseteq R_t$, a contradiction. Let us assume that $\xi(\mathfrak{v}_1)\geq 1-t$. Since $t<\xi(\mathfrak{v}_1)\circledast\xi(\mathfrak{v}_2)$, then $t<1-\xi(\mathfrak{v}_1)+\xi(\mathfrak{v}_2)$. Choose $\epsilon>0$ such that $\xi(\mathfrak{v}_2)-\epsilon>t-1+\xi(\mathfrak{v}_1)\geq 0$. Thus

$$\begin{aligned} \xi(\mathfrak{v}_1)<\xi(\mathfrak{v}_2)-\epsilon-t+1 &\;\Rightarrow\; [\xi(\mathfrak{v}_2)-\epsilon-t+1,1]\notin\mathfrak{v}_1 \\ &\;\Rightarrow\; [0,\xi(\mathfrak{v}_2)-\epsilon-t+1[\in\mathfrak{v}_1 \end{aligned}$$

(EP), so Theorem 2.3 holds. Together with Proposition 2.1(3), we conclude that each compactly generated space is a coequalizer of a coproduct of compact Hausdorff spaces, and the category $(\mathbb{T},\mathsf{V})$-Cat_c of compactly generated spaces is cartesian closed.

Let us consider the categories of Table (1.i). For V-Cat, quotients and coproducts of discrete objects are discrete, so that V-$\mathsf{Cat}_c\cong\mathsf{Set}$.

In Top, a space (X,a) belongs to $(\mathbb{U},2)$-Cat_c if, and only if, it is a coequalizer of a coproduct of compact Hausdorff $(\mathbb{U},2)$-spaces. Then we recover the fact that a topological space is compactly generated if, and only if, it is a quotient of a disjoint sum of compact Hausdorff spaces. This is equivalent to being a quotient of a locally compact Hausdorff space. Furthermore, the category Top_c of compactly generated spaces and continuous functions is cartesian closed (see [ELS04]). Classical examples of compactly generated topological spaces are sequential spaces, topological manifolds and CW-complexes.

For (non-Archimedean) approach spaces, we recall the equivalences given by (1.xiii):

$$(\mathbb{U},\mathsf{P}_{\max})\text{-}\mathsf{Cat}_{\mathrm{CompHaus}}\cong(\mathbb{U},\mathsf{P}_{+})\text{-}\mathsf{Cat}_{\mathrm{CompHaus}}\cong\mathsf{Set}^{\mathbb{U}}.$$

The embedding of Top in App corestricts to an embedding into NA-App, and Top is coreflective in both categories [Low97, CVO17]. Therefore, compactly generated (non-Archimedean) approach spaces, which are the colimits of 0-compact Hausdorff approach spaces, are precisely the topological approach spaces induced by a compactly generated topological space. Furthermore, they form a cartesian closed category $\mathsf{App}_c=\mathsf{NA}$-$\mathsf{App}_c$.

Let us consider $\mathbb{T}=\mathbb{U}$ and $\mathsf{V}=[0,1]_\odot$. The quantale homomorphism $\iota\colon 2\to[0,1]_\odot$, defined by $\iota(\bot)=0$ and $\iota(\top)=1$, which is *compatible* with the respective lax extensions of $\mathbb{U}$ to $\mathsf{2}$-Rel and $[0,1]_\odot$-Rel (see [HST14, III-3.5]), induces the embedding $\mathsf{Top}\hookrightarrow(\mathbb{U},[0,1]_\odot)$-$\mathsf{Cat}$, where a topological space (X,a) is assigned to $(X,\iota\cdot a)$, with $\iota\cdot a(\mathfrak{x},x)=\iota(a(\mathfrak{x},x))\in[0,1]$, for each $\mathfrak{x}\in UX$, $x\in X$, and morphisms are unchanged. The homomorphism ι has a right adjoint $p\colon[0,1]_\odot\to 2$, given by $p(1)=\top$ and $p(u)=\bot$, for $u\neq 1$. The quantale homomorphism p is also compatible with the lax extensions of $\mathbb{U}$ to $[0,1]_\odot$-Rel and $\mathsf{2}$-Rel. Hence, by [HST14, III-Proposition 3.5.1], the adjunction $\iota\dashv p$ induces an adjunction

$$\mathsf{Top}\;\overset{\top}{\circlearrowleft}\;(\mathbb{U},[0,1]_\odot)\text{-}\mathsf{Cat}.$$

Therefore, Top is coreflective in $(\mathbb{U},[0,1]_{\odot})$-$\mathsf{Cat}$, what implies that $(\mathbb{U},[0,1]_{\odot})$-compactly generated spaces are $(\mathbb{U},[0,1]_{\odot})$-spaces induced by compactly generated topological spaces.

3.2. Alexandroff $(\mathbb{T},\mathsf{V})$-spaces. Let us consider $\mathcal{C}$ as the singleton set containing the Sierpiński $(\mathbb{T},\mathsf{V})$-space $(\mathsf{V},\hom_{\xi})$. By analogy with the case in Top [ELS04, Example (2) of Section 3], we call the $\mathcal{C}$-generated spaces, or $(\mathsf{V},\hom_{\xi})$-generated spaces, *Alexandroff* spaces. Hence a $(\mathbb{T},\mathsf{V})$-space is Alexandroff if, and only if, it is a coequalizer of a coproduct of copies of $(\mathsf{V},\hom_{\xi})$.

Next we wish to verify whether the set $\mathcal{C}=\{(\mathsf{V},\hom_{\xi})\}$ satisfies condition **(EP)**. When $T1=1$, $(\mathsf{V},\hom_{\xi})$ is an injective space, and, consequently, under the hypotheses of Theorem 1.3, it is exponentiable, and this is the case for the categories in Table (1.i) as observed in Remark 1.3. Let us consider the binary product $(\mathsf{V}\times\mathsf{V},\hom_{\xi}\times\hom_{\xi})$; we wish to verify whether this product is $(\mathsf{V},\hom_{\xi})$-generated.

Lemma 1 *If $\mathbb{T}=\mathbb{I}$ and V is integral and totally ordered, then $(\mathsf{V}\times\mathsf{V},\hom\times\hom)$ is an Alexandroff V-space.*

Proof. Let us set $d=\hom\times\hom$; denoting by d^c the Alexandroff structure on $\mathsf{V}\times\mathsf{V}$, we have $d^c\leq d$. Now let $(u,v),(u',v')\in\mathsf{V}\times\mathsf{V}$; we wish to verify that $d^c((u,v),(u',v'))\geq d((u,v),(u',v'))$. Consider the following cases:

$\boxed{u\leq u'}$ Since the quantale is integral, $\top\otimes u=u\leq u'$, what is equivalent to $\top\leq\hom(u,u')$, whence

$$d((u,v),(u',v'))=\hom(u,u')\wedge\hom(v,v')=\top\wedge\hom(v,v')=\hom(v,v').$$

Define the maps

$$\begin{array}{rccc} f_{u}\colon & \mathsf{V} & \longrightarrow & \mathsf{V}\times\mathsf{V} \\ & z & \longmapsto & (u,z) \end{array}\qquad\text{and}\qquad \begin{array}{rccc} f_{v'}\colon & \mathsf{V} & \longrightarrow & \mathsf{V}\times\mathsf{V} \\ & z & \longmapsto & (z,v') \end{array}$$

which are continuous, since constant maps are so. Hence

$$d^c((u,v),(u,v'))=d^c(f_{u}(v),f_{u}(v'))\geq\hom(v,v'),$$

and also $d^c((u,v'),(u',v'))\geq\hom(u,u')=\top$. From transitivity of d^c it follows

$$\begin{aligned} d^c((u,v),(u',v')) &\geq d^c((u,v),(u,v'))\otimes d^c((u,v'),(u',v')) \\ &\geq\hom(v,v')\otimes\top=d((u,v),(u',v')). \end{aligned}$$

$\mathfrak{x}\in UX$, $x\in X$,

$$d_{\#}(\mathfrak{x},x)=\sup_{A\in\mathfrak{x}}\inf_{x'\in A}d(x',x).$$

In terms of approach distances, for each $x\in X$, $A\subseteq X$,

$$d_{\#}(x,A)=\inf_{x'\in A}d(x',x)$$

(see [HST14, III-3.4.3(2),2.4.1(1)]). Expressly, if (X,d) is an Alexandroff metric space, then $(X,d_{\#})$ is an Alexandroff approach space; if (X,a) is an Alexandroff approach space, then $(X,a\cdot e_{X})$ is an Alexandroff metric space, and, furthermore, $a=d_{\#}$, with $d=a\cdot e_{X}$, so that, for each $\mathfrak{x}\in UX$, $x\in X$,

$$a(\mathfrak{x},x)=\sup_{A\in\mathfrak{x}}\inf_{x'\in A}a(e_{X}(x'),x).$$

In terms of approach distances, if (X,δ) is an Alexandroff approach space, then, for each $x\in X$, $A\subseteq X$,

$$\begin{aligned} \delta(x,A) &= \inf_{x'\in A}\sup_{B\in e_{X}(x')}\delta(x,B) \\ &= \inf_{x'\in A}\delta(x,\{x'\}), \end{aligned}$$

since $\{x'\}\subseteq B$ implies $\delta(x,B)\leq\delta(x,\{x'\})$. Therefore, Alexandroff approach spaces are the metric approach spaces [Low97, Theorem 3.1.11].

(2) Consider the category $(\mathbb{U},[0,1]_{\odot})$-$\mathsf{Cat}$. Let us verify that the conditions of Lemma 2 hold. Firstly, $\xi\colon U[0,1]\to[0,1]$ is defined by $\xi(\mathfrak{v})=\sup\{u\in[0,1]\mid[u,1]\in\mathfrak{v}\}$. Let us verify the commutativity of the diagram

$$\begin{array}{ccc} U([0,1]\times[0,1]) & \xrightarrow{U(\wedge)} & U[0,1] \\ {\scriptstyle\langle\xi\cdot U\pi_{1},\xi\cdot U\pi_{2}\rangle}\downarrow & \leq & \downarrow{\scriptstyle\xi} \\ [0,1]\times[0,1] & \xrightarrow[\wedge]{} & [0,1]. \end{array}$$

Let $\mathfrak{w}\in U([0,1]\times[0,1])$ and fix $\mathfrak{v}_{i}=U\pi_{i}(\mathfrak{w})$, $i=1,2$. Suppose that $\xi\cdot U(\wedge)(\mathfrak{w})<\xi(\mathfrak{v}_{1})\wedge\xi(\mathfrak{v}_{2})$. Hence there exists $t\in[0,1]$ with $\xi\cdot U(\wedge)(\mathfrak{w})<t<\xi(\mathfrak{v}_{1})\wedge\xi(\mathfrak{v}_{2})$. Proceeding in a similar way as in the first example, this means that $[t,1]\times[t,1]\notin\mathfrak{w}$ and, by definition of ξ, $[t,1]\in\mathfrak{v}_{i}$, $i=1,2$. Then $[t,1]\times[t,1]\in\mathfrak{w}$, a contradiction. Therefore $\xi(\mathfrak{v}_{1})\wedge\xi(\mathfrak{v}_{2})\leq\xi\cdot U(\wedge)(\mathfrak{w})$.

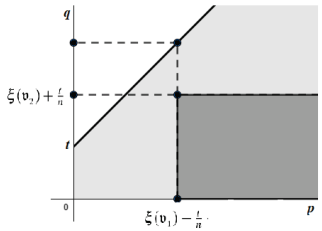
Let $t \in [0,\infty]$ be such that $\xi(\mathfrak{v}_2) \ominus \xi(\mathfrak{v}_1) < t < \xi(U\vec{\ominus}(\mathfrak{w}))$; then $S_t \notin \mathfrak{w}$. Since $\xi(\mathfrak{v}_2) < \xi(\mathfrak{v}_1) + t$, there exists $n \in \mathbb{N}$ such that $\xi(\mathfrak{v}_2) + \frac{t}{n} < \xi(\mathfrak{v}_1) + t$. Let us assume that $\xi(\mathfrak{v}_1) > 0$ so that we can choose $\frac{t}{n} < \xi(\mathfrak{v}_1)$. Hence

$$\xi(\mathfrak{v}_1) - \frac{t}{n} < \xi(\mathfrak{v}_1) \;\Rightarrow\; [0, \xi(\mathfrak{v}_1) - \frac{t}{n}] \notin \mathfrak{v}_1 \;\Rightarrow\;]\xi(\mathfrak{v}_1) - \frac{t}{n}, \infty] \in \mathfrak{v}_1$$
$$\Leftrightarrow\;]\xi(\mathfrak{v}_1) - \frac{t}{n}, \infty] \times [0,\infty] \in \mathfrak{w}$$

and

$$\xi(\mathfrak{v}_2) + \frac{t}{n} > \xi(\mathfrak{v}_2) \;\Rightarrow\; [0, \xi(\mathfrak{v}_2) + \frac{t}{n}] \in \mathfrak{v}_2 \;\Leftrightarrow\; [0,\infty] \times [0, \xi(\mathfrak{v}_2) + \frac{t}{n}] \in \mathfrak{w}.$$

Hence $]\xi(\mathfrak{v}_1) - \frac{t}{n}, \infty] \times [0, \xi(\mathfrak{v}_2) + \frac{t}{n}] \in \mathfrak{w}$, but $]\xi(\mathfrak{v}_1) - \frac{t}{n}, \infty] \times [0, \xi(\mathfrak{v}_2) + \frac{t}{n}] \subseteq S_t$, what implies that $S_t \in \mathfrak{w}$, a contradiction.



In the case $\xi(\mathfrak{v}_1) = 0$, we have that $\xi(\mathfrak{v}_2) + \frac{t}{n} < t$, whence $[0,\infty] \times [0, \xi(\mathfrak{v}_2) + \frac{t}{n}] \in \mathfrak{w}$, and $[0,\infty] \times [0, \xi(\mathfrak{v}_2) + \frac{t}{n}] \subseteq S_t$, so we obtain a contradiction.

Finally, for each set X, if $\mathfrak{w} \in U(X \times X)$ is such that, for $(x, x') \in X \times X$, $U\pi_1^X(\mathfrak{w}) = e_X(x)$ and $U\pi_2^X(\mathfrak{w}) = e_X(x')$, then $\{x\} \in U\pi_1^X(\mathfrak{w})$ is equivalent to $(\pi_1^X)^{-1}(\{x\}) = \{x\} \times X \in \mathfrak{w}$, and $\{x'\} \in U\pi_2^X(\mathfrak{w})$ is equivalent to $(\pi_2^X)^{-1}(\{x'\}) = X \times \{x'\} \in \mathfrak{w}$, whence

$$(\{x\} \times X) \cap (X \times \{x'\}) = \{(x, x')\} \in \mathfrak{w},$$

that is, $\mathfrak{w} = e_{X\times X}(x, x')$. Thus $(e_X \times e_X)^\circ \cdot \mathrm{can}_{X,X} \leq e_{X\times X}$.

Therefore we conclude that Alexandroff approach spaces form a cartesian closed subcategory of App. Moreover, Alexandroff approach spaces are precisely the images of Alexandroff metric spaces by the functor $A^\circ \colon \mathsf{Met} \to \mathsf{App}$, where, for each metric space (X, d), $A^\circ(X, d) = (X, d_\#)$ with, for each

The case $v \leq v'$ is analogous.

$\boxed{u > u' \;\&\; v > v'}$ Let us set

$$\gamma = d((u, v), (u', v')) = \hom(u, u') \wedge \hom(v, v'),$$

and observe that, similar to what we have in the first case, $\hom(u', u) \wedge \hom(v', v) = \top$. Consider the subset $\{\gamma, \top\} \subseteq \mathsf{V}$ endowed with the subspace V-structure. Define the map

$$f \colon \{\gamma, \top\} \to \mathsf{V} \times \mathsf{V}, \quad \gamma \mapsto (u', v'), \quad \top \mapsto (u, v).$$

Since $\gamma \leq \top$, we have $\hom(\gamma, \top) = \top$, whence

$$\hom(\gamma, \top) = \hom(u', u) \wedge \hom(v', v) = d(f(\gamma), f(\top));$$

also, by formula (1.vii), $\hom(\top, \gamma) = \bigvee\{w \in \mathsf{V} \mid w \otimes \top = w \leq \gamma\} = \gamma$, whence

$$\hom(\top, \gamma) = \hom(u, u') \wedge \hom(v, v') = d(f(\top), f(\gamma)).$$

Thus f is fully faithful, and since $(\mathsf{V} \times \mathsf{V}, \hom \times \hom)$ is a separated injective space, there exists a continuous map $\hat{f} \colon \mathsf{V} \to \mathsf{V} \times \mathsf{V}$ extending f along the embedding of $\{\gamma, \top\}$ into V:

$$\begin{array}{ccc} \{\gamma, \top\} & \hookrightarrow & \mathsf{V} \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle \hat{f}} & \\ & \mathsf{V} \times \mathsf{V}. & \end{array}$$

Hence

$$d^c((u, v), (u', v')) = d^c(\hat{f}(\top), \hat{f}(\gamma)) \geq \hom(\top, \gamma) = \gamma = d((u, v), (u', v')). \quad \blacksquare$$

Therefore, for $\mathbb{T} = \mathbb{I}$ and V integral and totally ordered, $\mathcal{C}$ satisfies condition **(EP)**, so that Alexandroff spaces form a cartesian closed subcategory of V-Cat. It is straightforward to verify that in Ord every space is Alexandroff. For Met, UltMet, and $\mathsf{B_1Met}$, Alexandroff spaces are the coequalizers of coproducts of copies of $([0,\infty], \ominus)$, $([0,\infty], \circledcirc)$, and $([0,1], \circledast)$, respectively (see (1.viii)).

For $(\mathbb{U}, 2)$-$\mathsf{Cat} \cong \mathsf{Top}$, as observed in [ELS04, Section 3], a topological space is Alexandroff in our sense if, and only if, it is Alexandroff in the classical sense, that is, if arbitrary intersections of open sets are open, which in turn is equivalent to each point to have a smallest open set containing it. This property trivially holds for the binary product $\mathbb{S} \times \mathbb{S}$ of Sierpiński spaces,

since its topology is finite, whence $\mathbb{S}\times\mathbb{S}$ is Alexandroff. We recover the fact that the subcategory of Alexandroff topological spaces is cartesian closed.

In fact, it is well-known that the subcategory of Alexandroff topological spaces is equivalent to Ord (see, for instance [HST14, II-5.10.5]). Motivated by this case, let us consider the pair of adjoint functors (see [CT03], [HST14, III-3.4,3.6])

$$\begin{array}{ccc} \mathsf{V}\text{-}\mathsf{Cat} & \underset{A_e}{\overset{A^\circ}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} & (\mathbb{T},\mathsf{V})\text{-}\mathsf{Cat} \\ & {\scriptstyle \mathrm{top.}}\searrow \quad \swarrow{\scriptstyle \mathrm{top.}} & \\ & \mathsf{Set}, & \end{array} \tag{3.i}$$

where, for each $(\mathbb{T},\mathsf{V})$-space (X,a), $A_e(X,a)=(X,a\cdot e_X)$, with $e_X\colon X\to TX$ the X-component of the natural transformation $e\colon \mathrm{Id}_{\mathsf{Set}}\to T$, and, for each V-space (Y,β), $A^\circ(Y,\beta)=(Y,\beta_\#)$, with $\beta_\#=e_Y^\circ\cdot T\beta$; on morphisms both functors are the identity, and the functors into Set are the forgetful functors which are topological.

When $\mathbb{T}=\mathbb{U}$ and $\mathsf{V}=\mathsf{2}$, the instance of adjunction (3.i) gives

$$\mathsf{Ord} \underset{A_e}{\overset{A^\circ}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathsf{Top},$$

where to each ordered set $(X,\leq)$ is assigned the space $(X,\tau_\leq)$, with $\tau_\leq$ the Alexandroff topology, that is, the topology that has as a basis the sets $\downarrow x$, $x\in X$; and to each topological space (X,τ) is assigned the ordered set $(X,\leq_\tau)$, where $\leq_\tau$ is the dual of the *specialization order*, that is,

$$x\leq_\tau y \iff \dot{x}\to y,$$

where $\dot{x}$ is the principal ultrafilter generated by $\{x\}$ and $\to$ denotes the convergence relation between ultrafilters and points defined by τ.

Moreover, Alexandroff topological spaces are precisely the spaces that are the image by A° of an ordered set [HST14, II-5.10.5, III-3.4.3(1)]. We wish to find conditions under which this fact holds in our general setting. Firstly, we must have that $(\mathsf{V},\hom_\xi)$ itself is the image by A° of some Alexandroff V-space. The natural candidate is $(\mathsf{V},\hom)$, so we wish to verify under which conditions we have $\hom_\xi=\hom_\#=e_\mathsf{V}^\circ\cdot T\hom$. By (1.vi), for each $\mathfrak{v}\in T\mathsf{V}$,

Therefore, under the conditions of Lemma 2, $\mathcal{C}=\{(\mathsf{V},\hom_\xi)\}$ satisfies condition **(EP)** and Alexandroff $(\mathbb{T},\mathsf{V})$-spaces form a cartesian closed subcategory of $(\mathbb{T},\mathsf{V})$-Cat.

Examples (1) Let us verify that, for the category $\mathsf{App}\cong(\mathbb{U},\mathsf{P}_+)$-$\mathsf{Cat}$, the conditions of Lemma 2 are satisfied. In this case, $\xi\colon U[0,\infty]\to[0,\infty]$ is given by $\xi(\mathfrak{v})=\inf\{u\in[0,\infty]\mid[0,u]\in\mathfrak{v}\}$, for each $\mathfrak{v}\in U[0,\infty]$ (see, for instance, [Hof14]). Moreover, with the same definition of ξ, $(\mathbb{U},\mathsf{P}_{\max},\xi)$ is a strict topological theory, hence the commutativity of the diagram

$$\begin{array}{ccc} U([0,\infty]\times[0,\infty]) & \xrightarrow{U(\max)} & U[0,\infty] \\ {\scriptstyle\langle\xi\cdot U\pi_1,\xi\cdot U\pi_2\rangle}\downarrow & & \downarrow{\scriptstyle\xi} \\ [0,\infty]\times[0,\infty] & \xrightarrow[\max]{} & [0,\infty] \end{array}$$

follows from (1.v). Consider the diagram

$$\begin{array}{ccc} U([0,\infty]\times[0,\infty]) & \xrightarrow{U\vec{\ominus}} & U[0,\infty] \\ {\scriptstyle\langle\xi\cdot U\pi_1,\xi\cdot U\pi_2\rangle}\downarrow & & \downarrow{\scriptstyle\xi} \\ [0,\infty]\times[0,\infty] & \xrightarrow[\vec{\ominus}]{} & [0,\infty]. \end{array}$$

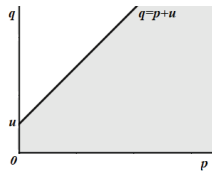
We know that $\vec{\ominus}\cdot\langle\xi\cdot U\pi_1,\xi\cdot U\pi_2\rangle\leq\xi\cdot U\vec{\ominus}$ (recall that the order on $[0,\infty]$ is $\geq$). Let us assume that $\vec{\ominus}\cdot\langle\xi\cdot U\pi_1,\xi\cdot U\pi_2\rangle<\xi\cdot U\vec{\ominus}$. Hence there exists $\mathfrak{w}\in U([0,\infty]\times[0,\infty])$ such that, fixing $\mathfrak{v}_1=U\pi_1(\mathfrak{w})$ and $\mathfrak{v}_2=U\pi_2(\mathfrak{w})$, we have

$$\xi(\mathfrak{v}_2)\ominus\xi(\mathfrak{v}_1)<\xi(U\vec{\ominus}(\mathfrak{w}))=\inf\{u\in[0,\infty]\mid[0,u]\in U\vec{\ominus}(\mathfrak{w})\}.$$

Here $[0,u]\in U\vec{\ominus}(\mathfrak{w})$ if, and only if, $(\vec{\ominus})^{-1}([0,u])\in\mathfrak{w}$, so that $\xi(U\vec{\ominus}(\mathfrak{w}))=\inf\{u\in[0,\infty]\mid S_u\in\mathfrak{w}\}$, where the set

$$S_u=(\vec{\ominus})^{-1}([0,u])=\{(p,q)\in[0,\infty]\times[0,\infty]\mid q\ominus p\leq u\}$$

can be depicted as the gray area in the graphic below.



Lemma 2 *For* V *integral and totally ordered, if the diagrams*

$$\begin{array}{ccc} T(\mathsf{V}\times\mathsf{V}) & \xrightarrow{T(\wedge)} & T\mathsf{V} \\ {\scriptstyle\langle \xi\cdot T\pi_1, \xi\cdot T\pi_2\rangle}\downarrow & \leq & \downarrow{\scriptstyle\xi} \\ \mathsf{V}\times\mathsf{V} & \xrightarrow[\wedge]{} & \mathsf{V} \end{array} \qquad \begin{array}{ccc} T(\mathsf{V}\times\mathsf{V}) & \xrightarrow{\overrightarrow{T\mathrm{hom}}} & T\mathsf{V} \\ {\scriptstyle\langle \xi\cdot T\pi_1, \xi\cdot T\pi_2\rangle}\downarrow & & \downarrow{\scriptstyle\xi} \\ \mathsf{V}\times\mathsf{V} & \xrightarrow[\overrightarrow{\hom}]{} & \mathsf{V} \end{array}$$

are (lax) commutative and the inequality (3.iii) *holds for* $X = \mathsf{V}$*, i.e.,* $(e_{\mathsf{V}} \times e_{\mathsf{V}})^\circ \cdot \mathrm{can}_{\mathsf{V},\mathsf{V}} \leq e^\circ_{\mathsf{V}\times\mathsf{V}}$*, then* $(\mathsf{V}\times\mathsf{V}, \hom_\xi \times \hom_\xi)$ *is an Alexandroff* $(\mathbb{T},\mathsf{V})$*-space.*

Proof. By Lemma 1 and the previous Theorem, it suffices to show that

$$(\mathsf{V}\times\mathsf{V}, \hom_\xi \times \hom_\xi) = A^\circ(\mathsf{V}\times\mathsf{V}, \hom\times\hom).$$

For each $(u,v),(z,w) \in \mathsf{V}\times\mathsf{V}$,

$$\begin{aligned} (\hom_\xi \times \hom_\xi)\cdot e_{\mathsf{V}\times\mathsf{V}}((u,v),(z,w)) &= \hom_\xi \times \hom_\xi(e_{\mathsf{V}\times\mathsf{V}}(u,v),(z,w)) \\ &= \hom_\xi(e_{\mathsf{V}}(u),z) \wedge \hom_\xi(e_{\mathsf{V}}(v),w) \\ &= \hom\times\hom((u,v),(z,w)). \end{aligned}$$

Hence $(\mathsf{V}\times\mathsf{V}, (\hom\times\hom)_\#) = A^\circ(\mathsf{V}\times\mathsf{V}, \hom\times\hom) = A^\circ\cdot A_e(\mathsf{V}\times\mathsf{V}, \hom_\xi \times \hom_\xi)$, so that $(\hom\times\hom)_\# \leq \hom_\xi \times \hom_\xi$, since the counit of $A^\circ \dashv A_e$ is an identity map. Conversely, for each $\mathfrak{w} \in T(\mathsf{V}\times\mathsf{V})$, $(u,v) \in \mathsf{V}\times\mathsf{V}$,

$$\begin{aligned} &\hom_\xi\times\hom_\xi(\mathfrak{w},(u,v)) = \\ &\quad= \hom_\xi(T\pi_1(\mathfrak{w}),u) \wedge \hom_\xi(T\pi_2(\mathfrak{w}),v) \\ &\quad= e^\circ_{\mathsf{V}}\cdot T\mathrm{hom}(T\pi_1(\mathfrak{w}),u) \wedge e^\circ_{\mathsf{V}}\cdot T\mathrm{hom}(T\pi_2(\mathfrak{w}),v) \\ &\quad= T\mathrm{hom}\times T\mathrm{hom}(\mathfrak{w}, e_{V}\times e_{\mathsf{V}}(u,v)) \\ &\quad= (e_{\mathsf{V}}\times e_{\mathsf{V}})^\circ\cdot(T\mathrm{hom}\times T\mathrm{hom})(\mathfrak{w},(u,v)) \\ &\quad= (e_{\mathsf{V}}\times e_{\mathsf{V}})^\circ\cdot(T\mathrm{hom} \oslash T\mathrm{hom})\cdot \mathrm{can}_{\mathsf{V},\mathsf{V}}(\mathfrak{w},(u,v)) \\ &\quad\leq (e_{\mathsf{V}}\times e_{\mathsf{V}})^\circ\cdot \mathrm{can}_{\mathsf{V},\mathsf{V}}\cdot T(\hom \oslash \hom)(\mathfrak{w},(u,v)) \qquad \text{(by (1.ix))} \\ &\quad= (e_{\mathsf{V}}\times e_{\mathsf{V}})^\circ\cdot \mathrm{can}_{\mathsf{V},\mathsf{V}}\cdot T(\hom\times\hom)(\mathfrak{w},(u,v)) \\ &\quad\leq e^\circ_{\mathsf{V}\times\mathsf{V}}\cdot T(\hom\times\hom)(\mathfrak{w},(u,v)) \qquad \text{(by hypothesis)} \\ &\quad= (\hom\times\hom)_\#(\mathfrak{w},(u,v)). \quad \blacksquare \end{aligned}$$

$v \in \mathsf{V}$,

$$\begin{aligned} e^\circ_{\mathsf{V}}\cdot T\mathrm{hom}(\mathfrak{v},v) &= T\mathrm{hom}(\mathfrak{v}, e_{\mathsf{V}}(v)) \\ &= \bigvee\{\xi\cdot \overrightarrow{T\mathrm{hom}}(\mathfrak{w}) \mid \mathfrak{w}\in T(\mathsf{V}\times\mathsf{V}),\ T\pi_1(\mathfrak{w}) = \mathfrak{v},\ T\pi_2(\mathfrak{w}) = e_{\mathsf{V}}(v)\}. \end{aligned}$$

Furthermore, by [Hof07, Lemma 3.2], lax commutativity of the diagram

$$\begin{array}{ccc} T(\mathsf{V}\times\mathsf{V}) & \xrightarrow{\overrightarrow{T\mathrm{hom}}} & T\mathsf{V} \\ {\scriptstyle\langle \xi\cdot T\pi_1, \xi\cdot T\pi_2\rangle}\downarrow & \geq & \downarrow{\scriptstyle\xi} \\ \mathsf{V}\times\mathsf{V} & \xrightarrow[\overrightarrow{\hom}]{} & \mathsf{V} \end{array}$$

is assured. Hence, for each $\mathfrak{w} \in T(\mathsf{V}\times\mathsf{V})$ such that $T\pi_1(\mathfrak{w}) = \mathfrak{v}$ and $T\pi_2(\mathfrak{w}) = e_{\mathsf{V}}(v)$, we have $\xi\cdot\overrightarrow{T\mathrm{hom}}(\mathfrak{w}) \leq \hom_\xi(\mathfrak{v},v)$, and consequently $e^\circ_{\mathsf{V}}\cdot T\mathrm{hom}(\mathfrak{v},v) \leq \hom_\xi(\mathfrak{v},v)$. Then we can see that the required condition is strict commutativity of the latter diagram.

Theorem *If the diagram below is commutative, then the functor* A° *preserves Alexandroff spaces.*

$$\begin{array}{ccc} T(\mathsf{V}\times\mathsf{V}) & \xrightarrow{\overrightarrow{T\mathrm{hom}}} & T\mathsf{V} \\ {\scriptstyle\langle \xi\cdot T\pi_1, \xi\cdot T\pi_2\rangle}\downarrow & & \downarrow{\scriptstyle\xi} \\ \mathsf{V}\times\mathsf{V} & \xrightarrow[\overrightarrow{\hom}]{} & \mathsf{V} \end{array} \tag{3.ii}$$

Proof. Commutativity of (3.ii) implies that $A^\circ(\mathsf{V},\hom) = (\mathsf{V},\hom_\xi)$. Let (X,α) be an Alexandroff V-space, and $(X,a) = A^\circ(X,\alpha)$. Let $h\colon (X,a) \to (Y,b)$ be a map such that, for every continuous map $f\colon (\mathsf{V},\hom_\xi) \to (X,a)$, the composite $h\cdot f$ is continuous. We wish to prove that h is a $(\mathbb{T},\mathsf{V})$-continuous map. Since $A^\circ \dashv A_e$, we only need to verify that $h\colon (X,\alpha) \to A_e(Y,b) = (Y, b\cdot e_Y)$ is a V-continuous map, which holds if, and only if, for each V-continuous map $f\colon (\mathsf{V},\hom) \to (X,\alpha)$, the composite $h\cdot f\colon (\mathsf{V},\hom) \to (Y, b\cdot e_Y)$ is V-continuous, since (X,α) is Alexandroff.

Each V-continuous map f from $(\mathsf{V},\hom)$ to (X,α) becomes a $(\mathbb{T},\mathsf{V})$-continuous map from $(\mathsf{V},\hom_\xi)$ to (X,a) by applying the functor A°. Therefore, $h\cdot f\colon (\mathsf{V},\hom_\xi) \to (Y,b)$ is a $(\mathbb{T},\mathsf{V})$-continuous map, whence

$$h\cdot f\colon (\mathsf{V},\hom) \to (Y, b\cdot e_Y)$$

is a V-continuous map. ∎

Proposition *(1) If (X,a) is an Alexandroff $(\mathbb{T},\mathsf{V})$-space, then $(X,a) = A^\circ \cdot A_e(X,a)$.*
(2) If $\mathbb{T}$ is such that, for each set X,

$$(e_X \times e_X)^\circ \cdot \mathrm{can}_{X,X} \leq e^\circ_{X\times X}, \tag{3.iii}$$

where $\mathrm{can}_{X,X} = \langle T\pi_1^X, T\pi_2^X\rangle \colon T(X\times X) \to TX \times TX$, then, for each V-space (X,α), we have $(X,\alpha) = A_e \cdot A^\circ(X,\alpha)$.

Proof. (1) Let us verify that $a = e^\circ_X \cdot Ta \cdot Te_X$. The equality $m_X \cdot Te_X = 1_{TX}$ implies the inequality $Te_X \leq m^\circ_X$, and $1_X \leq a \cdot e_X$ is equivalent to $e^\circ_X \leq a$, whence

$$e^\circ_X \cdot Ta \cdot Te_X \leq a \cdot Ta \cdot m^\circ_X \leq a \cdot m_X \cdot m^\circ_X \leq a$$

(see [HST14, III-3.4.2]). For the converse inequality, by the adjunction $A^\circ \dashv A_e$, each continuous map $f\colon (\mathsf{V},\hom_\xi) = A^\circ(\mathsf{V},\hom) \to (X,a)$ is continuous from $(\mathsf{V},\hom)$ to $A_e(X,a) = (X, a\cdot e_X)$, and applying A° we obtain a continuous map $f\colon (\mathsf{V},\hom_\xi) \to A^\circ(X,a\cdot e_X) = (X, e^\circ_X \cdot Ta \cdot Te_X)$. Since (X,a) is Alexandroff, the identity map 1_X is continuous,

$$\begin{array}{ccc} (\mathsf{V},\hom_\xi) & \xrightarrow{\ f\ } & (X,a) \\ & \searrow^{f} & \downarrow{\scriptstyle 1_X} \\ & & (X, e^\circ_X \cdot Ta \cdot Te_X) \end{array}$$

hence $a \leq e^\circ_X \cdot Ta \cdot Te_X$.
(2) For each set X, $\mathfrak{w} \in T(X\times X)$, $(x,x') \in X\times X$,

$$\begin{array}{rcl} e^\circ_{X\times X}(\mathfrak{w},(x,x')) = k & \iff & e_{X\times X}(x,x') = \mathfrak{w} \\ & \implies & \big(T\pi_1^X(\mathfrak{w}) = e_X(x) \quad \& \quad T\pi_2^X(\mathfrak{w}) = e_X(x')\big) \\ & \iff & \mathrm{can}_{X,X}(\mathfrak{w}) = e_X \times e_X(x,x') \\ & \iff & (e_X\times e_X)^\circ \cdot \mathrm{can}_{X,X}(\mathfrak{w},(x,x')) = \\ & & = \mathrm{can}_{X,X}(\mathfrak{w}, e_X \times e_X(x,x')) = k, \end{array}$$

hence $e^\circ_{X\times X} \leq (e_X\times e_X)^\circ \cdot \mathrm{can}_{X,X}$, and if (3.iii) holds, then it is an equality.

For each $(x,x') \in X\times X$, if $\mathfrak{w} \in T(X\times X)$ is such that $T\pi_1^X(\mathfrak{w}) = e_X(x)$ and $T\pi_2^X(\mathfrak{w}) = e_X(x')$, then $e_{X\times X}(x,x') = \mathfrak{w}$, and, for each V-space (X,α),

we calculate:

$$\begin{aligned} &T\alpha(e_X(x), e_X(x')) = \\ &\quad = \bigvee\{\xi \cdot T\vec{\alpha}(\mathfrak{w}) \mid \mathfrak{w} \in T(X\times X),\ T\pi_1^X(\mathfrak{w}) = e_X(x),\ T\pi_2^X(\mathfrak{w}) = e_X(x')\} \\ &\quad = \xi \cdot T\vec{\alpha} \cdot e_{X\times X}(x,x') \\ &\quad = \xi \cdot e_{\mathsf{V}} \cdot \vec{\alpha}(x,x') \qquad (e \text{ is a natural transformation}) \\ &\quad = \vec{\alpha}(x,x') \qquad (\text{because } \xi\colon T\mathsf{V} \to \mathsf{V} \text{ is a } \mathbb{T}\text{-algebra}). \end{aligned}$$

Therefore $(X,\alpha) = (X, e^\circ_X \cdot T\alpha \cdot e_X) = A_e \cdot A^\circ(X,\alpha)$. ∎

Corollary *If the diagram* (3.ii) *is commutative and $\mathbb{T}$ satisfies* (3.iii)*, then the Alexandroff $(\mathbb{T},\mathsf{V})$-spaces are precisely the images by A° of Alexandroff V-spaces.*

Proof. By the previous Theorem, commutativity of (3.ii) implies that the image by A° of any Alexandroff V-space is an Alexandroff $(\mathbb{T},\mathsf{V})$-space.

Conversely, let (X,a) be an Alexandroff $(\mathbb{T},\mathsf{V})$-space. Then, by item (1) of the previous Proposition, $(X,a) = A^\circ \cdot A_e(X,a)$. Let us verify that $(X,\alpha) = A_e(X,a)$ is an Alexandroff V-space. Let (Y,β) be a V-space and $f\colon (X,\alpha) \to (Y,\beta)$ be a map such that, for every V-continuous map $p\colon (\mathsf{V},\hom) \to (X,\alpha)$, the composite $f\cdot p\colon (\mathsf{V},\hom) \to (Y,\beta)$ is V-continuous.

$$\begin{array}{ccc} (\mathsf{V},\hom) & \xrightarrow{\ p\ } & (X,\alpha) \\ & \searrow^{f\cdot p} & \downarrow{\scriptstyle f} \\ & & (Y,\beta) \end{array}$$

We wish to prove that f is V-continuous. Since $\mathbb{T}$ satisfies (3.iii), by item (2) of the previous Proposition, $(Y,\beta) = A_e \cdot A^\circ(Y,\beta)$. Hence, by the adjunction $A^\circ \dashv A_e$, $f\colon (X,\alpha) \to A_e \cdot A^\circ(Y,\beta)$ is V-continuous if, and only if, $f\colon A^\circ(X,\alpha) = (X,a) \to A^\circ(Y,\beta)$ is $(\mathbb{T},\mathsf{V})$-continuous. Consider a $(\mathbb{T},\mathsf{V})$-continuous map $p\colon (\mathsf{V},\hom_\xi) = A^\circ(\mathsf{V},\hom) \to (X,a)$. Then $p\colon (\mathsf{V},\hom) \to (X,\alpha)$ is a V-continuous map. Hence, by hypothesis, the composite $f\cdot p\colon (\mathsf{V},\hom) \to (Y,\beta) = A_e \cdot A^\circ(Y,\beta)$ is V-continuous, and so

$$f\cdot p\colon A^\circ(\mathsf{V},\hom) = (\mathsf{V},\hom_\xi) \to A^\circ(Y,\beta)$$

is $(\mathbb{T},\mathsf{V})$-continuous. Since (X,a) is an Alexandroff $(\mathbb{T},\mathsf{V})$-space, we conclude that $f\colon (X,a) \to A^\circ(Y,\beta)$ is $(\mathbb{T},\mathsf{V})$-continuous. Therefore, (X,α) is $(\mathsf{V},\hom)$-generated. ∎