and $(q(a))^{-1}(\hat{H}) = \triangle a^{-1}(H)$. The initiality follows immediately, since, for every open set G of X, $\triangle G = \triangle \chi_G^{-1}(\{1\})$ for χ_G the characteristic function into the Sierpinski space.

Example 5.15. Let $\mathcal{K} = \Pi_0$, the category of T_0-topological spaces and continuous maps, and $\mathcal{A}$ consist of the finite T_0 spaces. The corresponding codensity monad is the prime open filter monad. More precisely, for every space X, TX is the set of all prime filters on $(\Omega X, \subseteq)$ with the topology having as a basis all sets of the form

$$\Box G = \{\mathcal{U} \in TX \mid G \in \mathcal{U}\}, \quad G \text{ open in } X.$$

The proof is analogous to the one for posets, using as cogenerator the Sierpinski space.

References

[1] J. Adámek, A. Brooke-Taylor, T. Campion, L. Positselski and J. Rosický, Colimit-dense subcategories, to appear in Comment. Math. Univ. Carolinae, arXiv:1812.10649.

[2] B. Banaschewski and E. Nelson, Tensor products and bimorphisms, *Canad. Math. Bull.* 19 (1976), 385–402.

[3] B.-P. Devlin, *Codensity, Compactness and Ultrafilters* PhD Thesis, Univ. of Edinburgh, 2015.

[4] F. Galvin and A. Horn, Operations preserving all equivalence relations, *Proc. AMS* 24 (1970), 521–523.

[5] P. Gabriel and F. Ulmer, *Lokal präsentierbare Kategorien*, Lect. Notes in Math. 221, Springer-Verlag, Berlin, 1971.

[6] J.R. Isbell, Adequate subcategories, *Illinois J. Math.* 4 (1960), 541–552.

[7] J. F. Kennison and D. Gildenhuys, Equational completion, model induced triples and pro-objects, *J. Pure Appl. Algebra* 1 (1971), 317–346.

[8] A. Kock, Strong functors and monoidal monads, *Archiv der Mathematik* 23 (1972), 113–120.

[9] T. Leinster, Codensity and the ultrafilter monad, *Theory and Applications of Categories* 28 (2013), 332–370.

[10] F. E. J. Linton, Autonomous equational categories, *J. Math. Mech.* 15 (1966), 637–642.

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D-ULTRAFILTERS AND THEIR MONADS

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Abstract: For a number of locally finitely presentable categories $\mathcal{K}$ we describe the codensity monad of the full embedding of all finitely presentable objects into $\mathcal{K}$. We introduce the concept of D-ultrafilter on an object, where D is a "nice" cogenerator of $\mathcal{K}$. We prove that the codensity monad assigns to every object an object representing all D-ultrafilters on it. Our result covers e.g. categories of sets, vector spaces, posets, semilattices, graphs and M-sets for finite commutative monoids M.

1. Introduction

We present a generalization of the concept of an ultrafilter on a set: for a number of categories $\mathcal{K}$ we define D-ultrafilters on an object of $\mathcal{K}$. Here D is a cogenerator of $\mathcal{K}$ with a special property; we speak about $*$-cogenerators, see below. For example $D = \{0,1\}$ is a $*$-cogenerator of Set, here D-ultrafilters are the usual ultrafilters. By a classical result of Kennison and Gildenhuys [7] the ultrafilter monad on Set (assigning to every set the set of all ultrafilters) is the codensity monad of the embedding $\mathrm{Set}_{fp} \hookrightarrow \mathrm{Set}$ of finite sets. We will prove that, in general, the corresponding monad of D-ultrafilters on $\mathcal{K}$ is the codensity monad of the embedding $\mathcal{K}_{fp} \hookrightarrow \mathcal{K}$ of finitely presentable objects of $\mathcal{K}$.

We consider closed monoidal categories, thus, the given cogenerator yields a contravariant endofunctor $[-, D]$. Our examples include all commutative varieties, for instance, vector spaces, semilattices or M-sets for finite commutative monoids M. Recall that a variety of algebras is closed monoidal with respect to the usual tensor product if and only if it is commutative (aka entropic), see [2]. Another sort of examples are cartesian closed categories such as posets or graphs.

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All of our examples (except the last section presenting some generalizations) are locally finitely presentable categories in the sense of Gabriel and Ulmer [5]. One of the most important features of locally finitely presentable categories $\mathcal{K}$ is that the full embedding

$$E : \mathcal{K}_{fp} \hookrightarrow \mathcal{K}$$

of all finitely presentable objects is dense, i.e. every object X is a canonical colimit of all morphisms $a : A \to X$ with A finitely presentable. More precisely: the forgetful functor $\mathcal{K}_{fp}/X \to \mathcal{K}$ of the coslice category has colimit X with the canonical colimit cocone.

Not surprisingly, finitely presentable objects are usually *not* codense. A measure of how "far away" a functor E is from being codense is the codensity monad $\mathbb{T}$ of E. This monad is given by the right Kan extension of E along itself:

$$T = \mathrm{Ran}_E E,$$

see below. For codense functors E, this is the trivial monad Id.

Recently, Leinster proved that the codensity monad of the embedding of finite-dimensional vector spaces into the category K-Vec of vector spaces over a field K is the *double-dualization monad*

$$TX = X^{**}.$$

And he asked for a general description of the codensity monad of $E : \mathcal{K}_{fp} \hookrightarrow \mathcal{K}$ for locally finitely presentable categories $\mathcal{K}$.

The purpose of our paper is to answer to Leinster's question. Not for general locally finitely presentable categories, but for quite some. Given a cogenerator D we denote by $(-)^* = [-, D]$ the contravariant endofunctor $X \mapsto [X, D]$; then D is a *$*$-cogenerator* if for every object X we have that X^* is a canonical colimit of objects A^* with A finitely presentable. We prove that every finitely presentable cogenerator is a $*$-cogenerator. The composite $(-)^{**}$ of $(-)^*$ with itself is the well-known double-dualization monad (relative to D).

We introduce the concept of a D-ultrafilter on an object X and form the corresponding D-ultrafilter monad on $\mathcal{K}$ as a submonad of the double-dualization monad. This turns out to be the desired codensity monad of E :

To prove additivity,

$$\overline{a_1 + a_2} = \overline{a_1} + \overline{a_2} \quad \text{for } a_1, a_2 \in X^*$$

consider the projections as morphisms

$$\pi_i : (K^2, < a_1, a_2 >) \to (K, a_i) \qquad (i = 1, 2)$$

of $X/\mathcal{A}$ which by compatibility yield

$$\pi_i \cdot \overline{< a_1, a_2 >} = \overline{a_i}.$$

That is,

$$\overline{< a_1, a_2 >} = < \overline{a_1}, \overline{a_2} > .$$

We also have a morphism

$$\pi_1 + \pi_2 : (K^2, < a_1, a_2 >) \to (K, a_1 + a_2)$$

therefore

$$(\pi_1 + \pi_2) \cdot \overline{< a_1, a_2 >} = \overline{a_1 + a_2}.$$

Since $(\pi_1 + \pi_2) \cdot < \overline{a_1}, \overline{a_1} > = \overline{a_1} + \overline{a_2}$, the proof is complete.

Example 5.14. Let $\mathcal{K} = \mathbb{T}$, the category of topological spaces and continuous maps, and $\mathcal{A} = \mathbb{T}_\mathrm{f}$ consist of all finite spaces. The corresponding codensity monad $\mathbb{T}$ is, as for sets, the ultrafilter monad. More precisely, for every space X, TX is the set of all ultrafilters on the underlying set of X with the topology τ having as a basis all sets of the form

$$\triangle G = \{\mathcal{U} \in TX \mid G \in \mathcal{U}\}, \quad G \text{ open in } X.$$

To see this, let $D = \{0, 1\}$ be equipped with the indiscrete topology. This is a cogenerator of $\mathbb{T}$, and the space $SX = D^{\mathbb{T}(X,D)}$ is the indiscrete space $\mathcal{P}\mathcal{P}X$ of all collections of subsets of X. The proof that the ultrafilters on the underlying set of a topological space X coincide with D-ultrafilters on X is completely analogous to that of Lemma 3.3.

To verify that τ is the topology of TX, we just need to show that τ makes all the morphisms $q(a)$ (see diagram (10) of Theorem 5.5) continuous and jointly initial. Indeed, the open sets of A_0 are of the form

$$\hat{H} = \{\mathcal{U} \in SX \mid a^{-1}(H) \in \mathcal{U}\} \qquad \text{for } H \text{ an open set of } A,$$

(a) If λ is an infinite cardinal, then the codensity monad is analogous to the above example of $\mathsf{Set}_\lambda \hookrightarrow \mathsf{Set}$, see [1]. A vector x in X^{**} is called *λ-complete* if for every linear decomposition $e : X \to A$ with $\dim A < \lambda$, $e^{**}(x)$ is an evaluation (a vector of $\eta_A[A]$). All λ-complete vectors form a submonad of $(-)^{**}$. And this is the codensity monad of $K\text{-}\mathsf{Vec}_\lambda \hookrightarrow K\text{-}\mathsf{Vec}$.

(b) For $\mathcal{A} = \{K\}$ the codensity monad is larger than $(-)^{**}$: it assigns to X all *homogeneous* functions from X^* to K (i.e., those preserving the scalar multiplication). More precisely, T is the subfunctor of $SX = K^{X^*}$ given by

$$TX = \text{ all homogeneous functions in } K^{X^*}.$$

Indeed, the diagram C_X given by $(X \xrightarrow{a} K) \mapsto K$ has the cone $\pi_a : TX \to K$ formed by restrictions of the projections of K^{X^*}. That is,

$$\pi_a(h) = h(a) \quad \text{for } h \in TX,\ a \in X^*.$$

To prove that this is a limit cone, let another cone with domain Z be given:

$$\frac{X \xrightarrow{a} K}{Z \xrightarrow{\bar{a}} K}$$

It is compatible, therefore, for every scalar $\lambda \in K$ the morphism $\lambda \cdot (-) : a \to \lambda a$ of $X/\{K\}$ yields

$$\lambda \cdot \bar{a} = \overline{\lambda \cdot a}.$$

Consequently, we can define a function $r : Z \to TX$ by taking $z \in Z$ and putting

$$r(z) : a \mapsto \bar{a}(z) \quad \text{for } a \in X^*.$$

Then $r(z)$ is homogeneous. This is the desired factorization: r is a linear function with

$$\pi_a \cdot r = \bar{a} \quad \text{for all } a \in X^*.$$

And it is clearly unique.

(2) In contrast, for $\mathcal{A} = \{K, K^2\}$ in $K\text{-}\mathsf{Vec}$ the codensity monad is $(-)^{**}$. Indeed, given a cone of C_X

$$\frac{X \xrightarrow{a} K^i}{Z \xrightarrow{\bar{a}} K^i} \qquad (i = 1, 2)$$

then we again define r by $r(z) : a \mapsto \bar{a}(z)$ for $a \in X^*$. We have to verify that each $r(z)$ is linear, the rest is as above. Homogeneity is verified as before.

$\mathcal{K}_{fp} \hookrightarrow \mathcal{K}$. Example: in the category of posets the 2-chain is a $*$-cogenerator. Here X^* is the poset of all $\uparrow$-sets of X, ordered by inclusion. Therefore X^{**} is the poset of all upwards closed collections $\mathcal{W}$ of $\uparrow$-sets, again ordered by inclusion. A D-ultrafilter on X is such a nonempty collection $\mathcal{W}$ which is

(i) closed under finite intersections,

and

(ii) *prime*, i.e., if it contains $R \cup S$, then it contains R or S, and it does not contain $\emptyset$.

This is analogous to the classical ultrafilters on sets, which are nonempty, upwards closed, prime collections of subsets, closed under finite intersections.

Analogously in all examples that our result covers: the codensity monad $\mathbb{T}$ assigns to every object X an object formed by all D-ultrafilters on X, and there is a close analogy between the latter and the classical ultrafilters.

On codensity monads. Recall that for every functor $E : \mathcal{A} \to \mathcal{K}$ the codensity monad is defined as the right Kan-extension along itself, $T = \mathrm{Ran}_E E$. That is, T is an endofunctor endowed with a natural transformation $\tau : TE \to E$ universal among natural transformations from $(-) \cdot E$ to E. Applying the universal property to $\mathrm{id} : \mathrm{Id} \cdot E \to E$ we get a unique natural transformation $\eta : \mathrm{Id} \to T$. And applying it to $\tau \cdot T\tau : TTE \to E$ we get a unique natural transformation $\mu : TT \to T$. Then (T, η, μ) is a monad, see [9].

If $\mathcal{A}$ (like $\mathcal{K}_{fp}$) is an essentially small full subcategory of a complete category $\mathcal{K}$, then the codensity monad of the embedding $E : \mathcal{A} \to \mathcal{K}$ is obtained by the following *limit formula*: for every object X denote by

$$C_X : X/\mathcal{A} \to \mathcal{K}$$

the functor assigning to every arrow $a : X \to A$ the codomain A, and put

$$TX = \lim C_X.$$

We have a limit cone denoted by $\psi_a : TX \to A$ for $(A,a) \in X/\mathcal{A}$. On morphisms $f : X \to Y$, Tf is defined as follows: there exists a unique morphism $Tf : TX \to TY$ with

$$\psi_a \cdot Tf = \psi_{a\cdot f} \qquad \text{for all } a : Y \to A \text{ in } Y/\mathcal{A}.$$

The unit $\eta^T_X : X \to TX$ is the unique morphism given by

$$\psi_a \cdot \eta^T_X = a \qquad \text{for all } a : X \to A \text{ in } X/\mathcal{A}$$

and the multiplication is defined by the following commutative triangles

$$\begin{array}{ccc} TTX & \xrightarrow{\mu^T_X} & TX \\ & {\scriptstyle \psi_{\psi_a}} \searrow \quad \swarrow {\scriptstyle \psi_a} & \\ & A & \end{array} \qquad \text{for all } a : X \to A \text{ in } X/\mathcal{A}.$$

Related work. As mentioned already, our paper was inspired by that of Leinster [9]. A related topic was discussed in the PhD thesis of Barry-Patrick Devlin [3]. He also aimed to describe codensity monads of embeddings of "finite-objects", and he also introduced a concept of ultrafilter on an object. However his thesis is fundamentally disjoint from our paper. For example, the categories he works with are varieties whose monads contain that of abelian groups as a submonad – the only example on our list above with this property is K-Vec. This is a meeting point of Devlin's work and ours, see Example 3.6 below.

2. $*$-cogenerators

Throughout we work with a symmetric monoidal closed category $(\mathcal{K}, \otimes, I)$ with a specified object D.

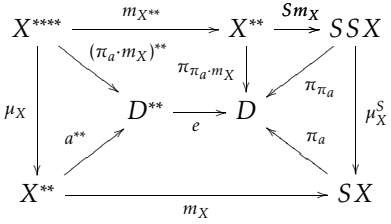
The functor $[-,D] : \mathcal{K} \to \mathcal{K}^{\mathrm{op}}$ is denoted by $(-)^*$. Since it is left adjoint to its dual, we obtain a monad $(-)^{**}$ on $\mathcal{K}$ given by

$$X^{**} = [[X,D],D]$$

called the *double-dualization monad*. Its unit

$$\eta_X : X \to [[X,D],D]$$

(iv) To prove that m preserves multiplication, consider the following diagram:



The upper left-hand part and the lower part commute due to the definition of m. The right-hand upper triangle expresses the definition of S on morphisms, and the lower one commutes due to Remark 5.1. Therefore, the outside square commutes.

■

Example 5.11. Let $\mathcal{K} = \mathsf{Set}$ and $\mathcal{A} = \mathsf{Set}_\lambda$, sets of power less than λ.

(a) Leinster observed in [9] that the ultrafilter monad is the codensity monad of $\mathsf{Set}_4 \hookrightarrow \mathsf{Set}$ (sets of at most 3 elements). In contrast, $\mathsf{Set}_3 \hookrightarrow \mathsf{Set}$ has the codensity monad defined by

TX = collections of nonempty sets of $\mathcal{P}X$ with either Y or $\overline{Y}$ for every $Y \subseteq X$.

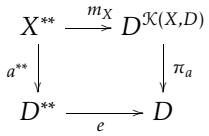
(b) For every infinite cardinal λ let $\mathbb{U}_\lambda$ be the submonad of the ultrafilter monad $\mathbb{U}$ of all λ-complete ultrafilters $\mathcal{F}$. Recall that this means that in every disjoint decomposition $e : X \twoheadrightarrow A$ with $|A| < \lambda$ one component lies in $\mathcal{F}$.

The codensity monad of $\mathsf{Set}_\lambda \hookrightarrow \mathsf{Set}$ is the submonad $\mathbb{U}_\lambda$ of $\mathbb{U}$ on all λ-complete ultrafilters, see [1].

Remark 5.12. Recall that a cardinal λ is measurable if there exists a non-principle λ-complete ultrafilter. Set_λ is codense in Set (i.e., has the trivial codensity monad Id) iff λ is not measurable. This was proved by Isbell in [6].

Example 5.13. Let $\mathcal{K} = K$-Vec and $\mathcal{A} = K\text{-Vec}_\lambda$, spaces of dimension less than λ.

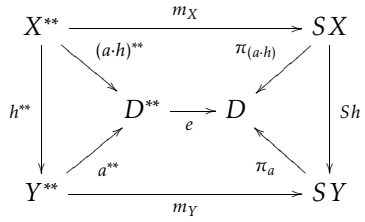
e is as in Definition 5.7):



Lemma 5.10. *Let D be a well-behaved $*$-cogenerator. Then $(-)^{**}$ is a submonad of $\mathbb{S}$ (for $\mathcal{A} = \mathcal{K}_{fp}$) via the above natural transformation $m : (-)^{**} \to \mathbb{S}$.*

Proof: We use the notation $\left((-)^{**}, \mu, \eta\right)$ and $\left(\mathbb{S}, \mu^S, \eta^S\right)$ for the corresponding monad structures.

(i) Naturality is seen from the following diagram where a ranges over $\mathcal{K}(A, D)$:



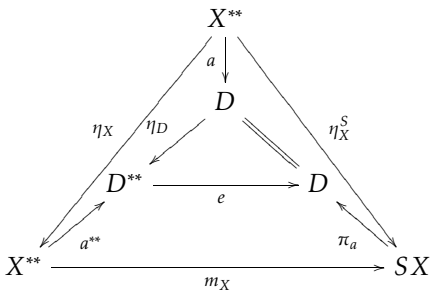
The right-hand triangle is the definition of Sh.

(ii) Each m_X is monic. This is clear since the cone of all $e \cdot a^{**}$ is monic.

(iii) m preserves units. The unit η^S of S has components $\eta^S_X : X \to D^{\mathcal{K}(X,D)}$ defined by

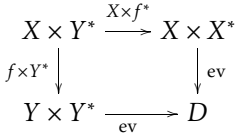
$$\pi_a \cdot \eta^S_X = a \qquad \text{for all } a : X \to D.$$

Thus, we obtain the following commutative diagram



is the mate of the evaluation map $[X, D] \otimes X \to D$ precomposed by the braiding $X \otimes [X, D] \xrightarrow{\cong} [X, D] \otimes X$.

Remark 2.1. $(-)^*$ is defined on morphisms $f : X \to Y$ by yielding the unique morphism $f^* : Y^* \to X^*$ for which the square below commutes:



Examples 2.2. Most of our examples are *commutative varieties* of finitary algebras. Recall that a variety $\mathcal{K}$ is called commutative (or entropic) if for each of its n-ary operation symbols σ and every algebra $K \in \mathcal{K}$ we have a homomorphism $\sigma_K : K^n \to K$. Let $|-|$ denote the forgetful functor. Every variety is symmetric monoidal w.r.t. the usual tensor product:

$$A \otimes B \text{ represents bimorphisms from } |A| \otimes |B|$$

and the unit

$$I = \text{free algebra on one generator.}$$

As proved by Banaschewski and Nelson [2], this is a monoidal closed category iff it is a commutative variety. Then, for arbitrary objects A and B, all morphisms in $\mathcal{K}(A, B)$ form a subalgebra of the power $B^{|A|}$ which yields the object $[A, B]$. Another equivalent formulation, as observed by Linton [10], is that the monad associated with $\mathcal{K}$ is commutative in Kock's sense [8].

Here are our leading examples of commutative varieties with a specified finitely presentable cogenerator D.

(a) Set with $D = \{0, 1\}$. Here $(-)^*$ is the contravariant power-set functor $\mathcal{P}$, thus $X^{**} = \mathcal{P}\mathcal{P}X$ consists of all collections of subsets of X. For a function $f : X \to Y$ the function f^{**} takes a collection $\mathcal{U} \subseteq \mathcal{P}X$ to

$$f^{**}(\mathcal{U}) = \{R \subseteq Y \mid f^{-1}(R) \in \mathcal{U}\}.$$

And η_X assigns every element x of X the trivial ultrafilter $\eta_X(x) = \{R \subseteq X \mid x \in R\}$.

(b) Par, the category of sets and partial functions, with $D = \{1\}$. This is completely analogous, $X^{**} = \mathcal{P}\mathcal{P}X$.

(c) K-Vec, the category of vector spaces over a field K. This example was the motivation for our notation: X^* is the usual dual space (of all linear forms on X). Thus X^{**} is the double-dual. For a linear function $f : X \to Y$, the function f^{**} assigns to every $a : X^* \to K$ in X^{**} the element $a \cdot f^* : Y^* \to K$ of Y^{**}. And $\eta_X : X \to X^{**}$ assigns to $x \in X$ the evaluation-at-x of linear forms.

(d) JSL, the category of join-semilattices (i.e., posets with finite joins) and homomorphisms, with $D = \mathbb{2}$, the chain $0 < 1$. Observe that homomorphisms preserve 0, the join of $\emptyset$. Given a semilattice X, every homomorphism $f : X \to \mathbb{2}$ defines a subset of X by $f^{-1}(1)$. This is an $\uparrow$-set which is *prime*, i.e., it does not contain 0 and whenever it contains $x_1 \vee x_2$, then it contains x_1 or x_2. Conversely, every prime $\uparrow$-set R of X defines a homomorphism $f_R : X \to \mathbb{2}$ by $f_R(x) = 1$ iff $x \in R$. We can thus identify

$$X^* = \text{ all prime } \uparrow\text{-sets of } X$$

ordered by inclusion. The least element of X^* is $\emptyset$. Consequently,

$$X^{**} = \text{ all prime upwards closed collections of prime } \uparrow\text{-sets of } X.$$

Here a collection is called prime if it does not contain the empty set and whenever it contains $R_1 \cup R_2$, then it contains R_1 or R_2. X^{**} is also ordered by inclusion. Its smallest element is the empty collection.

(e) M-Set, the category of sets with an action of a monoid M. We assume that M is commutative (so that M-Set is a commutative variety) and finite. We need the latter assumption to have a $*$-cogenerator, see Example 2.10 below. A cogenerator of M-Set is the power-set

$$D = \mathcal{P}M,$$

with the monoid action

$$mR = \{x \in M \mid mx \in R\} \quad \text{for } R \subseteq M,\ m \in M.$$

To see that this is indeed a cogenerator, observe that M-set homomorphisms $f : X \to \mathcal{P}M$ correspond bijectively to subsets (not only

Given a $*$-cogenerator D, how is the present description related to that of the last section? We would like to see the codensity monad of Section 4 as a submonad of $\mathbb{S}$ with the limit property. For that we need $(-)^{**}$ to be a submonad of $\mathbb{S}$. This holds for the examples of Section 4. Indeed, this is the consequence of the fact that the $*$-cogenerators D considered in those examples are well-behaved in the following sense:

Definition 5.7. Let $\mathcal{K}$ be a complete, symmetric monoidal closed category with a $*$-cogenerator D. We say that D is *well-behaved* if there exists a morphism $e : D^{**} \to D$ which satisfies the following conditions:

(1) $e \cdot \eta_D = \mathrm{id}_D$,
(2) $e \cdot \mu_D = e \cdot e^{**}$;
(3) the morphisms $e \cdot a^{**}$, $a \in X/\mathcal{K}_{fp}$, are jointly monic.

Example 5.8. In the examples of Section 4 the $*$-cogenerator D is well-behaved. Indeed, in all those examples, for every object X, the underlying set of $X^* = [X, D]$ is $\mathcal{K}(X, D)$ and $\eta_X : X \to X^{**}$ is defined by $(\eta_X(x))(a) = a(x)$. Furthermore, the counit of the adjunction $[-, D] \dashv [-, D] : \mathcal{K}^{\mathrm{op}} \to \mathcal{K}$ is just the dual of the unit, thus $\mu_X = \eta^*_{X^*}$. Let

$$e = \eta_{D^*}(\mathrm{id}_D) : D^{**} \to D.$$

It is clear that $e \cdot \eta_D = \mathrm{id}_D$. To verify (2), given $v \in D^{****}$, that is, a morphism $v : D^{***} \to D$, we have that:

$$(e \cdot \mu_D)(v) = e(v \cdot \eta_{D^*}) = (v \cdot \eta_{D^*})(\mathrm{id}_D) = v(\eta_{D^*}(\mathrm{id}_D)) = v(e)$$

as well as

$$(e \cdot e^{**})(v) = e(v \cdot e^*) = (v \cdot e^*)(\mathrm{id}_D) = v(e^*(\mathrm{id}_D)) = v(e).$$

Finally, for (3) given $u, v \in X^{**}$, $(e \cdot a^{**})(u) = (e \cdot a^{**})(v)$ is equivalent to $e(u \cdot a^*) = e(v \cdot a^*)$, that is, $(u \cdot a^*)(\mathrm{id}_D) = (v \cdot a^*)(\mathrm{id}_D)$, which means that $u(a) = v(a)$. Since this holds for all $a \in X^*$, we conclude that $u = v$.

Notation 5.9. Given a well-behaved $*$-cogenerator D, for every object X denote by $m_X : X^{**} \to D^{\mathcal{K}(X,D)}$ the unique morphism making the following square commutative (where π_a denotes the projection w.r.t. $a : X \to D$ and

for $\mathcal{A} \hookrightarrow \mathcal{K}$ if, instead of the double-dualization monad $(-)^{**}$, we take the monad $\mathbb{M}$. Thus, the codensity monad can be obtained from $\mathbb{M}$ by using the intersection of derived subalgebras analogous to that described in Definition 3.1 and Lemma 4.2. In particular, the codensity monad is a submonad of $\mathbb{M}$. For $\mathbb{M} = \mathbb{S}$, we deduce that the codensity monad $\mathbb{T}$ is a submonad of $\mathbb{S}$.

(2) Let $\mathbb{T}$ be the monad defined analogously to Theorem 4.5 with $\mathbb{S}$ replacing $(-)^{**}$ everywhere. Thus, for every object X, TX is the intersection of the preimages of η^S_A (see Remark 5.1) under Sa for all $a : X \to A$ in $X/\mathcal{K}$:

$$\begin{array}{ccc} TX & & \\ {\scriptstyle q(\bar{a})}\downarrow & \searrow^{i_X} & \\ A_0 & \xrightarrow{a_0} & SX \\ {\scriptstyle p(a)}\downarrow & & \downarrow{\scriptstyle Sa} \\ A & \xrightarrow[\eta^S_A]{} & SA \end{array} \tag{10}$$

This defines a functor T, its action on morphisms is defined precisely as in Lemma 4.2.

Then $\mathbb{T}$ is a submonad of $\mathbb{S}$ via the monad morphism $i : \mathbb{T} \to \mathbb{S}$ with the above components i_X.

(3) Moreover, this works in a entirely similar way for every submonad $\overline{\mathbb{S}}$ of $\mathbb{S}$ with the limit property, showing that the codensity monad is a submonad of any such $\overline{\mathbb{S}}$.

Since the codensity monad has the limit property, the proof is concluded. ∎

Example 5.6. Let $\mathcal{K}$ be a locally finitely presentable category with a cogenerating set $(D_i)_{i\in I}$ in $\mathcal{K}_{fp}$. Then the codensity monad of the embedding of $\mathcal{K}_{fp}$ into $\mathcal{K}$ is the smallest submonad of the monad $SX = \prod_{i\in I} D_i^{\mathcal{K}(X,D_i)}$ with the limit property. This is actually quite analogous to the description of Section 4, just the desired subobjects are now related to $\mathbb{S}$ rather than $(-)^{**}$ (see the proof above). However, in the concrete situations of Section 4 the description using $*$-cogenerators is more illustrative.

subalgebras!) of X: to every subset $Y \subseteq X$ assign f_Y defined by

$$f_Y(x) = \{m \in M \mid mx \in Y\} \text{ for all } x \in X.$$

The inverse assignment takes every $g : X \to \mathcal{P}M$ to $Y = \{mx \mid m \in g(x), x \in X\}$.

Thus for every M-set X we conclude that

$$X^* = \mathcal{P}X$$

is the power-set of the (underlying set of) X with the monoid action $mR = \{x \in X \mid mx \in R\}$. And the monoid action of $X^{**} = \mathcal{P}\mathcal{P}X$ assigns to $\mathcal{U} \subseteq \mathcal{P}X$ and $m \in M$ the result $m\mathcal{U} = \{R \subseteq X \mid mR \in \mathcal{U}\}$.

Examples 2.3. Further we consider some cartesian closed categories.

(a) Pos, the category of posets and monotone maps, with $D = \mathbb{2}$, the chain $0 < 1$. Here $[A, B] = \mathsf{Pos}(A, B)$ ordered pointwise. Thus

$$X^* = \text{ all } \uparrow\text{-sets of } X$$

(ordered by inclusion) and

$$X^{**} = \text{ all upwards closed collections of } \uparrow\text{-sets,}$$

also ordered by inclusion.

(b) Gra, the category of undirected graphs and homomorphisms. Thus an object (V, E) consists of a set V of vertices and a symmetric relation $E \subseteq V \times V$ of edges. In case $E = V \times V$ we speak about the complete graph on V. Gra has a cogenerator D, the complete graph on $\{0, 1\}$. Given graphs A and B, then

$$[A, B] = \mathsf{Set}(V_A, V_B)$$

consists of all functions, not only homomorphisms, and its edges are all pairs (f, g) with $(a, a') \in E_A \Rightarrow (f(a), g(a')) \in E_B$, for all $a, a' \in A$. We conclude that

$$X^* = \text{ complete graph on } \mathcal{P}V_X$$

and

$$X^{**} = \text{ complete graph on } \mathcal{P}\mathcal{P}V_X.$$

(c) Σ-Str, the category of relational structures, where Σ is a signature of finitely many finitary symbols. Objects X, Σ-structures, consist of a set V_X and an n-ary relation $\sigma_X \subseteq V_X^n$ for every $\sigma \in \Sigma$ n-ary. Analogously to (b) we choose as D the complete structure on $\{0,1\}$, that is, $\sigma_D = \{0,1\}^n$ for every n-ary symbol n. Then

$$X^* = \text{ complete } \Sigma\text{-structure on } \mathcal{P}V_X.$$

Remark 2.4. As explained in the introduction we want to describe the codensity monad of the full embedding

$$\mathcal{K}_{fp} \hookrightarrow \mathcal{K}$$

of the subcategory of finitely presentable objects. Recall that in case $\mathcal{K}$ is locally finitely presentable, $\mathcal{K}_{fp}$ is colimit-dense: every object X is the canonical colimit of the diagram $\mathcal{K}_{fp}/X \to \mathcal{K}$ assigning to every morphism $a : A \to X$ with $A \in \mathcal{K}_{fp}$ the domain. We are, however, not assuming that $\mathcal{K}$ is locally finitely presentable. Instead, we need that every object X^* is a canonical colimit of all A^* with A finitely presentable:

Notation 2.5. Recall that, for every object X, the diagram $C_X : X/\mathcal{K}_{fp} \to \mathcal{K}$ assigns to $a : X \to A$ with $A \in \mathcal{K}_{fp}$ the codomain. We denote the composite $(-)^* \cdot \left(C_X\right)^{\mathrm{op}}$ by C_X^*. That is,

$$C_X^* : (X/\mathcal{K}_{fp})^{\mathrm{op}} \to \mathcal{K} \quad \text{with} \quad C_X^*(A,a) = A^*.$$

Definition 2.6. An object D is called a *$*$-object* provided that for all objects X we have $X^* = \mathrm{colim} C_X^*$ with the canonical colimit cocone. If D is a cogenerator, we speak about *$*$-cogenerator*.

Proposition 2.7. *Every finitely presentable object D of a commutative variety is a $*$-object.*

Proof: Let X be an arbitrary object and suppose that a cocone of C_X^* with codomain Z is given as follows

$$\frac{X \xrightarrow{a} A}{A^* \xrightarrow{\bar{a}} Z} \qquad \text{for } (a,A) \in X/\mathcal{K}_{fp}.$$

(2) In a symmetric monoidal closed complete category $\mathcal{K}$, for every $*$-object D, the double-dualization monad $(-)^{**} = [D,[D,-]]$ has the limit property, since $[-,D] : \mathcal{K}^{\mathrm{op}} \to \mathcal{K}$ is a right adjoint.

Lemma 5.4. *The monad $\mathbb{S}$ has the limit property.*

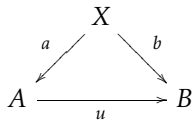
Proof: Since $S = R \cdot L$ and R preserves limits, it is sufficient to prove that the diagram $L \cdot C_X$ has in $\left(\mathrm{Set}^I\right)^{\mathrm{op}}$ the limit $\left(\mathcal{K}(X,D_i)\right)_{i\in I}$ with respect to the canonical cone of all maps $(-)\cdot a : \left(\mathcal{K}(A,D_i)\right)_{i\in I} \to \left(\mathcal{K}(X,D_i)\right)_{i\in I}$ with $a : X \to A$. We can work with the components individually, thus, let $i \in I$ be fixed. Hence in Set, rather than $\mathrm{Set}^{\mathrm{op}}$, we are to prove that the cocone

$$\mathcal{K}(A,D_i) \xrightarrow{(-)\cdot a} \mathcal{K}(X,D_i) \qquad (\text{for } a : X \to A,\ A \in \mathcal{A})$$

is a colimit cocone. Indeed, let another cocone

$$z_a : \mathcal{K}(A,D_i) \to Z$$

be given. Compatibility means that given a commutative triangle



in $\mathcal{K}$, then $z_b(t) = z_a(tu)$ for all $t : B \to D_i$. The function

$$z : \mathcal{K}(X,D_i) \to Z, \quad z(t) = z_t(\mathrm{id}_{D_i}),$$

for all $t : X \to D_i$, is the desired factorization.

Indeed, the equality $z_a = z \cdot \left((-)\cdot a\right)$ means that

$$z_a(r) = z(r \cdot a) = z_{ra}(\mathrm{id}_{D_i}) \qquad \text{for all } r : A \to D_i$$

by choosing $t = \mathrm{id}_{D_i}$ and $u = r$ (thus $b = ra$).

The uniqueness of z is clear. ∎

Theorem 5.5. *The codensity monad of the embedding $\mathcal{A} \hookrightarrow \mathcal{K}$ is the smallest submonad of $\mathbb{S}$ with the limit property.*

Proof: (1) Let $\mathbb{M}$ be a monad on $\mathcal{K}$ with the limit property and with a monic unit $\eta : \mathrm{Id} \to \mathbb{M}$. Looking at the proof of Theorem 4.5, we see that it works

5. Further Examples

In this section we consider a more general setting: a complete category $\mathcal{K}$ and a small, full subcategory $\mathcal{A}$, and we discuss the codensity monad of the embedding $\mathcal{A} \hookrightarrow \mathcal{K}$.

Given a set $\{D_i\}_{i\in I}$ of cogenerators of $\mathcal{K}$ lying in $\mathcal{A}$ we obtain a monad $\mathbb{S}$ on $\mathcal{K}$ from the well-known adjunction $L \dashv R : \left(\mathsf{Set}^I\right)^{\mathrm{op}} \to \mathcal{K}$ where

$$LX = \Big(\mathcal{K}(X, D_i)\Big)_{i\in I} \text{ and } R(M_i)_{i\in I} = \prod_{i\in I} D_i^{M_i}.$$

We can characterize the codensity monad of $\mathcal{A} \hookrightarrow \mathcal{K}$ as the smallest submonad of $\mathbb{S}$ with a property called the *limit property* below. We continue using the notation of Introduction:

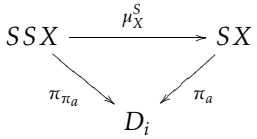
$$C_X : X/\mathcal{A} \to \mathcal{K},\ (X \xrightarrow{a} A) \mapsto A.$$

Remark 5.1. The above monad $\mathbb{S}$ is given on objects X by $SX = \prod_{i\in I} D_i^{\mathcal{K}(X,D_i)}$ with the unit $\eta^S : \mathrm{Id} \to S$ defined by the projections π_f $(f : X \to D_i)$ as follows

$$\pi_f \cdot \eta^S_X = f.$$

Thus η^S_X is monic, since (D_i) is a cogenerating set.

The multiplication μ^S is determined by the commutativity of the triangles

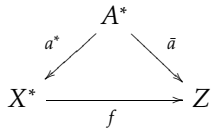


for all $a : X \to D_i$ and $i \in I$.

Definition 5.2. A monad $\mathbb{T}$ on $\mathcal{K}$ has the *limit property* (with respect to the embedding $\mathcal{A} \hookrightarrow \mathcal{K}$) if for every object X we have $TX = \lim TC_X$ with the canonical limit cone of all Ta for $a \in X/\mathcal{A}$.

Example 5.3. (1) The codensity monad of $\mathcal{A} \hookrightarrow \mathcal{K}$ has the limit property: use the limit formula.

We prove that there exists a unique morphism f making the following triangles



$(a, A) \in X/\mathcal{K}_{fp}$

commutative.

(1) Uniqueness. The cocone of all a^*'s is collectively surjective, hence, collectively epic: given $b : X \to D$ in X^*, then $(b, D) \in X/\mathcal{K}_{fp}$ and $b = b^*(\mathrm{id})$.

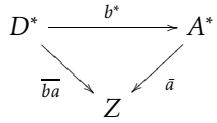
(2) Existence. For every $b \in X^*$ we have the corresponding $\bar{b} : D^* = [D, D] \to Z$. We define f by

$$f(b) = \bar{b}(\mathrm{id}_D).$$

To prove the equality

$$f \cdot a^* = \bar{a} \tag{1}$$

observe that every $b \in A^*$ is a morphism of $X/\mathcal{K}_{fp}$ from $X \xrightarrow{a} A$ to $X \xrightarrow{ba} D$. The compatibility of the above cocone $a \mapsto \bar{a}$ implies that the following triangle



commutes. This applied to $\mathrm{id}_D \in D^*$ yields

$$\bar{a}(b) = \overline{ba}(\mathrm{id}_D)$$

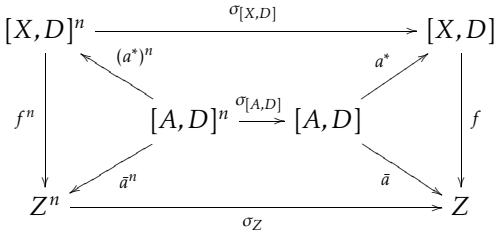
hence

$$f \cdot a^*(b) = f(ba) = \bar{a}(b)$$

as desired.

It remains to prove that f is a morphism of $\mathcal{K}$. Let Σ be a signature in which $\mathcal{K}$ is equationally specified. For every n-ary symbol $\sigma \in \Sigma$ we are to

prove that f preserves σ, i.e., the outward square in the following diagram



(2)

commutes. For the triangles use the above equality (1). The upper part commutes since a^*, being a morphism of $\mathcal{K}$, preserves σ. Analogously for the lower part. ∎

Proposition 2.8. *Every finite object of the categories* Pos, Gra *or* Σ-Str *is a* $*$*-object.*

Proof: (1) For Pos this is completely analogous to the proof of Proposition 2.7, except the part that f, defined by $f(b) = \bar{b}(\mathrm{id}_D)$, is a morphism of Pos. For that it is sufficient to prove that every comparable pair $b_1 \leq b_2$ in $[X,D]$ is the image under some a^* of a comparable pair in $[A,D]$. By the definition of $[X,D]$ we have $b_1(x) \leq b_2(x)$ for every $x \in X$. Let A be the subposet of D^2 on all comparable pairs. Then we conclude that $< b_1, b_2 >: X \to D^2$ has a codomain restriction

$$a : X \to A.$$

The projections $\pi_1, \pi_2 : A \to D$ fulfil

$$\pi_1 \leq \pi_2 \text{ in } A^* \text{ and } b_i = \pi_i \cdot a.$$

Therefore $b_i = a^*(\pi_i)$, as required.

(2) For Σ-Str again the proof is analogous to 2.7, just the proof of that f is a morphism needs to be modified. Let σ be an n-ary symbol of Σ. It is our task to prove that f^n restricted to $\sigma_{[X,D]}$ factorizes through σ_Z:

(2) Analogously for K-Vec. We thus obtain another proof of Leinster's result that $(-)^{**}$ is the codensity monad.

Remark 4.9. The last corollary gives a characterization that does not need the technical concept of $*$-cogenerator or D-ultrafilter.

It is an open problem whether it holds for arbitrary finitely presentable cogenerators in arbitrary symmetric monoidal closed categories that are locally finitely presentable.

4.10. Summarizing all our examples, here is a survey of the codensity monad $\mathbb{T} = (T, \mu^T, \eta^T)$ for embeddings $\mathcal{K}_{fp} \hookrightarrow \mathcal{K}$. In each case we describe the action of T on an arbitrary object X; in the table we just name the underlying set $|TX|$ of TX consisting of all D-ultrafilters, its structure as an object of $\mathcal{K}$ follows from Example 4.4. For morphisms $f : X \to Y$ the map Tf is always given by assigning to a collection $\mathcal{U}$ of subsets of TX the collection $\{R \subseteq |TY|;\ f^{-1}(R) \in \mathcal{U}\}$.

Category	D	D-ultrafilters on an object
Set	$\{0,1\}$	ultrafilters
Par	$\{0\}$	ultrafilters
Pos		nonempty, prime collections of $\uparrow$-sets closed under upper sets and finite intersections
JSL		prime, upwards closed collections of prime $\uparrow$-sets
Gra		ultrafilters on the set of vertices
Σ-Str	$\{0,1\}$ complete	ultrafilters on the underlying set
K-Vec	K	vectors of the double-dual space
M-Set	$\mathcal{P}M$	ultrafilters on the underlying set

Corollary 4.7. *The codensity monad of the embedding $\mathcal{K}_{fp} \hookrightarrow \mathcal{K}$ is the largest submonad of $(-)^{**}$ whose unit has invertible components at all finitely presentable objects.*

Proof: We show that every submonad

$$j : (\widehat{T}, \hat{\mu}, \hat{\eta}) \to ((-)^{**}, \mu, \eta)$$

with $\hat{\eta}_A$ invertible for all $A \in \mathcal{K}_{fp}$ factorizes through i. Indeed, it is sufficient to verify that for every object X and all $a : X \to A$ in $X/\mathcal{K}_{fp}$

$$j_X \text{ factorizes through } a'.$$

This implies that j_X factorizes through i_X, i.e., we have $u_X : \widehat{T}X \to TX$ with $j_X = i_X \cdot u_X$. Since i and j are monic monad morphisms, it follows easily that $u : \widehat{T} \to T$ is also a monad morphism.

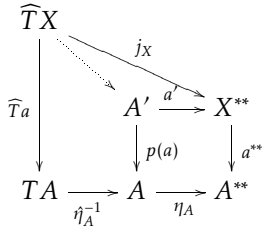
For every $a : X \to A$ in $X/\mathcal{K}_{fp}$ we have $\eta_A = j_A \cdot \hat{\eta}_A$, thus,

$$j_A = \eta_A \cdot (\hat{\eta}_A)^{-1}.$$

Since j is natural, we derive from $a^{**} \cdot j_X = j_A \cdot \widehat{T}a$ that

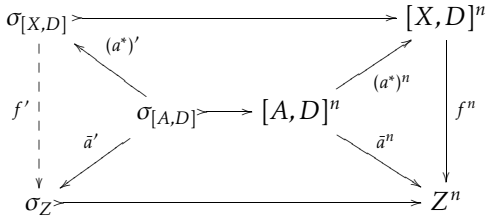
$$a^{**} \cdot j_X = \eta_A \cdot \hat{\eta}_A^{-1} \cdot \widehat{T}a.$$

This yields the desired factorization of j_X through a':



■

Example 4.8. (1) The codensity monad of the embedding of finite semilattices into JSL is the (full) double-dual monad. Indeed, for every finite semilattice A the dual A^* is isomorphic to A^{op}: to every prime $\uparrow$-set $M \subseteq A$ (see Example 2.2(d)) assign its meet in A to get a dual isomorphism $A^* \xrightarrow{\sim} A^{\mathrm{op}}$. Thus A^{**} is isomorphic to A, and it is easy to see that $\eta_A : A \to A^{**}$ is indeed an isomorphism.



In the above diagram $(a^*)^n$ has the depicted restriction $(a^*)'$ because a^* is a morphism of Σ-Str, analogously for $\bar{a}^n$. Thus, to prove that f' making the diagram commutative exists, we just need to show that every n-tuple

$$(b_1, \ldots, b_n) \in \sigma_{[X,D]}$$

lies in the image of $(a^*)'$ for some $a : X \to A$ in $X/\mathcal{K}_{fp}$. By the definition of $\sigma_{[X,D]}$ the morphisms $b_i : X \to D$ fulfil $(b_1(x), \ldots, b_n(x)) \in \sigma_D$ for every $x \in X$. Let A be the strong subobject of D^n on the subset σ_D, then $< b_i >: X \to D^n$ has a codomain restriction

$$a : X \to A.$$

The projections $\pi_i : A \to D$ fulfil $(\pi_1, \ldots, \pi_n) \in \sigma_{[A,D]}$ and $b_i = \pi_i \cdot a$. Therefore, $(b_1, \ldots, b_n) = (a^*)^n(\pi_1, \ldots, \pi_n)$, as required.

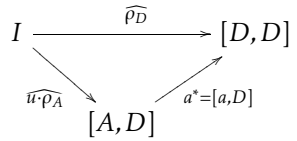
(3) The proof for Gra uses the same diagram as in (2). ■

Observe that in all examples of 2.2 and 2.3 the unit object I is finitely presentable. For commutative varieties, where I is the free algebra on one generator, this is automatic. In the cartesian closed categories Pos and Gra this also holds. For Σ-Str the terminal object $I = 1$ is finitely presentable iff Σ is finite.

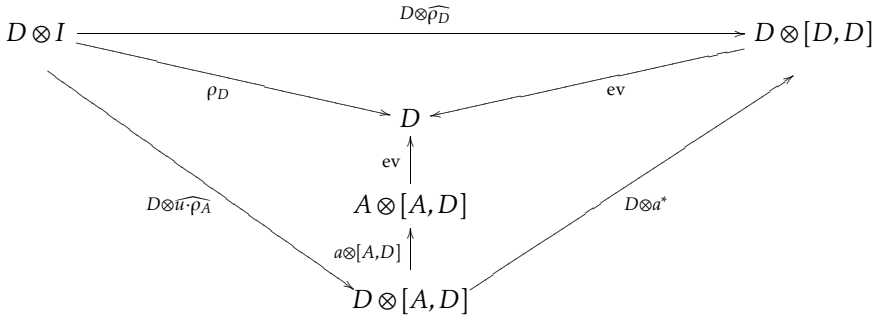
Proposition 2.9. *All $*$-objects are finitely presentable, assuming that I is.*

Proof: Denote by $\rho_A : A \otimes I \to A$ the right unitor isomorphism. If D is a $*$-object, then D^* itself is a filtered colimit of the diagram $(D/\mathcal{K}_{fp})^{\mathrm{op}} \to \mathcal{K}$ with the colimit cocone $a^* : A^* \to D^*$. The mate $\widehat{\rho_D} : I \to [D,D]$ of $\rho_D : D \otimes I \to D$ factorizes, if I is finitely presentable, through one of the colimit maps a^*. The factorizing morphism from I to $[A,D]$ is a mate $\widehat{u \cdot \rho_A} : A \otimes I \to$

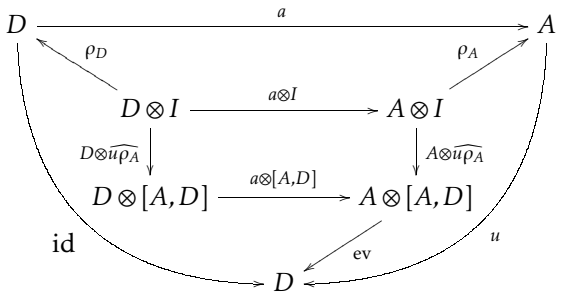
D for a morphism $u : A \to D$:



We obtain a commutative triangle by multiplying the above one with D:

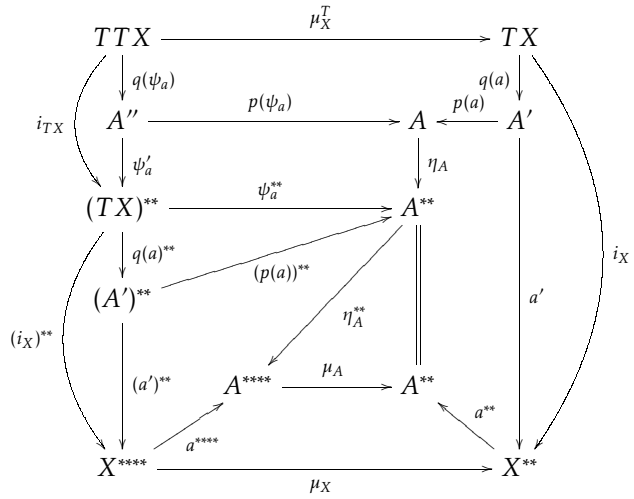


Moreover, the upper triangle commutes by the definition of mate, and the right-hand one does by Remark 2.1. Consequently, the left-hand triangle also commutes. Consider the following diagram, using the above triangle in its left-hand part:



The right-hand part commutes by the definition of mate, the upper part by naturality of ρ, and the middle square commutes since both passages yield $a\otimes\widehat{u\cdot\rho_A}$. This proves $u\cdot a = \mathrm{id}$. Thus D is a split quotient of $A\in\mathcal{K}_{fp}$, concluding the proof. ∎

where the derived subobject of $\psi_a : TX \to A$ is denoted by $\psi'_a : A'' \to (TX)^{**}$. This follows from the commutative diagram below:



All inner parts commute: for the upper one see (9), the lowest one is the naturality of μ, and the triangle above it is the monad law $\mu\cdot\eta^{**} = \mathrm{id}$. All the other parts commute by definition of a' and ψ'_a. Consequently, the desired outward square commutes when postcomposed by a^{**}. Once again apply (i) to see that the proof is complete. ∎

Observation 4.6. The components $\eta^T_A : A \to TA$ of the unit of the codensity monad are invertible for all finitely presentable objects A. Indeed, recall from the Introduction the formula $\psi_a\cdot\eta^T_X = a$. The case $a = \mathrm{id}_A : A \to A$, gives

$$\psi_{\mathrm{id}_A}\cdot\eta^T_A = \mathrm{id}_A.$$

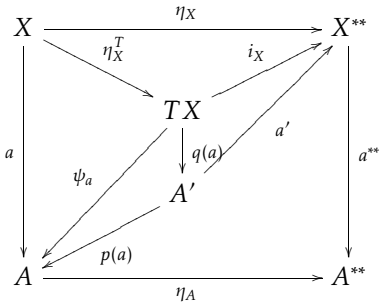
On the other hand, for every $b : A \to B$ in $A/\mathcal{K}_{fp}$, we have

$$\psi_b\cdot\eta^T_A\cdot\psi_{\mathrm{id}_A} = b\cdot\psi_{\mathrm{id}_A} = \psi_b.$$

The morphisms ψ_b are the components of a limit, therefore they are collectively monic and we get

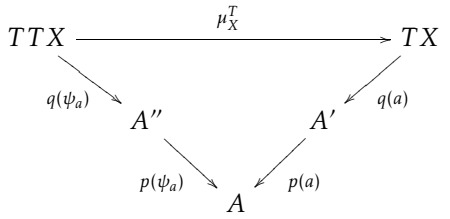
$$\eta^T_A\cdot\psi_{\mathrm{id}_A} = \mathrm{id}_A.$$

To prove that i preserves the unit, consider the following diagram for every object X and all (A,a) in $X/\mathcal{K}_{fp}$:



The left-hand triangle is the definition of η^T_X, see Introduction. All the other inner parts except the upper triangle commute by Notation 4.1, (3) and (6). Since the outward square commutes, this proves that the desired triangle, when prolonged by a^{**}, commutes. From (i) we conclude that the triangle commutes.

To prove that i preserves multiplication, recall from Introduction that μ^T_X is defined by the following commutative triangles



(9)

$(A,a) \in X/\mathcal{K}_{fp}$

Consider the desired equality

$$i_X \cdot \mu^T_X = \mu_X \cdot i^{**}_X \cdot i_{TX}.$$

which in view of Notation 4.1 means

$$a' \cdot q(a) \cdot \mu^T_X = \mu_X \cdot (a')^{**} \cdot q(a)^{**} \cdot \psi'_a \cdot q(\psi_a)$$

Example 2.10. A commutative variety with a cogenerator does not have to possess a $*$-cogenerator. An example is the variety

Un

of unary algebras on one operation. This is equivalent to $\mathbb{N}$-Set for the additive monoid $\mathbb{N}$ of natural numbers. It has a cogenerator analogous to that of Example 2.2(e): take $\mathcal{P}\mathbb{N}$ with the unary operation sending $V \subseteq \mathbb{N}$ to $\{n-1 \mid n \in V, n \neq 0\}$.

Assuming that Un has a $*$-cogenerator D, we derive a contradiction as follows:

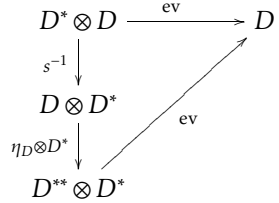
The operation of D forms some cycles, and since D is by Proposition 2.9 finitely generated, there exists a prime n such that all cycles of D have lengths smaller than n. But then D is not a cogenerator: if A is an algebra consisting of a cycle of length n, there exists no non-constant homomorphism from A to D.

Proposition 2.11. *For every cogenerator D the unit of the double-dualization monad is monic.*

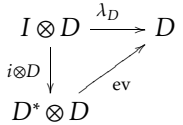
Proof: (1) η_D is monic. Indeed, by definition, η_D is the mate of the composite

$$D \otimes D^* \xrightarrow{s} D^* \otimes D \xrightarrow{\text{ev}} D$$

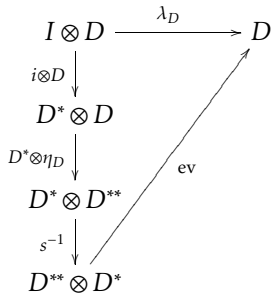
where s is the braiding. Thus we have a commutative triangle as follows:



Denote by $i : I \to D^*$ the mate of the left unitor isomorphism $\lambda_D : I \otimes D \to D$:



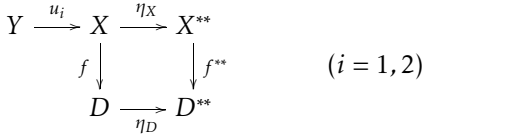
Thus the following diagram commutes (due to naturality of s):



Therefore, $i \otimes \eta_D$ is a split monomorphism (with splitting $\lambda_D^{-1} \cdot \mathrm{ev} \cdot s^{-1}$).

Consequently, given morphisms $u_1, u_2 : Y \to D$ with $\eta_D \cdot u_1 = \eta_D \cdot u_2$, then $I \otimes u_1 = I \otimes u_2$ (since $i \otimes \eta_D$ merges that last pair) which proves $u_1 = u_2$, since $I \otimes - \cong \mathrm{Id}_{\mathcal{K}}$.

(2) For every object X the morphism η_X is monic. Indeed, given $u_1, u_2 : Y \to X$ with $u_1 \neq u_2$, there exists, since D is a cogenerator, a morphism $f : X \to D$ with $f \cdot u_1 \neq f \cdot u_2$. Hence, by (1), $\eta_D \cdot f \cdot u_1 \neq \eta \cdot f \cdot u_2$. The following commutative diagrams



$(i = 1, 2)$

prove $\eta_X \cdot u_1 \neq \eta_X \cdot u_2$. ∎

3. D-ultrafilters

We assume that a finitely presentable cogenerator D in a symmetric monoidal category $\mathcal{K}$ with preimages is given. Recall from Proposition 2.11 that each $\eta_A : A \to A^{**}$ is monic.

Definition 3.1. (1) Given a morphism $a : X \to A$ with A finitely presentable, we call the preimage of η_A under $a^{**} : X^{**} \to A^{**}$ the *derived subobject* a' of a.

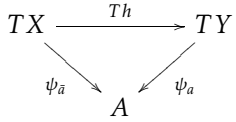
all inner parts commute. Thus, by using the square above we get

$$\eta_A \cdot (\psi_a \cdot k) = a^{**} \cdot k_0 = \eta_A \cdot \tilde{a} \tag{8}$$

By Proposition 2.11, η_A is monic, so k is the desired factorization.

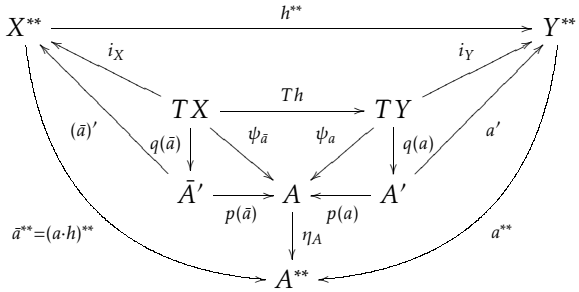
Given a factorization $\hat{k}$, we prove $\hat{k} = k$. Let $\hat{k}_0 = i_X \cdot \hat{k}$, then we get $a^{**} \cdot \hat{k}_0 = \eta_A \cdot \tilde{a}$. Comparing this with (8) yields $a^{**} \cdot \hat{k}_0 = a^{**} \cdot k_0$. From (i) we conclude $\hat{k}_0 = k_0$. Since i_X is monic, this proves $\hat{k} = k$.

(iii) For every morphism $h : X \to Y$ we need to verify that the definition of Th (see Lemma 4.2) agrees with the definition in the Introduction, i.e., the triangles



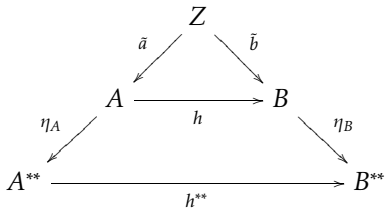
$a : Y \to A$ in $Y/\mathcal{K}_{fp}$

commute for $\bar{a} = a \cdot h$. For that consider the following diagram in which we denote, for $a : Y \to A$, by $\bar{a}' : \bar{A}' \to X^{**}$ the derived subobject of $\bar{a}$:



Its inner parts, except the desired triangle, commute by (3), 4.1, definition of ψ and naturality of i. The outward triangle also commutes. Thus, the desired triangle commutes since η_A is monic by Proposition 2.11.

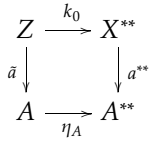
(iv) T has the structure of a monad, namely, the codensity monad of the embedding $\mathcal{K}_{fp} \hookrightarrow \mathcal{K}$. It remains to verify that it is a submonad of $(-)^{**}$, more precisely, that $i : T \to (-)^{**}$ is a monad morphism. We denote by η^T and μ^T the monad structure of T and by η and μ that of $(-)^{**}$.



Since by (i) X^{**} is the limit of C_X^{**}, we obtain a unique morphism

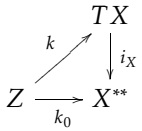
$$k_0 : Z \to X^{**}$$

making the following squares



for all $X \xrightarrow{a} A$ in $X/\mathcal{K}_{fp}$

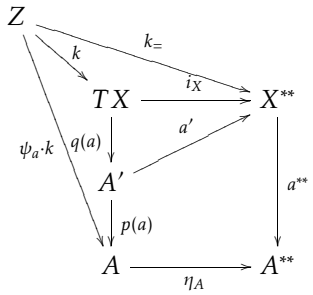
commutative. This implies that k_0 factorizes through the preimage a' of η_A under a^{**}. Hence, it factorizes through $i_X = \cap a'$:



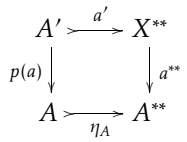
This is the desired factorization, i.e., we have

$$\psi_a \cdot k = \tilde{a} \qquad \text{for all } X \xrightarrow{a} A \text{ in } X/\mathcal{K}_{fp}.$$

Indeed in the following diagram

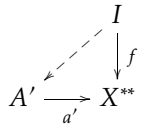


We use the following notation for the corresponding pullback:



(3)

(2) A *D-ultrafilter* on an object X is an external element of X^{**} lying in every derived subobject. That is, a morphism $f : I \to X^{**}$ factorizing as follows:



for all (A, a) in $X/\mathcal{K}_{fp}$

Example 3.2. In Set with $D = \{0,1\}$, this is precisely an ultrafilter on X. Recall that this means a nonempty collection $\mathcal{U}$ of subsets that is upwards closed, closed under finite intersections and *prime* (i.e., $\emptyset \notin \mathcal{U}$ and if $\mathcal{U}$ contains $R \cup S$ then it contains R or S).

Why do ultrafilters and $\{0,1\}$-ultrafilters coincide? Recall that $\eta_A(t)$ is the collection $\mathcal{U}$ of all $Z \subseteq A$ with $t \in Z$. And a^{**} takes every collection of subsets to the collection of their preimages under a. Thus, a collection of subsets $\mathcal{U}$ lies in the derived subobject A' iff there exists $t \in A$ such that

$$a^{-1}(Z) \in \mathcal{U} \quad \text{iff} \quad t \in Z \qquad (\text{for all } Z \subseteq A). \tag{4}$$

We are going to prove that this holds iff $\mathcal{U}$ is an ultrafilter. This can be derived from the result of Galvin and Horn [4] which states that $\mathcal{U}$ is an ultrafilter iff for every finite disjoint decomposition of X precisely one member lies in $\mathcal{U}$. We provide a full (short) proof since we need modifications of it below.

Lemma 3.3. *Let* $\mathcal{K} =$ Set *with* $D = \{0,1\}$, *or* $\mathcal{K} =$ Par *with* $D = \{1\}$. *Then a D-ultrafilter on a set is precisely an ultrafilter on it.*

Proof: To give an external element $f : 1 \to X^{**} = \mathcal{P}\mathcal{P}X$ means precisely to give a collection $\mathcal{U}$ of subsets of X. It is clear that (4) holds whenever $\mathcal{U}$ is an ultrafilter: in the finite decomposition $X = \cup_{t \in a[X]} a^{-1}(t)$ we have a unique $t \in A$ with $a^{-1}(t) \in \mathcal{U}$, then (4) follows.

Conversely, suppose $\mathcal{U}$ is a $\{0,1\}$-ultrafilter. From (4) we immediately see that $\mathcal{U} \neq \emptyset$ (it contains $a^{-1}(\{t\})$). Given subsets $R, S \subseteq X$ expressed by their characteristic functions, we put

$$A = \{0,1\}^2 \text{ and } a =< \chi_R, \chi_S >: X \to A.$$

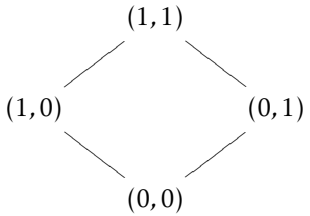
(i) If $R \subseteq S$ and $R \in \mathcal{U}$, then $S \in \mathcal{U}$. We see that $R = a^{-1}(\{(1,1)\})$, thus in (4) we have $t = (1,1)$. Consequently, S lies in $\mathcal{U}$, since it is $a^{-1}(Z)$ for $Z = \{(1,1), (0,1)\}$.

(ii) If $R, S \in \mathcal{U}$, then $R \cap S \in \mathcal{U}$, since this is $a^{-1}(\{(1,1)\})$.

(iii) If $R \cup S \in \mathcal{U}$, then $R \in \mathcal{U}$ or $S \in \mathcal{U}$. Indeed, assuming $R = a^{-1}(\{(1,0), (1,1)\})$ does not lie in $\mathcal{U}$, then t in (4) is $(0,1)$: it cannot be $(0,0)$ since $a^{-1}(\{(0,0)\}) = \emptyset$. Consequently, $S = a^{-1}(\{(0,1), (1,1)\})$ lies in $\mathcal{U}$. And $\emptyset \notin \mathcal{U}$ since we can choose $a : A \to 1$. ∎

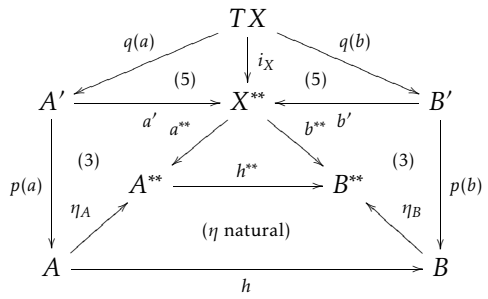
Example 3.4. Let $\mathcal{K} = \mathsf{Pos}$ and $D = \mathbb{2}$. A D-ultrafilter on a poset X is precisely a prime nonempty collection $\mathcal{U}$ of $\uparrow$-sets of X which is closed under upper sets and finite intersections. Here *prime* means that $\emptyset \notin \mathcal{U}$ and whenever $R \cup S \in \mathcal{U}$, then $R \in \mathcal{U}$ or $S \in \mathcal{U}$ (for all $\uparrow$-sets R, S).

The proof is completely analogous to that of the above lemma. To give an external element $f : 1 \to X^{**}$ means, by Example 2.3(a), to give an upwards closed collection of $\uparrow$-sets $\mathcal{U}$. If it is nonempty, prime, and closed under finite intersections, then for every morphism $a : X \to A$ with A finite, the collection $\hat{\mathcal{U}} = \{Z \in X^* \mid a^{-1}(Z) \in \mathcal{U}\}$ also has those properties, thus $\cap_{Z \in \hat{\mathcal{U}}} Z = \uparrow t \ \in \hat{\mathcal{U}}$ for some $t \in A$. Then $Z \in \hat{\mathcal{U}}$ iff $t \in Z$, ensuring that $\mathcal{U}$ is a D-filter. The rest is the same, just the set $A = \{0,1\}^2$ is substituted by the poset D^2:



Example 3.5. Let $\mathcal{K} = \mathsf{JSL}$ with $D = \mathbb{2}$. A D-ultrafilter on a semilattice X is precisely a prime, upwards closed collection of prime $\uparrow$-sets of X. Indeed,

of $X/\mathcal{K}_{fp}$, then $h \cdot \psi_a = \psi_b$. Indeed, the following diagram

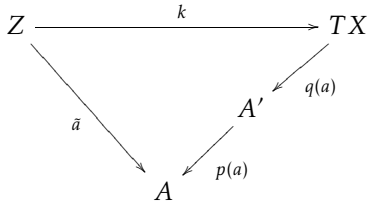


commutes.

Next suppose a cone of C_X with domain Z is given:

$$\frac{X \xrightarrow{a} A}{Z \xrightarrow{\tilde{a}} A} \qquad \text{for } (A,a) \in X/\mathcal{K}_{fp}.$$

We prove that there is a unique morphism k making the following triangles



commutative. The diagram $C_X^{**} = (-)^{**} \cdot C_X$ has the following cone:

$$\frac{X \xrightarrow{a} A}{Z \xrightarrow{\tilde{a}} A \xrightarrow{\eta_A} A^{**}}$$

Indeed, this is compatible with C_X^{**}, since given a morphism (7) of $X/\mathcal{K}_{fp}$ we have the following commutative diagram

where

$$mR = \{x \in X;\ mx \in R\}.$$

See Examples 3.7 and 2.2(e).

Theorem 4.5. *Let $\mathcal{K}$ be a complete, symmetric monoidal closed category with a $*$-cogenerator D. Then the D-ultrafilter monad is a submonad of $(-)^{**}$ which is the codensity monad of the embedding $\mathcal{K}_{fp} \hookrightarrow \mathcal{K}$.*

Proof: Since the natural transformation $i : T \to (-)^{**}$ is monic, there is at most one monad structure making i a monad morphism. We are going to prove that this structure exists, and that the resulting monad fulfils, for the embedding $E : \mathcal{K}_{fp} \to \mathcal{K}$, the limit formula for codensity monads (see Introduction).

(i) For every object X the cone

$$a^{**} : X^{**} \to A^{**} \quad (\text{for all } (A, a) \in X/\mathcal{K}_{fp})$$

is collectively monic. Indeed, since D is a $*$-object, we have $X^* = \operatorname{colim} C_X^*$, see Notation 2.5. Now $(-)^* : \mathcal{K}^{\mathrm{op}} \to \mathcal{K}$ is a right adjoint, thus, it takes the colimit to a limit cone $a^{**} : X^{**} \to A^{**}$ in $\mathcal{K}$.

(ii) Recall the notation $p(a)$ from Definition 3.1 and $q(a)$ from Notation 4.1. We are going to prove that for the embedding $E : \mathcal{K}_{fp} \to \mathcal{K}$ we have the limit formula of the Introduction

$$TX = \lim C_X$$

with the following limit cone

$$\psi_a \equiv TX \xrightarrow{q(a)} A_0 \xrightarrow{p(a)} A \qquad (a \in X/\mathcal{K}_{fp}). \tag{6}$$

First, ψ_a is a cone of C_X, i.e., given a morphism h

$$\begin{array}{ccc} & X & \\ {\scriptstyle a}\swarrow & & \searrow{\scriptstyle b} \\ A & \xrightarrow[h]{} & B \end{array} \tag{7}$$

every element $\mathcal{U}$ of X^{**} is a D-ultrafilter. To see this, given a morphism $X \xrightarrow{a} A$, put $\hat{\mathcal{U}} = \{Z \in A^* \mid a^{-1}(Z) \in \mathcal{U}\}$. We want to prove that there is a unique t_0 in A such that $Z \in \hat{\mathcal{U}}$ iff $t_0 \in Z$. If $\hat{\mathcal{U}} = \emptyset$, then $t_0 = 0$. If $\hat{\mathcal{U}} \neq \emptyset$, every $Z \in \hat{\mathcal{U}}$ is of the form $Z = \uparrow u_1 \cup \cdots \cup \uparrow u_k$ with $u_1, \ldots, u_k$ incomparable elements of A. Since $\mathcal{U}$ is prime, so is $\hat{\mathcal{U}}$, therefore some $\uparrow u_i$ belongs to $\hat{\mathcal{U}}$. Thus, there are incomparable elements of A, $t_1, \ldots, t_n$, such that $\hat{\mathcal{U}}$ consists of all sets $\uparrow t_i$, $i = 1, \ldots, n$, and all sets of A^* containing some of them. It is easily seen that $t_0 = t_1 \vee \cdots \vee t_n$ is as desired.

The rest is analogous to Pos, using that the above poset D^2 is a semilattice.

Example 3.6. Let $\mathcal{K} = K$-Vec and $D = K$. A D-ultrafilter on a vector space X is a vector of the double-dual space X^{**}. Indeed, for every finite-dimensional space A the unit $\eta_A : A \to A^{**}$ is well-known to be invertible. Thus, the derived subobject is all X^{**}.

It turns out that there is a close analogy between ultrafilters on a set and vectors of the double-dual of a space. It is based on the following observation made in [1]:

(i) To give an ultrafilter on a set X means precisely to give a choice, for every finite decomposition $a : X \twoheadrightarrow n$ ($n \in \mathbb{N}$) of a class $a^{-1}(i)$, $i \in n$, which is *compatible*. That is, if $b : X \twoheadrightarrow m$ is a coarser decomposition (one factorizing through a) then the chosen class for b contains $a^{-1}(i)$.

(ii) To give a vector of X^{**} for a space X means precisely to give a choice, for every finite-dimensional decomposition $a : X \twoheadrightarrow K^n$ ($n \in \mathbb{N}$) of a class $a^{-1}(i)$, $i \in K^n$, which is compatible.

A different analogy between X^{**} and ultrafilters was presented in Devlin's thesis [3].

Example 3.7. Let M be a finite commutative monoid. For $D = \mathcal{P}M$, a D-ultrafilter on an M-set $(X, \cdot)$ is precisely an ultrafilter on X. Indeed, we know from Example 2.2(e) that an external element of X^{**} is a collection $\mathcal{U}$ of subsets of X. $\mathcal{U}$ is a D-ultrafilter iff for every homomorphism $a : X \to A$ with A finite (= finitely presentable) there exists $t \in A$ such that (4) holds. Therefore, every D-ultrafilter is an ultrafilter. The proof of the converse is analogous to Lemma 3.3. We just use, instead of the function χ_R there, the

function $f_R : X \to \mathcal{P}M$ of Example 2.2(e). Thus, we work with $a =< f_R, f_S >: X \to (\mathcal{P}M)^2$.

4. The codensity monad of the embedding $\mathcal{K}_{fp} \hookrightarrow \mathcal{K}$

In this section $\mathcal{K}$ is a complete, symmetric monoidal closed category with a $*$-cogenerator D.

Notation 4.1. For every object X we denote by $i_X : TX \to X^{**}$ the wide intersection of all derived subobjects. (Thus, the external elements of TX are precisely the D-ultrafilters on X.) The factorizing morphisms are denoted by $q(a)$ for all $a : X \to A$, $A \in \mathcal{K}_{fp}$:

$$\begin{array}{ccc} & & A' \\ & \overset{q(a)}{\nearrow} & \downarrow{\scriptstyle a'} \\ TX & \xrightarrow[i_X]{} & X^{**} \end{array} \qquad \text{for } (A,a) \in X/\mathcal{K}_{fp} \tag{5}$$

Lemma 4.2. *The morphisms $i_X : TX \to X^{**}$ carry a subfunctor T of $(-)^{**}$.*

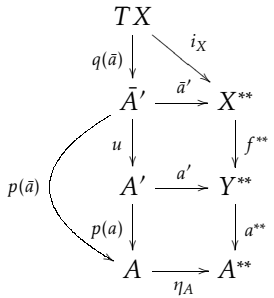
Proof: The definition of T on morphisms $f : X \to Y$ follows automatically from the naturality of $i : T \to (-)^{**}$. Indeed, given a morphism $f : X \to Y$, in order to verify that a (necessarily unique) morphism Tf exists making the following square

$$\begin{array}{ccc} TX & \xrightarrow{i_X} & X^{**} \\ {\scriptstyle Tf}\downarrow & & \downarrow{\scriptstyle f^{**}} \\ TY & \xrightarrow[i_Y]{} & Y^{**} \end{array}$$

commutative, we just need to observe that f^{**} factorizes through all derived subobjects of Y^{**}. Indeed, for all $a : Y \to A$ with A finitely presentable put

$$\bar{a} = a \cdot f : X \to A.$$

Use the universal property of the pullback a' (of η_A along a^{**}) to define a morphism u as follows:



Then $u \cdot q(\bar{a})$ is the desired factorization. ∎

Remark 4.3. The functor T of Lemma 4.2 carries a monad $\mathbb{T}$ which is a submonad of $(-)^{**}$ via (i_X). This is proved in the next theorem. $\mathbb{T}$ is called the *D-ultrafilter monad*.

Examples 4.4. (a) For $\mathcal{K} = \mathsf{Set}$ we see that $\mathbb{T}$ is the ultrafilter monad, for $\mathcal{K} = K\text{-}\mathsf{Vec}$ it is the double-dualization monad. In both cases, $\mathbb{T}$ is the codensity monad of $\mathcal{K}_{fp} \hookrightarrow \mathcal{K}$ (see Introduction).

(b) In case $\mathcal{K} = \mathsf{Pos}$ the monad $\mathbb{T}$ assigns to every poset X the poset of all nonempty, prime collections of $\uparrow$-sets closed under upper sets and finite intersections. It is ordered by inclusion, see Examples 3.4 and 2.3(a).

(c) In $\mathcal{K} = \mathsf{JSL}$ for every semilattice X the semilattice TX consists of all prime, upwards closed collections of nonempty prime $\uparrow$-sets. And the semilattice operation is the set-theoretic union. See Example 3.5.

(d) For $\mathcal{K} = \mathsf{Par}$, Gra, $\Sigma\text{-}\mathsf{Str}$ or $M\text{-}\mathsf{Set}$, the underlying set of TX is that of all ultrafilters on X.

In Gra, TX is a complete graph (all pairs of ultrafilters form an edge), see Examples 3.5 and 2.3(b). Analogously in $\Sigma\text{-}\mathsf{Str}$.

In $M\text{-}\mathsf{Set}$ the monoid action assigns to every ultrafilter $\mathcal{U}$ on X and every element $m \in M$ the ultrafilter

$$m\mathcal{U} = \{R \subseteq X;\ mR \in \mathcal{U}\}$$