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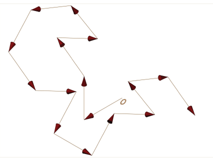
ON THE PROBABILITY TO BE AFTER n RANDOM JUMPS OF UNIT LENGTH IN SPACE WITHIN A DISTANCE OF RADIUS r FROM THE START: THE PROBLEM OF RANDOM FLIGHTS

PEDRO BARATA DE TOVAR SÁ AND ALEXANDER KOVAČEC

Abstract: It is shown that the probability for a particle to be after n random jumps of unit length in three dimensional space within a distance r from the start is given by a piecewise polynomial function of r. The solution is self contained and uses integration over polytopes.

Keywords: random walks, integration over polytopes.
Math. Subject Classification (2010): 60G50.

1. Introduction



Assume a particle, at instant 0 at the origin of three dimensional euclidean space jumps at each tick of the clock exactly one unit from its current position into a random direction. (Here the directions are defined as position vectors to uniformly distributed points of the origin-centered unit sphere.)

Question: What is - as a function of r - the probability to encounter the particle after exactly n random jumps within the 0-centered ball $B = B(0, r)$ of radius r?

For this problem an elementary solution by L. R. G. Treloar and solutions which use Fourier transforms and discontinuous factors are known. Thanks to an answer given by an anonymous person [Kh], we learned about relevant literature and the fact that the proper search word would be Random flights only after we had found the essence of our solution. As [Dt, p. 352] notes, in well known books on random walks little attention is paid to continuous

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random walks. About those earlier solutions we report in sections 6 and 7 of this article.

In this paper we give a new elementary solution which unlike Treloar's uses the multivariate change of variable rule and integrations over a polytopes. This possibility might come unexpected since the expressions to which we are led at first are highly nonlinear.

In Section 2 we give the recursive formula for the random variable R_n defined as the distance of the particle from the starting place after n steps and show that the wanted probability distribution $r \mapsto \text{prob}(R_n \leq r)$ can be computed from a certain volume. Then Corollary 3.3 shows

$$\text{prob}(R_n \leq r) = 2^{-(n-1)} \int_0^r x f_{n-1}(x) dx,$$

where the functions $f_n(x)$ are obtained as certain multiple integrals dependent on a parameter x, whose values we determine in Section 4 where we will recognize them as depending piecewise polynomially from x. The results in Section 5 give explicit formulas for the probability distribution mentioned above and we compute the distribution function $r \mapsto \text{prob}(R_4 \leq r)$ for illustration.

In this preprint we add furthermore in Section 6 the solution of Treloar. We give a complete proof which as it stood till now had to be composed from various sources. In the present authors' view it also left too many details to be filled by the reader. In Section 7 we report on the history and further results on random flights to the extent they are close to the ones given here.

2. Some preliminary considerations

One of the tools we need is a very old theorem. It is the base to one of the area preserving maps of the Earth, known as cylindrical projection or Lambert projection (1772), see e.g. Kreyszig [Kr, p. 210].

Theorem 2.1. *(Archimedes). The area of the region of a sphere $\mathbb{S}$ contained within two parallel planes (which both intersect $\mathbb{S}$) depends only on the distance between the planes and not on their position with respect to $\mathbb{S}$. Thus if $\mathbb{S}$ is the unit sphere and d the distance between the planes, the area is given by $2d\pi$.*

We call the region referred a *spherical ring of height d*. We use it only for the unit sphere. In the following lemma we allow the geometric language afforded by the concept of an affine space. See e.g. [Enc, Article 7A]. For

Now $r \mapsto \Re\sqrt{R^2 - r^2}$ is just the function which is $\sqrt{R^2 - r^2}$ when $|r| < R$ and 0 otherwise. It follows that $r \mapsto (\Re\sqrt{R^2 - r^2})^{d-3}$ is a truncated polynomial of degree $d - 3$ and from this one sees easily the following

Theorem 7.3. (García-Pelayo) *The convolution $r \mapsto s_d^{\otimes n}(r)$ is in case of odd dimension d a piecewise polynomial with support $[-nR, nR]$ of degree $nd - 2n - d$.*

Readers of this paper might wish to estimate how much they can trust its authors concerning novelty of their ideas and historical correctness. Thus it might be useful to mention the present authors absorbed in considerable detail in [Ch], [Dt], [G-P], [KS], [Qu], [Sch], [Trl], [Ra2] the parts related to their problem. They did not see [Kl] but constructed complete arguments from what they read about his solution in [Dt]. [KS] reports about [Ha]. We assume what [Ra2] writes in §42a is the essence of what can be found in [Ra1]. Where our 'story' touches on [Ra3], we rely on [Ch] and [Dt]. In [La] authors learned important facts about Fourier Transforms and convolutions. [KS] teaches examples on characteristic functions. From wikipedia they learned a version of the Abel transform for understanding relevant parts of [G-P].

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Theorem 7.2. *The probability* $W_N(\mathbf{R})d\mathbf{R}$ *that the position* $\mathbf{R}$ *of the particle will be found in the interval* $[\mathbf{R}, \mathbf{R}+d\mathbf{R}]$ *after* n *displacements in 3-space is given by (his notation)*

$$W_N(\mathbf{R}) = \frac{1}{2\pi^2|\mathbf{R}|}\int_0^\infty \sin(|\rho||\mathbf{R}|)\prod_{j=1}^{N}\frac{\sin(|\rho|l_j)}{|\rho|l_j}|\rho|d|\rho|$$

This, Chandrasekhar says, is a formula of Rayleigh [Ra3] (at least in the case that all $l_j = l$) and he follows Rayleigh in explicitly giving $W_N(\mathbf{R})$ for $N = 3, 4, 5, 6$ as piecewise polynomial functions.

In a 1947 paper of mere two pages, putting $R = |\mathbf{R}|$, Quenouille took Chandrasekhar's formula in the form

$$W_N(\mathbf{R}) = \frac{1}{2\pi^2 R}\int_0^\infty \sin(Rx)\left(\frac{\sin(lx)}{lx}\right)^N xdx$$

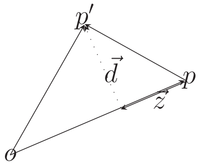
and proved for the quantities $I_N(R) = 2^{N+1}\pi l^N\Gamma(N-1)RW_N(R)$ a recursive differential equation from which he gets explicit piecewise polynomial expressions for $I_N(R)$ for all N. Evidently this result must be equivalent to Treloar's 1946 result but Quenouille does not mention Treloar, just as Treloar didn't know of Chandrasekhar and after all was in error (as Quenouille shows) of saying that 'Rayleigh's method is impracticable for large n'. Chandrasekhar relied on Fourier transforms and discontinuous factors, and we understand that Rayleigh's method was similar.

In 2012 apparently the first really new result concerning elementary representability of the density functions in dimensions higher than three was found. García-Pelayo [G-P] writes s_d for the isotropic probability density whose support is the $d-1$ dimensional surface of the sphere of radius R. From the known surface area of spheres it follows that via Dirac's δ this can be expressed as $s_d(r) = \frac{\Gamma(d/2)\delta(r-R)}{2\pi^{d/2}r^{d-1}}$. In this language the probability density for the n-step random flight problem in dimension d is given by the n-fold convolution of s_d, that is by $s_d^{\otimes n}$. The convolution theorem for Fourier transforms allows to say that $s_d^{\otimes n} = \mathcal{F}^{-1}((\mathcal{F}(s_d))^n)$. Using a 1963 result of Kingman according to which convolution and certain projections of the sphere onto lower dimensional disks commute, and developing a higher dimensional version of the Abel transform, [G-P] gets in case that d is odd that

$$s_d^{\otimes n}(r) = \left(\frac{1}{2^{(d-1)/2}}\frac{(d-2)\cdots 1}{(d-3)\cdots 2}\frac{1}{R^{d-2}}\right)^n\left(-\frac{1}{2\pi r}\frac{d}{dr}\right)^{\frac{d-1}{2}}\left((\Re\sqrt{R^2-r^2})^{d-3}\right)^{\otimes n}.$$

computational purposes, $\vec{op}$, p and $\vec{p}$ are the same but these notations elicit different ways to think about point p. Recall that $\vec{pp'} = p' - p$. By $|\vec{p}|$ we denote the euclidean norm of p.

Lemma 2.2. *Let* p *and* p' *be two points of distance 1 in Euclidean 3-space* $(E, \langle,\rangle)$ *with origin denoted* o. *Let* $\vec{z}$ *be the orthogonal projection of* $\vec{pp'}$ *onto* $\vec{op}$. *Then* $|\vec{p'}| = \sqrt{1+|\vec{p}|^2+2\langle\vec{p},\vec{z}\rangle}$.

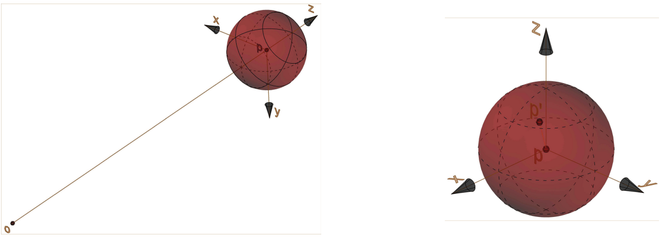


Proof. We may write $p' = p + \vec{pp'} = p + (\vec{z}+\vec{d})$ where $\vec{d} \perp \vec{z}$. It follows that

$$\begin{aligned}|\vec{p'}|^2 &= \langle\vec{p'},\vec{p'}\rangle\\ &= \langle\vec{p}+\vec{z}+\vec{d}, \vec{p}+\vec{z}+\vec{d}\rangle\\ &= |\vec{p}|^2+|\vec{z}|^2+|\vec{d}|^2+2\langle\vec{p},\vec{z}\rangle+2\langle\vec{p},\vec{d}\rangle+2\langle\vec{z},\vec{d}\rangle\\ &= |\vec{p}|^2+1+2\langle\vec{p},\vec{z}\rangle+0+0\end{aligned}$$

Here we used Phytagoras' theorem, and the parallelity or antiparallelity of $\vec{p}$ with $\vec{z}$ and the perpendicularity of $\vec{d}$ with respect to the latter two vectors. □

Now consider the following situation: a fixed point p, and a sphere of radius r with origin in o containing the o-centered ball $B(o,r)$. From p a particle jumps a unit length in a random direction to a point $p' \in \mathbb{S}_p$, the local unit sphere centered at p. It is assumed that the random unit vector is uniformly distributed, so that any two patches of same area on the local sphere $\mathbb{S}_p$ have equal probability to receive p'.



The figures show the local sphere $\mathbb{S}_p$ from far and near.

By leb' we shall denote the Lebesgue measure of appropriate dimension.

Lemma 2.3. *Under the described condition, the probability that* $p' \in B(o,r)$ *is given by*

$$\tfrac{1}{2}\text{leb}(\{z \in [-1,1] : \sqrt{1+|p|^2+2z|p|} \le r\}).$$

Proof. The hypothesis on the random direction implies by Archimedes' theorem that any two spherical rings of $\mathbb{S}_p$ of equal height receive the particle with the same probability. This in turn implies that the projections of the points p' onto the local z-axis (the axis through o and p) are uniform, when suitably constrained: any two given intervals of equal length on the z-axis and contained within $\mathbb{S}_p$ have the same chance to receive the projection of a point p'. This projection is given by $\vec{p}+\vec{z}$ in the previous figure. Since $\vec{z}$ has constant orientation in space, we have that $\langle\vec{p},\vec{z}\rangle = |\vec{p}|\langle\frac{\vec{p}}{|\vec{p}|},\vec{z}\rangle =: |\vec{p}|t$ is uniformly distributed, hence t is uniformly distributed in $[-1,1]$. Since $[-1,1]$ has Lebesgue measure 2, the probability that $p' \in B(o,r)$ is found by Lemma 1 to be the normalized Lebesgue measure of all $t \in [-1,1]$ such that $\sqrt{1+|\vec{p}|^2+2t|\vec{p}|} \leq r$ happens. This can evidently be written as done in the statement of the lemma. □

Using above formulas and the two examples below, lead us in a natural manner to a first formalization of our problem.

The distance that the particle has from the origin after n random jumps is a random variable which we denote R_n. It is clear that R_1 is trivial: $R_1 = 1$ and its distribution function is singular; for $\text{Prob}(R_1 \leq r)$ is 0 or 1 according to if $r < 1$ or $r \geq 1$, respectively.

Example. What is $\text{prob}(R_2 \leq r)$, that is, what is the probability that a particle doing two random jumps of unit length will remain within distance r of the starting place?

Solution: Since after one jump our particle has distance exactly 1 from the starting place, it has after two jumps by above considerations the distance $R_2(z_2) = \sqrt{1+1+2z_2 1} = \sqrt{2+2z_2}$ and therefore by Lemma 2, we find

$$\begin{aligned}\text{prob}(R_2 \leq r) &= \tfrac{1}{2}\text{leb}(\{z \in [-1,1] : \sqrt{2+2z} \leq r\}) \\ &= \tfrac{1}{2}\text{leb}(\{z \in [-1,1] : z \leq \tfrac{r^2-2}{2}\}) \\ &= \tfrac{1}{2}\text{leb}([-1,1] \cap]-\infty, \tfrac{r^2-2}{2}]) \\ &= \begin{cases} r^2/4 & \text{if } 0 \leq r \leq 2 \\ 1 & \text{if } r > 2 \end{cases}\end{aligned}$$

Example. The same question as in the previous example but for three random jumps: find $\text{prob}(R_3 \leq r)$.

In our temporary solution we reduce the problem again to the calculation of measure. If the particle is after two random jumps at a distance $R_2 = R_2(z_2)$ from the start, then it is after three random jumps at one of the distances

differential equations and here his method does not give precise results for given n.

It was a year after Pearson's question that the Belgian mathematician J.C. Kluyver provided an answer for the associated distribution function via an integral representation using Bessel functions J_0, J_1.

Theorem 7.1. (Kluyver, 1906). *Given fixed positive numbers l_i and r and real random variables θ_i independently and uniformly distributed in the interval $]-\pi,\pi]$, $i = 1,...,n$. Define associated vector random vectors $X_i = l_i(\cos\theta_i, \sin\theta_i)$ and their sum $S_n = X_1 + \cdots + X_n$.*
a. Then

$$\text{prob}(|S_n| \leq r) = \int_0^\infty rJ_1(rt)\prod_{m=1}^n J_0(l_m t)dt.$$

b. In particular, if $l_1 = \cdots = l_n = l$

$$\text{prob}(|S_n| \leq r) = \int_0^\infty rJ_1(rt)J_0(lt)^n dt.$$

Kluyver's ingenious solution is based on Weber's discontinuous factor expressing the indicator function $\mathbb{1}_{[0,r]}$ via an integral of an expression involving Bessel functions and another sophisticated formula for Bessel functions due to C.G. Neumann. To this day not even for the special case (b) an 'elementary' solution is known and for the evaluation of the integrals in the decades after Kluyver, big efforts where expended. It is shown in [Dt, p.11] that Kluyver's theorem can also be deduced from the method of characteristic functions.

In 1919 Rayleigh [Ra3] came back to his problem and it was he that called the problem of Pearson the problem of Random Flights. Chandrasekhar [Ch] tells us that he got essentially the results we announce below.

In view of the non-elementaricity of the 2-dimensional random flight problem, it is interesting that the three-dimensional case admits an elementary solution and such was Treloar's found in 1946 and related in the section before.

The potential of the method of discontinuous factors was recognized by Dirichlet in 1839 for the evaluation of multiple integrals and publicized in 1912 by A. Markoff in his book on probability theory. It was shown to have much potential in other problems of this sort.

In 1943 Chandrasekhar generalized Markoff's method to dimensions higher than 2 and solved various problems on random flights - all for three dimensional space - in the first chapter of his long paper [Ch]: One finds in that paper the following

From $*$ then, Treloar gets his final equation

$$P(r)=\frac{r}{2l^2}\frac{n^{n-2}}{(n-2)!}\sum_{s=0}^{\lfloor\frac{n}{2}(1-\frac{r}{nl})\rfloor}(-1)^s\binom{n}{s}(\frac{1}{2}-\frac{r}{nl}-\frac{s}{n})^{n-2}.$$

It is now elementary to see that this, written by replacing letter r by x and putting $l=1$ is precisely equal to our function $2^{-(n-1)}f_{n-1}(x)$.

7. More on the history of the random flight problem and related results

Many variations of the problem we treated make sense and an article by Dutka [Dt] surveys probably much of what was done till about 1985. Influenced by this article we mention the facts most relevant with respect to our article essentially in chronological order. We looked furthermore in depth into the original sources mentioned in the final paragraph below. The rest of the story we tell here is told as we understand it and hence might not be completely correct.

Formally, the problem of random flights began with a 1905 question by the later famous statistician K. Pearson who inquired about the analogue to our problem in 2 dimension. Mathematically speaking he asked the following question for the special case that $l_1=\cdots=l_n=l$.

Consider in the plane a random walk obtained by adding vectors of length $l_1,l_2,...,l_n$ in random directions. What is the probability density of the property that the sum of these vectors has a norm $<r$.?

However, already in 1880, Lord Rayleigh, interested in acoustics, was interested in finding 'the resultant of isoperimetric vibrations of equal amplitude' when the phases are chosen randomly. More precisely he was interested in the relative frequency with which the amplitude of sums $\sum_{j=1}^n\exp(\sqrt{-1}\theta_j)$ falls within a given interval when the θ_j are chosen randomly. Identifying the exponentials with unit vectors in the plane, this question is mathematically the same as Pearson's original question. Rayleigh also inquires about this problem in three dimensions. In §42a of his book [Ra2] he shows that in the 2-dimensional case the probability of a resultant amplitude between r and $r+dr$ for large n is given approximately by the formula $\frac{2}{n}e^{-r^2/n}rdr$ and in the 3-dimensional case this probability is $3\sqrt{\frac{6}{\pi n^3}}e^{-r^2\big/(2/3n)}r^2dr$. The relevance or existence of Rayleigh's results was apparently unknown to Pearson and Kluyver (below). Rayleigh argued statistically with difference and

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$$\begin{aligned}R_3(z_2,z_3)&=\sqrt{1+R_2(z_2)^2+2z_3R_2(z_2)}\\&=\sqrt{3+2z_2+2z_3\sqrt{2+2z_2}}\end{aligned}$$

Now since z_2,z_3 can be seen as random variables which are uniformly distributed on $[-1,1]$ (and 0 outside), and since $[-1,1]\times[-1,1]$ has twodimensional Lebesgue measure equal to 4, we find that

$$\mathrm{prob}(R_3\le r)=\tfrac{1}{4}\mathrm{leb}(\{(z_2,z_3)\in[-1,1]^2:\sqrt{3+2z_2+2z_3\sqrt{2+2z_2}}\le r\}).$$

This is evidently a more complex problem than before. But it is now clear where we are headed to: we need to compute for any natural n the $n-1$-dimensional Lebesgue measure of a set which has a complex definition in view of the nested square roots which define $R_n=R_n(z_2,...,z_n)$:

Letting

$$\mathcal{R}_n(r)=\{(z_2,...,z_n)\in[-1,1]^{n-1}:R_n(z_2,...,z_n)\le r\},$$

we have to compute the $n-1$ dimensional volume of $\mathcal{R}_n(r)$, since

$$\mathrm{prob}(R_n\le r)=\frac{1}{2^{n-1}}\mathrm{vol}(\mathcal{R}_n(r)).$$

This is the aim of the next sections. After the second author had determined by arduous computations the volumes of these sets for n up to 4, the first author made the quite surprising discovery that it is possible to reduce the computation of these volumes to the computation of integrals of a simple function over a polytope.

3. Reduction of the computation of vol$(\mathcal{R}_n(r))$ to an integral of a simple function over a polytope

From the considerations of the previous section we get that the quantities $R_1,R_2,...,R_i,...$ can be defined inductively as follows:

$$\begin{aligned}R_1&=1;\\R_{i+1}&=\sqrt{1+R_i^2+2z_{i+1}R_i},\end{aligned}$$

where we admit for $z_2,z_3,z_4,...$ only reals in the interval $[-1,1]$. One sees that the R_i will satisfy the inequalities $|1-R_i|\le R_{i+1}\le 1+R_i$, $i=1,2,...$. Explicitly, the first few R_i have the following aspect:

$$\begin{aligned}R_1&=1\\R_2&=\sqrt{2+2z_2}\\R_3&=\sqrt{3+2z_2+2z_3\sqrt{2+2z_2}}\\R_4&=\sqrt{4+2z_2+2z_3\sqrt{2+2z_2}+2z_4\sqrt{3+2z_2+2z_3\sqrt{2+2z_2}}}\\&\vdots\end{aligned}$$

It is clear that $R_i = R_i(z_2, z_3, ..., z_i)$ for $i = 2, 3, 4, ...$. If n is given from context, we define $z = (z_2, ..., z_n)$ for brevity. Then $R_i = R_i(z)$ but it does not depend on $z_{i+1}, ..., z_n$. The next two propositions give a first hint how to compute $\mathrm{vol}(\mathcal{R}_n(r))$ in practice.

Let $n \in \mathbb{Z}_{\geq 1}$, and $0 \leq a < b \leq n+1$. Consider the set

$$P_n(a,b) = \{x = (x_1, x_2, ..., x_n) : x \text{ satisfies the system S of inequalities below } \}.$$

$$\begin{gathered}\text{System S:}\\ |1-x_2| \leq x_1 \leq \min\{1+x_2, 2\}\\ |1-x_3| \leq x_2 \leq \min\{1+x_3, 3\}\\ \vdots\\ |1-x_{n-1}| \leq x_{n-2} \leq \min\{1+x_{n-1}, n-1\}\\ |1-x_n| \leq x_{n-1} \leq \min\{1+x_n, n\}\\ a \leq x_n \leq b.\end{gathered}$$

Proposition 3.1. *The set $P_n(a,b)$ is a nonempty polytope and one has for any continuous function $f : P_n(a,b) \to \mathbb{R}$, that*

$$\int\limits_{P_n(a,b)} f dx_{1:n} = \int\limits_a^b \int\limits_{|1-x_n|}^{\min\{1+x_n,n\}} \int\limits_{|1-x_{n-1}|}^{\min\{1+x_{n-1},n-1\}} \cdots \int\limits_{|1-x_2|}^{\min\{1+x_2,2\}} f(x)\, dx_1 \cdots dx_{n-2} dx_{n-1} dx_n.$$

Proof. Note that an inequality of the form $|1-x| \leq x'$ is equivalent to the conjunction of inequalities $-x-x' \leq -1$ & $x-x' \leq 1$ and $x' \leq \min\{1+x, a\}$ is equivalent to $x'-x \leq 1$ & $x' \leq a$. It follows that the whole system S is equivalent to some matrix inequalitiy of the form $Ax \leq b$ where A is a real matrix, $x = (x_1, x_2, ..., x_n)^\top$ an $n-1$-uple of variables and b real column. It is obvious that $P_n(a,b)$ is a bounded and closed set. Thus $P_n(a,b)$ satisfies the defining criteria for a polytope.

Next note that the last inequality of S guarantees $0 \leq x_n \leq 1+n$ and this guarantees that the inequality $|1-x_n| \leq \min\{1+x_n, n\}$ will hold. Thus there will exist x_{n-1} satisfying the penultimate inequality of S and we will have $0 \leq x_{n-1} \leq n$. This in turn implies that there will exist x_{n-2} satisfying the pen-penultimate inequality of S and we will have $0 \leq x_{n-2} \leq -1+n$. Continuing this reasoning, we see that $P_n(a,b)$ will be nonempty.

Turning now to the second part of the theorem, we note that the integration over a polytope is usually not a trivial task; see Schechter [Sch]. But in our case, the system S determines bounds for x_n, and given x_n, bounds for x_{n-1}, etc. in such a happy way that integration of f over $P_n(a,b)$ can be directly

one has to shift the previous function by 1/2 to the left, that is substitute t by $t+\frac{1}{2}$ (having now support $[-1/2, 1/2]$) yielding $\sqrt{n}f(\frac{n}{2}+nt)$ and now to stretch this function by the factor $2nl$ to get support $[-nl, nl]$ and, finally, to maintain the value 1 of the integral divide it by $2nl$. Treloar opts to exploit also the symmetry of the function and obtains (in his notation) the function

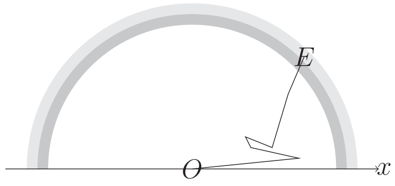
$$x \overset{p}{\mapsto} \frac{1}{2l}\frac{n^{n-1}}{(n-1)!} \sum_{s=0}^{\lfloor \frac{n}{2}(1-\frac{x}{nl}) \rfloor} (-1)^s \binom{n}{s} (\frac{1}{2} - \frac{x}{nl} - \frac{s}{n})^{n-1}$$

for the distribution (density) of the x-component of end points of random polygons with n links of length l emanating from 0.

The question however is: what is the density of the end point distances (from the origin) of the random polygons? In one more step difficult to comprehend from the few words Treloar dedicates to it, he says that the density $P(r)$ of the end point distances is related to the distribution function above by the formula

$$*:\ -\frac{dp}{dx}|_{x=r} = \frac{1}{2r}P(r).$$

Here is our justification: Analogously to the case for a single link, a polygon with endpoint distance r projects its endpoint onto the interval $[-r, r]$ of length $2r$ and the endpoints E of the polygons with $|\overrightarrow{OE}| = r$ will be found with equal probability on any patch of the O-centered sphere of radius r. Thus again analogously to the case of a single link, the projections of these Es to the x-axis will induce a uniform distribution over the interval $[-r, r]$ on the x-axis.



The density of these points on the x axis is hence found to be constant $\frac{1}{2r}P(r)$ in $[-r, r]$ and 0 outside. So $\frac{1}{2r}P(r)dr$ is precisely the increment of density of points that the shell defined by radii r and $r+dr$ contributes to the cumulative density p (which is the 'sum' of all such densities) on the x-axis.

The figure shows just two shells (of many) of thickness dr of one half of a ball. Since a shell intersects the x-axis along an interval of length dr, we may write $dx = dr$. At the outer fringes of length dr of the interval obtained by projecting the outer shell, only the outer shell contributes to p. It follows that on the x-axis at distance r from the origin we have $dp = -\frac{1}{2r}P(r)dr = -\frac{1}{2r}P(r)dx$ and hence the above differential equation.

To meet the requirement that the integral over a density function be 1, one needs now to normalize this function by a positive multiplicative constant c so that the integral $\int_0^n cf(z)dz = 1$. This is one more detail missing in the arguments: it is claimed but not proved that $c = 1/\sqrt{n}$. To prove this we may compute as follows

$$\begin{aligned}\int_0^n f(z)\frac{dz}{\sqrt{n}} &= \sum_{\mu=0}^{n-1}\int_\mu^{1+\mu}\frac{f(z)}{\sqrt{n}}\\ &= \sum_{\mu=0}^{n-1}\int_\mu^{1+\mu}\frac{1}{(n-1)!}\sum_{\nu=0}^{\mu}(-1)^\nu\binom{n}{\nu}(z-\nu)^{n-1}dz\\ &= \frac{1}{(n-1)!}\sum_{\mu=0}^{n-1}\sum_{\nu=0}^{\mu}(-1)^\nu\binom{n}{\nu}\int_\mu^{1+\mu}(z-\nu)^{n-1}dz\\ &= \frac{-1}{n!}\sum_{\mu=0}^{n-1}\sum_{\nu=0}^{\mu}(-1)^\nu\binom{n}{\nu}((1+\mu-\nu)^n-(\mu-\nu)^n)\\ &= \frac{1}{n!}\sum_{\nu=0}^{n-1}(-1)^\nu\binom{n}{\nu}\sum_{\mu=\nu}^{n-1}((1+\mu-\nu)^n-(\mu-\nu)^n)\\ &= \frac{1}{n!}\sum_{\nu=0}^{n-1}(-1)^\nu\binom{n}{\nu}(n-\nu)^n,\end{aligned}$$

because the inner sum telescopes.

Now since $(n-x)^n$ is a polynomial in x of degree n and we can evidently replace $\sum_{\nu=0}^{n-1}\ldots$ by $\sum_{\nu=0}^{n}\ldots$ the fact we mentioned in the second part of the proof of Theorem 1 allows us to say that the above sum is 1. Therefore the variable Z follows the law (has the density function) $z \mapsto f(z)/\sqrt{n}$.

Consequently the density for the variable Z/n (the mean) is obtained by squeezing the latter function by a factor n in the direction of z and multiplying by n to maintain the integral equal to 1. Explicitly, thus, the mean follows the law

$$t \mapsto \sqrt{n}f(nt) = \frac{n^n}{(n-1)!}\sum_{i=0}^{\lfloor nt\rfloor}(-1)^i\binom{n}{i}(t-\frac{i}{n})^{n-1}$$

The mean has, hence, as is also intuitively to be expected, a density function symmetric with respect to $1/2$, and support $[0,1]$.

Treloar starts from this formula. To adapt it to a distribution which covers the interval $[-nl, nl]$ as is the case with our polygons with n links of length l

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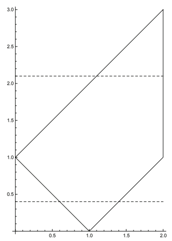
translated into a multiple integral of the form given in the theorem. To eliminate any doubts, we give a proof by induction over n.:



The polytope $P_2(a,b)$ is defined by the last two inequalities of S with indices 1,2 instead of $n-1, n$:

$$|1-x_2| \le x_1 \le \min\{1+x_2, 2\}$$
$$a \le x_2 \le b$$

A picture of this polytope for the case $a = 0, b = 3$ is shown at the left. If we have general a, b at heights $0 < a < b < 3$ these would yield a truncated polytope as indicated by the dashed lines. It is evident that in this case $\int_{P_2(a,b)} f(x)dx_{1:2} = \int\limits_a^b \int\limits_{|1-x_2|}^{\min\{1+x_2,2\}} f dx_1 dx_2$.

This starts the induction. Now assume the theorem already shown for $n-1$ in place of n. By the general recursive formula for computation of multiple integrals (a version of Fubini's theorem),

$$\int\limits_{P_n(a,b)} f dx = \int\limits_a^b \int\limits_{P_n(a,b)\cap(\mathbb{R}^{n-1}\times\{x_n\})} f(x_{1:n-1}, x_n)dx_{1:n-1}dx_n.$$

Now we know $0 < x_n < 1+n$ for any x_n here involved; and if we put for a fixed such x_n, $a' := |1-x_n|$, $b' := \min\{1+x_n, n\}$ we have $0 \le a' < b' \le n$. Therefore, $P_n(a,b) \cap (\mathbb{R}^{n-1}\times\{x_n\})$ is defined by the first $n-2$ inequalities of system R together with the inequality $a' \le x_{n-1} \le b'$, ; i.o.w. this intersection is $P_{n-1}(a', b')$. By the induction hypothesis,

$$\int\limits_{P_{n-1}(a',b')} f(x_{1:n}, x_n)dx_{1:n-1} =$$
$$\int\limits_{a'}^{b'} \int\limits_{|1-x_{n-1}|}^{\min\{1+x_{n-1},n-1\}} \int\limits_{|1-x_{n-2}|}^{\min\{1+x_{n-2},n-2\}} \cdots \int\limits_{|1-x_3|}^{\min\{1+x_3,3\}} \int\limits_{|1-x_2|}^{\min\{1+x_2,2\}} f(x)\, dx_1 dx_2 \cdots dx_{n-3}dx_{n-2}dx_{n-1}.$$

The theorem follows by substituting this expression in the displayed formula before; and substituting a', b' by their definitions. □

Proposition 3.2. *For any real number $0 < r < n$, one has that* $P(r) := \{(R_2(z), R_3(z), \ldots, R_n(z)) : -1 \le z_2, z_3, ..., z_n \le 1, R_n(z) \le r\}$ *equals the $n-1$-dimensional polytope $P_{n-1}(0, r)$ of the previous proposition.*

Proof. Fix an r with $0 < r < n$ and define
$P' = P'(r) =$
$\{(x_2, ..., x_n) : 0 \le x_2 \le 2, |1-x_2| \le x_3 \le 1+x_2, ..., |1-x_{n-1}| \le x_n \le 1+x_{n-1},\ x_n \le r\}$. We have already observed that the inequalities $0 \le R_2 \le 2$

and $|1-R_i| \leq R_{i+1} \leq 1+R_i$, $i=1,2,...$ hold so that it is clear that $P(r) \subseteq P'(r)$. At the other hand, if $(x_2,...,x_n) \in P'(r)$, then it is easy to see that we can find inductively $z_2, z_3, ..., z_n \in [-1,1]$ so as to satisfy $x_2 = R_2(z_2), x_3 = R_2(z_2,z_3), ..., x_n = R_n(z_2,z_3,...,z_n)$. Hence $P'(r) \subseteq P(r)$. So $P'(r) = P(r)$ and $P(r)$ is a polyhedron. Now note the following equivalences:

$$|1-u| \leq v \leq 1+u \Leftrightarrow \begin{array}{c} 1-u \leq v \\ u-1 \leq v \\ v \leq 1+u \end{array} \Leftrightarrow \begin{array}{c} 1-v \leq u \\ u \leq 1+v \\ v-1 \leq u \end{array} \Leftrightarrow |1-v| \leq u \leq 1+v.$$

$$\begin{array}{c} 0 \leq x_2 \leq 2 \\ |1-x_3| \leq x_2 \leq 1+x_3 \\ \vdots \\ |1-x_{n-1}| \leq x_{n-2} \leq 1+x_{n-1} \\ |1-x_n| \leq x_{n-1} \leq 1+x_n \\ 0 \leq x_n \leq r \end{array}$$

These imply we can describe $P'(r)$ alternatively by the inequalities at the left. But by the original inequalities for $P'(r)$ we see that $x_i \leq i$, so that we can substitute each $1+x_{i+1}$ at the right hand side of the new system by $\min\{1+x_{i+1}, i\}$. Done this, we can cancel the first of the inequalities, $0 \leq x_2 \leq 2$.

Finally we make the system in $x_2,...,x_n$ so obtained into a system in $x_1,...,x_{n-1}$ simply by subtracting 1 one from each index i in x_i. Then emerges precisely $P_{n-1}(0,r)$. □

At this point we recall the

Multivariate Change of Variables Rule. Let $M \subseteq \mathbb{R}^n$ be a measurable subset of $\mathbb{R}^n$ and let g be defined on an open subset containing the closure of M so that g is invertible of class C^1 on M and for the determinant of the Jacobian we have $\det \frac{dg}{dx} \neq 0$ on M. Then for any continuous function $f : g(M) \to \mathbb{R}$ there holds

$$\int\limits_{g(M)} f(x)dx = \int\limits_{M} f(g(x)) \left|\det \frac{dg}{dx}\right| dx; \text{ in particular } \mathrm{vol}(g(M)) = \int\limits_{M} \left|\det \frac{dg}{dx}\right| dx.$$

□

We show next that there is a bijective differentiable map between $P(r) = P_{n-1}(0,r)$ and $\mathcal{R}_n(r)$. To this we will apply the substitution rule.

Since $R_j^2 = 1 + R_{j-1}^2 + 2z_jR_{j-1}$ we get

$$z_j = \tfrac{1}{2}\left(-R_{j-1} + \tfrac{R_j^2-1}{R_{j-1}}\right) \text{ for } j = 2,3,4,....$$

Now one can compute by means of the principle of inclusion exclusion as follows:

$$\begin{aligned} &\mathrm{leb}(H(z) \cap [0,1]^n) \\ &= \mathrm{leb}(H(z)) - \mathrm{leb}(\{x \in H(z) : x \geq e_1 \text{ or } x \geq e_2 \text{ or ... or } x \geq e_n\}) \\ &= \mathrm{leb}(H(z)) - \mathrm{leb}(\bigcup_{i=1}^{n}\{x \in H(z) : x \geq e_i\}) \\ &= \mathrm{leb}(H(z)) - \sum_{i=1}^{n}(-1)^{i-1}(\sum_{I\subseteq\{1,...,n\},|I|=i} \mathrm{leb}(\bigcap_{i\in I}\{x \in H(z) : x \geq e_i\})) \\ &= \mathrm{leb}(H(z)) - \sum_{i=1}^{n}(-1)^{i-1}(\sum_{I\subseteq\{1,...,n\},|I|=i} \mathrm{leb}(\{x \in H(z) : x \geq e_I\}) \\ &= \mathrm{leb}(H(z)) - \sum_{i=1}^{n}(-1)^{i-1}\binom{n}{i}\mathrm{leb}(H(z-i)) \\ &= \sum_{i=0}^{n}(-1)^{i}\binom{n}{i}\mathrm{leb}(H(z-i)). \end{aligned}$$

Using that $H(z) = \mathrm{conv}\{ze_1,...,ze_n\}$ is an $(n-1)$-dimensional simplex in n-dimensional space, Kendall and Stuart (following probably Hall) go on to compute $\mathrm{leb}(H(z))$. This can be computed using that the volume of a simplex is the measure of its base (itself a simplex) times the height relatively to that base times 1 over the dimension of the simplex. The reader who wishes to fill the details in [KS]'s account may check that the center of the 'base' of $H(z)$ which is defined by the plane $\mathrm{conv}\{ze_1,...,ze_{n-1}\}$ is $c = \frac{1}{n-1}\sum_{i=1}^{n-1} ze_i$ and check that the vectors $\overrightarrow{c\,ze_n}$ and $\vec{c0}$ are perpendicular to that base and hence can be used as the heights of $H(z)$ and the simplex $\mathrm{conv}\{ze_1,...,ze_{n-1},0\}$ with respect to the same base. Since the euclidean norms of the mentioned vectors stand in the relation $|\overrightarrow{c\,ze_n}| : |\vec{c0}| = \sqrt{n} : 1$, the volumes of the mentioned simplexes stand in the same relation. Now for the simplex $\mathrm{conv}\{ze_1,...,ze_{n-1},0\}$ we considered $\mathrm{conv}\{ze_1,...,ze_{n-1}\}$ also as the base and thus have diminished the problem by one dimension. This gives a recursive formula from which $\mathrm{leb}(H_n(z)) = \frac{\sqrt{n}}{(n-1)!}z^{n-1}$ can be derived.

It follows that the function $z \overset{f}{\mapsto} \mathrm{leb}(H(z) \cap [0,1]^n)$ is given by

$$f(z) = \frac{\sqrt{n}}{(n-1)!}\sum_{i=0}^{\lfloor z \rfloor}(-1)^i\binom{n}{i}(z-i)^{n-i}.$$

We explain the main ideas of his solution (and those of a paper he uses) at certain points in some detail. We hope this will smooth the way to understanding the papers of Treloar and the respective pages in [KS] for readers interested in them.

Treloar begins by observing that in a previous paper of his, which he cites as I, he had found that the probability that a single randomly oriented link of length l, fixed with one end at the origin has an x-component between x and $x + dx$ given by $p_1(x)dx = \frac{dx}{2l}$ for $|x| < l$. As we would say nowadays, the density function of the projection is rectangular of height $1/2l$ spread over the interval $[-l, l]$. This observation corresponds to our application of Archimedes' theorem in Lemma 2. It follows that the x-component of the endpoint of a random polygon of n links of lengths l will be distributed just in the same manner as the random variable

$$Z = X_1 + X_2 + \cdots + X_n$$

will be, if random variables $X_1, ..., X_n$ are independent and have the rectangular distribution function $p_1(x)$ just explained. After his paper had been published, Treloar says, he learned from a colleague that a 1927 paper of P. Hall [Ha] should be of help. In Kendall and Stuart [KS] an account of Hall's paper is given which we will now explain in somewhat different notation and by invoking general principles.

It is easy to see that if $X_1, ..., X_n$ are independent random variables that all are uniformly distributed over the interval $[0, 1]$, then the variable $Z = X_1 + \cdots + X_n$ will follow a probability law that is essentially given by the Lebesgue measure (leb) of the region that the hyperplane $z = \sum_{i=1}^n x_i$ defines when intersecting it with the hypercube $[0, 1]^n$. More precisely, define

$$H = H(z) = \{x \in \mathbb{R}^n_{\geq 0} : z = \sum_{i=1}^{n} x_i\}.$$

Note that this is a simplex imbedded into the said hyperplane. Then the function $z \mapsto \text{leb}(H(z) \cap [0, 1]^n)$ will up to normalization follow the law of Z. Let $e_1, ..., e_n$ be the standard vectors in $\mathbb{R}^n$ and more generally for $I \subseteq \{1, 2, ..., n\}$ let e_I be the 01-n-tuple which has 1s exactly in the positions $i \in I$. Using translation invariance of Lebesgue measure one has

$$\text{leb}(\{x \in H(z) : x \geq e_I\}) = \text{leb}(H(z) - e_I) = \text{leb}(H(z - |I|)).$$

defining the $n - 1$-dimensional map

$$P(r) \ni \begin{pmatrix} R_2 \\ R_3 \\ \vdots \\ R_n \end{pmatrix} \overset{g}{\mapsto} \begin{pmatrix} \frac{1}{2}\left(-R_1 + \frac{R_2^2-1}{R_1}\right) \\ \frac{1}{2}\left(-R_2 + \frac{R_3^2-1}{R_2}\right) \\ \vdots \\ \frac{1}{2}\left(-R_{n-1} + \frac{R_n^2-1}{R_{n-1}}\right) \end{pmatrix} =: \begin{pmatrix} g_2(\underline{R}) \\ g_3(\underline{R}) \\ \vdots \\ g_n(\underline{R}) \end{pmatrix} = \begin{pmatrix} z_2 \\ z_3 \\ \vdots \\ z_n \end{pmatrix} \in \mathcal{R}_n(r).$$

The ij-entry of the Jacobian $\frac{dg}{dR} = (\frac{\partial g_i}{\partial R_j})_{i,j=2,...,n}$ is given by $\frac{\partial}{\partial R_j}(\frac{1}{2}(-R_{i-1} + \frac{R_i^2-1}{R_{i-1}}))$, showing that row i of the Jacobian may have nontrivial entries only for $j = i - 1, i$. Indeed row i of the Jacobian will be

$$[0, 0, ..., 0, \tfrac{-R_{i-1}^2 - 2R_i^2 + 2}{2R_{i-1}^2}, \tfrac{R_i}{R_{i-1}}, 0, ..., 0],$$

where the first and last zeros are counted as at positions 2 and n respectively, and R_i/R_{i-1} is at position i. The determinant of this $n \times n$ matrix is hence the product of its diagonal entries, i.e. $R_2/R_1 \cdot R_3/R_2 \cdot \cdots \cdot R_n/R_{n-1} = R_n$, for, recall, $R_1 = 1$.

It follows from these considerations that

$$\text{vol}(\mathcal{R}_n(r)) = \text{vol}(g(P(r))) = \textstyle\int_{P(r)} R_n d\underline{R}.$$

Corollary 3.3. *The volume of $\mathcal{R}_n(r)$ is given by*

$$\text{vol}(\mathcal{R}_n(r)) = \int\limits_0^r x_{n-1} \int\limits_{|1-x_{n-1}|}^{\min\{1+x_{n-1},n-1\}} \int\limits_{|1-x_{n-2}|}^{\min\{1+x_{n-2},n-2\}} \cdots \int\limits_{|1-x_2|}^{\min\{1+x_2,2\}} dx_1 \cdots dx_{n-3} dx_{n-2} dx_{n-1}.$$

Proof. It is sufficient to remind that $P(r) = P_{n-1}(0, r)$; that in the language of propositions 1,2, R_n goes over into x_{n-1}, and that the function $f(x_{1:n-1}) = x_{n-1}$ depends only on x_{n-1}. Thus we can pull it out of from all but the first integral sign in Proposition 1 and get the formula claimed above. □

We will wish to use such expressions for different n. So in the next section, we determine the functions

$$f_n(x_n) := \int\limits_{|1-x_n|}^{\min\{1+x_n,n\}} \int\limits_{|1-x_{n-1}|}^{\min\{1+x_{n-1},n-1\}} \cdots \int\limits_{|1-x_2|}^{\min\{1+x_2,2\}} dx_1 \cdots dx_{n-2} dx_{n-1}$$

in general and we will see they are piecewise polynomial. The computation of integrals as in the Corollary 4 then boils down to integrals of the form

$\int_0^r xf_{n-1}(x)dx$ which are of course easy to compute. We thus get explicit expressions for the function $r \mapsto \mathrm{vol}(\mathcal{R}_n(r))$ or, equivalently, for the probability that our particle after n unit jumps lies within a sphere of radius r.

4. Determination of the functions $f_n(x)$.

To understand the problem of the determination of the functions $f_n(x)$ better, note first that in consequence of the fact $R_n \leq n$, $\mathrm{vol}(\mathcal{R}_n(r))$ reaches its maximum at the point $r = n$ and stays from thereon constant. Our problem to find $r \mapsto \mathrm{vol}(\mathcal{R}_n(r))$ or equivalently to find $r \mapsto \int_0^r xf_{n-1}(x)dx$ is of interest to us only for $0 \leq r \leq n$ and we will treat it only in this context. The innermost integral $\int_{|1-x_2|}^{\min\{1+x_2,2\}} 1\,dx_1$ is a function of variable x_2, which we write $f_2(x_2)$; the two innermost integrations, $\int_{|1-x_3|}^{\min\{1+x_3,3\}} \int_{|1-x_2|}^{\min\{1+x_2,2\}} 1\,dx_1dx_2$ can thus be written as $\int_{|1-x_3|}^{\min\{1+x_3,3\}} f_2(x_2)dx_2$ and yield a function of x_3 which we write $f_3(x_3)$, etc. In this terminology we are interested to find for every $n \geq 2$ the function $f_n = f_n(x_n)$, and the fact that we fix in the volume computation for $\mathcal{R}_{n+1}(r)$ r to be in $[0, n+1]$ will imply that within the integrations we always will have that the bounds at the lower end of the integral sign are not larger than those at the upper end. In fact it might be useful to remark that for $i = 2, 3, ...$ one can easily prove that $\{x : |1-x| \leq \min\{1+x, i\}\} = [0, 1+i]$.

Also, for $0 \leq x \leq 1+i$ and $i \geq 2$, we have for any continuous f that, in dependence of the value of x, the integral $\int_{|1-x|}^{\min\{1+x,i\}} f(t)dt$ is $\int_{1-x}^{1+x} f(t)dt$, or $\int_{x-1}^{1+x} f(t)dt$, or $\int_{x-1}^{i} f(t)dt$, according to the cases where $0 \leq x < 1$ or $1 \leq x \leq i-1$ or $i-1 \leq x \leq 1+i$ respectively. Thus the functions $f_1, f_2, ...$ can be defined alternatively by induction as follows:

$$f_1(t) = \begin{cases} 1 & \text{if } 0 \leq t \leq 2 \\ 0 & \text{if } 2 < t \end{cases} ;$$

and if f_i is already defined, for $i \geq 1$, define

$$f_{i+1}(x) = \begin{cases} \text{for } 0 \leq x < 1 \text{ by:} & \int_{1-x}^{1+x} f_i(t)dt; \\ \text{for } 1 \leq x < i \text{ by:} & \int_{x-1}^{1+x} f_i(t)dt; \\ \text{for } i \leq x \leq 2+i \text{ by:} & \int_{x-1}^{1+i} f_i(t)dt; \\ \text{for } x \notin [0, 2+i] \text{ by:} & 0. \end{cases}$$

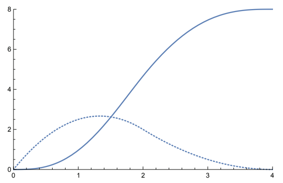
Example. Compute $r \mapsto \mathrm{vol}(\mathcal{R}_4(r))$; or equivalently $r \mapsto \mathrm{prob}(R_4 \leq r)$!
Solution: By the above formula one finds after a little computation

$$\begin{aligned} &\int_r^4 xf_3(x)dx \\ &= -\frac{1}{24}\sum_{\nu=0}^{\lceil 1-\frac{r}{2}\rceil}(-1)^\nu\binom{4}{\nu}(-4+2\nu+r)^3(4-2\nu+3r) \\ &= \begin{cases} 8 - \frac{4}{3}r^3 + \frac{3}{8}r^4 & \text{if } 0 \leq r \leq 2 \\ \frac{32}{3} - 4r^2 + \frac{4}{3}r^3 - \frac{1}{8}r^4 & \text{if } 2 \leq r \leq 4. \end{cases} \end{aligned}$$

Consequently

$$8\,\mathrm{prob}(R_4 \leq r) = 8 - \int_r^4 xf_3(x)dx = \begin{cases} \frac{4}{3}r^3 - \frac{3}{8}r^4 & \text{if } 0 \leq r \leq 2 \\ -\frac{8}{3} + 4r^2 - \frac{4}{3}r^3 + \frac{1}{8}r^4 & \text{if } 2 \leq r \leq 4. \end{cases}$$

The graphics shows the functions f_3 (dotted) and $r \mapsto \mathrm{vol}(\mathcal{R}_4(r))$ on $[0, 4]$.



6. Report on L. R. G. Treloar's elementary solution

There exists one other solution for the Random flight problem in three dimensions which remains entirely within the realm of real analysis and elementary probability theory. After the well-known physicist Lord Rayleigh had given a solution for polygons of up to ca 6 links of equal length, Treloar in 1945 came up with a formula which works for all n and which coincides with our formula for $2^{-(n-1)}xf_{n-1}(x)$ for the density of the distances of the end-points of polygons to the origin. His solution was worked out for the British rubber industry and formulated in terms of the end-to-end distances of long chain molecules.

$$\sum_{m=0}^{\bar m-1}\int\limits_{n-1-2m}^{n+1-2m} x f_n(x)dx$$
$$= \frac{(-1)^{n+1}}{(n+1)!}\sum_{m=0}^{\bar m-1}\sum_{\nu=0}^{m}(-1)^\nu\binom{n+1}{\nu}(t(\nu,m)-t(\nu,m+1))$$
$$= \frac{(-1)^{n+1}}{(n+1)!}\sum_{\nu=0}^{\bar m-1}(-1)^\nu\binom{n+1}{\nu}\sum_{m=\nu}^{\bar m-1}(t(\nu,m)-t(\nu,m+1))$$
$$= \frac{(-1)^{n+1}}{(n+1)!}\sum_{\nu=0}^{\bar m-1}(-1)^\nu\binom{n+1}{\nu}\cdot -t(\nu,\bar m),$$

because the inner sum is telescoping and $t(\nu,\nu)=0$.

A small adaption of the pen-penultimate computation also yields

$$\int\limits_{r}^{n+1-2\bar m} x f_{n|[n-1-2\bar m,n+1-2\bar m]}(x)dx$$
$$= \tfrac{(-1)^{n+1}}{(n+1)!}\sum_{\nu=0}^{\bar m}(-1)^\nu\tbinom{n+1}{\nu}(t(\nu,\bar m)-(-n-1+2\nu+r)^n(n+1-2\nu+nr)).$$

Now we find, summing the terms of $\sum_{m=0}^{\bar m-1}\dots$ in reverse order,

$$\int\limits_{r}^{n+1} x f_n(x)dx$$
$$= \int\limits_{r}^{n+1-2\bar m} x f_{n|[n-1-2\bar m,n+1-2\bar m]}(x)dx + \sum_{m=0}^{\bar m-1}\int\limits_{n-1-2m}^{n+1-2m} x f_{n|[n-1-2m,n+1-2m]}(x)dx$$
$$= \frac{(-1)^{n+1}}{(n+1)!}\left(\sum_{\nu=0}^{\bar m}(-1)^\nu\binom{n+1}{\nu}(t(\nu,\bar m)-(-n-1+2\nu+r)^n(n+1-2\nu+nr))+\right.$$
$$\left.\sum_{\nu=0}^{\bar m-1}(-1)^\nu\binom{n+1}{\nu}\cdot -t(\nu,\bar m)\right)$$
$$= -\frac{(-1)^{n+1}}{(n+1)!}\sum_{\nu=0}^{\bar m}(-1)^\nu\binom{n+1}{\nu}(-n-1+2\nu+r)^n(n+1-2\nu+nr).$$

We have now all tools at hand for finding specific distribution functions $r\mapsto \operatorname{prob}(R_n\le r)$.

From these definitions one proves readily that the functions f_i, $i\ge 2$ are all continuous on $\mathbb{R}$ and nonnegative on $]0,1+i[$. We give now the fine description of these functions.

Theorem 4.1. *The functions f_n are piecewise polynomial. For $n=1,2,3,\dots$ and $m=0,1,2,\dots,\lceil\frac{n-1}{2}\rceil$ the restriction of f_n to interval $[n-1-2m,n+1-2m]$ is given by*

$$f_{n_{|[n-1-2m,n+1-2m]}}(x)=\frac{(-1)^{n+1}}{(n-1)!}\sum_{\nu=0}^{m}(-1)^\nu\binom{n+1}{\nu}(-n-1+2\nu+x)^{n-1},$$

where in case that n even, the interval $[-1,1]$ (occurring when $m=\lceil\frac{n-1}{2}\rceil$) has to be understood as $[0,1]$.

Proof. For $n=1$ the only possibility admitted for m is $m=0$. In this case the formula says we should have

$$f_{1_{|[0,2]}}=\tfrac{(-1)^2}{(1-1)!}\textstyle\sum_{\nu=0}^{0}(-1)^\nu\tbinom{1+1}{\nu}(-2+2\nu+x)^{1-1}=1,$$

and this is indeed the case, by the definition of f_1.

In the case $n=2$, we have by putting in our inductive definition of the fs, $i=1$, that

$$f_{2_{|[0,1]}}(x)=\textstyle\int_{1-x}^{1+x}f_1(t)dt=\int_{1-x}^{1+x}1dt=t|_{1-x}^{1+x}=2x$$

and

$$f_{2_{|[1,3]}}(x)=\textstyle\int_{x-1}^{1+i}f_1(t)=\int_{x-1}^{1+1}1dt=t|_{x-1}^{2}=3-x.$$

At the other hand the theorem admits $m=0,1$ and claims for $m=0$ that

$$f_{2|[1,3]}=\tfrac{(-1)^3}{(2-1)!}\sum_{\nu=0}^{0}(-1)^\nu\tbinom{2+1}{\nu}(-2-1+2\nu+x)^{2-1}=-\tbinom{3}{0}(-3+x)^1=(3-x),$$

while for $m=1$ the formula yields

$$\begin{aligned} f_{2|[0,1]} &= -\sum_{\nu=0}^{1}(-1)^\nu\tbinom{2+1}{\nu}(-2-1+2\nu+x)^{2-1}\\ &= -(\tbinom{3}{0}(-3+x)^1+(-1)^1\tbinom{3}{1}(-3+2+x)^1)\\ &= (3-x)+3(-1+x)\\ &= 2x.\end{aligned}$$

So again the formula yields the directly computed results.

Having won this way some confidence into the formula we are going to prove it in general. So we assume the claim valid for a given n and $m=0,1,\dots,\lceil\frac{n-1}{2}\rceil$ as above and prove the corresponding claim for $n+1$ in place of n. In the case $m=0$, we need to find the representation of $f_{n+1_{|[n,n+2]}}$. Now if $n\le x\le 2+n$ then, by definition,

$f_{n+1}(x) = \int_{x-1}^{1+n} f_n(t)dt$. Noting that $x-1 \in [n-1, 1+n]$, the integration parameter t ranges in a subset of $[n-1, n+1]$, and for such t we know by induction assumption that $f_n(t) = \frac{(-1)^{n+1}}{(n-1)!}(-n-1+t)^{n-1}$.

Hence

$$\begin{aligned} f_{n+1}(x) &= \tfrac{(-1)^{n+1}}{(n-1)!} \int_{x-1}^{1+n} (-n-1+t)^{n-1} dt \\ &= \tfrac{(-1)^{n+1}}{(n-1)!} \cdot \tfrac{1}{n} (-n-1+t)^n \big|_{x-1}^{1+n} \\ &= \tfrac{(-1)^{n+1}}{n!} \cdot (0^n - (-n-1+x-1)^n) \\ &= \tfrac{(-1)^{n+2}}{n!} \cdot (-n-2+x)^n, \end{aligned}$$

and this is precisely the result the formula of the theorem also gives for $n+1$ in place of n and $m=0$.

We next show the formula for f_{n+1} in the the case that $m \geq 1$ and $1 \leq n-2m$; or, equivalently, $m = 1, ..., \lceil \frac{n}{2} \rceil - 1$.

To find $f_{n+1|[n-2m, n+2-2m]}$ note that for $x \in [n-2m, n+2-2m]$, we then have $1 \leq x \leq n$ and $n-2m-1 \leq x-1 \leq n-2m+1 \leq 1+x \leq n+3-2m$. Consequently by the definition of f_{n+1} for this case,

$$f_{n+1}(x) = \int\limits_{x-1}^{x+1} f_n dt = \int\limits_{x-1}^{n+1-2m} f_n dt + \int\limits_{n+1-2m}^{1+x} f_n dt$$

and $f_n(t)$ in the first of these integrals equals by induction hypothesis

$$f_{n|[n-1-2m, n+1-2m]}(t) = \tfrac{(-1)^{n+1}}{(n-1)!} \sum_{\nu=0}^{m} (-1)^\nu \tbinom{n+1}{\nu} (-n-1+2\nu+t)^{n-1};$$

while $f_n(t)$ in the second of the integrals equals by induction hypothesis

$$f_{n|[n+1-2m, n+3-2m]}(t) = \tfrac{(-1)^{n+1}}{(n-1)!} \sum_{\nu=0}^{m-1} (-1)^\nu \tbinom{n+1}{\nu} (-n-1+2\nu+t)^{n-1}.$$

Using that $\int (-n-1+2\nu+t)^{n-1} dt = \frac{1}{n}(-n-1+2\nu+t)^n + C$, for the first and the second of the integrals above we get

$$\begin{aligned} \int\limits_{x-1}^{n+1-2m} f_n dt &= \tfrac{(-1)^{n+1}}{n!} \sum_{\nu=0}^{m} (-1)^\nu \tbinom{n+1}{\nu} (-n-1+2\nu+t)^n \big|_{x-1}^{n+1-2m} \\ &= \tfrac{(-1)^{n+1}}{n!} \sum_{\nu=0}^{m} (-1)^\nu \tbinom{n+1}{\nu} ((2\nu-2m)^n - (-n-2+2\nu+x)^n), \\ \int\limits_{n+1-2m}^{1+x} f_n dt &= \tfrac{(-1)^{n+1}}{n!} \sum_{\nu=0}^{m-1} (-1)^\nu \tbinom{n+1}{\nu} (-n-1+2\nu+t)^n \big|_{n+1-2m}^{1+x} \\ &= \tfrac{(-1)^{n+1}}{n!} \sum_{\nu=0}^{m-1} (-1)^\nu \tbinom{n+1}{\nu} ((-n+2\nu+x)^n - (2\nu-2m)^n). \end{aligned}$$

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5. Solution of the problem: a formula for prob$(R_n \leq r)$.

We know from the ends of sections 1 and 2 that

$$\text{prob}(R_n \leq r) = 2^{-(n-1)} \int\limits_0^r x f_{n-1}(x)$$

and since $\max\{R_n(z_{2:n}) : z_{2:n} \in [-1,1]^{n-1}\} = n$, we get $\text{prob}(R_n \leq n) = 1$, so that for any $0 \leq r \leq n$, we will have $\int\limits_0^r x f_{n-1}(x)dx = 2^{n-1} - \int\limits_r^n x f_{n-1}(x) dx$. In this section we use the piecewise polynomial representations of f_n given in Theorem 4.1 to compute the integral at the right explicitly and to illustrate the theory we give the distribution function $r \mapsto \text{prob}(R_4 \leq r)$.

Given any $0 < r < 1+n$, we have
$r \in]n-1-2m, n+1-2m[\;\Leftrightarrow\; n-1-2m < r < n+1-2m \;\Leftrightarrow\; n-1-r < 2m < n+1-r$
from which it follows that $\bar{m} := m(r) := \lceil \frac{n-1-r}{2} \rceil$ is the parameter value of m defining the interval containing r.

It is direct to check that

$$\int x(a+x)^{n-1} dx = (n(n+1))^{-1}(a+x)^n(nx-a) + C$$

so that, defining

$$t(\nu, m) := (2\nu-2m)^n(n^2+2n+1-2\nu-2nm), \quad \text{and} \quad a = -n-1+2\nu,$$

we find

$$\int\limits_{n-1-2m}^{n+1-2m} x(-n-1+2\nu+x)dx = (n(n+1))^{-1}(t(\nu,m) - t(\nu, m+1)),$$

and therefore by Theorem 3.1 for $m = 0, 1, 2, ..., \lceil \frac{n-1}{2} \rceil$,

$$\begin{aligned} &\int\limits_{n-1-2m}^{n+1-2m} x f_{n|[n-1-2m, n+1-2m]}(x) dx \\ &= \frac{(-1)^{n+1}}{(n-1)!} \sum_{\nu=0}^{m} (-1)^\nu \binom{n+1}{\nu} \int\limits_{n-1-2m}^{n+1-2m} x(-n-1+2\nu+x)dx \\ &= \frac{(-1)^{n+1}}{(n+1)!} \sum_{\nu=0}^{m} (-1)^\nu \binom{n+1}{\nu} (t(\nu,m) - t(\nu, m+1)), \end{aligned}$$

and so

$$\begin{aligned}
&\sum_{\nu=0}^{\bar m}(-1)^{\nu}\tbinom{2\bar m+2}{\nu}(-2\bar m-2+2\nu+x)^{2\bar m}+\sum_{\nu=0}^{\bar m-1}(-1)^{\nu}\tbinom{2\bar m+1}{\nu}(-2\bar m+2\nu+x)^{2\bar m}\\
&+\sum_{\nu=0}^{\bar m}(-1)^{\nu-1}\tbinom{2\bar m+1}{\nu}(-2\bar m+2\nu-x)^{2\bar m}\\
=\ &\sum_{\nu=0}^{\bar m}(-1)^{\nu}\left(\tbinom{2\bar m+2}{\nu}-\tbinom{2\bar m+1}{\nu-1}\right)(-2\bar m-2+2\nu-x)^{2\bar m}\\
&\quad+\sum_{\nu=0}^{\bar m}(-1)^{\nu-1}\tbinom{2\bar m+1}{\nu}(-2\bar m+2\nu-x)^{2\bar m}\\
=\ &\sum_{\nu=0}^{\bar m}(-1)^{\nu}\tbinom{2\bar m+1}{\nu}(2\bar m+2-2\nu-x)^{2\bar m}\\
&\quad+\sum_{\nu=\bar m+1}^{2\bar m+1}(-1)^{\nu}\tbinom{2\bar m+1}{\nu}(2\bar m+2-2\nu-x)^{2\bar m}\\
=\ &\sum_{\nu=0}^{2\bar m+1}(-1)^{\nu}\tbinom{2\bar m+1}{\nu}(2\bar m+2-2\nu-x)^{2\bar m}\\
=\ &0
\end{aligned}$$

In view of similar previous justifications it will be sufficient to note that the second equality is justified by replacing the summation index in the second sum by $2\bar m+1-\nu$.

To show that $f_{n+1_{|[1,2]}}$ has the same polynomial representation as $f_{n+1_{|[0,1]}}$ we prove the following

Claim. If n is even, then $f_{n_{|[0,1]}}$ is a polynomial with only odd degree monomials.

$[\rhd$ We know by direct computation that $f_{2_{|[0,1]}}(t)=2t$. Now for $0\le x\le 1$, there holds $0\le 1-x\le 1\le 1+x\le 2$ and so by the inductive definitions, $f_n(x)=\int_{1-x}^{1+x} f_{n-1_{|[0,2]}}(t)dt$. Now since $n-1$ is odd the function $f_{n-1_{|[0,2]}}$ is polynomial. Thus the associated stem function $F(t):=\int f_{n-1_{|[0,2]}}dt$ is a polynomial, and we see $f_n(x)=F(1+x)-F(1-x)$ is an antisymmetric polynomial. Hence it has only monomials of odd degree. $\lhd]$

Now by definition for $x\in[1,2]$, $f_{n+1}(x)=\int_{x-1}^{1} f_{n_{|[0,1]}}(t)dt+\int_{1}^{1+x} f_{n_{|[1,3]}}(t)dt$ but by the claim just proved the stem function $\int f_{n_{|[0,1]}}(t)dt$ is a polynomial in t with monomials of only even degree. Hence $\int_{x-1}^{1} f_{n_{|[0,1]}}(t)dt=\int_{1-x}^{1} f_{n_{|[0,1]}}(t)dt$ and it follows by comparing with the sum-of-integrals representation defining $f_{n+1_{|[0,1]}}$, we gave before that indeed $f_{n+1_{|[1,2]}}$ and $f_{n+1_{|[0,1]}}$ have the same polynomial representations. This concludes the proof of the theorem □

By adding these expressions we get isolating in the sum $\sum_{\nu=0}^{m}\ldots$ the term associated to $\nu=m$,

$$\begin{aligned}
f_{n+1}(x)\ =\ &\tfrac{(-1)^{n+1}}{n!}\underbrace{\sum_{\nu=0}^{m-1}(-1)^{\nu}\binom{n+1}{\nu}((-n+2\nu+x)^n-(-n-2+2\nu+x)^n)}_{I}\\
&+\tfrac{(-1)^{n+1}}{n!}(-1)^m\tbinom{n+1}{m}(0-(-n-2+2m+x)^n).
\end{aligned}$$

Now changing the summation-index,

$$\begin{aligned}
I=\ &\sum_{\nu=1}^{m}(-1)^{\nu-1}\tbinom{n+1}{\nu-1}(-n-2+2\nu+x)^n+\sum_{\nu=0}^{m-1}(-1)^{\nu-1}\tbinom{n+1}{\nu}(-n-2+2\nu+x)^n\\
=\ &\sum_{\nu=1}^{m-1}(-1)^{\nu-1}\left(\tbinom{n+1}{\nu-1}+\tbinom{n+1}{\nu}\right)(-n-2+2\nu+x)^n\\
&\ +(-1)^{m-1}\tbinom{n+1}{m-1}(-n-2+2m+x)^n+(-1)^{0-1}\tbinom{n+1}{0}(-n-2+x)^n\\
=\ &\sum_{\nu=1}^{m-1}(-1)^{\nu-1}\tbinom{n+2}{\nu}(-n-2+2\nu+x)^n+(-1)^{m-1}\tbinom{n+1}{m-1}(-n-2+2m+x)^n\\
&-(-n-2+x)^n\\
=\ &\sum_{\nu=0}^{m-1}(-1)^{\nu-1}\tbinom{n+2}{\nu}(-n-2+2\nu+x)^n+(-1)^{m-1}\tbinom{n+1}{m-1}(-n-2+2m+x)^n.
\end{aligned}$$

Thus

$$\begin{aligned}
f_{n+1}(x)\ &=\ \tfrac{(-1)^{n+1}}{n!}I+\tfrac{(-1)^{n+1}}{n!}(-1)^{m-1}\tbinom{n+1}{m}(-n-2+2m+x)^n\\
&=\ \tfrac{(-1)^{n+1}}{n!}\sum_{\nu=0}^{m}(-1)^{\nu-1}\tbinom{n+2}{\nu}(-n-2+2\nu+x)^n.
\end{aligned}$$

Now the right hand side of the latter expression is precisely what is claimed in the theorem when n is replaced by $n+1$.

Finally there remains to show that the formula for f_{n+1} (obtained from the theorem when n is replaced by $n+1$) is also correct in the case that $m=\bar m:=\left\lceil\frac{(n+1)-1}{2}\right\rceil=\lceil\frac{n}{2}\rceil$. Then the interval to be considered for f_{n+1} is $[(n+1)-1-2\bar m,(n+1)+1-2\bar m]=[n-2\bar m,n+2-2\bar m]$ and this is in case n odd equal to $[-1,1]$ to be read as $[0,1]$; in case n even equal to $[0,2]$.

Unfortunately a detailed treatment of these rather special cases takes more space than we would like to allow. We will make use of the following

Fact. If $p=\sum_{k=0}^{n}a_kx^k$ is a real polynomial, then

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}p(k)=(-1)^n n!a_n.$$

For a proof of this, see [GKP, Section 5.3].

Case n odd. In this case $\bar{m}=\frac{n+1}{2}, 2\bar{m}=n+1$. As mentioned we have to show the formula of the theorem for $f_{n+1}(x)$ with $0\le x\le 1$ and by definition then $f_{n+1}(x)=\int_{1-x}^{1+x} f_n(t)dt$. We note $[1-x,1+x]\subseteq[0,2]=[n-1-2(\bar{m}-1), n+1-2(\bar{m}-1)]$ and so we know by induction hypothesis $f_{n_{|[0,2]}}(t)=\frac{(-1)^{n+1}}{(n-1)!}\sum_{\nu=0}^{\bar{m}-1}(-1)^\nu\binom{n+1}{\nu}(-n-1+2\nu+t)^{n-1}$ and

$$\begin{aligned}
&f_{n+1_{|[0,1]}}(x)\\
&=\int\limits_{1-x}^{1+x} f_{n_{|[0,2]}}(t)dt\\
&=\tfrac{(-1)^{n+1}}{n!}\sum_{\nu=0}^{\bar{m}-1}(-1)^\nu\tbinom{n+1}{\nu}(-n-1+2\nu+t)^n\big|_{1-x}^{1+x}\\
&=\tfrac{1}{(2\bar{m}-1)!}\sum_{\nu=0}^{\bar{m}-1}(-1)^\nu\tbinom{2\bar{m}}{\nu}\big((-2\bar{m}+1+2\nu+x)^{2\bar{m}-1}-(-2\bar{m}+1+2\nu-x)^{2\bar{m}-1}\big),
\end{aligned}$$

while the formula of the theorem claims on the interval $[0,1]$ that

$$\begin{aligned}
f_{n+1_{|[0,1]}}(x) &= \tfrac{(-1)^{n+2}}{n!}\sum_{\nu=0}^{\bar{m}}(-1)^\nu\tbinom{n+2}{\nu}(-(n+1)-1+2\nu+x)^n\\
&= \tfrac{-1}{(2\bar{m}-1)!}\sum_{\nu=0}^{\bar{m}}(-1)^\nu\tbinom{2\bar{m}+1}{\nu}(-2\bar{m}-1+2\nu+x)^{2\bar{m}-1}.
\end{aligned}$$

To show that the two formulas for $f_{n+1_{|[0,1]}}$ agree it is sufficient to check the following calculation which after multiplication with $-(2\bar{m}-1)!$ computes the second of the formulas minus the first of the formulas and transforms the result gradually into a form to which we can apply the fact mentioned above.

$$\begin{aligned}
&\sum_{\nu=0}^{\bar{m}}(-1)^\nu\tbinom{2\bar{m}+1}{\nu}(-2\bar{m}-1+2\nu+x)^{2\bar{m}-1}+\sum_{\nu=0}^{\bar{m}-1}(-1)^\nu\tbinom{2\bar{m}}{\nu}(-2\bar{m}+1+2\nu+x)^{2\bar{m}-1}\\
&-\sum_{\nu=0}^{\bar{m}-1}(-1)^\nu\tbinom{2\bar{m}}{\nu}(-2\bar{m}+1+2\nu-x)^{2\bar{m}-1}\\
&\quad=(-2\bar{m}-1+x)^{2\bar{m}-1}+\sum_{\nu=1}^{\bar{m}}(-1)^\nu\tbinom{2\bar{m}+1}{\nu}(-2\bar{m}-1+2\nu+x)^{2\bar{m}-1}\\
&\qquad+\sum_{\nu=1}^{\bar{m}}(-1)^{\nu-1}\tbinom{2\bar{m}}{\nu-1}(-2\bar{m}-1+2\nu+x)^{2\bar{m}-1}\\
&\qquad-\sum_{\nu=\bar{m}+1}^{2\bar{m}}(-1)^\nu\tbinom{2\bar{m}}{2\bar{m}-\nu}(2\bar{m}+1-2\nu-x)^{2\bar{m}-1}\\
&\quad=\sum_{\nu=0}^{\bar{m}}(-1)^\nu\tbinom{2\bar{m}}{\nu}(-2\bar{m}-1+2\nu+x)^{2\bar{m}-1}\\
&\qquad-\sum_{\nu=\bar{m}+1}^{2\bar{m}}(-1)^\nu\tbinom{2\bar{m}}{2\bar{m}-\nu}(2\bar{m}+1-2\nu-x)^{2\bar{m}-1}
\end{aligned}$$

$$\begin{aligned}
&=\sum_{\nu=0}^{2\bar{m}}(-1)^\nu\tbinom{2\bar{m}}{\nu}(-2\bar{m}-1+2\nu+x)^{2\bar{m}-1}\\
&=0.
\end{aligned}$$

Here the first equality is obtained by isolating in the first sum the case $\nu=0$; by incrementing in the second sum the summation index ν by 1 and by replacing in the third sum ν by $2\bar{m}-\nu$. The second equality is justified by using $\binom{2\bar{m}+1}{\nu}-\binom{2\bar{m}}{\nu-1}=\binom{2\bar{m}}{\nu}$ and reincorporating the isolated leftmost term as the case $\nu=0$ in the first sum at the right of the equality sign. The third equality follows from multiplying $2\bar{m}+1-2\nu-x$ by -1, taking oddness of $2\bar{m}-1$ into account and then joining the sums. That this is said to be zero in the fourth inequality is a consequence of the displayed fact above noting that $(-2\bar{m}-1+2\nu+x)^{2\bar{m}-1}$ is a polynomial in ν of degree less than $2\bar{m}$.

Case n even: In this case $\bar{m}=\frac{n}{2}$, $2\bar{m}=n$. We have to show the formula of the theorem for $f_{n+1}(x)$ with $0\le x\le 2$. We begin with the case $0\le x\le 1$. Then by definiton $f_{n+1}(x)=\int_{1-x}^{1+x} f_n(t)dt$ and $0\le 1-x\le 1\le 1+x\le 2$. Unfortunately for the current case, the induction hypothesis tells us that f_n has polynomial representations in $[-1,1]$ and $[1,3]$ which differ. Therefore we compute $f_{n+1_{|[0,1]}}$ according to the following first line and then continue with methods we have exercised above:

$$\begin{aligned}
f_{n+1}(x) &= \int_{1-x}^{1} f_{n_{|[0,1]}}(t)dt+\int_1^{1+x} f_{n_{|[1,3]}}(t)dt\\
&= \tfrac{-1}{(2\bar{m})!}\Bigg(\sum_{\nu=0}^{\bar{m}}(-1)^\nu\tbinom{2\bar{m}+1}{\nu}\big((-2\bar{m}+2\nu)^{2\bar{m}}-(-2\bar{m}+2\nu-x)^{2\bar{m}}\big)\\
&\quad+\sum_{\nu=0}^{\bar{m}-1}(-1)^\nu\tbinom{2\bar{m}+1}{\nu}\big((-2\bar{m}+2\nu+x)^{2\bar{m}}-(-2\bar{m}+2\nu)^{2\bar{m}}\big)\Bigg)\\
&= \tfrac{-1}{(2\bar{m})!}\Bigg(\sum_{\nu=0}^{\bar{m}-1}(-1)^\nu\tbinom{2\bar{m}+1}{\nu}\big((-2\bar{m}+2\nu+x)^{2\bar{m}}\\
&\quad-(-2\bar{m}+2\nu-x)^{2\bar{m}}\big)+(-1)^{\bar{m}+1}\tbinom{2\bar{m}+1}{\bar{m}}x^{2\bar{m}}\Bigg)
\end{aligned}$$

At the other hand the theorem claims

$$f_{n+1}(x)=\tfrac{1}{(2\bar{m})!}\sum_{\nu=0}^{\bar{m}}(-1)^\nu\tbinom{2\bar{m}+2}{\nu}(-2\bar{m}-2+2\nu+x)^{2\bar{m}}$$

The equality of the two expressions is similarly as in the previous case shown by multiplying both expressions by $(2\bar{m})!$ and computing the second expression minus the first one: