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ON BORNOLOGICAL SEMI-ABELIAN ALGEBRAS

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Abstract: If $\mathbb{T}$ is a semi-abelian algebraic theory, we prove that the category $\mathsf{Born}^{\mathbb{T}}$ of bornological $\mathbb{T}$-algebras is homological with semi-direct products. We give a formal criterion for the representability of actions in $\mathsf{Born}^{\mathbb{T}}$ and, for a bornological $\mathbb{T}$-algebra X, we investigate the relation between the representability of actions on X as a $\mathbb{T}$-algebra and as a bornological $\mathbb{T}$-algebra. We investigate further the algebraic coherence and the algebraic local cartesian closedness of $\mathsf{Born}^{\mathbb{T}}$ and prove in particular that both properties hold in the case of bornological groups.

Keywords: Semi-abelian algebraic theory, bornology, bornological algebra, bornological group, action representability, algebraic coherence, local algebraic cartesian closedness.
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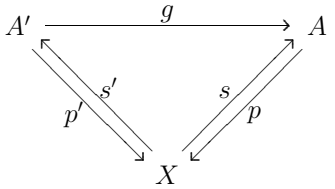
Introduction

Let $\mathcal{C}$ be a category with pullbacks. We consider its fibration of points (see [5]) whose stalk at an object $X \in \mathcal{C}$ has for objects the pairs $s, p\colon A \underset{p}{\overset{s}{\leftrightarrows}} X$ with $ps = \mathsf{id}_X$. The arrows are the morphisms $g\colon A' \longrightarrow A$ making both triangles commute.



$$pg = p',\ gs' = s \qquad\qquad p's' = \mathsf{id}_X = ps$$

The inverse image functor f^* of that fibration along a morphism $f\colon Y \longrightarrow X$

$$f^*\colon \mathsf{Pt}(X) \longrightarrow \mathsf{Pt}(Y)$$

is simply pulling back along f.

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- The category $\mathcal{C}$ is *protomodular* (see [5]) when these functors f^* reflect isomorphisms.
- The category $\mathcal{C}$ is *algebraically coherent* (see [10]) when these functors f^* preserve jointly strongly epimorphic pairs.
- The category $\mathcal{C}$ is locally algebraically cartesian closed when these functors f^* have a right adjoint.

Let us recall that, in the presence of coproducts, a family $(f_i\colon A_i \longrightarrow B)_{i\in I}$ is jointly strongly epimorphic when the corresponding factorization $f\colon +_{i\in I} A_i \longrightarrow B$ of $(f_i)_{i\in I}$ is a strong epimorphism. Since a functor with a right adjoint preserves colimits, local algebraic cartesian closedness implies at once algebraic coherence.

Semi-abelian categories have been introduced in [14]; in that case the functors f^* above are monadic. In a semi-abelian category $\mathcal{C}$, considering for f a morphism $\mathbf{0} \longrightarrow X$ from the zero object, an action of the object X on an object Z is just an f^*-action in the sense of monads. Actions on Z are representable when the functor $\mathsf{Act}(-, Z)\colon \mathcal{C} \longrightarrow \mathsf{Set}$, mapping an object X on the set of actions of X on Z, is representable. In the case of groups, actions on Z are represented by the group $\mathsf{Aut}(Z)$ of automorphisms and in the case of Lie algebras, by the Lie algebra $\mathsf{Der}(Z)$ of derivations.

Algebraic coherence is studied in [10], where it is proved in particular that all categories of interest in the sense of Orzech (see [16]) are algebraically coherent. Local algebraic cartesian closedness is studied in [13], where it is shown in particular that the categories of groups and Lie algebras satisfy that property. In all these cases, it is proved that, with the notation above, it suffices to require the property in the case $Y = \mathbf{0}$, in which case the inverse image functor f^* is simply the kernel functor

$$\mathsf{Ker} : \mathsf{Pt}(X) \longrightarrow \mathsf{Set}^{\mathbb{T}},\ \big(p, s\colon A \rightleftarrows X\big) \mapsto \mathsf{Ker}\, p$$

A *bornology* on a set X consists of a family of so-called *bounded subsets* of X, so that subsets and finite unions of bounded sets are bounded, and containing the singletons. The morphisms between bornological sets are the mappings preserving boundedness. We write Born for the corresponding category. Given an algebraic theory $\mathbb{T}$, a bornological $\mathbb{T}$-algebra is a $\mathbb{T}$-algebra provided with a bornology which makes the $\mathbb{T}$-operations bounded. We write $\mathsf{Born}^{\mathbb{T}}$ for the category of bornological $\mathbb{T}$-algebras and bounded $\mathbb{T}$-homomorphisms.

We prove first that, given a semi-abelian algebraic theory $\mathbb{T}$, the category $\mathsf{Born}^{\mathbb{T}}$ is homological in the sense of [6], and thus, in particular, regular and

protomodular. We prove further the monadicity of the functors f^* as above and investigate the representability of the corresponding actions. We investigate as well the algebraic coherence and the local algebraic cartesian closedness of $\mathsf{Born}^{\mathbb{T}}$. In fact, we look for criteria telling us when these properties carry over from $\mathsf{Set}^{\mathbb{T}}$ to $\mathsf{Born}^{\mathbb{T}}$. We prove that this is in particular the case when $\mathbb{T}$ is the theory of groups.

1. The category of bornological sets

This section lists some useful basic properties of the category of bornological sets; various results are borrowed from the first chapter of [3].

Definition 1. *A bornological set $(X, \mathcal{B})$ is a set X provided with a family $\mathcal{B}$ (called a bornology) of so-called* bounded subsets*, in such a way that:*

[B1] *a subset of a bounded subset is bounded;*
[B2] *a finite union of bounded subsets is bounded;*
[B3] *the bounded subsets cover X.*

A bounded map $f\colon (X, \mathcal{B}) \longrightarrow (Y, \mathcal{C})$ between bornological sets is a set theoretical mapping $f\colon X \longrightarrow Y$ preserving bounded subsets: $B \in \mathcal{B}$ implies $f(B) \in \mathcal{C}$. We shall write Born *for the category of bornological sets and bounded maps.*

Of course Axiom [B3] could equivalently be replaced by the requirement that every point is bounded. A finite subset is thus bounded for whatever bornology and moreover, the finite subsets constitute the smallest bornology on a given set. The largest bornology on a set X is the set $\mathcal{P}(X)$ of all its subsets. Moreover, the bornology generated by a family $\mathcal{D}$ of subsets of X is the set of all subsets of finite unions of singletons and elements of $\mathcal{D}$.

Proposition 2. *The category* Born *of bornological sets is topological over the category* Set *of sets.*

Proof: For every family $(X_i, \mathcal{B}_i)_{i\in I}$ of bornological sets, every set Y and every family of mappings $f_i\colon Y \longrightarrow X_i$, the initial bornology on Y is constituted by those subsets $U \subseteq Y$ which are mapped on a bounded subset by each f_i, just because direct images commute with (finite) unions.

On the other hand, for every family $(X_i, \mathcal{B}_i)_{i\in I}$ of bornological sets, every set Y and every family of mappings $f_i\colon X_i \longrightarrow Y$, the final bornology on Y is that generated by all the $f_i(U)$, for all $i \in I$ and all $U \in \mathcal{B}_i$. ∎

Proposition 2 implies thus in particular that, in $\mathsf{Born}^{\mathbb{T}}$, limits and colimits are computed as in Set and provided, respectively, with the initial or the final bornology for the canonical morphisms of the (co)limit.

Corollary 3. *In the category of bornological sets,*

(1) *the regular monomorphisms are the injections $s\colon S \rightarrowtail X$ where S is equipped with the induced bornology;*
(2) *the regular epimorphisms are the surjections $p\colon (X,\mathcal{B}) \twoheadrightarrow (Y,\mathcal{C})$ such that the bounded subsets in $\mathcal{C}$ are exactly the subsets $p(U)$ for all bounded subsets $U \in \mathcal{B}$.*

Proof: By "induced bornology" on S, we mean of course the bornology on S constituted by the traces of the bounded subsets of X. ∎

Proposition 4. *In the category* Born *of bornological sets, finite limits commute with filtered colimits.*

Proof: Let $\mathcal{D}$ be a finite diagram shape and $\mathcal{E}$ a filtered diagram shape. Given a functor $F\colon \mathcal{D}\times\mathcal{E} \longrightarrow \mathsf{Born}$, we must prove that the canonical comparison morphism in Born

$$\mathsf{colim}_{E\in\mathcal{E}} \lim_{D\in\mathcal{D}} F(D,E) \longrightarrow \lim_{D\in\mathcal{D}} \mathsf{colim}_{E\in\mathcal{E}} F(D,E)$$

is an isomorphism. But we know already that it is an isomorphism of sets, since limits and colimits in Born are computed as in Set. So it remains to prove that a bounded subset of the right-hand object is also bounded in the left-hand object.

By Proposition 2, a subset U in the finite limit is bounded precisely when each $p_D(U)$ is bounded in the $\mathsf{colim}_{E\in\mathcal{E}} F(D,E)$. By the same result, each $p_D(U)$ is contained in a finite union of images of bounded subsets arising each one from some $F(D,E)$. But the finiteness of these unions, together with the finiteness of $\mathcal{D}$, implies the existence of some step $E \in \mathcal{E}$ where the whole situation can be lifted and a corresponding bounded subset of $\lim_{D\in\mathcal{D}} F(D,E)$, whose image in the left-hand colimit contains U. Thus U is bounded in that colimit. ∎

Proposition 5. *The category of bornological sets is regular and every regular equivalence relation is the kernel pair of its cokernel.*

Proof: Consider the following pullback, where p is a regular epimorphism.

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Corollary 45. *Let $\mathbb{T}$ be a semi-abelian theory such that $\mathsf{Set}^{\mathbb{T}}$ is locally algebraically cartesian closed and $\mathsf{Born}^{\mathbb{T}}$ is algebraically coherent. Then $\mathsf{Born}^{\mathbb{T}}$ is locally algebraically cartesian closed as well.*

Proof: With the notation of Lemma 42, we have this time by assumption that (u, v) is an isomorphism in $\mathsf{Set}^{\mathbb{T}}$ and a regular epimorphism in $\mathsf{Born}^{\mathbb{T}}$. Hence (u, v) is both a monomorphism and a regular epimorphism in $\mathsf{Born}^{\mathbb{T}}$, thus an isomorphism. As the comment at the beginning of Section 8 indicates, this means precisely that the functor Ker preserves binary coproducts. One concludes by Proposition 44. ∎

Once more, in the case of bornological groups, we get so:

Theorem 46. *The category $\mathsf{Born}^{\mathsf{Gr}}$ of bornological groups is locally algebraically cartesian closed.*

Proof: By Corollary 45, Theorem 43 and [13], where it is proved that the category of groups is locally algebraically cartesian closed. ∎

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$$\begin{array}{ccc} (P,\mathcal{E}) & \xrightarrow{\ g\ } & (X,\mathcal{B}) \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ (Z,\mathcal{D}) & \xrightarrow[\ f\ ]{} & (Y,\mathcal{C}) \end{array}$$

Of course, q is a surjection. Given a bounded subset $W \subseteq Z$, we have $f(W) \in \mathcal{C}$ and, by Corollary 3, $f(W) = p(U)$ for some $U \in \mathcal{B}$. Therefore $W \times_Y U$ is bounded in $(P, \mathcal{E})$ and is mapped on W by q.

Next, coequalizers and kernel pairs are computed as in Set and kernel pairs in Born have the bornology induced by that of the product. One concludes by Corollary 3. ∎

It should be noticed that Born is not an exact category, since an equivalence relation provided with a bornology weaker than that induced by the product cannot be a kernel pair.

Proposition 6. *The category of bornological sets admits a classifier of regular subobjects.*

Proof: This is simply the two points sets $\Omega = \{0, 1\}$ with its only possible bornology: every (finite) subset is bounded. For every morphism $\varphi\colon X \longrightarrow \Omega$ in Born, pulling back the "true morphism" along φ yields a subobject of $s\colon S \rightarrowtail X$ provided with the induced bornology, thus a regular subobject.

$$\begin{array}{ccc} S & \longrightarrow & \mathbf{1} \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle t} \\ X & \xrightarrow[\ \varphi\ ]{} & \Omega \end{array} \qquad t(\star) = 1$$

Conversely, every regular subobject is recaptured in that way when taking for φ its characteristic mapping. ∎

Proposition 7. *The category of bornological sets is cartesian closed and, even, locally cartesian closed.*

Proof: Cartesian closedness is proved in [3]. Given two bounded sets Y and Z, Z^Y is the set $\mathsf{Born}(Y, Z)$ of bounded maps equipped with the *equiboundedness bornology*: a subset $H \subseteq \mathsf{Born}(Y, Z)$ is bounded if and only if, for every

bounded subset $B \subseteq Y$,

$$H(B) = \bigcup_{h \in H} h(B)$$

is bounded in Z.

Let us infer the local cartesian closedness. Let us view a mapping $p\colon X \longrightarrow I$ as a family $(X_i)_{i \in I}$ of subsets, with of course $X_i = p^{-1}(i)$. In Born/I, the product is the pullback $X \times_I Y \longrightarrow I$. For $r\colon Z \longrightarrow I$ and $q\colon Y \longrightarrow I$ we define

$$(Z,r)^{(Y,q)} = \coprod_{i \in I} \mathsf{Born}(Y_i, Z_i)$$

where $Y_i \subseteq Y$ and $Z_i \subseteq Z$ are equipped with the induced bornologies. We choose as bounded subsets $H \subseteq (Z,r)^{(Y,q)}$ those H satisfying, for every bounded subset B of Y, that

$$H(B) = \coprod_{i \in I} \bigcup_{h_i \in H \cap \mathsf{Born}(Y_i, Z_i)} h_i(B)$$

is bounded in Z. This is trivially a bornology.

Consider now a morphism $f\colon (X,p) \times (Y,q) \longrightarrow (Z,r)$ in Born/I, which restricts thus as a bounded morphism $f_i\colon X_i \times Y_i \longrightarrow Z_i$ for each $i \in I$. By cartesian closedness, this yields corresponding bounded morphisms

$$\bar{f}_i\colon X_i \longrightarrow \mathsf{Born}(Y_i, Z_i).$$

Glueing these together over I, we obtain a mapping $\bar{f}\colon (X,p) \longrightarrow (Z,r)^{(Y,q)}$ and we must prove that it is bounded. Given a bounded subset $A \subseteq X$

$$\bar{f}(A) = \coprod_{i \in I} \{f_i(a,-) \mid a \in X_i \cap A\}.$$

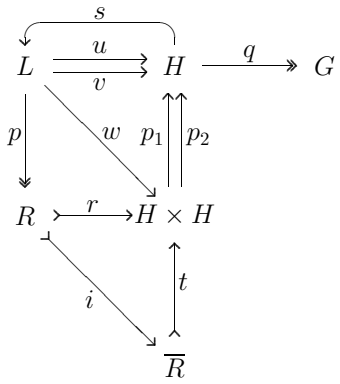
To prove that $\bar{f}(A)$ is bounded, choose a bounded subset $B \subseteq Y$.

$$\bar{f}(A)(B) = \coprod_{i \in I} f_i(A \cap X_i \times B \cap Y_i) = f(A \times_I B).$$

This subset is bounded because f is bounded. Conversely, if $\bar{f}$ is bounded, it suffices to read this last formula from right to left to conclude that f is bounded. ∎

All this allows us to conclude that:

Theorem 8. *The category of bornological sets is a quasi-topos.* ∎



We write p_1, p_2 for the two projections of the product $H \times H$ in $\mathsf{Pt}(X)$ and w for the factorization yielding $p_1 w = u$, $p_2 w = v$. We consider further the image factorization $w = rp$ of w and write $\overline{R}$ for the sub-$\mathbb{T}$-algebra R of $H \times H$ provided with the induced bornology. Thus i, as a mapping is the identity on the set R.

Since s is a common section of u and v, the diagonal of H is contained in R, thus also in $\overline{R}$. Thus R and $\overline{R}$ are equivalence relations, because every protomodular category (see Proposition 18) is a Mal'tsev category (see Proposition 3.1.19 in [5]). But p (regular epimorphism) and i (bijection) are epimorphisms in $\mathsf{Born}^{\mathbb{T}}$, thus $q = \mathsf{Coker}\,(u,v)$ implies $q = \mathsf{Coker}\,(p_1 t, p_2 t)$. And since $\overline{R}$ is an equivalence relation provided with the induced bornology, by exactness of $\mathsf{Set}^{\mathbb{T}}$, it is the kernel pair of its coequalizer. We know already that the functor Ker preserves the coequalizer $q = \mathsf{Coker}\,(p_1 t, p_2 t)$ of that kernel pair and transforms p into a regular epimorphism. But in $\mathsf{Set}^{\mathbb{T}}$, the functor Ker transforms the isomorphism i into an isomorphism, proving that in $\mathsf{Born}^{\mathbb{T}}$, $\mathsf{Ker}\, i$ is both a monomorphism and an epimorphism. Thus $\mathsf{Ker}\,(ip)$ is an epimorphism in $\mathsf{Born}^{\mathbb{T}}$ and therefore

$$\begin{aligned}
\mathsf{Coker}\big(\mathsf{Ker}\,u, \mathsf{Ker}\,v\big) &= \mathsf{Coker}\big(\mathsf{Ker}\,(p_1 tip), \mathsf{Ker}\,(p_2 tip)\big) \\
&= \mathsf{Coker}\big(\mathsf{Ker}\,(p_1 t) \circ \mathsf{Ker}\,(ip), \mathsf{Ker}\,(p_2 t) \circ \mathsf{Ker}\,(ip)\big) \\
&= \mathsf{Coker}\big(\mathsf{Ker}\,(p_1 t), \mathsf{Ker}\,(p_2 t)\big) \\
&= \mathsf{Ker}\big(\mathsf{Coker}\,(p_1 t, p_2 t)\big) \\
&= \mathsf{Ker}\, q \\
&= \mathsf{Ker}\big(\mathsf{Coker}\,(u,v)\big)
\end{aligned}$$

□

introduction) that this property is stronger than algebraic coherence. Let us first reduce the problem to a property of finite coproducts.

Proposition 44. *Given a semi-abelian algebraic theory* $\mathbb{T}$*, the category* $\mathsf{Born}^{\mathbb{T}}$ *is locally algebraically cartesian closed if and only if every kernel functor*

$$\mathsf{Ker} : \mathsf{Pt}(X) \longrightarrow \mathsf{Born}^{\mathbb{T}},\ (p, s\colon \xleftarrow{\longrightarrow} X) \mapsto \mathsf{Ker}\, p$$

preserves binary coproducts.

Proof: The local algebraic cartesian closedness of $\mathsf{Born}^{\mathbb{T}}$ means thus that these kernel functors have a right adjoint, which implies at once that they preserve finite coproducts.

Conversely, notice first that the categories $\mathsf{Pt}(X)$ in $\mathsf{Born}^{\mathbb{T}}$ are trivially cocomplete, since so is $\mathsf{Born}^{\mathbb{T}}$. They admit a generator, namely

$$X + F(\mathbf{1}) \underset{(\mathsf{id}_X, 0)}{\overset{\sigma_X}{\xleftarrow{\longrightarrow}}} X$$

where $F(\mathbf{1})$ is the free $\mathbb{T}$-algebra on one generator, provided with the bornology of finite subsets. These categories $\mathsf{Pt}(X)$ are also well-powered since so is $\mathsf{Born}^{\mathbb{T}}$ (see Proposition 14). Thus by the special adjoint functor theorem, the existence of a right adjoint to the functors Ker reduces to the preservation of colimits. Let us recall also that the categories $\mathsf{Pt}(X)$ remain homological (see [5]).

Since finite limits commute with filtered colimits in $\mathsf{Born}^{\mathbb{T}}$ (see Proposition 10), the same arguments as in the proof of Theorem 28 reduce the problem to proving the preservation of the coequalizers of those pairs of morphisms admitting a common section.

First, since the category $\mathsf{Born}^{\mathbb{T}}$ is regular (see Proposition 12) and since taking the kernel is computing the pullback over $\mathbf{0}$, the kernel functors preserve regular epimorphisms. But the kernel functors preserve trivially kernel pairs; together with the preservation of regular epimorphisms, this forces already the preservation of coequalizers of kernel pairs.

Let us now work in the protomodular category $\mathsf{Pt}(X)$ and consider $q = \mathsf{Coker}\,(u, v)$, with s a common section of u and v, thus $us = \mathsf{id}_H = vs$.

Let us make clear that Theorem 8 is not the one proved in [1] and [12]: what is called *bornological set* in these papers is generally called *Kolmogorov set* in functional analysis, that is, a bornological set with the additional requirement that a subset is bounded as soon as every sequence in it is bounded. The corresponding category is then a *Grothendieck quasi-topos*, that is, a category of separated objects on some site. This is not the case of the category Born, as the following counter-example shows.

Counterexample 9. *The category of bornological sets is not locally presentable.*

Proof: If Born was locally presentable, every bornological set X would be a quotient of a coproduct $\coprod_{i\in I} G_i \twoheadrightarrow X$ of objects G_i chosen in a fixed family $\mathcal{G}$ of strong (thus regular) generators. Choosing a regular cardinal α strictly bigger than the cardinal of whatever generator in the family $\mathcal{G}$, the bornology of X would be generated by subsets of cardinality strictly smaller than α. This is impossible if X is chosen with cardinal bigger than α, with all its subsets bounded. ∎

2. Bornological algebras

Proposition 10. *Let* $\mathbb{T}$ *be an algebraic theory. The category* $\mathsf{Born}^{\mathbb{T}}$ *of bornological* $\mathbb{T}$*-algebras is topological over the category* $\mathsf{Set}^{\mathbb{T}}$ *of* $\mathbb{T}$*-algebras, while the forgetful functor* $\mathsf{Born}^{\mathbb{T}} \longrightarrow \mathsf{Born}$ *is monadic and preserves filtered colimits.*

Proof: Only the case of filtered colimits requires a proof; the rest follows from Proposition 2 and the Taut Lift Theorem (see [18]).

Consider a filtered diagram $(X_i, \mathcal{B}_i)_{i\in I}$ in $\mathsf{Born}^{\mathbb{T}}$ and its colimit $(X, \mathcal{B})$ in Born. Let us observe that $(X, \mathcal{B})$ is at once a bornological $\mathbb{T}$-algebra. First X is a $\mathbb{T}$-algebra, because filtered colimits in $\mathsf{Set}^{\mathbb{T}}$ are computed as in Set. It remains to prove that all $\mathbb{T}$-operations on X are bounded. If α is an n-ary $\mathbb{T}$-operation, we must prove that $\alpha(B_1, \ldots, B_n)$ is bounded, for all choices B_i in $\mathcal{B}$. But each B_i is contained in a finite union of $B_{i,j_i} \in \mathcal{B}_{j_i}$. By filteredness, all these B_{i,j_i} can be mapped at a same level B_k, where the bounded condition for α holds. This forces the conclusion. ∎

Corollary 11. *Let* $\mathbb{T}$ *be an algebraic theory. The free* $\mathbb{T}$*-algebra on one generator, equipped with the bornology of finite subsets, is a generator in* $\mathsf{Born}^{\mathbb{T}}$. ∎

Proposition 12. *Let $\mathbb{T}$ be an algebraic theory. The category $\mathsf{Born}^{\mathbb{T}}$ of bornological $\mathbb{T}$-algebras is regular and the forgetful functor $\mathsf{Born}^{\mathbb{T}} \longrightarrow \mathsf{Born}$ preserves coequalizers of kernel pairs.*

Proof: This is Theorem 5.11 in [4]. ∎

Corollary 13. *Let $\mathbb{T}$ be an algebraic theory. A morphism $f\colon A \longrightarrow B$ in $\mathsf{Born}^{\mathbb{T}}$ is a regular epimorphism if and only if the underlying mapping is a surjective $\mathbb{T}$-homomorphism and B is provided with the quotient bornology, that is, the bounded subsets of B are exactly the direct images of the bounded subsets in A.*

Proof: By Propositions 10, 12 and Corollary 3. ∎

Proposition 14. *Let $\mathbb{T}$ be an algebraic theory. The category $\mathsf{Born}^{\mathbb{T}}$ of bornological $\mathbb{T}$-algebras is well-powered and co-well-powered.*

Proof: By Proposition 10, monomorphisms are injective. Thus a bornological $\mathbb{T}$-subalgebra of a bornological $\mathbb{T}$-algebra A is (up to isomorphism) constructed on a subset of A. There is only a set of such subsets and, for each of them, only a set of bornologies on it. Thus $\mathsf{Born}^{\mathbb{T}}$ is well-powered.

Next if $f\colon A \longrightarrow B$ is an epimorphism in $\mathsf{Born}^{\mathbb{T}}$, it is an epimorphism in $\mathsf{Set}^{\mathbb{T}}$ by Proposition 2. But the category $\mathsf{Set}^{\mathbb{T}}$ is locally finitely presentable and therefore, co-well powered (see 1.58 in [2]). So again, up to isomorphism, we have only a set of possible $\mathbb{T}$-algebras B and, on each of them, only a set of bornologies. ∎

Let us now provide an explicit description of the coproduct bornology in $\mathsf{Born}^{\mathbb{T}}$.

Proposition 15. *Let $\mathbb{T}$ be an algebraic theory and $(A_i, \mathcal{B}_i)_{i\in I}$ a family of bornological $\mathbb{T}$-algebras. The bornology of the coproduct $A = +_{i\in I}(A_i, \mathcal{B}_i)$ in $\mathsf{Born}^{\mathbb{T}}$ is the bornology $\mathcal{B}$ in A generated by the subsets*

$$\alpha(B_1, \ldots, B_m) = \big\{\alpha\big(\sigma_{i_1}(b_1), \ldots, \sigma_{i_m}(b_m)\big) \mid \forall j\ b_j \in B_j\big\}$$

where

- *α is an m-ary operation of the theory $\mathbb{T}$;*
- *$\sigma_{i_j}\colon A_{i_j} \longrightarrow +_{i\in I} A_i$ is the canonical morphism of the coproduct in $\mathsf{Set}^{\mathbb{T}}$, with $i_j \in I$ and $j = 1, \ldots, m$;*
- *$B_j \subseteq A_{i_j}$ is $\mathcal{B}_{i_j}$-bounded, with $i_j \in I$ and $j = 1, \ldots, m$.*

and these operations are bounded mappings, since X is a bornological group. This allows defining a new bounded subset

$$E_X = D_X \cup (D_X + D_X) + (D_X + D_X + D_X) + \cdots + (D_X + \cdots + D_X) \subseteq X,$$

where the last parenthesis has n terms.

But on A we also have the operation

$$\beta\colon A \times A \longrightarrow A,\ (a_1, a_2) \mapsto a_1 + a_2 - a_1.$$

That operation is bounded since A is a bornological group. But H (via the inclusion k) and X (via the inclusion s) are subgroups of A with the induced bornology: indeed k is a kernel and s has the retraction p. And since H is a normal subgroup of A, $\beta(a_1, a_2) \in H$ as soon as $a_2 \in H$. The same considerations can be made with C. Since H and L have the bornologies induced by those of A and C, we obtain new bounded subsets

$$E_H = D_H \cup \beta(E_X, D_H) \subseteq H,\ E_C = D_L \cup \beta(E_X, D_L) \subseteq L.$$

Let us come back to $y \in V$, which is a formal sum of n elements in D_X, D_H, D_L. In the expression as above of the element y, all the individual lines that we have exhibited, when computed in A or C, yield elements of H or L, since these are normal subgroups. The formal sum of these lines, computed in A or C, yields thus a new element $y' \in H + L \subseteq X + H + L$, different from y in $X + H + L$, but whose formal expression in $A +_X C$ is exactly the same as that of y; thus $r(y') = r(y) = z$.

We have just seen that replacing D_X, D_H, D_L by E_X, E_H or E_L, we still obtain a bounded subset $W = E_1 + \cdots + E_n$ of $X + H + L$ (with, again, each E_i equal to E_X, E_H or E_L, according to the case). But in our expression of y as above, each individual line, computed in A or C, lies now in E_H or E_C, according to the case. And thus $y' \in E_H + E_L \subseteq W \cap (H + L)$. Since $H + L$ has the bornology induced by that of $X + H + L$ (σ_{H+L} is a strong monomorphism), $W \cap (H + L)$ is bounded in $H + L$ and $z = (u, v)(y') \in (u, v)\big(W \cap (H + L)\big)$. This proves that $U \subseteq (u, v)\big(W \cap (H + L)\big)$, with $W \cap (H + L)$ bounded in $H + L$. ∎

9. Local algebraic cartesian closedness

We recall that $\mathsf{Born}^{\mathbb{T}}$ is locally algebraically cartesian closed when all inverse image functors of its fibration of points admit a right adjoint. And it suffices to check this in the case of the inverse image along a morphism $\mathbf{0} \longrightarrow X$, that is, for the kernel functors on the categories $\mathsf{Pt}(X)$. We recall (see the

where each D_i is D_X, D_H or D_L, according to the case.

Choose now an element $z \in U = r(V)$. There exists thus an element $y \in V$ such that $r(y) = z$. Since (u, v) is surjective, there exists also an element $y' \in H+L$ such that $(u, v)(y') = z$, but in fact we need to construct explicitly such a y' from the given y, in order to handle properly the arguments of boundedness.

Since $V \subseteq D_1 + \cdots + D_n$, the element y can be written as a formal sum of n elements in D_X, D_H, D_L. Let us write further each of the n terms of the sum under the form $x_i + h_i + l_i$, with thus two of the three terms equal to zero:

$$y = (x_1 + h_1 + l_1) + \ldots + (x_n + h_n + l_n),\ x_i \in D_X,\ h_i \in D_H,\ l_i \in D_L.$$

But by construction

$$r(y) = s(x_1) + h_1 + l_1 + \cdots + s(x_n) + h_n + l_n.$$

Since H, K, L are the kernels of p', p, p'' and $ps = \mathsf{id}_X$, we obtain further

$$0 = p(z) = pr(y) = x_1 + \cdots + x_n$$

since $z \in K$.

It is now an easy game to compute that in $X + H + L$

$$\begin{aligned} y &= x_1 + h_1 + l_1 + \ldots + x_n + h_n + l_n \\ &= (x_1 + h_1 - x_1) + (x_1 + l_1 - x_1) \\ &\quad + \big((x_1 + x_2) + h_2 - (x_1 + x_2)\big) + \big((x_1 + x_2) + l_2 - (x_1 + x_2)\big) \\ &\quad + \big((x_1 + x_2 + x_3) + h_3 - (x_1 + x_2 + x_3)\big) + \cdots \\ &\quad \cdots \\ &\quad \cdots + \big((x_1 + \cdots + x_n) + l_n - (x_1 + \cdots + x_n)\big) \\ &\quad + (x_1 + \cdots x_n). \end{aligned}$$

Observe that the last sum does not matter, since $x_1 + \cdots + x_n = 0$ because $z \in K$. Of course in A or C, each individual line would yield an element of H or L, since these are normal subgroups. But in $X + H + L$ each line remains a formal composite of elements in X and H, or X and L.

Now on X we have the operations

$$x'_1 + x'_2,\ x'_1 + x'_2 + x'_3,\ \ldots\ x'_1 + \cdots + x'_n$$

Proof: First, we must prove that the operations of A are bounded. For this choose a k-ary operation β. An elementary bounded subset B of A^k has the form (see Proposition 2)

$$B = \alpha_1\left(B_1^{(1)}, \ldots, B_{m_1}^{(1)}\right) \times \cdots \times \alpha_k\left(B_1^{(k)}, \ldots, B_{m_k}^{(k)}\right)$$

with each α_t a m_t-ary operation and each $B_{m_j}^{(t)}$ bounded in A_{m_j}. We must prove that $\beta(B)$ is bounded in A. It suffices for that to consider the operation

$$\gamma\left(x_1^{(1)}, \ldots, x_{m_1}^{(1)}, \ldots, x_1^{(k)}, \ldots, x_{m_k}^{(k)}\right) =$$

$$\beta\Big(\alpha_1\big(x_1^{(1)}, \ldots, x_{m_1}^{(1)}\big), \ldots, \alpha_k\big(x_1^{(k)}, \ldots, x_{m_k}^{(k)}\big)\Big)$$

and observe that

$$\beta(B) = \gamma\left(B_1^{(1)}, \ldots, B_{m_1}^{(1)}, \ldots, B_1^{(k)}, \ldots, B_{m_k}^{(k)}\right).$$

Next the canonical morphisms $\sigma_i \colon A_i \longrightarrow +\, A_i$ are bounded: it suffices to apply the construction above to the 1-ary identity operation of the theory.

Finally given a bornological $\mathbb{T}$-algebra A and bounded $\mathbb{T}$-homomorphisms $f_i \colon G_i \longrightarrow A$, we must prove that the corresponding factorization $f \colon +_{i\in I} A_i \longrightarrow A$ is bounded. It suffices to prove that the image by f of a basic bounded subset $\alpha(B_1, \ldots, B_m)$ of $+_{i\in I} A_i$ as above is bounded in A. But

$$\begin{aligned} f\big(\alpha(B_1, \ldots, B_m)\big) &= \Big\{ f\Big(\alpha\big(\sigma_{i_1}(b_1), \ldots, \sigma_{i_m}(b_m)\big)\Big) \,\Big|\, \forall j\ b_j \in B_j \Big\} \\ &= \Big\{ \alpha\big(f\sigma_{i_1}(b_1), \ldots, f\sigma_{i_m}(b_m)\big) \,\Big|\, \forall j\ b_j \in B_j \Big\} \\ &= \Big\{ \alpha\big(f_{i_1}(b_1), \ldots, f_{i_m}(b_m)\big) \,\Big|\, \forall j\ b_j \in B_j \Big\} \\ &= \alpha\big(f_{i_1}(B_1), \ldots, f_{i_m}(B_m)\big) \end{aligned}$$

and this last subset is bounded in A because each f_{i_j} is bounded and the operation α is bounded on A. ∎

In some cases of interest, the general description as in Proposition 15 can be further improved. This is the case for groups.

Lemma 16. *The bornology of a bornological group G is generated by the so-called "symmetric" bounded subsets $B \subseteq G$, that is, those bounded subsets B containing 0 and stable for the opposite operation.*

Proof: The opposite operation is bounded, thus given a bounded subset B, the subset $-B = \{-b \mid b \in B\}$ is bounded. Therefore $\overline{B} = B \cup \{0\} \cup (-B)$ is bounded and contains B. ■

Lemma 17. *The bornology of a finite coproduct $G_1 + \cdots + G_n$ of bornological groups is generated by the subsets*

$$B_1 + \cdots + B_m = \{b_1 + \cdots + b_m \in G_1 + \cdots + G_n \mid b_1 \in B_1, \ldots, b_m \in B_m\},$$

where each B_i is a symmetric bounded subset of some G_j.

Proof: Let us make clear that n and m are two arbitrary natural numbers: several B_i – or at the opposite, none – can be chosen bounded in a same given G_j.

The result follows at once from Proposition 15 and Lemma 17. ■

3. Semi-abelian bornological algebras

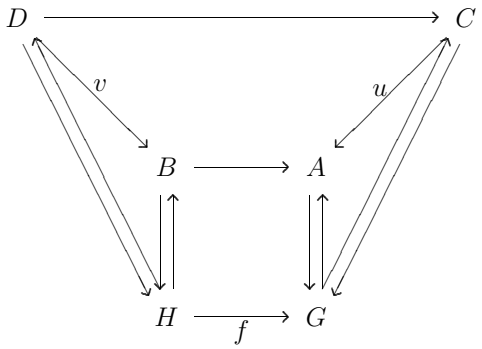
Theorem 18. *Let $\mathbb{T}$ be a semi-abelian algebraic theory. The category $\mathsf{Born}^{\mathbb{T}}$ of bornological $\mathbb{T}$-algebras is homological (see* [6]*).*

Proof: The category $\mathsf{Born}^{\mathbb{T}}$ is regular by Proposition 12 and admits the singleton as a zero-object. Since protomodularity is a purely finite limit condition, it follows by a classical Yoneda argument; let us recall it.

Given $f\colon H \longrightarrow G$ in $\mathsf{Born}^{\mathbb{T}}$, we must prove that the pullback functor

$$f^*\colon \mathsf{Pt}(G) \longrightarrow \mathsf{Pt}(H),$$

between the corresponding categories of split epimorphisms, reflects isomorphisms. So let u be such that $v = f^*(u)$ is an isomorphism.



itself the final $\mathbb{T}$-bornology for (ι_A, ι_C), this proves that $y \in \mathsf{Born}^{\mathbb{T}}$. That is, finality of r follows from finality of the pair (ι_A, ι_C) and of the morphisms (s', h), (s'', l).

It remains to observe that the diagram in the statement is commutative, which is immediate when composing with the two injections of the coproduct. As a kernel, the monomorphism k is strong, while σ_{H+L} admits a retraction, because the category $\mathsf{Born}^{\mathbb{T}}$ has a zero object, and thus σ_{H+L} is strong as well. ■Lemma 42 forces the expected result in the case of bornological groups:

Theorem 43. *The category $\mathsf{Born}^{\mathsf{Gr}}$ of bornological groups is algebraically coherent.*

Proof: We apply Lemma 42 to prove that (u, v) is a strong epimorphism in $\mathsf{Born}^{\mathsf{Gr}}$, that is, K has the final $\mathbb{T}$-bornology for (u, v). Let $U \subseteq K$ be a bounded subset: we must prove that it is the image of a bounded subset of $H + L$. By stability of boundedness for subsets, this is equivalent to proving the existence of a bounded subset of $H + L$ whose image contains U. And since $H + L$ has the bornology induced by that of $X + H + L$, the challenge is finally to exhibit a bounded subset $W \subseteq X + H + L$ such that $U \subseteq (u, v)\big(W \cap (H + L)\big)$.

For simplicity of the notation, let us write h, k, l and σ_{H+L} as set theoretical inclusions. Since U is bounded in K, U is bounded in $A +_X C$. Since r yields the final $\mathbb{T}$-bornology, there exists a bounded subset $V \subseteq X + H + L$ such that $r(V) = U \subseteq K$.

By Lemma 15, V is contained in the union of finitely many bounded subsets V_i of the form $B_1 + \cdots + B_n$, where each B_i is a symmetric bounded subset of X, H or L. Putting $U_i = U \cap r(V_i)$, we can argue on each of these subsituations and perform at the end the union of the finitely many results so obtained. In other words, there is no restriction in assuming at once that $U = r(V)$ with $V \subseteq B_1 + \cdots + B_n$ and each B_i is bounded symmetric in X, H or L.

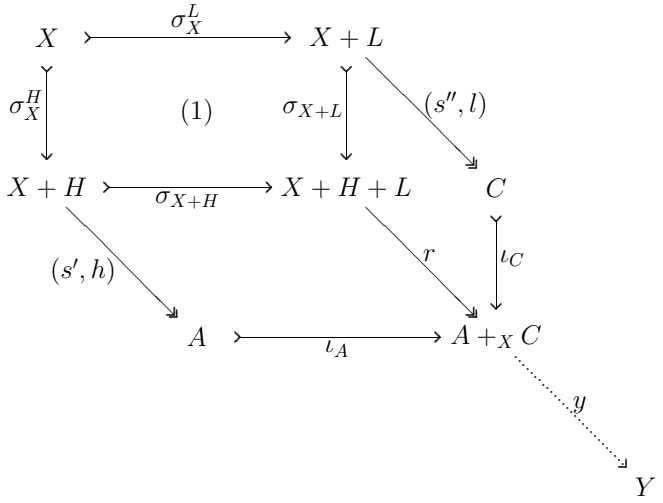
But since the bounded subset W we are looking for can be as big as we want (we need $U \subseteq W \cap (H + L)$), there is no restriction in increasing further the size of the B_i's. We define D_X, D_H and D_L to be each the (finite) union of all the B_i corresponding respectively to a bounded subset of X, H, L. And so we consider further

$$V \subseteq B_1 + \cdots + B_n \subseteq D_1 + \cdots + D_n$$

and therefore the pairs (h, s'), (k, s) and (l, s'') are jointly strongly epimorphic (see [5], Lemma 3.1.22). The following arrows are thus strong epimorphisms in $\mathsf{Born}^{\mathbb{T}}$:

$$(s', h)\colon X + H \twoheadrightarrow A,\ (s, k)\colon X + K \twoheadrightarrow A +_X C,$$
$$(s'', l)\colon X + L \twoheadrightarrow C.$$

This allows constructing further the following commutative diagram



where all the morphisms σ indicate canonical inclusions of coproducts. In particular, the rectangle (1) is a pushout. Notice that r is the composite

$$X + H + L \xrightarrow{\mathsf{id}_X + (u, v)} X + K \xrightarrow{(s, k)} A +_X C.$$

But since $\mathsf{Set}^{\mathbb{T}}$ is algebraically coherent, (u, v) is a strong epimorphism in $\mathsf{Set}^{\mathbb{T}}$; thus also $\mathsf{id}_X + (u, v)$ is a strong epimorphism in $\mathsf{Set}^{\mathbb{T}}$. So we know already that this composite r is a surjective morphism in $\mathsf{Born}^{\mathbb{T}}$.

For r being a strong epimorphism in $\mathsf{Born}^{\mathbb{T}}$, it remains to prove that $A +_X C$ has the final $\mathbb{T}$-bornology for r with respect to the topological functor $\mathsf{Born}^{\mathbb{T}} \longrightarrow \mathsf{Set}^{\mathbb{T}}$. For this choose $y \in \mathsf{Set}^{\mathbb{T}}$ such that $yr \in \mathsf{Born}^{\mathbb{T}}$. Then

$$y\iota_A(s', h) = yr\sigma_{X+H},\ y\iota_C(s'', l) = yr\sigma_{X+L}$$

from which $y\iota_A$ and $y\iota_C$ lie in $\mathsf{Born}^{\mathbb{T}}$, because (s', h) and (s'', l), as strong epimorphisms, yield final $\mathbb{T}$-bornologies. And since the pushout $A +_X C$ has

Viewing this situation in the semi-abelian category $\mathsf{Set}^{\mathbb{T}}$, we know already that u is an algebraic isomorphism. We must prove that it is also an isomorphism at the level of bornologies.

For every bornological set X and every bornological $\mathbb{T}$-algebra E, the set $\mathsf{Born}(X, E)$ of bounded mappings, provided with the pointwise operations, is at once a $\mathbb{T}$-algebra. This process $\mathsf{Born}(X, -)$ preserves limits so that, applied to the diagram above, we get that $\mathsf{Born}(X, u)$ is an isomorphism in the protomodular category $\mathsf{Set}^{\mathbb{T}}$, because so is $\mathsf{Born}(X, v)$. It remains to choose $X = A$ to get that

$$\mathsf{Born}(A, u)\colon \mathsf{Born}(A, C) \longrightarrow \mathsf{Born}(A, A)$$

is an isomorphism in $\mathsf{Set}^{\mathbb{T}}$. Therefore the identity on A corresponds to a bounded mapping $w\colon A \longrightarrow C$ such that $uw = \mathsf{id}_A$. Since u is a bijection, we get $w = u^{-1}$ proving that u^{-1} is bounded. ∎

Theorem 19. *Let $\mathbb{T}$ be a semi-abelian algebraic theory. The category $\mathsf{Born}^{\mathbb{T}}$ has semi-direct products.*

Proof: Considering the category $\mathsf{Pt}(G)$ of points (= split epimorphisms) over an object $G \in \mathsf{Born}^{\mathbb{T}}$, it suffices to observe that for every morphism $f\colon H \longrightarrow G$, the pullback functor $f^*\colon \mathsf{Pt}(G) \longrightarrow \mathsf{Pt}(H)$ is monadic. (The case $H = \mathbf{1}$ suffices already; see [5].)

The functor f^* has a left adjoint, namely, the pushout along f. It reflects isomorphisms by protomodularity of $\mathsf{Born}^{\mathbb{T}}$ (see Theorem 18). By the Beck criterion, it remains to prove that f^* preserves the coequalizer of reflexive pairs. This is the case because f^* preserves these coequalizers in $\mathsf{Set}^{\mathbb{T}}$ and preserves regular epimorphisms in $\mathsf{Born}^{\mathbb{T}}$. ∎

4. On the representability of actions

Given two objects G, X in an homological category with semi-direct products, the actions of G on X are the M_G-algebra structures on X for the monad M_G of Theorem 19. These actions can equivalently be presented as the split extensions of X by G, that is the situations

$$X \xrightarrow{k} A \underset{p}{\overset{s}{\leftrightarrows}} G,\ k = \mathsf{Ker}\, p,\ p = \mathsf{Coker}\, k,\ ps = \mathsf{id}_G.$$

Let us write $\mathsf{SplEx}(G, X)$ for the isomorphism classes of such split extensions. This extends at once as a contravariant functor $\mathsf{SplEx}(-, X)$, acting by pullbacks on the variable G. The actions on X are said to be representable when

this functor $\mathsf{SplEx}(-,X)$ is represented by an object $\mathsf{Act}(X)$. In the case of the category of groups, $\mathsf{Act}(X)$ is the group of automorphisms of X while in the case of Lie algebra $\mathsf{Act}(X)$ is the Lie algebra of derivations. The representability of actions for bornological groups and bornological Lie algebras follows from more general consideration in [7].

We shall first generalize to $\mathsf{Born}^{\mathbb{T}}$ the following result, which can be found in [8].

Theorem 20. *Let $\mathcal{C}$ be a locally presentable semi-abelian category in which finite limits commute with filtered colimits. The following conditions are equivalent:*

(1) *The actions on an object X are representable.*
(2) *The functor $\mathsf{SplEx}(-,X)$ transforms binary coproducts in binary products.*

Proof: See Theorem 5.8 in [8]. ∎

Of course, when we say that the contravariant functor $\mathsf{SplEx}(-,X)$ preserves some colimit, we mean that it transforms that colimit into a limit.

Lemma 21. *Let $\mathbb{T}$ be a semi-abelian algebraic theory and X a bornological $\mathbb{T}$-algebra. The functor $\mathsf{SplEx}(-,X)$ preserves the initial object.*

Proof: The only possible split extension of X by $\mathbf{0}$ is

$$X = X \leftrightarrows \mathbf{0}$$

from which the result. ∎

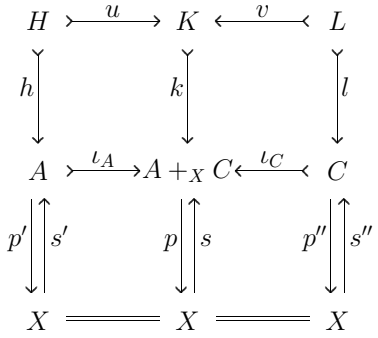
Lemma 22. *Let $\mathbb{T}$ be a semi-abelian algebraic theory and X, G two bornological $\mathbb{T}$-algebras. Given a split extension of X by G in $\mathsf{Set}^{\mathbb{T}}$,*

$$X \overset{k}{\rightarrowtail} A \underset{p}{\overset{s}{\leftrightarrows}} G$$

there exists at most one bornology on A which turns this diagram into a split extension of X by G in $\mathsf{Born}^{\mathbb{T}}$.

Proof: Computing the kernel of p is pulling back the zero subobject along p.

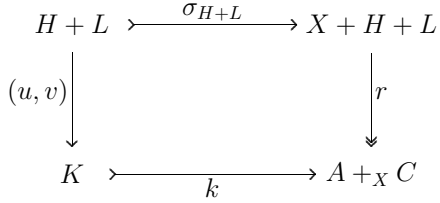
$$\begin{tikzcd}
H \arrow[r, tail, "u"] \arrow[d, tail, "h"'] & K \arrow[d, tail, "k"'] & L \arrow[l, tail, "v"'] \arrow[d, tail, "l"] \\
A \arrow[r, tail, "\iota_A"] \arrow[d, shift right, "p'"'] & A +_X C \arrow[d, shift right, "p"'] & C \arrow[l, tail, "\iota_C"'] \arrow[d, shift right, "p''"'] \\
X \arrow[r, equal] \arrow[u, shift right, "s'"'] & X \arrow[r, equal] \arrow[u, shift right, "s"'] & X \arrow[u, shift right, "s''"']
\end{tikzcd}$$



the pair (u,v) is jointly strongly epimorphic. In this diagram, $A +_X C$ indicates the pushout in $\mathcal{C}$ of s' and s'', which yields at once the coproduct $(p, s\colon A +_X C \leftrightarrows X)$ of $(p', s'\colon A \leftrightarrows X)$ and $(p'', s''\colon C \leftrightarrows X)$ in $\mathsf{Pt}(X)$.

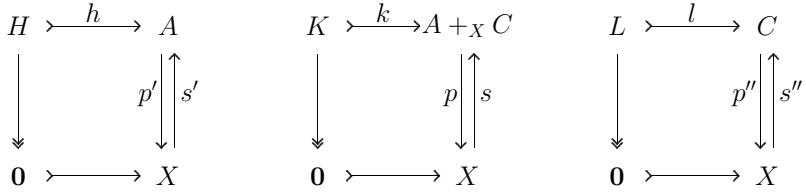
Lemma 42. *Consider a semi-abelian algebraic theory $\mathbb{T}$ such that $\mathsf{Set}^{\mathbb{T}}$ is algebraically coherent. Given in $\mathsf{Born}^{\mathbb{T}}$ the diagram above, there exists further in $\mathsf{Born}^{\mathbb{T}}$ a commutative diagram*

$$\begin{tikzcd}
H + L \arrow[r, tail, "\sigma_{H+L}"] \arrow[d, "(u,v)"'] & X + H + L \arrow[d, two heads, "r"] \\
K \arrow[r, tail, "k"'] & A +_X C
\end{tikzcd}$$



where r is a regular epimorphism, k and σ_{H+L} are strong monomorphisms, and (u,v) is surjective.

Proof: In the homological category $\mathsf{Born}^{\mathbb{T}}$ (see Proposition 18) we have the pullbacks



$$\begin{tikzcd}
H \arrow[r, tail, "h"] \arrow[d, two heads] & A \arrow[d, shift right, "p'"'] \\
\mathbf{0} \arrow[r, tail] & X \arrow[u, shift right, "s'"']
\end{tikzcd}
\qquad
\begin{tikzcd}
K \arrow[r, tail, "k"] \arrow[d, two heads] & A +_X C \arrow[d, shift right, "p"'] \\
\mathbf{0} \arrow[r, tail] & X \arrow[u, shift right, "s"']
\end{tikzcd}
\qquad
\begin{tikzcd}
L \arrow[r, tail, "l"] \arrow[d, two heads] & C \arrow[d, shift right, "p''"'] \\
\mathbf{0} \arrow[r, tail] & X \arrow[u, shift right, "s''"']
\end{tikzcd}$$

if $a_m = x \in X \subseteq A_1$ and you need to consider x as being in A_2, just observe that exactly the same sum appears as another element of W_1, but this time with $x \in X \in A_2$: this is just because W_1 has been constructed using all the possible sequences of indices 1, 2, while E_1 and E_2 contain the same elements of X. So, inside W_1, using a different element having exactly the same formal expression, you can perform a first reduction of that element. And of course you can always perform a reduction by adding two consecutive terms appearing in the same A_i. Let us thus perform one reduction (just one) on each element of W_1 ... every time this is possible. If no reduction is possible, it does not matter.

But the addition $+ \colon A_i \times A_i \longrightarrow A_i$ is bounded, so that

$$F_i = E_i \cup (A_i + A_i) \subseteq A_i$$

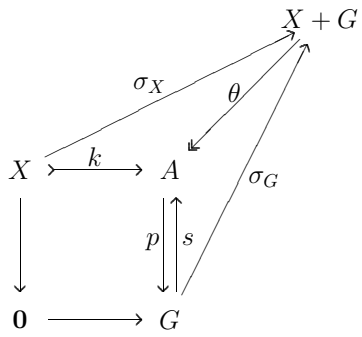
remains bounded. Perform on F_1, F_2 the same constructions as earlier on C_1, C_2 and get so a second bounded subset $W_2 \subseteq A + B$. All the formal elements of $A + B$ obtained via a first reduction (= adding two elements of A_1 or two elements of A_2, including the case of elements in X which have possibly been re-labeled), all these elements are now in W_2. And everything has been done to allow a second reduction, every time such a reduction is possible.

And you repeat the process iteratively, with successive bounded subsets W_3, W_4, and so on. Of course after n steps, no element of the original W_1, thus no element of V, will still admit a reduction. This means that if a formal element of V, when viewed in $q(V)$, turns out to be an element of $a \in A_i$, then the formal element of $A + B$ reduced to the single letter a belongs to W_n. This is precisely saying that $q(V) \cap A_i \subseteq W_n \cup A_i$, that is, $U \cap A_i \subseteq W_n \cap A_i$. ∎

Corollary 41. *The category* $\mathsf{Born}^{\mathsf{Gr}}$ *is action representative.* ∎

8. Algebraic coherence

Proposition 3.13 in [10] proves that in the context in which we are working, namely, finitely cocomplete homological categories, algebraic coherence reduces to the following property. Given the following commutative diagram, where the three columns are kernels of split epimorphisms

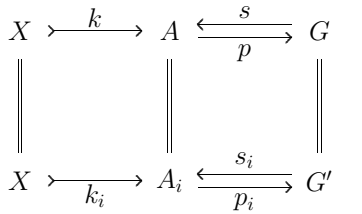


In a protomodular category, when pulling back a split epimorphism (p, s), the pair (k, s) is strongly epimorphic (see Lemma 3.1.22 in [5]). In other words, the corresponding factorization θ of k and s through $X + G$ is a strong epimorphism. Considering such a situation in $\mathsf{Born}^{\mathbb{T}}$, θ is a strong, thus regular, epimorphism (see Proposition 5). Therefore the bornology on A is necessarily the quotient bornology corresponding to θ, which depends only on the bornologies of X and G. ∎

Let us make clear that Lemma 22 says that the bornology indicated is the only possible one, but it does not say that it always produces a split extension. This lemma is just a "uniqueness result".

Corollary 23. *Let* $\mathbb{T}$ *be a semi-abelian algebraic theory and* X *a bornological* $\mathbb{T}$*-algebra. The functor* $\mathsf{SplEx}(-, X)$ *transforms bijections into injections.*

Proof: Of course there is no restriction in supposing that the bijection $G \longrightarrow G'$ below is a set theoretical identity. We must thus consider the following situation in $\mathsf{Born}^{\mathbb{T}}$ where $i = 1, 2$, G and G' have the same underlying $\mathbb{T}$-algebra, and the same upper row is obtained by pullbacks, from both lower rows.



This implies at once $A_1 = A = A_2$, $p_1 = p = p_2$, $s_1 = s = s_2$. By Lemma 22, the two lower rows are isomorphic in $\mathsf{Born}^{\mathbb{T}}$. ∎

Lemma 24. *Let $\mathbb{T}$ be a semi-abelian algebraic theory and X a bornological $\mathbb{T}$-algebra. The functor $\mathsf{SplEx}(-,X)$ preserves coequalizers of kernel pairs.*

Proof: Consider a kernel pair (u_1,u_2) and its coequalizer q

$$R \underset{u_2}{\overset{u_1}{\rightrightarrows}} G \xrightarrow{\ q\ } H.$$

We must prove to have an equalizer in Set

$$\mathsf{SplEx}(H,X) \xrightarrow{\mathsf{SplEx}(q,X)} \mathsf{SplEx}(G,X) \underset{\mathsf{SplEx}(u_2,X)}{\overset{\mathsf{SplEx}(u_1,X)}{\rightrightarrows}} \mathsf{SplEx}(R,X).$$

The proof of Lemma 5.6 in [8] applies as such. Let us recall it.

We prove first the injectivity of $\mathsf{SplEx}(q,X)$. For that we consider the following diagram

$$\begin{array}{ccccc}
X & \overset{k_i}{\rightarrowtail} & A_i & \underset{p_i}{\overset{s_i}{\leftrightarrows}} & R \\
\| & & \downarrow{\scriptstyle u_1'}\ \downarrow{\scriptstyle u_2'} & & \downarrow{\scriptstyle u_1}\ \downarrow{\scriptstyle u_2} \\
X & \overset{k'}{\rightarrowtail} & A' & \underset{p'}{\overset{s'}{\leftrightarrows}} & G \\
\| & & \downarrow{\scriptstyle q'} & & \downarrow{\scriptstyle q} \\
X & \underset{k}{\rightarrowtail} & A & \underset{p}{\overset{s}{\leftrightarrows}} & H
\end{array}$$

where $i=1,2$ and the rows are split extensions of X, each being obtained by pullback from the previous one. Since $qu_1 = qu_2$, the two upper rows, for $i=1,2$, coincide. But q' is a regular epimorphism by Proposition 5, while (u_1',u_2') is the kernel pair of q' because (u_1,u_2) is the kernel pair of q. Therefore $q' = \mathsf{Coker}\,(u_1',u_2')$. So the lower row is recaptured by a coequalizer process from the upper part of the diagram. This proves that if two lower rows yield the same upper part of the diagram, they are isomorphic. This is just rephrasing the injectivity of $\mathsf{SplEx}(q,X)$.

To check the universal property of the equalizer, let us work on the same diagram, starting with the central row yielding isomorphic upper rows by pullbacks. Of course we construct the lower row by coequalizers. We get at once $ps = \mathsf{id}_H$. Next, by a well-known Barr–Kock result (see Example 6.10 in [4]), the square $qp' = pq'$ is a pullback since so are the upper squares. But

bounded subset as described in Proposition 15. Such a basic bornological subset has thus the form $\alpha(B_{i_1},\dots,B_{i_n})$ with B_{i_k} bounded in A_{j_k} $(j_k=1,2)$. Now since we are looking for an inclusion $U\cap A_i \subseteq W\cap A_i$, there is no restriction at all to enlarge the size of W, thus the size of the various B_{i_k}. To achieve this consider first

$$C_1 = \bigcup\{B_{i_k} \mid j_k = 1\},\ C_2 = \bigcup\{B_{i_k} \mid j_k = 2\}.$$

Next since $-\colon A_i \longrightarrow A_i$ is bounded, $-C_i$ is bounded and we define further

$$D_i = C_i \cup (-C_i) \cup \{0\}.$$

We still want D_1 and D_2 to contain the same elements of X. But since k_i is a strong monomorphism, thus yields the induced bornology, $D_i \cap X$ is bounded in X and thus bounded in both A_1, A_2. We consider therefore

$$E_1 = D_1 \cup (D_2 \cap X),\ E_2 = D_2 \cup (D_1 \cap X).$$

We have so a bounded subset E_i to be used in all the components corresponding to an occurrence of A_i. E_i contains 0, is stable under the opposite $-a$ and E_1, E_2 contain the same elements of X.

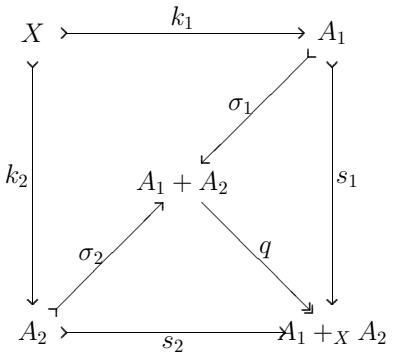
But it is not yet enough. Instead of the original sequence of A_{j_k} with corresponding bounded subsets B_{i_k}, we consider all the possible sequences $(i_1,\dots,i_n)$, with $i_k = 1,2$, and we define

$$W_1 = \bigcup_{(i_1,\dots,i_n)} E_{i_1} + \cdots + E_{i_n},\ i_k = 1,2.$$

This is thus a finite union of basic bounded subsets as in Proposition 15. Since E_i is stable under opposites, we have simply used the addition at each place instead of addition and subtraction. Oh yes, W has been given an index 1 ... because this is only the first step of the construction. Notice already that $V \subseteq W$ thus $U = q(V) \subseteq q(W)$.

We know also how $q(W)$ is constructed: take all the formal expressions of the elements of W as sums of n elements chosen in the various E_{i_k}, but allow now to combine the elements of X with both the elements of A_1 and A_2. In $A_1 + A_2$ this is impossible: you have two copies of each element of X, one in A_1 which can only be combined with elements of A_1, a second one in A_2 which can only be combined with elements of A_2. But given an element

$$a_1 + \ldots + a_n \in E_{i_1} + \cdots + E_{i_n}$$



with k_1 and k_2 strong monomorphisms, that is, inclusions with the induced bornology. By the classical amalgamation property for groups, we know already that the bounded mappings s_1, s_2 are injective: we must prove that they give rise as well to induced bornologies. For this we shall use the construction of the pushout as a quotient of the coproduct: σ_1 and σ_2 are the canonical inclusions of the coproduct and $q = \mathsf{Coker}\,(\sigma_1 k_1, \sigma_2 k_2)$ is the quotient morphism. The morphisms σ_i are split monomorphisms, because the category is pointed, thus are strong monomorphisms and yield the induced bornology.

The challenge is to prove that given $U \subseteq A_1 +_X A_2$ bounded, then $U \cap A_i \subseteq A_i$ is bounded as well $(i = 1, 2)$. But $A_1 +_X A_2$ carries the final bornology for q, thus $U = q(V)$ with $V \subseteq A_1 + A_2$ bounded. Of course

$$U \cap A_i = s_i^{-1}(U) = \sigma_i^{-1} q^{-1}(U) = \sigma_i^{-1} q^{-1} q(V) = \sigma_i^{-1}(\overline{V})$$

where $\overline{V}$ is the saturation of V for the equivalence relation given by the kernel pair of q. Unfortunately, $\overline{V}$ has no reason to be bounded: it can of course be presented as a union of "translations" of V, but a priori an infinite union. Since σ_i carries the induced bornology, the thesis reduces to proving the existence of some bounded subset $W \subseteq A_1 + A_2$ such that $U \cap A_i \subseteq W \cap A_i$; $\overline{V}$ cannot work since it has no reason to be bounded, so we are faced with the challenge of constructing an adequate bounded subset W.

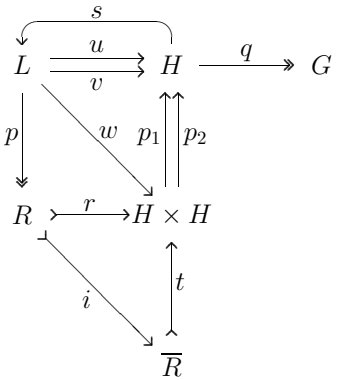
By Proposition 15, the bounded subset $V \subseteq A_1 + A_2$ is contained in a finite union of basic bounded subsets. We can of course split U in as many subsets, each one contained in one of the basic subsets of the union, work separately on each piece and perform the union of the various (finitely many!) results. In other words, there is no restriction in supposing that V is contained in a basic

then $\mathsf{Ker}\,p'$ is isomorphic to $\mathsf{Ker}\,p$, yielding the lower left-hand square with $k = \mathsf{Ker}\,p$. Thus the bottom row is a split extension of X by H mapped on the middle row by $\mathsf{SplEx}(q, X)$. ∎

Corollary 25. *Let $\mathbb{T}$ be a semi-abelian algebraic theory and X a bornological $\mathbb{T}$-algebra. The functor $\mathsf{SplEx}(-, X)$ preserves regular epimorphisms (i.e. transforms them into regular monomorphisms).* ∎

Lemma 26. *Let $\mathbb{T}$ be a semi-abelian algebraic theory and X a bornological $\mathbb{T}$-algebra. The functor $\mathsf{SplEx}(-, X)$ preserves the coequalizers of those pairs of morphisms which admit a common section.*

Proof: In $\mathsf{Born}^{\mathbb{T}}$, let $q = \mathsf{Coker}\,(u, v)$ with s a common section of u and v, thus $us = \mathsf{id}_H = vs$.



We write p_1, p_2 for the two projections of the product $H \times H$ and w for the factorization yielding $p_1 w = u$, $p_2 w = v$. We consider further the image factorization $w = rp$ of w and write $\overline{R}$ for the sub-$\mathbb{T}$-algebra R of $H \times H$ provided with the induced bornology. Thus i, as a mapping, is the identity on the set R.

Since s is a common section of u and v, the diagonal of H is contained in R, thus also in $\overline{R}$. Thus R and $\overline{R}$ are equivalence relations, because every protomodular category is a Mal'tsev category (see Proposition 3.1.19 in [5]). Since p and i are epimorphisms (see Proposition 10), $q = \mathsf{Coker}\,(u, v)$ implies $q = \mathsf{Coker}\,(p_1 t, p_2 t)$. And since $\overline{R}$ is a regular equivalence relation, by Propositions 5 and 10, it is the kernel pair of its coequalizer.

We know by Lemma 24 that the functor $\mathsf{SplEx}(-, X)$ preserves the coequalizer $q = \mathsf{Coker}\,(p_1 t, p_2 t)$. By Corollaries 25 and 23, $\mathsf{SplEx}(-, X)$ transforms

both morphisms p and i into monomorphisms. Therefore

$$\begin{aligned}\mathsf{Ker}\big(\mathsf{SplEx}(u,X),\mathsf{SplEx}(v,X)\big) &= \mathsf{Ker}\big(\mathsf{SplEx}(qp_1,X),\\ \mathsf{SplEx}(qp_2,X)\big) &= \mathsf{SplEx}(q,X).\end{aligned}$$

This concludes the proof. ∎

Lemma 27. *Let $\mathbb{T}$ be a semi-abelian algebraic theory and X a bornological $\mathbb{T}$-algebra. The functor $\mathsf{SplEx}(-,X)$ preserves filtered colimits.*

Proof: This is just rephrasing the fact that a filtered colimit of split extensions remains a split extension, just by Proposition 4 and because filtered colimits commute with kernels and pullbacks in $\mathsf{Set}^{\mathbb{T}}$. ∎

We are now ready to prove the expected formal representability criterion.

Theorem 28. *Let $\mathbb{T}$ be a semi-abelian algebraic theory and X a bornological $\mathbb{T}$-algebra. The following conditions are equivalent.*

(1) *The actions on X are representable.*
(2) *The functor $\mathsf{SplEx}(-,X)$ preserves binary coproducts.*

Proof: A covariant functor $F\colon \mathcal{C}\longrightarrow\mathsf{Set}$ to the category of sets is representable when the singleton $\mathbf{1}$ admits a universal reflection along F:

$$\mathcal{C}\big(\rho(\mathbf{1}),X\big)\cong\mathsf{Set}\big(\mathbf{1},F(X)\big)\cong F(X).$$

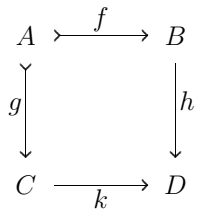
We know that this is the case when $\mathcal{C}$ is complete, F preserves limits, and the solution set condition holds. But via the so-called "special adjoint functor theorem", the solution set condition can be replaced by well-poweredness and the existence of a cogenerator.

In our case, the functor $\mathsf{SplEx}(-,X)$ is contravariant. We know that the category $\mathsf{Born}^{\mathbb{T}}$ is cocomplete (see Proposition 10), co-well powered (see Proposition 14) and admits a generator (see Corollary 11). Thus the representability of $\mathsf{SplEx}(-,X)$ reduces to the preservation of colimits.

First, an arbitrary coproduct is the filtered colimit of its finite sub-coproducts. By Lemmas 21 and 27, the preservation of all coproducts reduces thus to the preservation of binary (and thus finite) coproducts.

One concludes by Lemma 26 and the classical theorem attesting that every limit can be reconstructed via two coproducts and the coequalizer of a pair of morphisms between them, admitting a common section. The existence of this common section is generally not pointed out in the classical presentation of this theorem, but is trivial.

Definition 38. *A category with pushouts is said to satisfy* the amalgamation property for strong monomorphisms *when, given a pushout diagram*



if f and g are strong monomorphisms, then h and k are strong monomorphisms as well.

For the needs of this paper, one could even restrict our attention to the case where f and g are kernels.

We have thus proved the following theorem:

Theorem 39. *Let $\mathbb{T}$ be a semi-abelian theory giving rise to a "category $\mathsf{Set}^{\mathbb{T}}$ of interest" in the sense of Orzech (see* [16]*). When $\mathsf{Born}^{\mathbb{T}}$ satisfies the amalgamation property for strong monomorphisms, $\mathsf{Born}^{\mathbb{T}}$ is action representative.* ∎

7. Amalgamation of bornological subgroups

This section is devoted to proving the amalgamation property for strong monomorphisms in the category of bornological groups. Via Theorem 39, this provides an alternative proof (see [7]) of the representability of actions. But the amalgamation property is in itself interesting and worth being mentioned.

Proposition 40. *The category of bornological groups satisfies the amalgamation property for strong monomorphisms.*

Proof: Let us consider the following pushout square in the category of bornological groups

Next if $z\colon S \longrightarrow Z$ is such that $zn = 0$, then

$$zt_1k_1 = zn = 0,\ zt_2k_2 = zn = 0$$

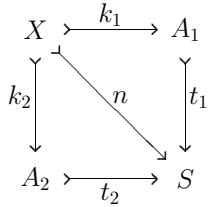
from which one gets factorizations $z_i\colon G_i \longrightarrow Z$ such that $z_ip_i = zt_i$. This yields a further morphism $(z_1, z_2)\colon G_1 + G_2 \longrightarrow Z$ such that

$$(z_1, z_2) \circ (\sigma_1 p_1, \sigma_2 p_2) = z.$$

So indeed $(\sigma_1 p_1, \sigma_2 p_2) = \mathsf{Coker}\, n$ and since n is a kernel, $n = \mathsf{Ker}\,(\sigma_1 p_1, \sigma_2 p_2)$. So the middle row is a split extension of X. Now since the three rows are short exact sequences and the left vertical morphisms are isomorphisms, the right-hand squares are pullbacks.

So, given two split extensions of X by G_1 and G_2, we have constructed a split extension of X by $G_1 + G_2$. Such an extension is unique, by the uniqueness in $\mathsf{Set}^{\mathbb{T}}$ and Lemma 22. This proves that the functor $\mathsf{SplEx}(X)$ transforms the coproduct $G_1 + G_2$ into a product and we conclude by Theorem 28. ∎

Corollary 36. *Let $\mathbb{T}$ be a semi-abelian theory. In $\mathsf{Born}^{\mathbb{T}}$, suppose that for every pushout square*



if k_1, k_2 are kernels of split epimorphisms, then the diagonal n is a kernel as well. Then $\mathsf{Born}^{\mathbb{T}}$ is action representative.

Proof: Just observe that, in the proof of Theorem 35, we have only used the fact that n is a kernel in $\mathsf{Born}^{\mathbb{T}}$. ∎

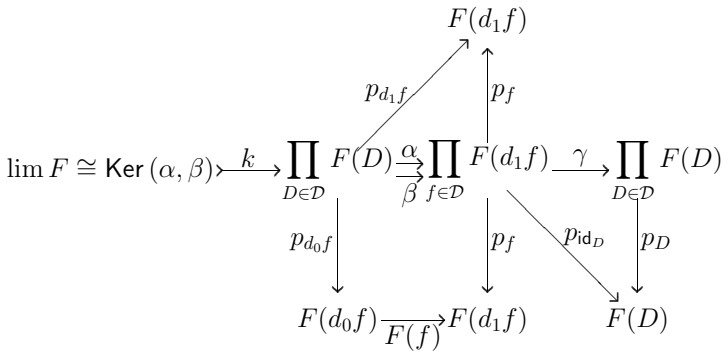
Corollary 37. *Let $\mathbb{T}$ be a semi-abelian theory giving rise to a "category $\mathsf{Set}^{\mathbb{T}}$ of interest" in the sense of Orzech (see* [16]*). Then $\mathsf{Born}^{\mathbb{T}}$ is action representative if and only if the necessary condition of Proposition 33 holds.*

Proof: This follows from Example 6.8 in [8]. ∎

In particular, Corollary 37 applies to the cases of groups, Lie algebras, rings, and so on ... and to every theory obtained from these by adding axioms.

So probably one could introduce the following definition.

Indeed writing the statement as one usually does, that is in the case of the limit of a functor $F\colon \mathcal{D} \longrightarrow \mathcal{C}$, the diagram that one constructs is



where d_0f and d_1f indicate respectively the domain and the codomain of f. It is well-known that $\lim F$ can be recaptured as $\mathsf{Ker}\,(\alpha, \beta)$. But α and β admit trivially the common retraction γ. ∎

5. A criterion for lifting the representability of actions

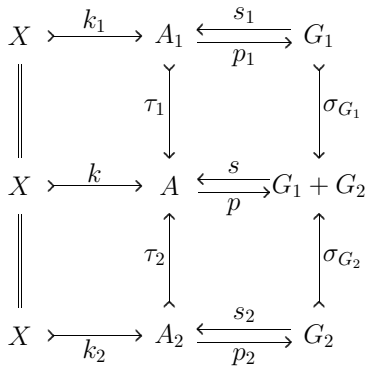
Putting together Theorems 20 and 28, we have thus, given a semi-abelian algebraic theory $\mathbb{T}$ and a bornological $\mathbb{T}$-algebra X:

- the actions on the $\mathbb{T}$-algebra X are representable if and only if the functor
$$\mathsf{SplEx}(-, X)\colon \mathsf{Set}^{\mathbb{T}} \longrightarrow \mathsf{Set}$$
preserves binary coproducts;
- the actions on the bornological $\mathbb{T}$-algebra X are representable if and only if the functor
$$\mathsf{SplEx}(-, X)\colon \mathsf{Born}^{\mathbb{T}} \longrightarrow \mathsf{Set}$$
preserves binary coproducts.

One would like to investigate when the representability of actions on X in $\mathsf{Set}^{\mathbb{T}}$ implies the representability of actions on X in $\mathsf{Born}^{\mathbb{T}}$.

To facilitate the language, let us introduce a point of terminology.

Definition 29. *Given a semi-abelian algebraic theory $\mathbb{T}$ and a bornological $\mathbb{T}$-algebra X, let us call "test diagram on X" a commutative diagram in $\mathsf{Born}^{\mathbb{T}}$ having the following shape:*



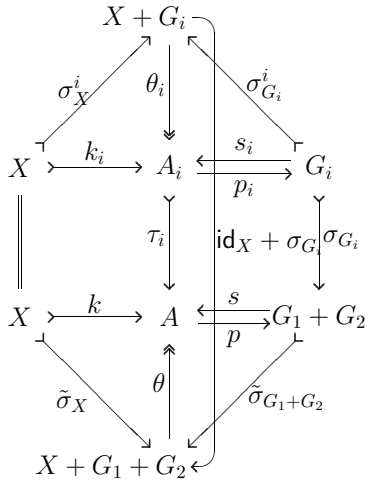
and such that

- *the upper and the lower rows are split extensions in* $\mathsf{Born}^{\mathbb{T}}$*;*
- *A is equipped with the quotient bornology for the morphism*

$$(k, s)\colon X + G_1 + G_2 \twoheadrightarrow A$$

  (as in the proof of Lemma 22);
- *the central row is a split extension in* $\mathsf{Set}^{\mathbb{T}}$*;*
- *in* $\mathsf{Set}^{\mathbb{T}}$*, the upper and the lower rows are obtained by pullbacks from the central one.*

It is probably useful to draw further the following diagrams, for $i = 1, 2$:



the diagonal n is a normal monomorphism in $\mathsf{Set}^{\mathbb{T}}$*. Under these conditions, actions on X are representable in* $\mathsf{Set}^{\mathbb{T}}$*.* ∎

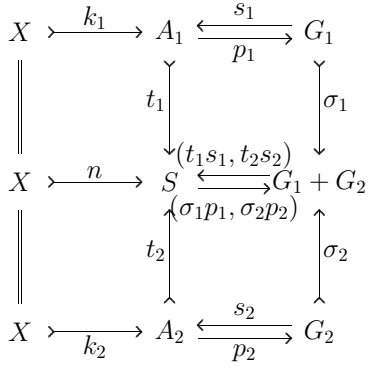
Of course as observed in [8], in the situation of Proposition 34, a result analogous to our Proposition 33 forces t_1, t_2 to be monomorphisms.

The situation of Proposition 34 implies a first interesting step in the search of a criterion for the representability of actions in $\mathsf{Born}^{\mathbb{T}}$.

Theorem 35. *Let* $\mathbb{T}$ *be a semi-abelian theory satisfying the conditions of Proposition 34. Then the necessary condition of Proposition 33 is also sufficient for the representability of actions on X in* $\mathsf{Born}^{\mathbb{T}}$*.*

Proof: Let us thus assume the necessary condition of Proposition 33 and the conditions of Proposition 34. Together, these conditions imply that n is a normal monomorphism in $\mathsf{Set}^{\mathbb{T}}$ providing X with the bornology induced by that of S. Thus n is a kernel morphism in $\mathsf{Born}^{\mathbb{T}}$ and therefore, n is in $\mathsf{Born}^{\mathbb{T}}$ the kernel of its cokernel.

We can now consider the diagram



One has of course

$$(\sigma_1 p_1, \sigma_2 p_2) \circ (t_1 s_1, t_2 s_2) = \mathsf{id}_{G_1+G_2}$$

as observed when composing with σ_1, σ_2. One has also

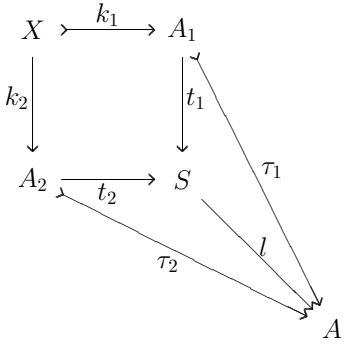
$$(\sigma_1 p_1, \sigma_2 p_2) = \mathsf{Coker}\, n.$$

Indeed, first

$$(\sigma_1 p_1, \sigma_2 p_2) n = (\sigma_1 p_1, \sigma_2 p_2) t_1 k_1 = \sigma_1 p_1 k_1 = 0.$$

the pair (k_1, k_2) satisfies the following regular amalgamation property: the pushout of (k_1, k_2) is constituted of two monomorphisms in $\mathsf{Born}^{\mathbb{T}}$ *which are regular in* Born*, that is, carry the induced bornology.*

Proof: Let us go back to Criterion 31 and the *test diagram* of Definition 29, which we build from the ingredients in the statement of the present proposition; let us write (S, t_1, t_2) for the pushout. We get a factorization l



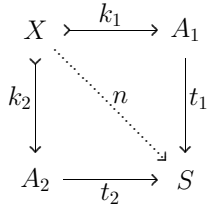
Thus $lt_i = \tau_i$ with τ_i a monomorphism, proving that t_i is a monomorphism. But still by Criterion 31, A_i has the bornology induced by that of A, thus is a regular monomorphism in Born. But Born, as a quasi-topos, is coregular (see Theorem 8 and [15]); so $\tau_i = lt_i$ is regular, t_i is a regular monomorphism in Born and thus A_i has the bornology induced by that of S. ∎

Let us recall the following result, which is Proposition 6.3 in [8].

Proposition 34. *Let $\mathbb{T}$ be a semi-abelian theory. Suppose that for every two split extensions of X in* $\mathsf{Set}^{\mathbb{T}}$

$$X \rightarrowtail^{k_i} A_i \underset{p_i}{\overset{s_i}{\leftrightarrows}} G_i, \ i = 1, 2.$$

given the pushout (S, t_1, t_2) of (k_1, k_2)



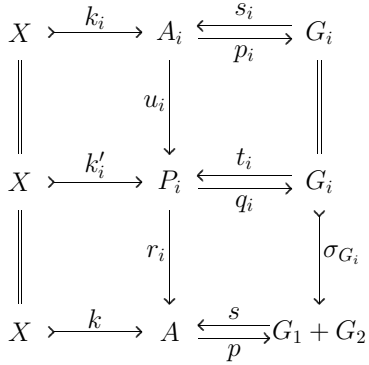
where thus $\theta_i = (k_i, s_i)$ and $\theta = (k, s)$.

Lemma 30. *Let $\mathbb{T}$ be a semi-abelian algebraic theory and X a bornological $\mathbb{T}$-algebra such that the actions on X are representable in* $\mathsf{Set}^{\mathbb{T}}$*. Given a "test diagram on X", the following conditions are equivalent:*

(1) *the central row is a split extension in* $\mathsf{Born}^{\mathbb{T}}$*;*
(2) *the bornology of X is that induced by the bornology of A;*
(3) *the two right-hand squares are pullbacks in* $\mathsf{Born}^{\mathbb{T}}$*;*
(4) *the bornology of A_i is induced by that of A.*

Proof: Of course if the central row is a split extension in $\mathsf{Born}^{\mathbb{T}}$, $k = \mathsf{Ker}\, p$ and X is provided with the bornology induced by that of A (see Corollary 3 and Proposition 10).

If the bornology of X is induced by that of A, consider the following diagram



where the lower right square is a pullback, $k_i' = \mathsf{Ker}\, q_i$ and u_i is the factorization of the commutative square $p\tau_i = \sigma_{G_i} p_i$ through this pullback. By the short five lemma, u_i is an isomorphism and thus the squares $p\tau_i = \sigma_{G_i} p_i$ are pullbacks.

Next if these squares are pullbacks and $U \subseteq A_i$ is such that $\tau_i(U)$ is bounded in A, then $\sigma_{G_i} p_i(U) = p\tau_i(U)$ is bounded in $G_1 + G_2$. But since σ_{G_i} admits a retraction (namely, the factorization of the identity on G_i and the zero morphism on G_j), σ_{G_i} is a regular monomorphism and thus G_i has the bornology induced by that of $G_1 + G_2$. And since $\sigma_{G_i} p_i(U)$ is bounded, so is $p_i(U)$. So $\tau_i(U)$ and $p_i(U)$ are bounded, so that U is bounded (see Proposition 2). This proves that A_i has the bornology induced by that of A.

Finally if A_i has the bornology induced by that of A, since X itself has the bornology induced by that of A_i, then X has the bornology induced by that of A. But in $\mathsf{Set}^{\mathbb{T}}$ $k = \mathsf{Ker}\, p$ and $p = \mathsf{Coker}\, k$, while in $\mathsf{Born}^{\mathbb{T}}$ k is now a regular monomorphism and p a regular epimorphism. Thus $k = \mathsf{Ker}\, p$ and $p = \mathsf{Coker}\, k$ in $\mathsf{Born}^{\mathbb{T}}$. ∎

Criterion 31. *Let $\mathbb{T}$ be a semi-abelian algebraic theory and X a bornological $\mathbb{T}$-algebra such that the actions on X are representable in $\mathsf{Set}^{\mathbb{T}}$. The following conditions are equivalent:*

(1) *the actions on X are representable in $\mathsf{Born}^{\mathbb{T}}$;*
(2) *for every "test diagram on X", the equivalent conditions of Lemma 30 are satisfied.*

Proof: Giving the top and bottom row of a "test diagram on X" is giving in $\mathsf{Born}^{\mathbb{T}}$ two elements in $\mathsf{SplEx}(G_i, X)$, $i = 1, 2$. By Theorem 28, we must prove that this corresponds to a unique element in $\mathsf{SplEx}(G_1 + G_2, X)$, that is to a middle row in $\mathsf{Born}^{\mathbb{T}}$, recapturing the other two rows by pullbacks in $\mathsf{Born}^{\mathbb{T}}$. By assumption on X, such a middle row exists and is unique in $\mathsf{Set}^{\mathbb{T}}$.

By Lemma 22, the only possibility for a bornology on A making the central row a split extension of X is precisely that as indicated. Let us first observe that, when A is provided with that bornology, all the arrows of the diagram are morphisms in $\mathsf{Born}^{\mathbb{T}}$, so that the diagram is a "test diagram" for X. Since $k = \theta\tilde{\sigma}_X$ and $s = \theta\tilde{\sigma}_{G_1+G_2}$, k and s are bounded. To prove that p is bounded, it suffices to prove that $p\theta$ is bounded. For that, it suffices further to prove that the composites with the canonical morphisms of the coproduct are bounded. And indeed

$$p\theta\tilde{\sigma}_X = pk = 0,\ p\theta\tilde{\sigma}_{G_1+G_2} = ps = \mathsf{id}_{G_1+G_2}$$

thus p is bounded. To prove that τ_i is bounded, it suffices to prove that $\tau_i\theta_i$ is bounded. This is the case since

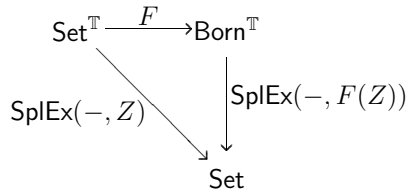
$$\tau_i \circ \theta_i = \theta \circ (\mathsf{id}_X + \sigma_{G_i}).$$

So, all the arrows of the diagram are bounded and we have a "test diagram" for X.

The equivalent conditions in Lemma 30 are now precisely the required condition for the representability of actions on X in $\mathsf{Born}^{\mathbb{T}}$ (see Theorem 28). ∎

Corollary 32. *Let $\mathbb{T}$ be a semi-abelian algebraic theory. The representability of actions in $\mathsf{Set}^{\mathbb{T}}$ is a necessary condition for the representability of actions in $\mathsf{Born}^{\mathbb{T}}$.*

Proof: Let us recall that the forgetful functor from $\mathsf{Born}^{\mathbb{T}} \longrightarrow \mathsf{Set}^{\mathbb{T}}$ is topological (see Proposition 10) and so admits a left adjoint F, namely, the functor which equips a $\mathbb{T}$-algebra Y with the bornology of finite subsets. As a left adjoint, this functor preserves colimits. Given a $\mathbb{T}$-algebra $Z \in \mathsf{Set}^{\mathbb{T}}$, we have the following situation



Observe that this diagram is commutative up to isomorphism, that is

$$\mathsf{SplEx}\big(F(H), F(Z)\big) \cong \mathsf{SplEx}(H, Z)$$

for every $\mathbb{T}$-algebra $H \in \mathsf{Set}^{\mathbb{T}}$. Indeed given a split extension in $\mathsf{Set}^{\mathbb{T}}$

$$Z \rightarrowtail C \underset{}{\overset{\longleftarrow}{\longrightarrow}} H$$

and writing $\mathcal{F}$ for the bornologies of finite subsets,

$$F(Z) = (Z, \mathcal{F}_Z) \rightarrowtail (C, \mathcal{F}_C) \underset{}{\overset{\longleftarrow}{\longrightarrow}} (H, \mathcal{F}_H) = F(H)$$

is a split extension in $\mathsf{Born}^{\mathbb{T}}$ and, by Lemma 22, this is the only possible one.

But we know that F preserves colimits. When $\mathsf{Born}^{\mathbb{T}}$ is action representative, $\mathsf{SplEx}(-, F(Z))$ is representable thus preserves colimits, so that the composite $\mathsf{SplEx}(-, X)$ preserves colimits as well. Therefore $\mathsf{Set}^{\mathbb{T}}$ is action representative (see Theorem 20). ∎

6. A necessary and a sufficient condition for the representability of actions

Let us first observe an obvious necessary condition.

Proposition 33. *Let $\mathbb{T}$ be a semi-abelian theory such that actions in $\mathsf{Born}^{\mathbb{T}}$ are representable. In that case, given two split extensions in $\mathsf{Born}^{\mathbb{T}}$*

$$X \overset{k_i}{\rightarrowtail} A_i \underset{p_i}{\overset{s_i}{\longleftrightarrow}} G_i,\ i = 1, 2$$