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CONTINUITY OF LOCALIC MAPS

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ABSTRACT: Mending the contravariance of the natural point-free representation of classical spaces and continuous maps one replaces the category of frames by its dual $\mathbf{Loc} = \mathbf{Frm}^{\rm op}$. To make $\mathbf{Loc}$ a concrete category one can replace frame homomorphisms by their right Galois adjoints (called then *localic maps*). This rather formal representation of generalized continuous maps turns out to be surprisingly geometrically satisfactory: we prove that a localic map is characterized among plain maps between underlying sets in terms of preserving closed and open subobjects by preimage. This is, a.o., another justification of defining open localic maps as those with open images (images being understood as the standard set images of plain maps) of open subobjects. We add a few remarks on the openness and completeness of localic maps.

KEYWORDS: Frame, locale, sublocale, coframe of sublocales, localic map, image, preimage, open map, complete Heyting homomorphism.

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Introduction

Classical topology is connected with the point-free one by the functor

$$\Omega\colon \mathbf{Top} \to \mathbf{Frm}$$

associating with a space X the frame (a special lattice, see 1.3) $\Omega(X)$ of open sets, and with a continuous map $f\colon X \to Y$ the frame homomorphism $\Omega(f) = (U \mapsto f^{-1}[U])\colon \Omega(Y) \to \Omega(X)$. Viewing point-free topology as an extension of the classical theory is justified (a.o.) by the fact that for a very wide class of spaces (the sober[1] ones), frame homomorphisms $h\colon \Omega(Y) \to \Omega(X)$ are precisely the $\Omega(f)$'s. Thus, Ω embeds a substantial part of the category $\mathbf{Top}$ in that of frames as a full subcategory. It does it contravariantly, but this

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[1] A space is *sober* (see [3]) if it is T_0 and $X \smallsetminus \overline{\{x\}}$ are the only prime elements (that is $U \neq X$ such that $U = V \cap W$ only if $U = V$ or $U = W$) in $\Omega(X)$; for instance every Hausdorff space is sober.

is easily mended by replacing the category **Frm** by its dual $\mathbf{Loc} = \mathbf{Frm}^{\mathrm{op}}$. Working with the formal category of "inverted arrows" has its drawbacks, but, luckily enough, a frame homomorphisms $h\colon L \to M$, as a mapping preserving all suprema, is naturally and uniquely associated with a mapping heading in the opposite direction, namely its right Galois adjoint $h_*\colon M \to L$. Thus we can represent **Loc** as a concrete category of *localic maps*, special meet-preserving maps (special because we have to take into account that not every *join-preserving* map between frames is a homomorphism – see 1.5.1 below).

This turns out to be technically very expedient, but one may wonder how much of the original geometric contents is left after such two rather formal representation steps (inverting arrows, and then replacing special join-preserving maps by their adjoints). Are we not rather far from the original continuity as a geometric phenomenon? Can we still think of the resulting localic maps as having something to do with continuity? In this note we will show that we can.

Frames have natural subobjects called *sublocales*, special subsets representing generalized subspaces. Among them we have the open and closed ones exactly extending the notions of open and closed sublocales of spaces (see 1.5.2, 1.6). We will prove that localic maps between frames are precisely those mappings between the underlying sets that preserve closedness and openness by preimages. This is fully in parallel with characterizing continuous maps. The fact that for those it suffices to require only one of the two is because the second comes for free: the preimages preserve complements; since the closed and open sublocales are not set-theoretic complements and we wish to formulate the results strictly in set-theoretic maps and preimages, we have to assume both explicitly (although preserving openness comes in a relaxed form[2] – see 1.3).

The paper is organized as follows. After Preliminaries containing necessary formal definitions and basic facts we devote Section 2 to comparing the image and preimage functions in set-theoretical and localic setting. The main result is presented in Section 3. In this perspective we now see how natural is the definition of open localic maps. They, and the related complete localic maps are then discussed in the last, fourth section.

[2]In fact by necessity: we wish to formulate the final result consistently in the language of underlying sets, plain maps and set-theoretic preimages that preserve openness in a relaxed way only.

1. Preliminaries

1.1. Posets. For a subset A of a poset $(X, \leq)$ we write

$$\uparrow A = \{x \in X \mid x \geq a \text{ for some } a \in A\}.$$

A join (supremum) of $A \subseteq (X, \leq)$ will be denoted by $\bigvee A$, and we write $a \vee b$ for $\bigvee\{a, b\}$; similarly we write $\bigwedge A$ and $a \wedge b$ for infima.

The smallest resp. largest element in a poset will be denoted by 0 resp. 1.

Our posets will be mostly complete lattices (that is, for each $A \subseteq (X, \leq)$ the join and meet will be supposed to exist).

1.2. Adjoint maps. If X, Y are posets we say that monotone maps $f\colon X \to Y$, $g\colon Y \to X$ are *adjoint, f to the left* and *g to the right* if

$$f(x) \leq y \quad \text{iff} \quad x \leq g(y).$$

Recall that this is characterized by $fg(y) \leq y$ and $x \leq gf(x)$, that left adjoints preserve all the existing suprema and the right ones preserve the infima. On the other hand,

1.2.1. *if X, Y are complete lattices then an $f\colon X \to Y$ preserving all suprema* (*resp. a $g\colon Y \to X$ preserving all infima*) *has a right* (*resp. left*) *adjoint.*

1.3. Frames and coframes. A *frame*, resp. *coframe*, is a complete lattice L satisfying the distributivity law

$$(\bigvee A) \wedge b = \bigvee\{a \wedge b \mid a \in A\}, \tag{frm}$$

$$\text{resp.} \quad (\bigwedge A) \vee b = \bigwedge\{a \vee b \mid a \in A\}, \tag{cofrm}$$

for all $A \subseteq L$ and $b \in L$; a *frame* (resp. *coframe*) *homomorphism* preserves all joins and all finite meets (resp. all meets and all finite joins). The lattice $\Omega(X)$ of all open subsets of a topological space X is an example of a frame, and if $f\colon X \to Y$ is continuous we obtain a frame homomorphism $\Omega(f)\colon \Omega(Y) \to \Omega(X)$ by setting $\Omega(f)(U) = f^{-1}[U]$. Thus we have a functor Ω from the category of topological spaces into that of frames.

Synonymously, we speak about frames as of *locales*, in particular when we think of them as of generalized spaces (see Introduction and 1.5 below)

1.4. The Heyting structure. The equality (frm) states that the maps $(x \mapsto x \wedge b)\colon L \to L$ preserve all joins. Hence, by 1.2.1, every frame is a Heyting algebra with the Heyting operation $\to$ satisfying

$$a \wedge b \leq c \quad \text{iff} \quad a \leq b \to c. \tag{$*$}$$

1.4.1. By 1.2 the mapping $b \to (-)$ preserves meets,

$$b \to (\bigwedge_i x_i) = \bigwedge_i (b \to x_i)$$

and since $a \le b \to c$ iff $a \wedge b \le c$ iff $b \le a \to c$, that is, $a \to c \le^{\mathrm{op}} b$ we also have that $(-) \to c$ sends joins to meets, that is,

$$(\bigvee_i x_i) \to c = \bigwedge_i (x_i \to c).$$

1.4.2. A few more Heyting rules. We will use some simple rules, all of them very easy consequences of $(*)$ (see also, e.g., [12, 11]):

(a) $a \le b \to a$,
(b) $a \le b$ iff $a \to b = 1$,
(c) $a \wedge (a \to b) = a \wedge b$,
(d) $a \to (b \to c) = (a \wedge b) \to c$,
(e) $a \to b = a \to c$ iff $a \wedge b = a \wedge c$.

1.5. The concrete category Loc. Recall the Introduction. The *category* **Loc** = **Frm**$^{\mathrm{op}}$ *of locales* is represented as a concrete category with morphisms the *localic maps* $f\colon L \to M$ represented as the right adjoints of frame homomorphisms.

1.5.1. A characteristic of localic maps. Localic maps are adjoints to maps preserving all joins and furthermore also finite meets. Hence they are specific meet preserving maps. In among the maps preserving all meets they are characterized by the conditions (f^* is the left Galois adjoint of f)

(lm1) $f(a) = 1 \;\Rightarrow\; a = 1$, and
(lm2) $f(f^*(a) \to b) = a \to f(b)$

(see e.g. [12]). An equation of the type (lm2) is often referred to as the *Frobenius identity* (see e.g. [1, 9]); we will follow this convention.

1.5.2. Sublocales. Subobjects of locales (generalizing the concept of subspaces of topological spaces) are the subsets $S \subseteq L$ such that the embedding $j\colon S \subseteq L$ is a localic map.[3] They are characterized by the conditions

(S1) $M \subseteq S$ implies $\bigwedge M \in S$, and
(S2) if $a \in L$ and $s \in S$ then $a \to s \in S$.

[3]This naturally follows from standard categorical reasoning: we wish for extremal monomorphism and those are one-to-one localic maps because extremal epimorphisms in **Frm** are precisely the onto frame homomorphisms.

Note. We use the terms "frame" and "locale" interchangeably and hence we do not hesitate to speak of a "sublocale of a frame". But it is important not to confuse "sublocales" and "subframes": the latter are subalgebras of a frame in the original definition, that is, an algebra with the operations $\bigvee$, $\wedge$ and 1.

1.6. The coframe of sublocales. (See e.g. [6, 12, 10].) The set of all sublocales ordered by inclusion, denoted by $\mathsf{S}(L)$, is a co-frame, with the lattice operations

$$\bigwedge_{i\in J} S_i = \bigcap_{i\in J} S_i \quad \text{and} \quad \bigvee_{i\in J} S_i = \{\bigwedge A \mid A \subseteq \bigcup_{i\in J} S_i\}.$$

The top of $\mathsf{S}(L)$ is L and the bottom is the set $\mathsf{O} = \{1\}$ (the *empty sublocale*).

1.6.1. Closed and open sublocales. We have closed resp. open sublocales

$$\mathfrak{c}(a) = {\uparrow}a \quad \text{resp.} \quad \mathfrak{o}(a) = \{x \mid a \to x = x\} = \{a \to x \mid x \in L\}$$

modeling closed resp. open subspaces (and corresponding precisely to the *closed* resp. *open parts* in [4]). They are complements of each other, and we have (see e.g. [12]):

- $\mathfrak{o}(0) = \mathsf{O},\ \mathfrak{o}(1) = L,\ \mathfrak{o}(a \wedge b) = \mathfrak{o}(a) \cap \mathfrak{o}(b),\ \mathfrak{o}(\bigvee a_i) = \bigvee \mathfrak{o}(a_i)$,
- $\mathfrak{c}(0) = L,\ \mathfrak{c}(1) = \mathsf{O},\ \mathfrak{c}(a \wedge b) = \mathfrak{c}(a) \vee \mathfrak{c}(b),\ \mathfrak{c}(\bigvee a_i) = \bigcap \mathfrak{c}(a_i)$.

Similarly like in spaces we have the *closure* $\overline{S}$, the smallest closed sublocale containing S, and the *interior* $\operatorname{int} S$, the largest open sublocale contained in S. There is a particularly simple formula for the closure, namely $\overline{S} = {\uparrow} \bigwedge S$.

1.6.2. Boolean sublocales. By (S2) each sublocale containing a has to contain

$$\mathfrak{b}(a) = \{x \to a \mid x \in L\}.$$

By the second identity in 1.4.1, $\bigwedge(x_i \to a) = (\bigvee x_i) \to a$, hence $\mathfrak{b}(a)$ also satisfies (S1) and we see it is a sublocale, the smallest sublocale containing a.

1.7. The Boolean part. The *Boolean part* of a distributive lattice L is the sublattice consisting of all the complemented elements, that is, of the $x \in L$ such that there is a $y \in L$ with $x \wedge y = 0$ and $x \vee y = 1$.

2. Images and preimages

The main fact we will prove in Section 3 below is a criterion choosing localic maps $L \to M$ in among the *plain maps* between the underlying sets of L and M as those that preserve closed and open sublocales by preimages, quite analogous to choosing continuous maps as those preserving closed (and open) subset by (standard set) preimages.[4] Getting ready for that we present here a short section on images and preimages in the localic context. The reader should note that although we will need a "localic preimage" in a technical step in the proofs, the final result will be strictly in terms of

– sublocales as subsets,
– set preimages,
– and maps as a priori non-structured ones between the carriers.

We will use the definitions from [12] and for the reader's convenience present the proofs of 2.2 and 2.4. The subsection 2.5 is new, though.

2.1. Lemma. *Let L, M be frames (more generally, complete lattices). Let $f\colon L \to M$ be a right adjoint (that is, let it preserve meets). If $S \subseteq L$ and $T \subseteq M$ are closed under meets then so are $f[S]$ and $f^{-1}[T]$.*

Proof: It is straightforward. ∎

2.2. Proposition. *Let L, M be frames and let $f\colon L \to M$ be a localic map adjoint to a frame homomorphism $h\colon M \to L$. Then:*

(1) *If $S \subseteq L$ is a sublocale then $f[S]$ is a sublocale of M;*
(2) *If $\mathfrak{c}(a) = {\uparrow}a$ is a closed sublocale of M then $f^{-1}[\mathfrak{c}(a)] = \mathfrak{c}(h(a))$.*

Proof: (1) If $s \in S$ and $x \in M$, we have by 1.5.1 $x \to f(s) = f(h(x) \to s)$ and $(h(x) \to s) \in S$.

(2) We have $f(x) \in \mathfrak{c}(a)$, that is, $f(x) \geq a$, and by the adjunction this is iff $x \geq h(a)$, that is, $x \in \mathfrak{c}(h(a))$. ∎

2.3. Localic preimage. Let A be a subset of a frame L closed under meets. Recalling the formula for a join of sublocales in 1.6 we see that there is a

[4] Set preimages preserve the set complements, but not the complements of sublocales in $S(L)$ resp. $S(M)$. Hence, in the standard criterion of continuity of maps between topological spaces requiring that preimages of closed sets are closed, we obtain preserving open sets for free; to preserve open sublocales by (set) preimages we will have to add an explicit condition concerning preimages of open sublocales.

largest sublocale of L contained in A, namely the join $\bigvee\{S \in \mathsf{S}(L) \mid S \subseteq A\}$; it will be denoted by

$$A_{\mathsf{sl}}.$$

In particular this holds for the preimages of sublocales. We will write

$$f_{-1}[S] = f^{-1}[S]_{\mathsf{sl}}$$

and speak of the *localic preimage* (as opposed to the *set preimage* $f^{-1}[S]$).

More generally, for any meet preserving f and any S closed under meets, $A = f^{-1}[S]$ is closed under meets and hence we have a sublocale $f_{-1}[S] = f^{-1}[S]_{\mathsf{sl}}$. We will use this notation in this more general context and to avoid confusion state that "the localic preimage $f_{-1}[S]$ makes sense".

2.3.1. From 1.6.2 we immediately infer that $a \in f_{-1}[S]$ iff $\mathfrak{b}(a) \subseteq f^{-1}[S]$, that is, iff for every $x \in L$, $f(x \to a) \in S$.

2.3.2. Notes. (1) From the standard set-theoretic adjunction between image and preimage we readily infer that we also have an adjunction

$$\begin{tikzcd} \mathsf{S}(L) \arrow[r, shift left, "f[-]"] \arrow[r, phantom, "\perp"] & \mathsf{S}(M), \arrow[l, shift left, "f_{-1}[-]"] \end{tikzcd}$$

that is, $f[S] \subseteq T$ iff $S \subseteq f_{-1}[T]$ for sublocales $S \subseteq L$ and $T \subseteq M$. In particular, $f[-]$ preserves joins and $f_{-1}[-]$ preserves meets (intersections) of sublocales.

(2) In fact, $f_{-1}[-]$ is a coframe homomorphisms and hence $f[-]$ is a colocalic map; this is, however, not quite so easy to prove.

2.4. Proposition. *Localic preimages of open sublocales are open. More precisely, we have*

$$f_{-1}[\mathfrak{o}(a)] = \mathfrak{o}(h(a))$$

where h is the adjoint frame homomorphism.

Proof: I. $\mathfrak{o}(h(a)) \subseteq f^{-1}[\mathfrak{o}(a)]$: For a general element $h(a) \to x$ of $\mathfrak{o}(h(a))$ we have by 1.5.1 $f(h(a) \to x) = a \to f(x) \in \mathfrak{o}(a)$.

II. Now let S be an arbitrary sublocale contained in $f^{-1}[\mathfrak{o}(a)]$; we will show that $S \subseteq \mathfrak{o}(h(a))$.

Let $s \in S$ be arbitrary. Set $b = h(a)$. We have $(b \to s) \to s \in S$ and hence $f((b \to s) \to s) \in \mathfrak{o}(a)$ so that, using 1.4.1 (d) and (c) we compute

$$\begin{aligned} f((b \to s) \to s) &= a \to f((b \to s) \to s) = f(h(a) \to ((b \to s) \to s)) = \\ &= f(b \to ((b \to s) \to s)) = f((b \wedge (b \to s)) \to s) = \\ &= f((b \wedge s) \to s) = f(1) = 1 \end{aligned}$$

and since for a localic map f, $f(x) = 1$ only if x =1 (recall (lm1) in 1.5.1) we see that $(b \to s) \to s = 1$, but then by 1.4.1(d), $(b \to s) \le s$ and since always $s \le b \to s$ (see 1.4.1(a)) we conclude that $s \in \mathfrak{o}(b)$. ■

2.5. Localic preimage versus set preimage. In 2.2 we saw that for closed sublocales the set and localic preimages coincide. This is an exception: for other sublocales they differ (and hence we have to keep in mind that the set preimages of sublocales we will have to work with later are typically not sublocales).

2.5.1. Proposition. *Let S be a sublocale of L that is not closed. Then there is a localic map $f\colon M \to L$ such that $f_{-1}[S] \neq f^{-1}[S]$.*

More specifically, one has $f_{-1}[S] \neq f^{-1}[S]$ for any f adjoint to a frame embedding $h\colon L \to M$ such that $h[L]$ is contained in the Boolean part of M.

Proof: Consider a frame embedding $h\colon L \to M$ such that $h[L]$ is contained in the Boolean part of M, for instance the standard embedding of L into $M = \mathsf{S}(L)^{\mathrm{op}}$. Since h is one-to-one we have for the adjoint localic map f

$$\forall x \in L, \quad f(h(x)) = x.$$

Let S not be closed. Then there is a $b > a = \bigwedge S$ such that $b \notin S$.

Suppose $f_{-1}[S] = f^{-1}[S]$ so that $f^{-1}[S]$ is a sublocale. Since $f(h(a)) = a \in S$, we have $h(a) \in f^{-1}[S]$ and consequently $x \to h(a) \in f^{-1}[S]$ for any x. Take $x = h(b)^*$. This element is in the Boolean part, that is, it is complemented, and hence $x \to h(a) = x^* \vee h(a)$, so

$$h(b) = h(b \vee a) = h(b) \vee h(a) = x \to h(a) \in f^{-1}[S],$$

and finally $b = f(h(b)) \in S$, a contradiction. ■

2.5.2. Corollary. *Let S be a sublocale of L. Then $f_{-1}[S] = f^{-1}[S]$ for all localic maps $f\colon M \to L$ if and only if S is closed.*

3. Localic maps as continuous ones

3.1. Monotone maps. It is a standard fact that for the quasi-discrete (Alexandroff) spaces associated with posets (the topology $\tau(\leq)$ consisting of all the up-sets in $(X, \leq)$)

> *the continuous maps* $f\colon (X, \tau(\leq_X)) \to (Y, \tau(\leq_Y))$ *are precisely the monotone maps* $f\colon (X, \leq_X) \to (Y, \leq_Y)$.

Thus, since the preimage preserves unions, monotone maps are precisely those $f\colon X \to Y$ such that

> *for each* $b \in Y$, $f^{-1}[\uparrow b]$ *is an up-set.*

3.2. Proposition. *Let* L, M *be complete lattices. Let* $f\colon L \to M$ *be such that*

$$\forall\ b \in M\ \exists a \in L,\ \textit{such that } f^{-1}[\uparrow b] = \uparrow a.$$

Then f *preserves meets.*

Proof: First, by 3.1 the map f is monotone. Now the a in $f^{-1}[\uparrow b] = \uparrow a$ is obviously uniquely determined; let us denote it by $\phi(b)$. The equality $\uparrow\phi(b) = f^{-1}[\uparrow b]$ can be rewritten as

$$\phi(b) \leq x \quad \text{iff} \quad b \leq f(x). \tag{$*$}$$

Realizing that ϕ is monotone (if $b \leq b'$ we have $\uparrow b \supseteq \uparrow b'$ and hence $\uparrow\phi(b) \supseteq \uparrow\phi(b')$ and $\phi(b) \leq \phi(b')$) we conclude that $(*)$ makes f a right Galois adjoint, hence a mapping preserving all meets. ■

3.3. Meet-preserving maps. Speaking more generally of the $\uparrow a$ as of the closed subobjects in arbitrary complete lattices we obtain

3.3.1. Corollary. *A map* $f\colon L \to M$ *between complete lattices preserves all meets iff the preimages of closed sets are closed.*

Proof: By 3.2 it suffices to prove that if f preserves meets then the preimages of closed sets are closed. Such an f has a left adjoint $\phi\colon M \to L$ and we have

$$x \in f^{-1}[\uparrow b] \quad \text{iff} \quad f(x) \geq b \quad \text{iff} \quad x \geq \phi(b) \quad \text{iff} \quad x \in \uparrow\phi(b),$$

that is, $f^{-1}[\uparrow b] = \uparrow\phi(b)$. ■

3.4. Theorem. *Let* L, M *be frames. Then* $f\colon L \to M$ *is a localic map iff*

> $f^{-1}[\mathsf{O}] = \mathsf{O}$,
> *for every closed* A, $f^{-1}[A]$ *is closed, and*
> *for every open* U, $f^{-1}[U] \supseteq f^{-1}[U^{\mathsf{c}}]^{\mathsf{c}}$.

($-^{\mathsf{c}}$ *stands for complementation.*)

Proof: I. If f is a localic map then the conditions are satisfied by 2.2 (taking also into account that $f_{-1}[-]\colon \mathsf{S}(M) \to \mathsf{S}(L)$ preserves complements – recall 2.3.2): If U is open then U^{c} is closed and

$$f^{-1}[U] \supseteq f_{-1}[U] = f_{-1}[U^{\mathsf{c}}]^{\mathsf{c}} = f^{-1}[U^{\mathsf{c}}]^{\mathsf{c}},$$

and

$$f^{-1}[\mathsf{O}] = f_{-1}[\mathsf{O}] = f_{-1}[M^{\mathsf{c}}] = L^{\mathsf{c}} = \mathsf{O}.$$

II. Now let the conditions hold. By 3.2 we already know that f is a right adjoint to ϕ. Hence we have to prove that

(a) $f(a) = 1$ only for $a = 1$, and that
(b) $f(\phi(a) \to x) = a \to f(x)$.

The first is in $f^{-1}[\mathsf{O}] = \mathsf{O}$.

Now consider a $B = {\uparrow}a$ so that $B^{\mathsf{c}} = \mathfrak{o}(a)$. Thus, $A = f^{-1}[B] = {\uparrow}\phi(a)$ and by the assumption we have $\mathfrak{o}(\phi(a)) \subseteq f^{-1}[\mathfrak{o}(a)]$. Consequently

$$f(\phi(a) \to x) = a \to y \tag{$*$}$$

for some y and we have to prove that $a \to y = a \to f(x)$, and hence, by 1.4.1(e), that

$$a \wedge y = a \wedge f(x).$$

$\leq$: Trivially, $f(x) \leq f(\phi(a) \to x) = a \to y$, hence $a \wedge f(x) \leq y$ and hence $a \wedge f(x) \leq a \wedge y$.
$\geq$: Using the adjunction inequality $\mathrm{id} \geq \phi f$, $(*)$ and by 1.4.1(a)

$$\begin{aligned}\phi(a) \to x \geq \phi(f(\phi(a) \to x)) = \phi(a \to y) = \\ = \phi(\textstyle\bigvee\{u \mid u \wedge a \leq y\} = \bigvee\{\phi(u) \mid u \wedge a \leq y\},\end{aligned}$$

hence $\bigvee\{\phi(u) \mid u \wedge a \leq y\} \leq \phi(a) \to x$ and so by 1.4.1(c)

$$\begin{aligned}\textstyle\bigvee\{\phi(a) \wedge \phi(u) \mid u \wedge a \leq y\} = \phi(a) \wedge \bigvee\{\phi(u) \mid u \wedge a \leq y\} \leq \\ \leq \phi(a) \wedge (\phi(a) \to x) \leq x.\end{aligned}$$

Consequently, $\phi(a \wedge y) \leq \phi(a) \wedge \phi(y) \leq x$ and finally $a \wedge y \leq f(x)$ and $a \wedge y \leq a \wedge f(x)$. ∎

3.5. Localic maps. Once we know that the preimages of closed sublocales are closed, we know by 3.3 that the localic preimages of sublocales make sense (recall 2.3). Hence we can translate the previous theorem as follows.

Proposition. *Let L, M be frames. Then $f\colon L \to M$ is a localic map iff*

for every closed A, $f^{-1}[A]$ is closed, and
for every open U, $f_{-1}[U] = f^{-1}[U^{\mathsf{c}}]^{\mathsf{c}}$ (and hence it is open).

Proof: The implication $\Rightarrow$ follows from the properties of the preimage as recalled in Section 2.

On the other hand if the conditions hold we have

$$f^{-1}[\mathsf{O}] = f_{-1}[\mathsf{O}] = f_{-1}[M^*] = f^{-1}[M]^* = L^* = \mathsf{O},$$

and

$$f^{-1}[U] \supseteq f_{-1}[U] = f^{-1}[U^{\mathsf{c}}]^{\mathsf{c}}. \quad \blacksquare$$

3.6. A more symmetric variant. The question naturally arises whether Theorem 3.4 has a "more symmetric variant" in which also the condition on the preimages of closed sets is relaxed. This we can prove for monotone maps f only.

Let us analyze the following condition considering a map $f\colon L \to M$ between locales.

$$\forall \text{ closed } A \subseteq M \ \ \exists \text{ largest sublocale } B \subseteq f^{-1}[A], \text{ and it is closed.} \qquad (*)$$

For $A = \mathfrak{c}(a)$ we have, then, $B = \mathfrak{c}(b)$ with obviously unique b which we will denote by $\phi(a)$. Thus, $\phi(a)$ is the smallest $y \in L$ with $\uparrow y \subseteq f^{-1}[\uparrow a]$, that is,

$$y \geq \phi(a) \quad \Rightarrow \quad f(y) \geq (a).$$

3.6.1. Lemma. *For any f satisfying $(*)$, ϕ is monotone and* $\mathrm{id} \leq f\phi$, *and f is monotone iff $\phi f \leq$* id.

Proof: I. If $a_1 \leq a_2$ then $\uparrow\phi(a_2) \subseteq f^{-1}[\uparrow a_2] \subseteq f^{-1}[\uparrow a_1]$ and hence, by $(*)$, $\uparrow\phi(a_2) \subseteq \uparrow\phi(a_1)$ so that $\phi(a_1) \leq \phi(a_2)$. Furthermore, from $\phi(a) \geq \phi(a)$ it follows that $f\phi(a) \geq a$.

II. Now if f is monotone then by the definition of ϕ, $\uparrow b \subseteq f^{-1}[\uparrow f(b)]$. On the other hand, if $\phi f \leq \mathrm{id}$ and $b_1 \leq b_2$ in L then $b_2 \geq b_1 \geq \phi f(b_1)$ and hence $f(b_2) \geq f(b_1)$. $\blacksquare$

From 3.6.1 and 3.3 we now obtain

3.6.2. Corollary. *If $f\colon L \to M$ satisfies $(*)$ then it is monotone iff ϕ is its left Galois adjoint. In particular, if it is monotone then $f^{-1}[\mathfrak{c}(a)] = \mathfrak{c}(\phi(a))$ for every $a \in M$.*

Now we can infer from 3.4 the following

3.6.3. Proposition. *Let $f\colon L \to M$ be a monotone map between locales. Then it is localic iff for every closed S, $f^{-1}[S]$ has the largest sublocale S' contained in it and S' is closed, and for every open S, $f^{-1}[S]$ contains the complement of $(S^{\mathsf{c}})'$.*

3.6.4. Note. In fact we have used less, namely that for a closed S, $f^{-1}[S]$ has a *largest closed sublocale* (a largest *general* sublocale need not be assumed).

4. Complete and open localic maps

We have seen that localic maps behave with respect to open and closed sublocales and preimages quite in parallel with the behavior of continuous maps, open and closed subsets and preimages. Consequently also the definition of an open localic map as a localic map preserving openness of sublocales by images, which might have been taken for just a formal analogy becomes more convincing. In this section we will compare the open and complete localic maps and present a proof of Joyal-Tierney Theorem ([8]) using the sublocalic technique.

4.1. Notation. $f\colon L \to M$ will be a localic map adjoint to a frame homomorphism $h\colon M \to L$. The left adjoint to h, if it exists, that is, if h is complete, will be denoted by the standard h^*. But often it will be a map obtained from an assumption or from a construction about which the adjunction property will be shown during a proof; then we will use an ad hoc symbol.

4.2. Completeness without Heyting. Joyal-Tierney Theorem, that we will discuss in 4.4.2 below, claims in our notation that every $f[\mathfrak{o}(a)]$ is open iff h is a complete Heyting homomorphism. It is also known that for a wide range of frames (at least for the subfit ones), but not always, this is equivalent with just completeness. We start this section with a necessary and sufficient condition for the completeness without the Heyting property.

4.2.1. Theorem. *The left adjoint h of a localic map f is complete iff for every open $U \subseteq L$ there is a unique minimal open $V \subseteq M$ such that $f[U] \subseteq V$.*

Morover, the mapping $a \mapsto b$ defined by the minimal $\mathfrak{o}(b)$ containing $f[\mathfrak{o}(a)]$ is the left adjoint to h.

Proof: $\Rightarrow$: Let h be complete. Set $U = \mathfrak{o}(a)$ and $b = h^*(a)$. Then $a \le h(b)$ and hence $\mathfrak{o}(a) \subseteq \mathfrak{o}(h(b))$. We will show first that $f[\mathfrak{o}(h(b)] \subseteq \mathfrak{o}(b)$. Indeed, take an element $y = h(b) \to x$ of $\mathfrak{o}(h(b)$. For its image $f(y)$ we obtain, using 1.4.1,

$$b \to f(y) = b \to f(h(b) \to x) = b \to (b \to f(x)) = \\ = (b \wedge b) \to f(x) = b \to f(x) = f(h(b) \to x) = f(y),$$

hence $f(y) \in \mathfrak{o}(b)$ and $f[\mathfrak{o}(a)] \subseteq f[\mathfrak{o}(h(b))] \subseteq \mathfrak{o}(b)$. This $\mathfrak{o}(b)$ is, moreover, the smallest such open sublocale: If $f[\mathfrak{o}(a)] \subseteq \mathfrak{o}(c)$ then $\mathfrak{o}(a) \subseteq f_{-1}[\mathfrak{o}(c)] = \mathfrak{o}(h(c)$, hence $a \le h(c)$ and finally $b = h^*(a) \le c$, and $\mathfrak{o}(b) \subseteq \mathfrak{o}(c)$.

$\Leftarrow$: Assume that for each $a \in L$ there is a unique minimal $\mathfrak{o}(b)$ with $f[\mathfrak{o}(a)] \subseteq \mathfrak{o}(b)$. Set $\phi(a) = b$. Hence $\phi(a) \le x$ iff $\mathfrak{o}(\phi(a)) \subseteq \mathfrak{o}(x)$ iff $f[\mathfrak{o}(a)] \subseteq \mathfrak{o}(x)$ iff $\mathfrak{o}(a) \subseteq f_{-1}[\mathfrak{o}(x)] = \mathfrak{o}(h(x)$ iff $a \le h(x)$. Thus, ϕ is a left adjoint of h. ■

4.2.2. Notes. A sublocale S is *complete* if the homomorphism adjoint to the embedding $j\colon S \subseteq L$ is complete. One of the properties of this concept studied in [2] now immediately follows. We have that

a sublocale $S \subseteq L$ is complete iff for every $\mathfrak{o}_S(a)$ open in S, $\mathfrak{o}(a)$ is the smallest open extension in L.

4.3. Frobenius identities for the Heyting property. Let h be the frame homomorphism adjoint to a localic map f and let it be complete. Then we have

Theorem. *The following statements are equivalent.*

(1) *h preserves the Heyting operation.*
(2) *For all $a \in L$ and $b \in M$, $h^*(a \wedge h(b)) = h^*(a) \wedge b$.*
(3) *For all $a \in L$ and $b \in M$, $f(a \to h(b)) = h^*(a) \to b$.*

Proof: Use 1.4.2. For (2), we have in case of a Heyting map $h^*(a \wedge h(b)) \le x$ iff $a \wedge h(b) \le h(x)$ iff $a \le h(b) \to h(x) = h(b \to x)$ iff $h^*(a) \le b \to x$ iff $h^*(a) \wedge b \le x$ and if we have the identity we obtain $x \le h(a) \to h(b)$ iff $x \to h(a) \le h(b)$ iff $h^*(x) \wedge b = h^*(x \to h(a)) \le b$ iff $h^*(x) \le a \to b$ iff $x \le h(a \to b)$.

Similary for (3) we have $y \le f(a \to h(b))$ iff $h(y) \le a \to h(b)$ iff $a \le h(y) \to h(b) = h(y \to b)$ iff $h^*(a) \le y \to b$ iff $y \le h^*(a) \to b$, and from the identity we have the Heyting property computing $y \leq h(a) \to h(b)$ iff $h(a) \leq y \to h(b)$ iff $a \leq f(y \to h(b)) = h^*(y) \to b$ iff $h^*(y) \leq a \to b$ iff $y \leq h(a \to b)$. ■

4.4.1. Lemma. *For a localic map $f\colon L \to M$ we have*

$$f[\mathfrak{o}(a) \cap \mathfrak{o}(h(b))] = f[\mathfrak{o}(a)] \cap \mathfrak{o}(b).$$

Proof: If $a \to x = x$ and $h(b) \to x = x$ then by 1.5.1

$$b \to f(x) = f(h(b) \to x) = f(x),$$

hence the inclusion $\subseteq$. Now if $y \in f[\mathfrak{o}(a)] \cap \mathfrak{o}(b)$ we have $y = f(a \to x)$ and $b \to y = y$, hence

$$\begin{aligned} y = b \to f(a \to x) = f(h(b) \to (a \to x)) = \\ = f((h(b) \wedge a) \to x) \in f[\mathfrak{o}(h(b) \wedge a)] = f[\mathfrak{o}(h(b)) \cap \mathfrak{o}(a)]. \end{aligned}$$ ■

4.4.2. Theorem. (*Joyal and Tierney* [8]) *A localic map $f\colon L \to M$ is open* (*that is, for every open sublocale U in L, $f[U]$ is open in M*) *iff the associated frame homomorphism h is a complete Heyting homomorphism.*

Proof: $\Rightarrow$: By 4.2.1 h is complete and has a left adjoint ϕ with $f[\mathfrak{o}(a)] \subseteq \mathfrak{o}[\phi(a)]$. By the minimality of $\mathfrak{o}[\phi(a)]$ in 4.2.1 and by the openness of f, we have $f[\mathfrak{o}(a)] = \mathfrak{o}[\phi(a)]$. Now, by Lemma 4.4.1, we have

$$\begin{aligned} \mathfrak{o}(\phi(a \wedge h(b))) = f[\mathfrak{o}(a \wedge h(b))] = f[\mathfrak{o}(a) \cap \mathfrak{o}(h(b))] = \\ = f[\mathfrak{o}(a)] \cap \mathfrak{o}(b) = \mathfrak{o}(\phi(a)) \cap \mathfrak{o}(b) = \mathfrak{o}(\phi(a) \wedge b), \end{aligned}$$

hence $\phi(a \wedge h(b)) = \phi(a) \wedge b$ and h preserves the Heyting operation by 4.3(2).

$\Leftarrow$: It suffices to check that $f[\mathfrak{o}(a)] = \mathfrak{o}[\phi(a)]$ for all a. Using 4.3(3) we obtain for any $x \in L$

$$\phi(a) \to f(a \to x) = f(a \to h(f(a \to x)) \leq f(a \to (a \to x)) = f(a \to x),$$

hence $\phi(a) \to f(a \to x) = f(a \to x)$ and $f(a \to x) \in \mathfrak{o}(\phi(a))$. On the other hand, each $\phi(a) \to x \in \mathfrak{o}(\phi(a))$ is by 1.5.1 equal to $f(a \to h(x)) \in f[\mathfrak{o}(a)]$. ■

4.5. A final remark. Similarly like in spaces we can characterize open maps by preserving interiors. With preserving closures the situation differs: only one of the implications holds. Since one has trivially, for any localic map,

$$\overline{f_{-1}[T]} \subseteq f_{-1}[\overline{T}] \quad \text{and} \quad f_{-1}[\operatorname{int} T] \subseteq \operatorname{int} f_{-1}[T]$$

we will have to discuss in both cases just one of the inclusions.

4.5.1. Proposition. *A localic map $f: L \to M$ is open iff for every sublocale $T \subseteq M$, $f_{-1}[\operatorname{int} T] = \operatorname{int} f_{-1}[T]$.*

Proof: Let f be open. If $\mathfrak{o}(a) \subseteq f_{-1}[T]$ then $f[\mathfrak{o}(a)] \subseteq T$, hence $f[\mathfrak{o}(a)] \subseteq \operatorname{int} T$, and hence $\mathfrak{o}(a) \subseteq f_{-1}[\operatorname{int} T]$. Thus, $\operatorname{int} f_{-1}[T] \subseteq f_{-1}[\operatorname{int} T]$.

Let the equality hold. Then $\mathfrak{o}(a) \subseteq \operatorname{int} f_{-1} f[\mathfrak{o}(a)] = f_{-1}[\operatorname{int} f[\mathfrak{o}(a)]]$ and hence $f[\mathfrak{o}(a)] \subseteq \operatorname{int} f[\mathfrak{o}(a)]$. ■

4.5.2. Lemma. *Let $f : L \to M$ be an open localic map. Then for every sublocale $T \subseteq M$ we have $h[T] \subseteq f_{-1}[T]$.*

Proof: Let $t \in T$ and let $x \in L$ be arbitrary. Then by Theorem 4.3, $f(x \to h(t)) = h^*(x) \to t \in T$ and hence $x \to h(t) \in f_{-1}[T]$. Use 2.3.1. ■

4.5.3. Proposition. *Let $f : L \to M$ be an open localic map. Then for every sublocale $T \subseteq M$ we have*

$$f_{-1}[\overline{T}] = \overline{f_{-1}[T]}.$$

Proof: Set $a = \bigwedge f_{-1}[T]$ and $b = h^*(a)$. Then

$$f_{-1}[\mathfrak{c}(b)] = \mathfrak{c}(h(b)) = \mathfrak{c}(hh^*(a)) \subseteq \mathfrak{c}(a) = \overline{f_{-1}[T]}, \qquad (*)$$

and for $t \in T$ we have by 4.5.2 $h(t) \in f_{-1}[T]$, hence $h(t) \geq a$ and using the adjunction $t \geq h^*(a) = b$, that is, $t \in \mathfrak{c}(b)$. So $T \subseteq \mathfrak{c}(b)$, hence $\overline{T} \subseteq \mathfrak{c}(b)$ and we conclude $f_{-1}[\overline{T}] \subseteq f_{-1}[\mathfrak{c}(b)] \subseteq \overline{f_{-1}[T]}$. ■

Note that the converse is not true, as the following example due to Johnstone [5] shows: the dense embedding

$$j \colon \mathfrak{b}(0) = \{x \to 0 \mid x \in L\} \longrightarrow L$$

is rarely an open map but it satisfies $(*)$, since in $\mathfrak{b}(0)$ (a Boolean algebra) every sublocale is closed and thus

$$\overline{j_{-1}[T]} = j_{-1}[T] = \mathfrak{b}(0) \cap T = \mathfrak{b}(0) \cap \overline{T}$$

(indeed, for each $x \to 0 \in \overline{T} = \uparrow \bigwedge T$, $x \wedge \bigwedge T = 0$ hence $x \to 0 = x \to (x \wedge \bigwedge T) = x \to \bigwedge T \in T$).

The localic maps with property $(*)$ are characterized by Johnstone in [7] and referred to as the *hereditarily skeletal maps*. It is also shown there that if the image of an hereditarily skeletal map is a complemented sublocale of the codomain, then the map is open.

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