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LAX COMMA 2-CATEGORIES AND ADMISSIBLE 2-FUNCTORS

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Abstract: This paper is a contribution towards a two dimensional extension of the basic ideas and results of Janelidze-Galois theory. In the present paper, we give a suitable counterpart notion to that of *absolute admissible Galois structure* for the lax idempotent context, compatible with the context of *lax orthogonal factorization systems*. As part of this work, we study lax comma 2-categories, giving analogue results to the basic properties of the usual comma categories. We show that each morphism of a 2-category induces a 2-adjunction between lax comma 2-categories and comma 2-categories, playing the role of the usual *change of base functors*. With these induced 2-adjunctions, we are able to show that each 2-adjunction induces 2-adjunctions between lax comma 2-categories and comma 2-categories, which are our analogues of the usual lifting to the comma categories used in Janelidze-Galois theory. We give sufficient conditions under which these liftings are 2-premonadic and induce a lax idempotent 2-monad, which corresponds to our notion of 2-admissible 2-functor. In order to carry out this work, we analyse when a composition of 2-adjunctions is a lax idempotent 2-monad, and when it is 2-premonadic. We give then examples of our 2-admissible 2-functors (and, in particular, simple 2-functors), specially using a result that says that all admissible (2-)functors in the classical sense are also 2-admissible (and hence simple as well). We finish the paper relating coequalizers in lax comma 2-categories and Kan extensions.

Keywords: change of base functor, comma object, Galois theory, Kock-Zöberlein monads, semi-left exact functor, lax comma 2-categories, simple 2-adjunctions, 2-admissible 2-functor.
Math. Subject Classification (2020): 18N10, 18N15, 18A05, 18A22, 18A40.

Introduction

Categorical Galois theory, originally developed by Janelidze [16, 4], gives a unifying setting for most of the formerly introduced Galois type theorems, even generalizing most of them. It neatly gives a common ground for Magid's Galois theory of commutative rings, Grothendieck's theory of étale covering of schemes, and central extension of groups. Furthermore, since its genesis

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Janelidze's Galois theory has found several developments, applications and examples in new settings (see, for instance, [6], [14], [8], [35], [17, Theorem 4.2], and [30, Theorem 9.8]).

The most elementary observation on factorization systems and Janelidze-Galois theory is that, in the suitable setting of finitely complete categories, the notion of absolute admissible Galois structure coincides with that of a semi-left-exact reflective functor/adjunction [4, Section 5.5].

Motivated by the fact above and the theory of *lax orthogonal factorization systems* [9, 10], we started a project whose aim is to investigate a two dimensional extension of the basic ideas and results of (absolute) Janelidze-Galois theory. We deal herein with a key step of this endeavor, that is to say, we develop the basics in order to give a suitable counterpart notion to that of *absolute admissible Galois structure*.

We adopt the *usual* viewpoint that the 2-dimensional analogue of an idempotent monad (full reflective functor) is that of a lax idempotent monad (pre-Kock-Zöberlein 2-functor). Therefore the concept of an admissible Galois structure within our context should be a lax idempotent counterpart to the notion of *semi-left exact reflective functor*. Namely, an appropriate notion of semi-left exact functor for the context of [9].

We study the lifting of 2-adjunctions to comma type 2-categories. We find two possible liftings which deserve interest. The underlying adjunction of the first type of lifting is the usual 1-dimensional case, while the other one, more relevant to our context, is a counterpart to the lifting of the 2-monad given in [9] by comma objects. The last one requires us to study the lax analogue notion for comma categories, the notion of *lax comma* 2*-categories* of the title.

We show that the lax comma 2-categories are isomorphic to the 2-category of coalgebras (and lax morphisms) of a suitable 2-comonad provided that the base 2-category has products. We also study the basic aspects of lax comma 2-categories. Among them, the 2-adjunction between the usual comma 2-category and the lax comma 2-category (for each object), and a counterpart for the usual change of base 2-functors, which comes into play as fundamental aspect of our work and, specially, to introduce the definition of 2*-admissible* 2*-adjunction*.

With these analogues of the change of base 2-functors, we are able to introduce the lifting of each 2-adjunction to a 2-adjunction between the lax comma 2-category and the comma 2-category as a composition of 2-adjunctions.

for hosting us at the UCL, Louvain la Neuve, in May/2018, when we started this project.

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Theorem 9.7. *Let $\mathbb{A}$ be a 2-category, and $\mathbb{B}$ a full sub-2-category. If $\mathbb{A}$ has terminal object 1, the obvious functor*

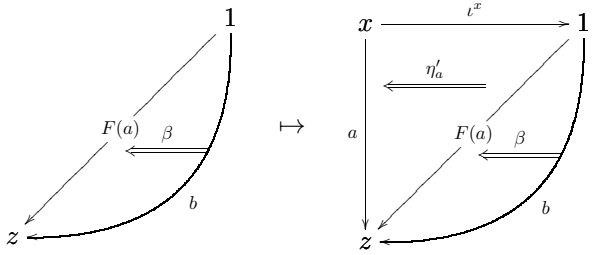
$$K : \mathbb{A}^{\mathrm{co}}(1, z) \to (\mathbb{B}//z)_0 ,$$

whose action on morphisms is given by $\alpha \mapsto (\iota^1, \alpha)$ has a left adjoint if and only if z is conical $\mathbb{B}$-complete. Moreover, in this case, the left adjoint takes each $a : x \to z$ (object of $(\mathbb{B}//z)_0$) to $\mathsf{lan}_{\iota^x} a$.

Proof: In this proof, by abuse of language, we denote by $\mathbb{B}//z$ the category $(\mathbb{B}//z)_0$. There is a left adjoint $F : \mathbb{B}//z \to \mathbb{A}^{\mathrm{co}}(1, z)$ if, and only if, for each morphism $a : x \to z$ such that $x \in \mathbb{B}$, there is a pair

$$(F(a), \eta_a : a \to KF(a))$$

of an object $F(a) : 1 \to z$ of $\mathbb{A}(1, z)$ and a morphism $\eta_a : a \to KF(a)$ in $\mathbb{B}//z$ defined by a pair (ι^x, η'_a) such that



(9.2.1)

gives a bijection

$$\mathbb{A}^{\mathrm{co}}(1, z)(F(a), b) \cong \mathbb{B}//z\,(a, K(b)) .$$

In order to complete the proof, it should be noted that $\mathbb{A}^{\mathrm{co}}(1, z)(F(a), b) \cong \mathbb{A}(1, z)(b, F(a))$ and $\mathbb{B}//z\,(a, K(b)) \cong \mathbb{A}(x, z)(b \circ \iota^x, a)$. Therefore, we proved that there is a left adjoint if, and only if, for each $a : x \to z$, there is a pair $(F(a), \eta_a : a \to KF(a))$ which is the right Kan extension of a along ι^x. ∎

Remark 9.8. The main observation of [1, Lemma 7.13] is actually given in codual form of the theorem above.

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Namely, the composition of a straightforward lifting to the lax comma 2-categories with a change of the base 2-functor induced by the appropriate component of the unit. Fully relying on the study of properties of compositions of 2-adjunctions, we investigate the properties of these liftings of the 2-adjunctions. Namely, we show under which conditions these liftings induce lax idempotent 2-monads (the simple 2-adjunctions of [9]), recovering one characterization given in [9] of their *simple 2-adjunctions*. We give also a characterization of the 2-functors whose introduced lifting is lax idempotent and 2-premonadic, the 2-*admissible* 2-*functors* within our context.

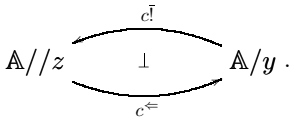
In Section 1 we recall basic aspects and terminology of 2-categories, such as 2-adjunctions and 2-monads, finishing the section giving aspects on *raris*, right adjoints right inverses (see Definition 1.1) within a 2-category. We also recall the universal properties of the main two dimensional limits used in our work in Section 2, that is to say, the definitions of conical 2-limits and comma objects.

In Section 3 we recall and show aspects on idempotent and lax idempotent 2-monads needed to our work on admissible and 2-admissible 2-functors, also introducing a characterization of the 2-adjunctions that induce lax idempotent 2-monads, called herein lax idempotent 2-adjunctions (see, for instance, Theorem 3.12).

In Section 4 we introduce the main concepts and results on composition of 2-adjunctions in order to introduce the notions of simple, admissible and 2-admissible 2-adjunctions (see, for instance, Definitions 4.3, 4.5, and 4.10). The results focus on characterizing and giving conditions under which the composition of 2-adjunctions is an idempotent/lax idempotent (full reflective/pre-Kock-Zöberlein) 2-adjunction (2-functor). Most of them are analogues for the simpler case of idempotent 2-adjunctions (see, for instance, Theorem 4.11 which characterizes when the composition of right 2-adjoints is pre-Kock-Zöberlein).

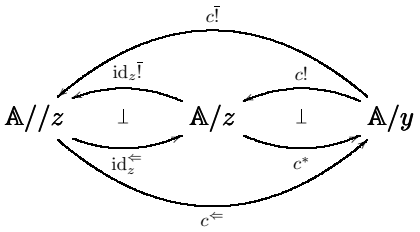
In Section 5 we introduce the notion of lax comma 2-categories $\mathbb{A}//y$, for each 2-category $\mathbb{A}$ and object $y \in \mathbb{A}$ (see Definition 5.1). This notion has already appeared in the literature (see, for instance, [32, Exercise 5, pag. 115] or [39, pag. 305]). We also prove that, provided that the 2-category $\mathbb{A}$ has products, the lax comma 2-category $\mathbb{A}//y$ is isomorphic to the 2-category of coalgebras and lax morphisms for the canonical 2-monad whose underlying endo-2-functor $(y \times -)$.

In Section 6 we introduce the change of base 2-functors for lax comma 2-categories. More precisely, we show that, for each morphism $c : y \to z$ in a 2-category $\mathbb{A}$ with comma objects, we have an induced 2-adjunction



between the lax comma 2-category $\mathbb{A}//z$ and the comma 2-category $\mathbb{A}/y$. We introduce this 2-adjunction using two approaches. Firstly, we get it via the adjoint triangle theorem for coalgebras (and lax morphisms), provided that $\mathbb{A}$ has products. Then we give the most general and (elementary) approach (see Theorem 6.7).

Provided that $\mathbb{A}$ has pullbacks and comma objects, these induced 2-adjunctions, together with the classical change of base 2-functors, give the 2-adjunctions



in which the composition of $c_! \dashv c^* : \mathbb{A}/z \to \mathbb{A}/y$ with $\mathrm{id}_z{}_!^{\overline{1}} \dashv id_z^{\Leftarrow} : \mathbb{A}//z \to \mathbb{A}/z$ is, up to 2-natural isomorphism, the 2-adjunction $c_!^{\overline{1}} \dashv c^{\Leftarrow} : \mathbb{A}//z \to \mathbb{A}/y$ (see Theorem 6.9). We finish Section 6 showing that, whenever it is well defined, $\mathrm{id}_y^{\Leftarrow}$ is pre-Kock-Zöberlein (Theorem 6.10).

The main point of Section 7 is to introduce our notions of admissibility and 2-admissibility (Definition 7.4), relying on the definitions previously introduced in Section 4. We also use the main results of Section 4 to characterize and give conditions under which a 2-functor is 2-admissible (see, for instance, Corollaries 7.10 and 7.11).

We finish Section 7 with a fundamental observation on admissibility and 2-admissibility, namely, Theorem 7.13. It says that, provided that $\mathbb{A}$ has comma objects, if $F \dashv G$ is admissible in the classical sense (called herein *admissible w.r.t. the basic fibration*), meaning that G itself is full reflective

This completes the proof of the result. ∎

9.2. Conical colimits in objects. We finish with a theorem on Kan extensions and lax comma 2-categories, that is to say, Theorem 9.7. It is related with Walter Tholen's talk in the International Conference on Category Theory of 2019 or, more precisely, it is a generalization of the main point of the proof of [1, Lemma 7.13].

We start by recalling a well-known observation w.r.t. Kan extensions in Cat. Below we denote by $\mathsf{colim}f$ the conical colimit of the functor f.

Lemma 9.4. *Let* 1 *be the terminal category in* Cat*, and, for each category* x*,* $\imath^x : x \to 1$ *the unique functor. Given a functor* $f : x \to z$*, we have that*

$$\mathsf{lan}_{\imath^x} f \cong \mathsf{colim} f,$$

either side existing if the other does.

This observation motivates the following definition.

Definition 9.5. [Conical cocomplete] Let $\mathbb{A}$ be a 2-category with terminal object, denoted by 1, and $\mathbb{B}$ a full sub-2-category of $\mathbb{A}$. An object z in $\mathbb{A}$ is conical $\mathbb{B}$-complete ($\mathbb{B}$-cocomplete) if $\mathsf{ran}_{\imath^x} j$ ($\mathsf{lan}_{\imath^x} j$) exists for any morphism $j : x \to z$, in which $\imath^x$ is the morphism $x \to 1$.

The need of considering a sub-2-category $\mathbb{B}$ above comes from the elementary facts about size of diagrams and limits in the classical context of Cat. More precisely, the notion of cocomplete category corresponds to the notion of conical cat-cocomplete category.

In order to establish Theorem 9.7, we need to consider the lax comma 2-categories of type $\mathbb{B}//y$ for each $y \in \mathbb{A}$, even if $y \notin \mathbb{B}$. More precisely:

Definition 9.6. Let $\mathbb{A}$ and $\mathbb{B}$ be 2-categories, and assume that $S : \mathbb{B} \to \mathbb{A}$ is a full inclusion. Given an object $y \in \mathbb{A}$, we denote by $(\mathbb{B}//y)_0$ or $S//y$ the underlying category of the full sub-2-category of $\mathbb{A}//y$ defined by the pullback of S along the forgetful 2-functor $\mathbb{A}//y \to \mathbb{A}$. Analogously, we denote by $(\mathbb{B}/y)_0$ the 2-category defined by the pullback of S along the forgetful functor $\mathbb{A}/y \to \mathbb{A}$ (that is to say, the comma object in 2-Cat of $y : 1 \to \mathbb{A}$ along S).

In the classical case, then, we can consider, for instance, $\mathsf{cat}//\mathsf{Set}$. In this case, if z is any complete category, as observed in [1, Lemma 7.13], there is a functor $\mathsf{cat}//z \to z^{\mathrm{op}}$ that takes each object of $\mathsf{cat}//z$ (that is to say, small diagram $x \to z$) to its limit.

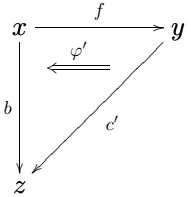
In the case of locally preordered 2-categories (or locally thin 2-categories) with coequalizers, the reciprocal of Theorem 9.2 holds. More precisely:

Theorem 9.3. *Let $\mathbb{A}$ be a 2-category with coequalizers. Assume that $a : w \to z$ is preterminal in $\mathbb{A}(w, z)$. Then $(f, \varphi) : b \to c$ is the coequalizer of the pair $(g, \gamma), (h, \zeta) : a \to b$ in $\mathbb{A}//z$ if, and only if, $f : x \to y$ is the coequalizer of the pair of morphisms $g, h : w \to x$ in $\mathbb{A}$ and (c, φ) is the right Kan extension of b along f.*

Proof: By Theorem 9.2, it is enough to prove one of the directions.

Assume that $a : w \to z$ is preterminal in $\mathbb{A}(w, z)$ and $(f, \varphi) : b \to c$ is the coequalizer of the pair $(g, \gamma), (h, \zeta) : a \to b$ in $\mathbb{A}//z$. By Theorem 9.1, $f : x \to y$ is the coequalizer of the pair of morphisms $g, h : w \to x$ in $\mathbb{A}$.

Finally, it remains to prove that (c, φ) is the right Kan extension $\mathsf{ran}_f b$. Given any 2-cell



(9.1.13)

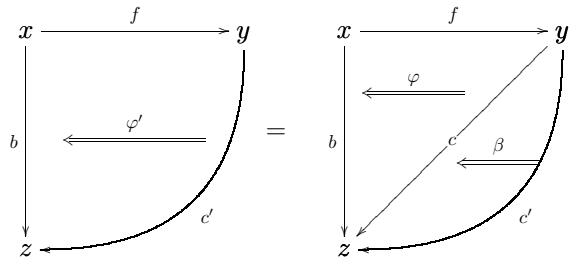
we get that, since a is preterminal in $\mathbb{A}(w, z)$,

$$(f, \varphi') \cdot (g, \gamma) = (f, \varphi') \cdot (h, \zeta)\,.$$

Therefore there is a unique morphism (s, β) in $\mathbb{A}//z$ such that $(s, \beta) \cdot (f, \varphi) = (f, \varphi')$. Since f is the coequalizer of h, g in $\mathbb{A}$, we have that $s = \mathrm{id}_y$ and, hence, this proves that

$$\beta : c' \Rightarrow c$$

is the unique 2-cell in $\mathbb{A}$ such that



(9.1.14)

and the compositions

$$\eta_y^* \circ \check{G} : \mathbb{A}/F(y) \to \mathbb{B}/y$$

are full reflective for all y, then G is 2-admissible, which means that the compositions

$$\eta_y^{\Leftarrow} \circ \check{G} : \mathbb{A}//F(y) \to \mathbb{B}/y$$

are pre-Kock-Zöberlein for all objects y.

We discuss examples of 2-admissible 2-functors (and hence also simple 2-functors) in Section 8. Most of the examples are about cocompletion of 2-categories, making use of Theorem 7.13.

Finally, in Section 9, we give two remarks about Kan extensions and lax comma 2-categories. The main remark is a sufficient condition in order to get coequalizers in the lax comma 2-category, which also gives a characterization of coequalizers via the universal property of the right Kan extension for locally preordered 2-categories.

1. Preliminaries

Let Cat be the cartesian closed category of categories in some universe. We denote the internal hom by

$$\mathsf{Cat}(-,-) : \mathsf{Cat}^{\mathrm{op}} \times \mathsf{Cat} \to \mathsf{Cat}.$$

A 2-category $\mathbb{A}$ herein is the same as a Cat-enriched category. We denote the enriched hom of a 2-category $\mathbb{A}$ by

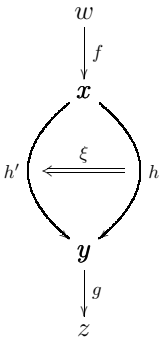
$$\mathbb{A}(-,-) : \mathbb{A}^{\mathrm{op}} \times \mathbb{A} \to \mathsf{Cat}$$

which, again, is of course a 2-functor. As usual, the composition of 1-cells (morphisms) are denoted by $\circ$, $\cdot$, or omitted whenever it is clear from the context. The vertical composition of 2-cells is denoted by $\cdot$ or omitted when it is clear, while the horizontal composition is denoted by $*$. Recall that, from the vertical and horizontal compositions, we construct the fundamental operation of *pasting* [23, 36].

Finally, if $f : w \to x$, $g : y \to z$ are 1-cells of $\mathbb{A}$, given a 2-cell $\xi : h \Rightarrow h' : x \to y$, motivated by the case of $\mathbb{A} = \mathsf{Cat}$, we use interchangeably the

notations

$$\mathrm{id}_g * \xi * \mathrm{id}_f \quad = \quad$$



$$\quad = \quad g\xi f \qquad (1.0.1)$$

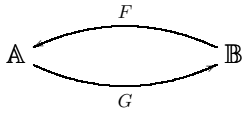
to denote the whiskering of ξ with f and g.

Henceforth, we consider the 3-category of 2-categories, 2-functors, 2-natural transformations and modifications, denoted by 2-Cat. We refer to [23, 37] for the basics on 2-dimensional category theory, and, more particularly, to the definitions of adjunctions, monads and Kan extensions inside a 2-category. Moreover, we also extensively assume aspects of 2-monad theory. The pioneering reference is [2], while we mostly follow the terminology (and results) of [29].

In this paper, we consider the *strict* versions of 2-dimensional adjunctions and monads: the concepts coincide with the Cat-enriched ones. A 2-*adjunction*, denoted by

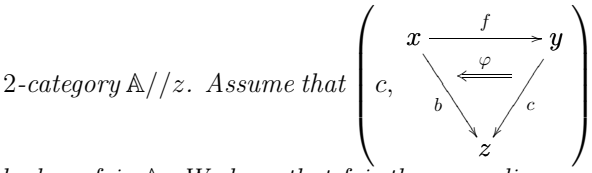
$$(F \dashv G, \varepsilon, \eta) : \mathbb{A} \to \mathbb{B},$$

consists of 2-functors



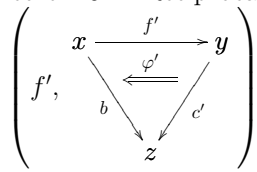
with 2-natural transformations $\varepsilon : FG \Longrightarrow \mathrm{id}_{\mathbb{A}}$ and $\eta : \mathrm{id}_{\mathbb{B}} \Longrightarrow GF$ playing the role of the *counit* and the *unit* respectively. More precisely, the equations of 2-natural transformations

morphisms in the lax comma 2*-category* $\mathbb{A}//z$. *Assume that*



is the right Kan extension of b *along* f *in* $\mathbb{A}$. *We have that* f *is the coequalizer of the morphisms* $h, g : w \to x$ *if and only if the morphism* $(f, \varphi) : (x, b) \to (y, c)$ *is the coequalizer of the pair of the morphisms* (9.1.11) *in* $\mathbb{A}//z$.

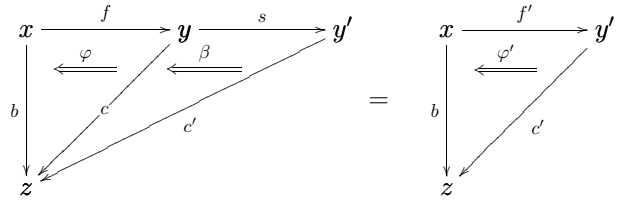
Proof: If (f, φ) is the coequalizer of (9.1.11) in $\mathbb{A}//z$, we get that f is the coequalizer of h, g in $\mathbb{A}$ by Theorem 9.1. Reciprocally, assume that f is the coequalizer of h, g in $\mathbb{A}$, and



is a morphism of $\mathbb{A}//z$ such that

$$(f', \varphi') \cdot (g, \gamma) = (f', \varphi') \cdot (h, \zeta)\,.$$

In this case, we have in particular that $f'g = f'h$ and, hence, by the universal property of the coequalizer, there is a unique morphism $s : y \to y'$ in $\mathbb{A}$ such that $sf = f'$. Moreover, by the universal property of the right Kan extension (c, φ), we get that there is a unique 2-cell $\beta : c' \cdot s \Rightarrow c$ in $\mathbb{A}$ such that the equation



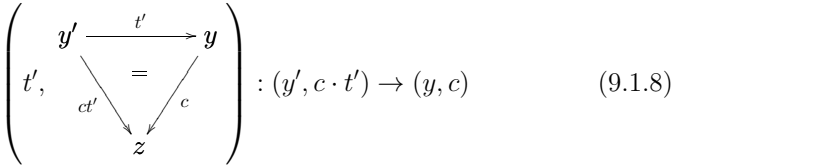
(9.1.12)

holds. This proves that

$$(s, \beta) : (y, c) \to (y', c')$$

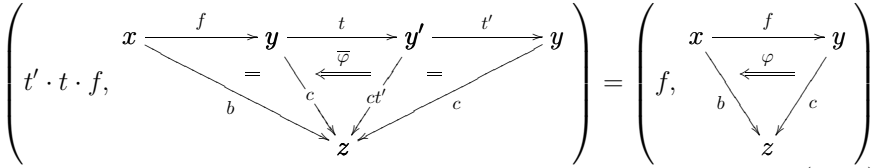
is the unique morphism in $\mathbb{A}//z$ such that $(s, \beta) \cdot (f, \varphi) = (f', \varphi')$. This completes the proof that (f, φ) is the coequalizer of the morphisms (9.1.11). ∎

of the morphisms of (9.1.1), since the morphism



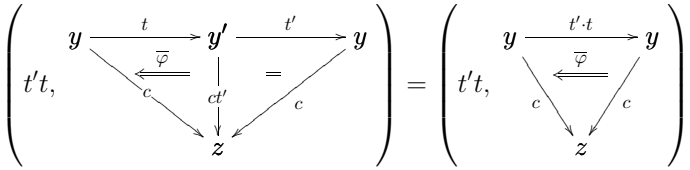
(9.1.8)

in $\mathbb{A}//z$ is such that the equation



(9.1.9)

holds, we get that

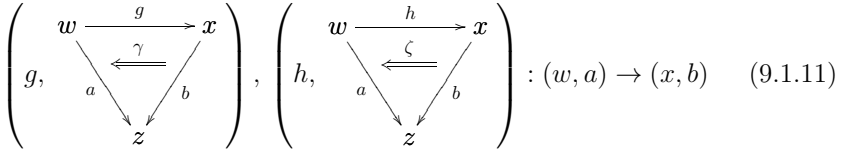


(9.1.10)

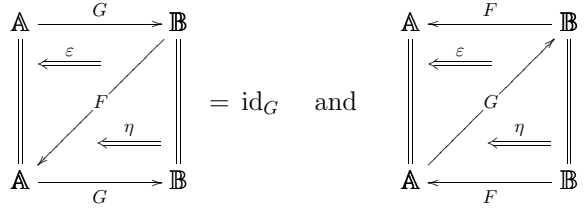
is equal to the identity $(\mathrm{id}_y, \mathrm{id}_c)$. In particular, we get that $t't = \mathrm{id}_y$ (and $\overline{\varphi}$ is the identity) which completes the proof that t' is an isomorphism. ∎

There is a relation between Kan extensions and coequalizers in lax comma 2-categories. More precisely, it gives a strong condition for coequalizers as stated below.

Theorem 9.2. *Let $\mathbb{A}$ be a 2-category with coequalizers, and*

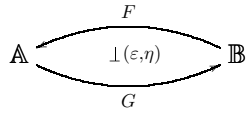


(9.1.11)



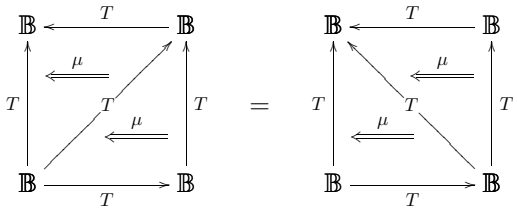
(triangle identities)

hold. We usually denote a 2-adjunction $(F \dashv G, \varepsilon, \eta) : \mathbb{A} \to \mathbb{B}$ by

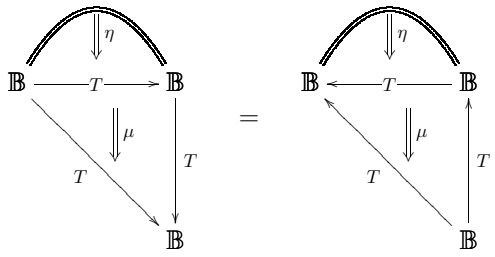


or by $F \dashv G : \mathbb{A} \to \mathbb{B}$ for short, when the counit and unit are already given.

A 2-*monad* on a 2-category $\mathbb{B}$ is a triple $\mathcal{T} = (T, \mu, \eta)$ in which $T : \mathbb{B} \to \mathbb{B}$ is an endo-2-functor and μ, η are 2-natural transformations playing the role of the multiplication and the unit respectively. That is to say, μ and η are 2-natural transformations such that the equations



(associativity of a 2-monad)



(identity of a 2-monad)

hold.

Since the notions above coincide with the Cat-enriched ones, it should be noted that the formal theory of monads applies to this case. More precisely, every 2-adjunction does induce a 2-monad, and we have the usual Eilenberg-Moore and Kleisli factorizations of a right 2-adjoint functor (*e.g* [37, Section 2] or [27, Section 3]), which give rise respectively to the notions of 2-monadic and Kleisli 2-functors. Furthermore, we also have (the enriched version of) Beck's monadicity theorem [12, Theorem II.2.1].

In this direction, we use expressions like *equivalence (or 2-equivalence)*, and *fully faithful 2-functor* to mean the (strict) Cat-enriched notions: that is to say, respectively, *equivalence* in the 2-category of 2-categories, and a 2-functor that is *locally an isomorphism*.

1.1. Lalis and ralis. Our terminology is similar to the terminology of [9] to refer to adjunctions with unit (or counit) being identities. More precisely:

Definition 1.1. Assume that $(f \dashv g, v, n)$ is an adjunction in a 2-category $\mathbb{A}$.

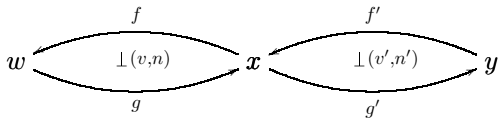
- If the counit v is the identity 2-cell, $(f \dashv g, v, n)$ is called a *rari adjunction (or rari pair)*, or a *lali adjunction*.

  If there is a rari adjunction $f \dashv g$, the morphism f is called a *lali (left adjoint and left inverse)*, while the morphism g is called a *rari (right adjoint and right inverse)*.
- If the unit n is the identity 2-cell, $(f \dashv g, v, n)$ is called a *rali adjunction*, or a *lari adjunction*.

  If there is a rali adjunction $f \dashv g$, the morphism f is called a *lari*, while the morphism g is called a *rali*.

Laris (ralis) are closed by composition, and have specific cancellation properties. We recall them below.

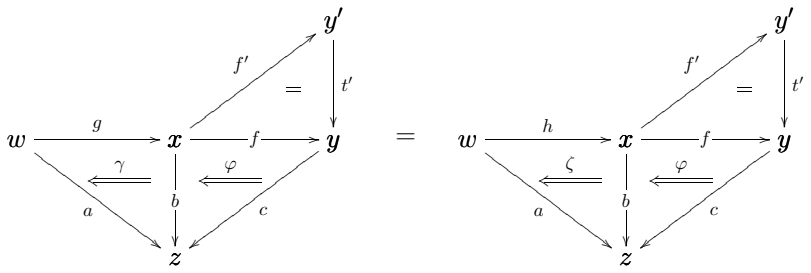
Lemma 1.2. *Assume that*



(1.1.1)

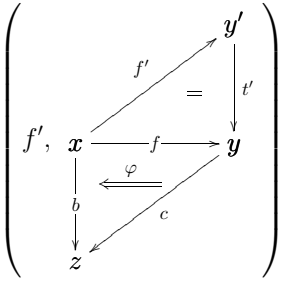
are adjunctions in $\mathbb{A}$.

We have that



(9.1.3)

which proves that the morphism



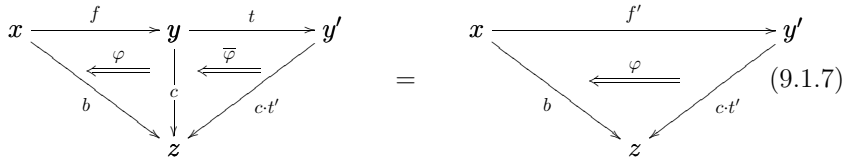
$: (x, b) \to (y', c \cdot t')$ (9.1.4)

in $\mathbb{A}//z$ is coequalized by the morphisms of (9.1.1) and, hence, there is a unique morphism

$$(t, \overline{\varphi}) : (y, c) \to (y', c \cdot t') \tag{9.1.5}$$

in $\mathbb{A}//z$ such that the equations

$$t \cdot f \;=\; f' \tag{9.1.6}$$



(9.1.7)

hold.

By the universal property of the coequalizer of g, h in $\mathbb{A}$, since $tt'f' = tf = f'$, we get that $tt' = \mathrm{id}_{y'}$. Finally, by the universal property of the coequalizer

is the *left Kan extension* of j along h if it corresponds to $\mathsf{ran}_h j$ in the 2-category $\mathbb{A}^{\mathrm{co}}$.

9.1. Coequalizers. If $\mathbb{A}$ has comma objects, since $\mathrm{id}_z^{\overline{!}} : \mathbb{A}/z \to \mathbb{A}//z$ has right 2-adjoint, it does preserve colimits. Since $\mathbb{A}/z \to \mathbb{A}$ does create all the colimits, we conclude that the colimits of "diagrams of strict morphisms" in $\mathbb{A}//z$ are pointwise. That includes coequalizers. The remaining problem is to study coequalizers of non-strict morphisms.

Theorem 9.1 (Preservation of coequalizers)**.** *Let $\mathbb{A}$ be a 2-category with coequalizers. Given any object $z \in \mathbb{A}$, the forgetful 2-functor*

$$\mathbb{A}//z \to \mathbb{A}$$

preserves coequalizers.

Proof: Let $\left(f, \begin{array}{ccc} x & \xrightarrow{f} & y \\ & {\scriptstyle b}\searrow \quad \swarrow{\scriptstyle c} & \\ & z & \end{array} \right) : (x,b) \to (y,c)$ and $f' : x \to y'$ be respectively the coequalizer of the pair $(g,\gamma), (h,\zeta)$ of morphisms $(w,a) \to (x,b)$ in $\mathbb{A}//z$

$$\left(g, \begin{array}{ccc} w & \xrightarrow{g} & x \\ & {\scriptstyle a}\searrow \quad \swarrow{\scriptstyle b} & \\ & z & \end{array} \right), \left(h, \begin{array}{ccc} w & \xrightarrow{h} & x \\ & {\scriptstyle a}\searrow \quad \swarrow{\scriptstyle b} & \\ & z & \end{array} \right) \tag{9.1.1}$$

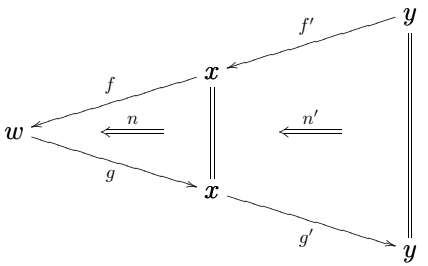
and the coequalizer of the pair of morphisms $g, h : w \to x$ in $\mathbb{A}$.

Since $fg = fh$, we get that there is a unique morphism $t' : y' \to y$ of $\mathbb{A}$ such that

$$t' \cdot f' = f. \tag{9.1.2}$$

a) Assuming that $f \dashv g$ is a lari adjunction: we have that $ff' \dashv g'g$ is a lari adjunction if, and only if, $f' \dashv g'$ is a lari adjunction as well.
b) Assuming that $f' \dashv g'$ is a lali adjunction: the adjunction $ff' \dashv g'g$ is a lali adjunction if, and only if, $f \dashv g$ is a lali adjunction as well.

Proof: Assuming that n is an isomorphism, we have that the unit

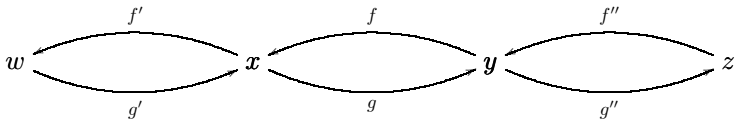


(1.1.2)

of the composition $ff' \dashv g'g$ is invertible if, and only if, n' is invertible. This proves b) and, dually, we get a). ∎

Of course, the situation is simpler when we consider isomorphisms. That is to say:

Corollary 1.3. *Assume that*



(1.1.3)

are morphisms in $\mathbb{A}$ such that $(f')^{-1} = g'$ and $(f'')^{-1} = g''$. There is a lali (rali) adjunction $f' \cdot f \cdot f'' \dashv g'' \cdot g \cdot g'$ if and only if there is a lali (rali) adjunction $f \dashv g$.

Proof: If $f \dashv g$ is a lali (rali) adjunction, since $f' \dashv g'$ and $f'' \dashv g''$ are of course lali and rali adjunctions, it follows that the composite

$$f' \cdot f \cdot f'' \dashv g'' \cdot g \cdot g'$$

is a lali (rali) adjunction by Lemma 1.2.

Reciprocally, if $f' \cdot f \cdot f'' \dashv g'' \cdot g \cdot g'$ is a lali (rali) adjunction, since $g' \dashv f'$ and $g'' \dashv f''$ are lali and rali adjunctions, we get that the composite

$$g' \cdot f' \cdot f \cdot f'' \cdot g'' \dashv f'' \cdot g'' \cdot g \cdot g' \cdot f',$$

which is $f \dashv g$, is a lali adjunction. ∎

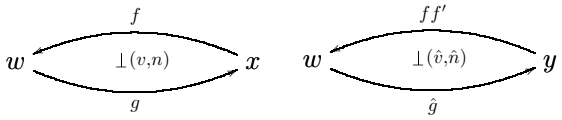
But we also have a stronger cancellation property:

Theorem 1.4 (Left cancellation property)**.** *Let $f : x \to w, f' : y \to x$ be morphisms of a 2-category $\mathbb{A}$.*

a) Assuming that $f : x \to w$ is a lari: the composite $ff' : y \to w$ is a lari if, and only if, $f' : y \to x$ is a lari as well.
b) Assuming that f is a rari: the composite ff' is a rari if and only if f' is a rari.

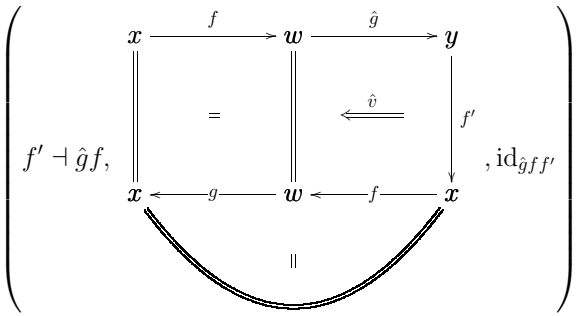
Proof: By Lemma 1.2, if f and f' are laris, the composite ff' is a lari as well.

Reciprocally, assume that f and ff' are laris. This means that there are adjunctions



in $\mathbb{A}$ such that $n = \mathrm{id}_{gf}$ and $\hat{n} = \mathrm{id}_{\hat{g}ff'}$.

We claim that



(1.1.4)

is a (lari) adjunction. In fact, the triangle identities follow from the facts that the equations $\hat{v}ff' = \mathrm{id}_{ff'}$ and $\hat{g}\hat{v} = \mathrm{id}_{\hat{g}}$ hold.

Finally, the statement b) is the codual of a). ∎

On the one hand, the *left cancellation property* of Theorem 1.4 does not hold for lalis or ralis. For instance, in Cat, we consider the terminal category

Corollary 8.9. *Let $\mathbb{A}$ be a lextensive 2-category. The 2-functor $I_{\mathbb{A}} : \mathbb{A} \to \mathsf{Fam}_{\mathsf{fin}}(\mathbb{A})$ is admissible w.r.t. the basic fibration and, hence, 2-admissible.*

Proof: In fact, since products of fully faithful 2-functors are fully faithful, we get that $\eta_Y^* I_{\mathbb{A}}$ is fully faithful by the 2-natural isomorphism (8.0.3). ∎

Remark 8.10. Definition 8.7 has an obvious infinite analogue, the definition of *infinitary lextensive 2-category*. For an infinitary lextensive 2-category $\mathbb{A}$, we have an analogous result w.r.t. $\mathsf{Fam}(\mathbb{A})$. More precisely,

$$I_{\mathbb{A}} : \mathbb{A} \to \mathsf{Fam}(\mathbb{A})$$

is admissible w.r.t. the basic fibration (and, hence, 2-admissible) whenever $\mathbb{A}$ is infinitary extensive.

9. Remarks on Kan extensions and colimits

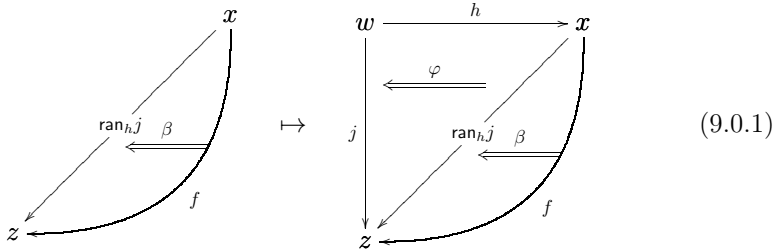
We finish the paper giving two remarks on the relation of Kan extension with lax comma 2-categories. In order to give these remarks, we recall the definition of Kan extensions. Let $j : w \to z$ and $h : w \to x$ be morphisms of a 2-category $\mathbb{A}$. The right Kan extension of j along h is, if it exists, the right reflection $\mathsf{ran}_h j$ of j along the functor

$$\mathbb{A}(h, z) : \mathbb{A}(x, z) \to \mathbb{A}(w, z).$$

This means that the right Kan extension is actually a pair

$$(\mathsf{ran}_h j : x \to z, \varphi : (\mathsf{ran}_h j) \circ h \Rightarrow j)$$

consisting of a morphism $\mathsf{ran}_h j$ and a 2-cell φ of $\mathbb{A}$ such that each morphism $f : x \to z$ of $\mathbb{A}$ defines a bijection $\mathbb{A}(x, z)(f, \mathsf{ran}_h j) \cong \mathbb{A}(w, z)(f \circ h, j)$:



(9.0.1)

A pair

$$(\mathsf{lan}_h j : x \to z, \varphi : j \Rightarrow (\mathsf{lan}_h j) \circ h)$$

the 2-functor

$$\begin{aligned}\prod_{j=1}^{n}\mathbb{A}/y_j &\to \mathbb{A}/\coprod_{j=1}^{n} y_j\\ (a_j : w_j \to y_j)_{j\in\{1,\dots n\}} &\mapsto \coprod_{j=1}^{n} a_j\end{aligned}$$

defined pointwise by the coproduct is a (Cat-)equivalence.

Theorem 8.8. *Let $\mathbb{A}$ be a lextensive 2-category. We consider the 2-adjunction*

$$\mathbb{A} \underset{I}{\overset{\bot\,(\varepsilon,\eta)\quad \mathsf{Fam_{fin}}\,(\mathbb{A})}{\rightleftarrows}}$$

in which the right 2-adjoint is the canonical inclusion. For each finite family $Y = (y_j)_{j\in\{1,\dots,n\}}$ of objects in $\mathbb{A}$, there is a (canonical) 2-natural isomorphism

$$\begin{array}{ccc}\prod_{j=1}^{n}\mathbb{A}/y_j & \xrightarrow{\prod_{j=1}^{n} I_{\mathbb{A}/y_j}} & \prod_{j=1}^{n}\mathsf{Fam_{fin}}\,(\mathbb{A}/y_j)\\ \simeq\downarrow & \cong & \downarrow\simeq\\ \mathbb{A}/\coprod_{j=1}^{n} y_j & \xrightarrow[\eta^{*}_{Y}\circ \tilde{I}_{\mathbb{A}}]{} & \mathsf{Fam_{fin}}\,(\mathbb{A})\,/\,(y_j)_{j\in\{1,\dots,n\}}\end{array} \tag{8.0.3}$$

Proof: The equivalence 2-functor

$$\prod_{j=1}^{n}\mathsf{Fam_{fin}}\,(\mathbb{A}/y_j) \to \mathsf{Fam_{fin}}\,(\mathbb{A})\,/\,(y_j)_{j\in\{1,\dots,n\}}$$

is such that each object

$$A = \left(\left(a_{(1,1)},\dots,a_{(1,m_1)}\right),\dots,\left(a_{(n,1)},\dots,a_{(n,m_n)}\right)\right)$$

is taken to

$$t^A = \left(t^A_l\right)_{l\in\{0,(1,1),\dots,(1,m_1),\dots,(n,m_n)\}}$$

in which $t^A_0(j,k) := j$ and $t^A_{(j,k)} := a_{(j,k)}$. The action on morphisms and 2-cells is then pointwise defined. ∎

$\mathsf{1}$ and the category $\mathsf{2}$ with two objects and only one nontrivial morphism between them. The morphisms

$$\mathsf{1} \underset{s^0}{\overset{s^0d^0}{\rightleftarrows}} \mathsf{2} \quad \mathsf{1} \tag{1.1.5}$$

are lalis. But the inclusion $d^0 : \mathsf{1} \to \mathsf{2}$ of the terminal object of $\mathsf{2}$ is not a lali, since it does not have a right adjoint. On the other hand, the dual of Theorem 1.4 gives a right cancellation property for ralis and lalis.

Corollary 1.5 (Right cancellation property)**.** *Let $f : x \to w, f' : y \to x$ be morphisms of a 2-category $\mathbb{A}$. If $f' : y \to x$ is a lali (rali): we have that $f : x \to w$ is a lali (rali) if, and only if, the composite $ff' : y \to w$ is a lali (rali) as well.*

2. Two dimensional limits

In this section, we recall basic universal constructions related to the results of this paper. Two dimensional limits [38] are the same as weighted limits in the Cat-enriched context [12]. We refer, for instance, to [38, 21] for the basics on 2-dimensional limits. We are particularly interested in *conical 2-(co)limits* and *comma objects*.

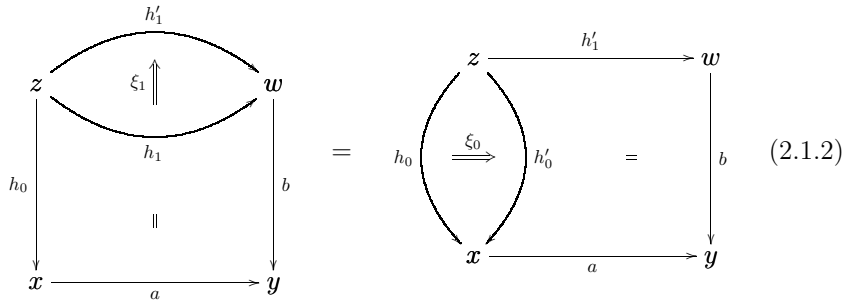
2.1. Conical 2-limits. Two dimensional conical (co)limits are just weighted (co)limits with a weight constantly equal to the terminal category $\mathsf{1}$. Henceforth, the words *(co)product*, *pullback/pushout* and *(co)equalizer* refer to the 2-dimensional versions of each of those (co)limits. For instance, if $a : x \to y$, $b : w \to y$ are morphisms of a 2-category $\mathbb{A}$, assuming its existence, the *pullback* of b along a is an object $x\times_{(a,b)}w$ together with 1-cells $a^*(b) : x\times_{(a,b)}w \to x$ and $b^*(a) : x \times_{(a,b)} w \to w$ making the diagram

$$\begin{array}{ccc}x \times_{(a,b)} w & \xrightarrow{b^*(a)} & w\\ {\scriptstyle a^*(b)}\downarrow & & \downarrow{\scriptstyle b}\\ x & \xrightarrow[a]{} & y\end{array} \tag{2.1.1}$$

commutative, and satisfying the following universal property. For every object z and every pair of 2-cells

$$(\xi_0 : h_0 \Rightarrow h_0' : z \to x,\ \xi_1 : h_1 \Rightarrow h_1' : z \to w)$$

such that the equation



(2.1.2)

holds, there is a unique 2-cell $\xi : h \Rightarrow h' : z \to x \times_{(a,b)} w$ satisfying the equations

$$\mathrm{id}_{a^*(b)} * \xi = \xi_0 \text{ and } \mathrm{id}_{b^*(a)} * \xi = \xi_1.$$

Remark 2.1. It is clear that the concept of *pullback* in locally discrete 2-categories coincides with the concept of (1-dimensional) *pullback* in the underlying categories.

Moreover, when a *pullback* exists in a 2-category, it is isomorphic to the (1-dimensional) *pullback* in the underlying category.

Finally, both the statements above are also true if *pullback* is replaced by any type of conical 2-limit with a locally discrete *shape* (domain).

2.2. Comma objects. If $a : x \to y$, $b : w \to y$ are morphisms of a 2-category $\mathbb{A}$, the comma object of a along b, if it exists, is an object $a \downarrow b$ with the following universal property. There are 1-cells $a^{\Rightarrow}(b) : a \downarrow b \to x$

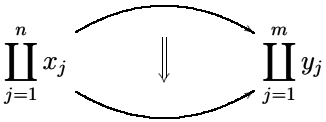
Proposition 8.5. *The fully faithful 2-functor*

$$I_{\mathbb{A}} : \mathbb{A} \to \mathsf{Fam}_{\mathsf{fin}}(\mathbb{A})$$

has a left 2-adjoint if and only if $\mathbb{A}$ has finite coproducts. In this case, the left 2-adjoint is given by the coproduct. More precisely, a 2-cell

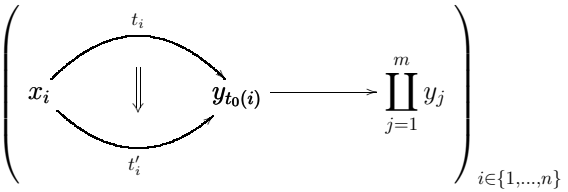
$$(\tau_1, \dots, \tau_n) : (t_0, \dots, t_n) \Longrightarrow (t_0', \dots, t_n') : (x_1, \dots, x_n) \to (y_1, \dots, y_m)$$

in $\mathsf{Fam}_{\mathsf{fin}}(\mathbb{A})$ is taken to the unique 2-cell



(8.0.1)

induced by the 2-cells



(8.0.2)

in which the second arrows are the components of the universal cocone that gives the coproduct.

Remark 8.6. If we replace *finite families* with *arbitrary families* in Definition 8.4, we get the concept of $\mathsf{Fam}(\mathbb{A})$ which corresponds to the free cocompletion of $\mathbb{A}$ under coproducts.

We say that a 2-category $\mathbb{A}$ *has finite limits* if it has finite products, pullbacks and comma objects. The well-known notion of extensive category has an obvious (strict) 2-dimensional analogue. In order to simplify the hypothesis on completion of the 2-category $\mathbb{A}$, we are going to consider lextensive 2-categories.

Definition 8.7. [Lextensive 2-category] A 2-category $\mathbb{A}$ is lextensive if it has finite limits and coproducts, and, for every finite family of objects $(y_1, \dots, y_n)$,

Proof: In fact $\mathbb{A} \to \overline{\mathbb{A}}$ has a left 2-adjoint if and only if $\mathbb{A}$ has initial object. Moreover, provided that $\mathbb{A}$ has initial object, we denote by η the unit of this 2-adjunction and by $\overline{0}$ the initial object freely added.

We have that η_x is invertible whenever $x \neq \overline{0}$. Therefore, in this case,

$$\eta_x^* \circ \check{G} : \mathbb{A}/x \to \overline{\mathbb{A}}/x$$

is fully faithful.

Moreover, $\eta_{\overline{0}}^* \circ \check{G} : \mathbb{A}/0 \to \overline{\mathbb{A}}/\overline{0}$ is clearly an isomorphism, since $\mathbb{A}/0$ and $\overline{\mathbb{A}}/\overline{0}$ are both empty.

This completes the proof that G is admissible w.r.t. the basic fibration and, hence, 2-admissible provided that it has comma objects. ∎

Another example is the free cocompletion of a 2-category under (finite) coproducts.

Definition 8.4. Let $\mathbb{A}$ be a 2-category. We define the 2-category $\mathsf{Fam}_{\mathsf{fin}}(\mathbb{A})$ as follows. The objects of $\mathsf{Fam}_{\mathsf{fin}}(\mathbb{A})$ are finite families of objects of $\mathbb{A}$, which can be seen as (possibly empty) lists of objects

$$(x_1, \dots, x_n).$$

In this case, a morphism $(x_1, \dots, x_n) \to (y_1, \dots, y_m)$ is a list $t = (t_0, \dots, t_n)$ in which

$$t_0 : \{1, \dots, n\} \to \{1, \dots, m\}$$

is a function, and, for $j > 0$,

$$t_j : x_j \to y_{t_0(j)}$$

is a morphism of $\mathbb{A}$. The composition and, hence, the identities are defined pointwise. Finally, given morphisms

$$t = (t_0, \dots, t_n), t' = (t'_0, \dots, t'_n) : (x_1, \dots, x_n) \to (y_1, \dots, y_m)$$

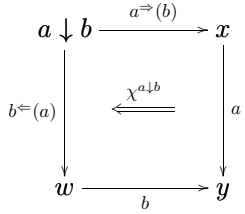
of $\mathsf{Fam}_{\mathsf{fin}}(\mathbb{A})$, there is no 2-cell $t \Rightarrow t'$, provided that $t_0 \neq t'_0$. Otherwise, a 2-cell $\tau : t \Rightarrow t'$ is a finite family of 2-cells

$$\left(\tau_j : t_j \Rightarrow t'_j : x_j \to y_{t_0(j)}\right)_{j \in \{1,\dots,n\}}$$

of $\mathbb{A}$. The horizontal and vertical compositions are again defined pointwise.

There is an obvious full faithful 2-functor $I_{\mathbb{A}} : \mathbb{A} \to \mathsf{Fam}_{\mathsf{fin}}(\mathbb{A})$ which takes each object x to the family (x). As observed above, the 2-category $\mathsf{Fam}_{\mathsf{fin}}(\mathbb{A})$ is the *free cocompletion* of $\mathbb{A}$ under finite coproducts. In particular, we have:

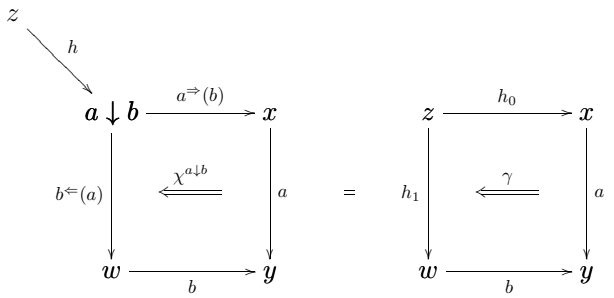
and $b^{\Leftarrow}(a) : a \downarrow b \to w$ and a 2-cell



(2.2.1)

such that:

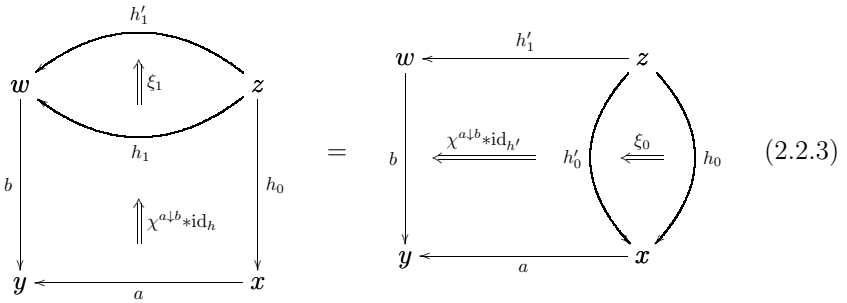
(1) For every triple $(h_0 : z \to x, h_1 : z \to w, \gamma : ah_0 \Rightarrow bh_1)$ in which h_0, h_1 are morphisms and γ is a 2-cell of $\mathbb{A}$, there is a unique morphism $h : z \to a \downarrow b$ such that the equations $h_0 = a^{\Rightarrow}(b) \cdot h$, $h_1 = b^{\Leftarrow}(a) \cdot h$ and



(2.2.2)

hold.

(2) For every pair of 2-cells $(\xi_0 : h_0 \Rightarrow h'_0 : z \to x, \xi_1 : h_1 \Rightarrow h'_1 : z \to w)$ such that



(2.2.3)

holds, there is a unique 2-cell $\xi : h \Rightarrow h' : z \to a \downarrow b$ such that $\mathrm{id}_{a \Rightarrow (b)} * \xi = \xi_0$ and $\mathrm{id}_{b \Leftarrow (a)} * \xi = \xi_1$.

Remark 2.2. If $\mathbb{A}$ is a locally discrete 2-category, the comma object of a morphism a along b has the same universal property of the pullback of a along b.

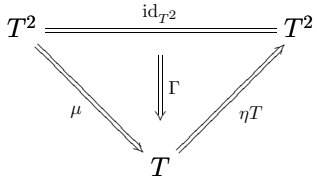
3. Lax idempotent 2-adjunctions

Herein, our standpoint is that the notion of *pre-Kock-Zöberlein 2-functor* is the 2-dimensional counterpart of the notion of *full reflective functor*. In this section, we recall the basic definitions and give basic characterizations, but we refer to [24, 34, 22] for fundamental aspects on lax idempotent 2-monads.

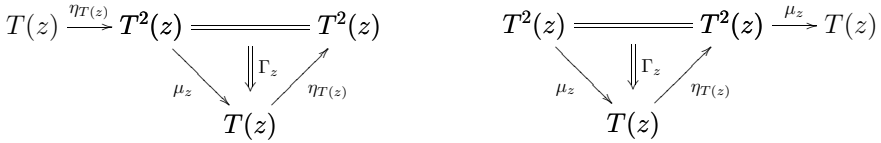
Definition 3.1. [Lax idempotent 2-monad] A *lax idempotent 2-monad* is a 2-monad $\mathcal{T} = (T, \mu, \eta)$ such that we have a rari adjunction $\mu \dashv \eta * \mathrm{id}_T$.

An *idempotent 2-monad* is a 2-monad $\mathcal{T} = (T, \mu, \eta)$ such that μ is invertible or, in other words, it is a lax idempotent 2-monad such that $\mu \dashv \eta * \mathrm{id}_T$ is a rali adjunction as well.

More explicitly, a 2-monad $\mathcal{T} = (T, \mu, \eta)$ on a 2-category $\mathbb{B}$ is lax idempotent if there is a modification



such that, for each object $z \in \mathbb{B}$,



are respectively the identity 2-cells on $\eta_{T(z)}$ and on μ_z.

Remark 3.2. [Dualities and self-duality] The concepts of lax idempotent and idempotent 2-monads are actually notions that can be defined inside any 3-category (or, more generally, tricategory [13]). Therefore they have eight dual notions each (counting the concept itself).

If $F \dashv G$ is admissible w.r.t. the basic fibration, we get that, for every $y \in \mathbb{B}$, $\eta^*_y \circ \check{G}$ is full reflective. Therefore $\eta^*_y \circ \check{G} \circ \mathrm{id}^{\Leftarrow}_{F(y)}$ is a pre-Kock-Zöberlein 2-functor by Corollary 4.13. By Proposition 7.12, this means that $F \dashv G$ is 2-admissible. ∎

8. Examples

The references [9, 10] provide several examples of simple 2-adjunctions/monads. In this section, we give examples of 2-admissible 2-adjunctions which, in particular, are also examples of simple 2-adjunctions.

Our first example of 2-admissible 2-adjunction is the identity. The result below follows directly from Theorem 6.10.

Lemma 8.1. *Let $\mathbb{A}$ be any 2-category with comma objects. The 2-adjunction $\mathrm{id}_{\mathbb{A}} \dashv \mathrm{id}_{\mathbb{A}}$ is 2-admissible.*

Of course, the identity is also an example of admissible 2-functor w.r.t. the basic fibration. Moreover, by Theorem 7.13, examples of admissible 2-functors w.r.t. the basic fibrations give us a wide class of examples of 2-admissible 2-functors.

Theorem 8.2. *Let* ord *be the 2-category of preordered sets, and* cat *the 2-category of small categories. The inclusion 2-functor* $\mathsf{ord} \to \mathsf{cat}$ *has a left 2-adjoint and it is admissible w.r.t. the basic fibration (and, hence, also 2-admissible).*

Proof: It is known that the underlying adjunction is admissible (w.r.t. the basic fibration) [40]. Since cat is a complete 2-category, we get that the 2-adjunction is admissible w.r.t. the basic fibration. ∎

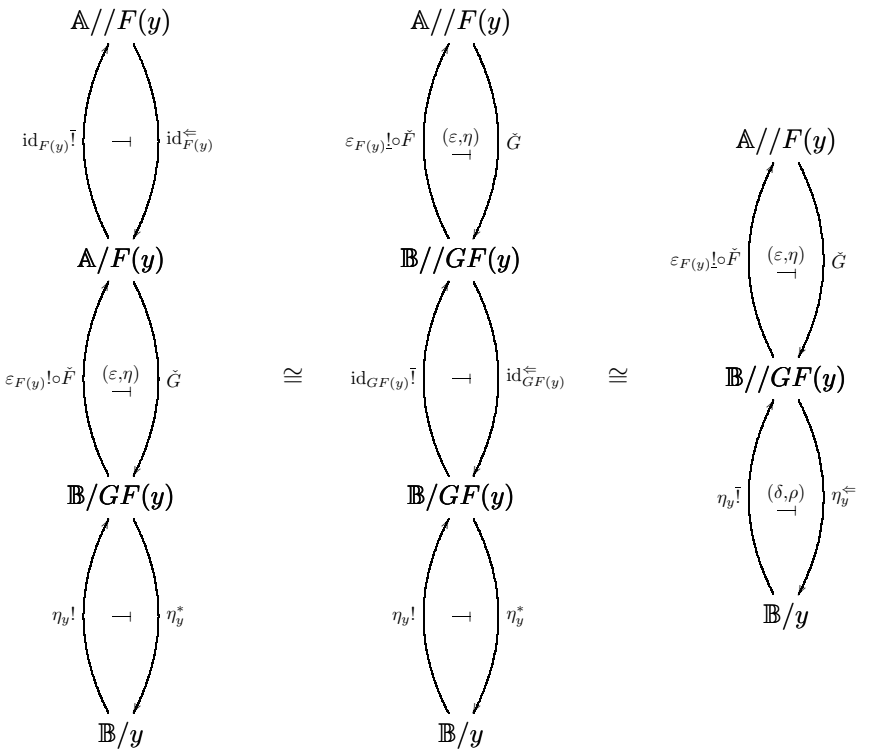
Free cocompletions of 2-categories also give us a good source for examples of admissibility w.r.t. the basic fibration. In particular, the most basic cocompletion is the free addition of the initial object.

Theorem 8.3. *Let $\mathbb{A}$ be a 2-category with pullbacks and an initial object 0. We denote by $\overline{\mathbb{A}}$ the free addition of an initial object. If $\mathbb{A}(-, 0) : \mathbb{A}^{\mathrm{op}} \to \mathsf{Cat}$ is constantly equal to the empty category, the canonical 2-functor*

$$G : \mathbb{A} \to \overline{\mathbb{A}}$$

is admissible w.r.t. the basic fibration (and, hence, if $\mathbb{A}$ has comma objects, it is 2-admissible as well).

comma objects and, hence, we get that



(7.0.10)

in which the second 2-natural isomorphism follows from Theorem 6.9. By the definitions of simplicity and 2-admissibility (see Definitions 4.10 and 4.5), the proof is complete. ∎

Theorem 7.13. *Provided that $\mathbb{A}$ has comma objects, if $(F \dashv G) : \mathbb{A} \to \mathbb{B}$ is admissible w.r.t. the basic fibration, then it is 2-admissible.*

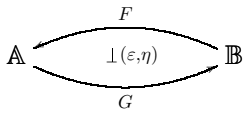
Proof: By Theorem 6.10, the 2-functor $\mathrm{id}^{\Leftarrow}_{F(y)}$ (the right 2-adjoint of (7.0.8)) is a pre-Kock-Zöberlein 2-functor for every $y \in \mathbb{B}$.

However, the notions of lax idempotent and idempotent 2-monads are self-dual, that is to say, the dual notion coincides with itself. More precisely, a triple $\mathcal{T} = (T, \mu, \eta)$ is a (lax) idempotent 2-monad in the 3-category 2-Cat if and only if the corresponding triple is also a (lax) idempotent 2-monad in the 3-category $(2\text{-}\mathsf{Cat})^{\mathrm{op}}$.

Furthermore, the notion of idempotent 2-monad is self-3-dual, meaning that the notion does not change when we invert the directions of the 3-cells (which are, in our case, the modifications). However the 3-dual of the notion of lax idempotent 2-monad is that of colax idempotent 2-monad.

Finally, the notions obtained from the inversion of the directions of the 2-cells, that is to say, the codual (or 2-dual) concepts, are those of lax idempotent and idempotent 2-comonads.

Henceforth, throughout this section, we always assume that a 2-adjunction



is given, and we denote by $\mathcal{T} = (T, \mu, \eta)$ the induced 2-monad $(GF, G\varepsilon F, \eta)$ on $\mathbb{B}$.

3.1. Idempotency. There are several useful well-known characterizations of idempotent (2-)monads (see, for instance, [3, pag. 196]).

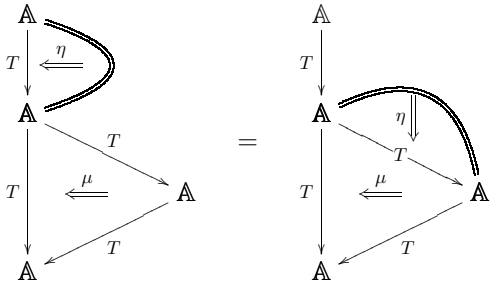
Lemma 3.3 (Idempotent 2-monad)**.** *The following statements are equivalent.*

i) $\mathcal{T}$ is idempotent;
ii) $T\eta$ (or ηT) is an epimorphism;
iii) μ is a monomorphism;
iv) $T\eta = \eta T$;
v) $a : T(x) \to x$ is a $\mathcal{T}$-algebra structure if, and only if, $a \cdot \eta_x = \mathrm{id}_x$;
vi) $a : T(x) \to x$ is a $\mathcal{T}$-algebra structure if, and only if, a is the inverse of η_x;
vii) the forgetful 2-functor $\mathcal{T}$-$\mathsf{Alg}_{\mathsf{s}} \to \mathbb{B}$ between the 2-category of (strict) $\mathcal{T}$-algebras (and strict $\mathcal{T}$-morphisms) and the 2-category $\mathbb{B}$ is fully faithful (that is to say, locally an isomorphism).

Proof: Since $\mu \cdot (\eta T) = \mu \cdot (T\eta) = \mathrm{id}_T$, we have the following chain of equivalences: μ is a monomorphism $\Leftrightarrow$ μ is invertible $\Leftrightarrow$ ηT or $T\eta$ is invertible $\Leftrightarrow$

ηT or $T\eta$ is an epimorphism. This proves the equivalence of the first three statements.

By the definition of monomorphism, iii) implies iv). Reciprocally, assuming that $T\eta = \eta T$, we have that $T^2\eta = T\eta T$ and, thus, we get that

$(T\eta)\cdot\mu \quad =$  $=$ $= \quad \mathrm{id}_{T^2}.$

Therefore $T\eta$ is the inverse of μ and, hence, μ is a monomorphism.

Assuming one of the first four equivalent statements (and hence all of them), we have that, given a morphism $a : T(x) \to x$ such that $a \cdot \eta_x = \mathrm{id}_x$, the equation

$$\eta_x \cdot a = T(a) \cdot \eta_{T(x)} = T(a \cdot \eta_x) = \mathrm{id}_{T(x)}. \tag{3.1.1}$$

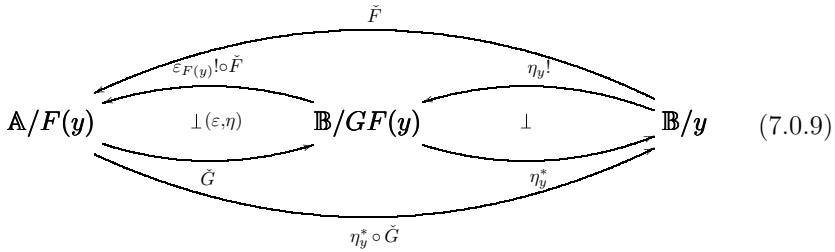
holds. Thus, since $\eta_{T(x)} \cdot \eta_x = T(\eta_x) \cdot \eta_x$ and $\mu = (T\eta)^{-1}$, we conclude that

$$a \cdot \mu_x = \left(\eta_{T(x)} \cdot \eta_x\right)^{-1} = \left(T\left(\eta_x\right) \cdot \eta_x\right)^{-1} = a \cdot T(a). \tag{3.1.2}$$

This proves that v) holds. Reciprocally, v) trivially implies iii) (and, hence, all of the first four equivalent statements), since, for each $x \in \mathbb{B}$, μ_x is a (free) $\mathcal{T}$-algebra structure for x. Moreover, by Equations (3.1.1) and (3.1.2), we conclude that the first four statements are also equivalent to vi).

Finally, recall that every forgetful functor $\mathcal{T}\text{-}\mathsf{Alg}_{\mathsf{s}} \to \mathbb{B}$ is faithful. Assuming vi), in order to verify that the forgetful functor is full, it is enough to see that, for any morphism $f : x \to y$ of $\mathbb{B}$, if $a : T(x) \to x$, $b : T(y) \to y$ are

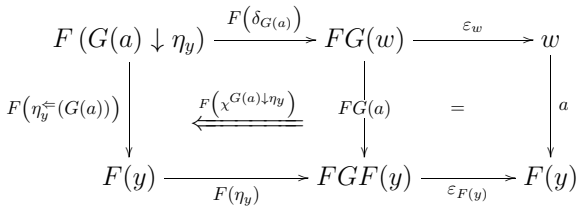
is simple (2-admissible) w.r.t. the composite of the 2*-adjunctions*



(7.0.9)

Proof: By definition, $F \dashv G$ is simple (2-admissible) if, and only if, for each object $y \in \mathbb{B}$, the composition of the 2-adjunctions of (7.0.5) is lax idempotent (pre-Kock-Zöberlein). Since G is right 2-adjoint, it preserves

and, for every object $y \in \mathbb{B}$ and every object $a : w \to F(y)$ of $\mathbb{A}//F(y)$, the morphism defined by



in $\mathbb{A}//F(y)$ is a regular epimorphism, i.e. the morphism defined by

$$\left(\varepsilon_w \cdot F\left(\delta_{G(a)}\right), \mathrm{id}_{\varepsilon_{F(y)}} * F\left(\chi^{G(a)\downarrow\eta_y}\right)\right) : \varepsilon_{F(y)}! \, \check{F}\, \eta_y^{\Leftarrow}\, \check{G}(a) \to a$$

is a regular epimorphism in $\mathbb{A}//F(y)$.

Proof: The result follows from Corollary 7.3 and Theorem 4.11. ∎

Corollary 7.11. *If G is pre-Kock-Zöberlein then $F \dashv G$ is 2-admissible, provided that, for each $y \in \mathbb{B}$, $\check{F}\delta\check{G}$ is invertible.*

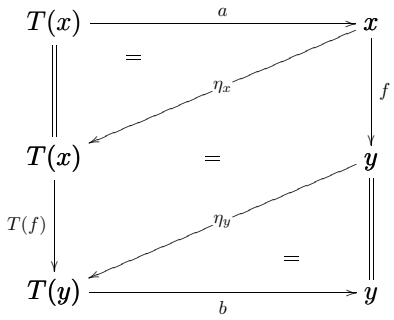
Proof: It follows from Corollary 7.3 and Corollary 4.13. ∎

It should be noted that by Lemma 3.17 we can conclude that the notion of simplicity w.r.t. the basic fibration (admissibility w.r.t. the basic fibration) coincides with the notion of simplicity (2-admissibility) if $\mathbb{A}$ and $\mathbb{B}$ are locally discrete. This shows that the notion of simplicity and 2-admissibility can be seen as generalizations of the classical notions of simplicity and admissibility/semi-left exact reflective functors [7, 4] when categories are seen as locally discrete 2-categories. Furthermore, Theorem 7.13 shows that classical admissibility implies 2-admissibility in the presence of comma objects.

Proposition 7.12. *Assume that $F \dashv G$ is pre-Kock-Zöberlein, and $\mathbb{A}$ has comma objects. The 2-adjunction $F \dashv G$ is simple (2-admissible) if, and only if, for each object $y \in \mathbb{B}$, the 2-adjunction*

$$\mathbb{A}//F(y) \underset{\mathrm{id}^{\Leftarrow}_{F(y)}}{\overset{\mathrm{id}_{F(y)}!}{\overset{\frown}{\underset{\smile}{\perp}}}} \mathbb{A}/F(y) \tag{7.0.8}$$

$\mathcal{T}$-algebra structures, we have that the pasting



is the identity 2-cell and, hence, the morphism f induces a morphism of algebras between (x, a) and (y, b).

Assuming vii), we get that, for any object $x \in \mathbb{B}$, $\eta_{T(x)}$ induces a morphism between the free $\mathcal{T}$-algebras $(T(x), \mu_x)$ and $\left(T^2(x), \mu_{T(x)}\right)$. That is to say,

$$\eta_{T(x)} \cdot \mu_x = \mu_{T(x)} \cdot T(\eta_{T(x)})$$

and, since the right side of the equation above is equal to the identity on $T^2(x)$, we conclude that μ_x is a split monomorphism. This proves that iii) holds. ∎

A 2-adjunction induces an idempotent 2-monad if, and only if, the induced 2-comonad is also idempotent. More generally:

Proposition 3.4. *The following statements are equivalent.*

i) $\mathcal{T}$ is idempotent;
ii) $F\eta$ (or ηG) is an epimorphism;
iii) εF (or $G\varepsilon$) is a monomorphism;
iv) The induced 2-comonad is idempotent.

Proof: Since, by the triangle identities, we have that

$$(\varepsilon F) \cdot (F\eta) = \mathrm{id}_F \text{ and } (G\varepsilon) \cdot (\eta G) = \mathrm{id}_G,$$

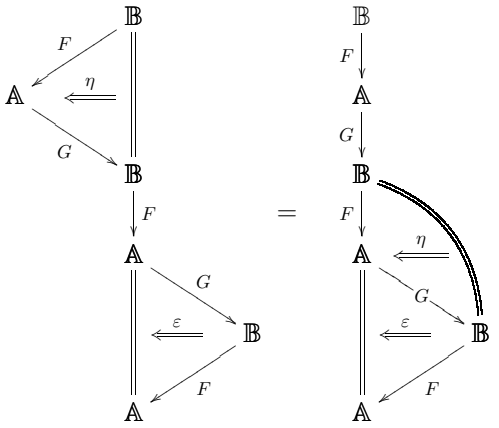
we get that ii) implies that εF or $G\varepsilon$ is invertible and, therefore, $G\varepsilon F = \mu$ is invertible. Analogously, iii) implies i).

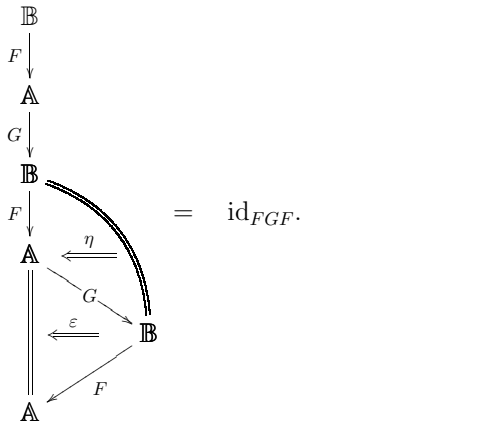
Moreover, if we assume that $\mathcal{T}$ is idempotent, by Lemma 3.3, we have that

$$GF\eta = \eta GF$$

which, together with one of the triangle identities, implies that

$(F\eta)\cdot(\varepsilon F) \quad = \quad = \quad = \quad \mathrm{id}_{FGF}.$





This proves that i) implies ii) and iii). Therefore we proved that i), ii) and iii) are equivalent statements.

Finally, since condition iii) is codual and equivalent to condition ii), we conclude that i) is equivalent to its codual – that is to say, to condition iv). ∎

Motivated by the result above, we say that a 2-adjunction is *idempotent* if it induces an idempotent 2-(co)monad.

Remark 3.5. If the 2-adjunction $F \dashv G : \mathbb{A} \to \mathbb{B}$ is such that the underlying category of $\mathbb{A}$ (or $\mathbb{B}$) is *thin*, then the induced 2-monad is idempotent by Proposition 3.4. In particular, seeing categories as locally discrete 2-categories and contravariant 2-functors as covariant ones defined in the dual of the respective domains, any *Galois connection* induces an idempotent (2-)(co)monad.

If the 2-adjunction $F \dashv G$ is idempotent and G is 2-monadic, G is called a *full reflective 2-functor*. This terminology is justified by the well-known characterization below.

Proposition 3.6 (Full reflective 2-functor)**.** *The following statements are equivalent.*

i) G is a full reflective 2-functor;
ii) $F \dashv G$ is idempotent and G is 2-premonadic;

$$(GF(b))^{\Rightarrow}(\eta_y)\cdot\alpha_b = \eta_w \qquad \text{and} \qquad \eta_y^{\Leftarrow}(GF(b))\cdot\alpha_b = b$$

hold.

Remark 7.7. The composition of $\varepsilon_{F(y)}!\circ\check{F}$ with $\eta_y!$ is given by $\check{F}$. More precisely, the diagrams

$$\mathbb{A}//F(y) \xleftarrow[\varepsilon_{F(y)}!\circ\check{F}]{} \mathbb{B}//GF(y) \xleftarrow[\eta_y!]{} \mathbb{B}/y \qquad \mathbb{A}/F(y) \xleftarrow[\varepsilon_{F(y)}!\circ\check{F}]{} \mathbb{B}/GF(y) \xleftarrow[\eta_y!]{} \mathbb{B}/y$$

commute.

As direct consequences of the main results of Section 4, we get the following corollaries.

Corollary 7.8 (Simplicity [9])**.** *Let G be pre-Kock-Zöberlein. The 2-adjunction*

$$(F \dashv G, \varepsilon, \eta) : \mathbb{A} \to \mathbb{B}$$

is simple if, and only if, for each $y \in \mathbb{B}$,

$$\mathrm{id}_T * \alpha \dashv \mu\cdot(\mathrm{id}_T * \delta * \mathrm{id}_T)$$

in which $(\mathrm{id}_T * \alpha)$ is pointwise defined by $(\mathrm{id}_T * \alpha)_b := T(\alpha_{(x,b)})$, and $\mu\cdot(\mathrm{id}_T * \delta * \mathrm{id}_T)$ is pointwise defined by

$$(\mu\cdot(\mathrm{id}_T * \delta * \mathrm{id}_T))_b \quad := \quad \begin{array}{ccccc} T\left(T(b)\downarrow\eta_y\right) & \xrightarrow{T\left(\delta_{T(b)}\right)} & TT(x) & \xrightarrow{\mu_x} & T(x) \\ \downarrow{\scriptstyle T\left(\eta_y^{\Leftarrow}(T(b))\right)} & \xleftarrow{T\left(\chi^{T(b)\downarrow\eta_y}\right)} & \downarrow{\scriptstyle TT(b)} & = & \downarrow{\scriptstyle T(b)} \\ T(y) & \xrightarrow[T(\eta_y)]{} & TT(y) & \xrightarrow{\mu_y} & T(y) \end{array}$$

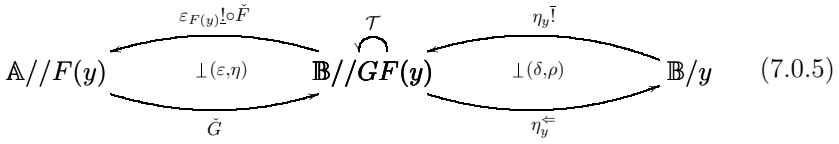
Proof: The result follows from Corollary 7.3 and Theorem 4.6. ∎

Corollary 7.9. *Assume that $F \dashv G$ is lax idempotent. We have that $F \dashv G$ is simple provided that, for each $y \in \mathbb{B}$, $\eta_y^{\Leftarrow}T\delta\check{G}$ or $F\delta T$ is invertible.*

Proof: It follows from Corollary 7.3 and Corollary 4.8. ∎

Corollary 7.10 (2-admissibility)**.** *Assume that G is pre-Kock-Zöberlein. The 2-adjunction $(F \dashv G, \varepsilon, \eta) : \mathbb{A} \to \mathbb{B}$ is 2-admissible if and only if it is simple*

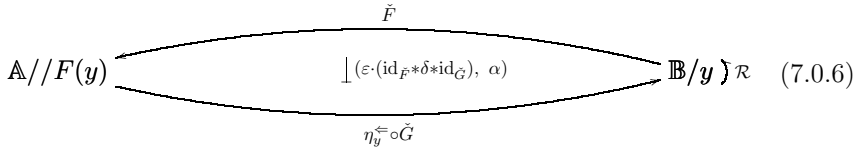
each y of $\mathbb{B}$, we consider the 2-adjunctions



(7.0.5)

in which, by abuse of language, we denote respectively by ε and η the counit and unit defined pointwise, and $\mathcal{T} = (T, \mu, \eta)$ the 2-monad induced by $\varepsilon_{F(y)}!\circ \check{F} \dashv \check{G}$.

In this case, the composition of 2-adjunctions above is given by



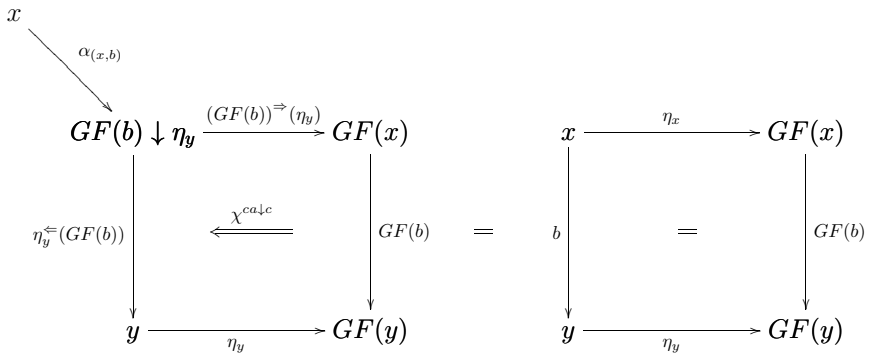
(7.0.6)

where $\alpha = \left(\mathrm{id}_{\eta_y^{\Leftarrow}} * \eta * \mathrm{id}_{\eta_y\bar{!}}\right)\cdot\rho$, and we denote by $\mathcal{R} = (R, v, \alpha)$ the 2-monad induced by $\check{F} \dashv \eta_y^{\Leftarrow}\circ\check{G}$.

Remark 7.6. [α] Given an object $(x, b) \in \mathbb{B}/y$,

$$\alpha_{(x,b)} : (x, b) \to \eta_y^{\Leftarrow}\check{G}\check{F}\,(x, b)$$

is defined by the unique morphism $\alpha_b : w \to GF(b) \downarrow \eta_y$ in $\mathbb{B}$ such that the equations



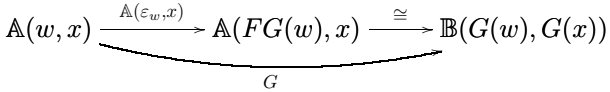
(7.0.7)

iii) G is fully faithful;
iv) ε is invertible.

Proof: Recall that a 2-functor is 2-premonadic if the (Eilenberg-Moore) comparison 2-functor is fully faithful (that is to say, locally an isomorphism).

We have that i) trivially implies ii). Moreover, since the forgetful 2-functor $\mathcal{T}\text{-}\mathsf{Alg}_s \to \mathbb{B}$ is fully faithful whenever $\mathcal{T}$ is idempotent, we have that ii) implies iii).

Since, for every pair of objects $w, x \in \mathbb{A}$, the diagram



commutes, iii) and iv) are equivalent.

Assuming iv), we have in particular that ε is a split epimorphism and G reflects isomorphisms, hence, G is 2-monadic (see Proposition at [33, pag. 236]). Furthermore, clearly, we also get that $G\varepsilon$ is a (split) monomorphism, which implies that $F \dashv G$ is idempotent by Proposition 3.4. Therefore iv) implies i). ∎

The dual notion of full reflective 2-functor in 2-Cat is called *full co-reflective* 2-*functor*. As a consequence of Proposition 3.6, we have:

Corollary 3.7. *If $F \dashv G$ is such that F is full co-reflective and G is full reflective, then $F \dashv G$ is a 2-adjoint equivalence.*

Remark 3.8. [Idempotent 2-adjunction vs. full reflective 2-functor] It should be noted that there are non-2-monadic idempotent 2-adjunctions. Remark 3.5 gives a way of constructing easy examples. For instance, given a 2-category $\mathbb{A}$, the unique 2-functor $\mathbb{A} \to \mathbf{1}$ has a left 2-adjoint if and only if $\mathbb{A}$ has an initial object. Assuming that $\mathbb{A}$ has an initial object and $\mathbb{A}$ is not (2-)equivalent to $\mathbf{1}$, the 2-functor $\mathbb{A} \to \mathbf{1}$ is not a reflective 2-functor, although the 2-adjunction is idempotent.

More generally, by Corollary 3.7 any full reflective 2-functor which is not an equivalence gives an example of an idempotent 2-adjunction such that the left 2-adjoint is not 2-comonadic. Dually, any non-equivalence full co-reflective 2-functor gives an idempotent 2-adjunction such that the right 2-adjoint is not a full reflective 2-functor.

3.2. Kleisli vs. idempotent adjunctions. Recall that a 2-adjunction $F \dashv G$ *is Kleisli if the Kleisli comparison* 2*-functor* is an equivalence. This fact holds if, and only if, F is essentially surjective on objects. Moreover, a Kleisli 2-adjunction is always premonadic, since the Kleisli 2-category is equivalent to the full sub-2-category of free algebras of the 2-category $\mathcal{T}\text{-}\mathsf{Alg}_{\mathsf{s}}$ of the (strict) algebras of the induced 2-monad.

It should be noted that, by Proposition 3.6, we have that, whenever a 2-adjunction $F \dashv G$ is idempotent, G *is* 2*-premonadic if and only if* G *is* 2*-monadic*. Therefore by Lemma 3.9 below, this means that, whenever $\mathcal{T}$ is idempotent, the Kleisli 2-category is (2-)equivalent to the 2-category $\mathcal{T}\text{-}\mathsf{Alg}_{\mathsf{s}}$.

Lemma 3.9. *The following statements are equivalent.*

i) The Kleisli 2*-category w.r.t.* $\mathcal{T}$ *is* 2*-equivalent to the* 2*-category of (strict)* $\mathcal{T}$*-algebras.*
ii) If $F' \dashv G'$ *induces* $\mathcal{T}$*, then* G' *is* 2*-premonadic if, and only if,* G' *is* 2*-monadic.*

By Proposition 3.6, we conclude the following well-known result:

Corollary 3.10. *An idempotent* 2*-adjunction* $F \dashv G$ *is* 2*-monadic if, and only if, it is Kleisli.*

3.3. Lax idempotency. For this part, we assume the definition of strict algebras and lax $\mathcal{T}$-morphisms between them, which can be found, for instance, in [31, Definition 2.2]. Theorem 3.11 is a well-known characterization of lax idempotent 2-monads [24]. We refer to [34, 22] for the proofs.

Theorem 3.11 (Lax idempotent 2-monad)**.** *The following statements are equivalent.*

i) $\mathcal{T}$ *is lax idempotent;*
ii) $\mathrm{id}_T * \eta \dashv \mu$ *is a rali adjunction;*
iii) $a : T(x) \to x$ *is a* $\mathcal{T}$*-algebra structure if, and only if, there is a rari adjunction* $a \dashv \eta_x$*;*
iv) $a : T(x) \to x$ *is a* $\mathcal{T}$*-pseudoalgebra structure if, and only if, there is an adjunction* $a \dashv \eta_x$*;*
v) the forgetful 2*-functor* $\mathcal{T}\text{-}\mathsf{Alg}_{\ell} \to \mathbb{B}$ *between the* 2*-category of (strict)* $\mathcal{T}$*-algebras (and lax* $\mathcal{T}$*-morphisms) and the* 2*-category* $\mathbb{B}$ *is fully faithful.*

are lax idempotent (premonadic) if, and only if, $F \dashv G$ *is lax idempotent (premonadic).*

Henceforth, we further assume that $\mathbb{B}$ has comma objects and pullbacks whenever necessary. Recall that, in this case, by Section 6, for each object y of $\mathbb{B}$, we have 2-adjunctions

$$\eta_y! \dashv \eta_y^* : \mathbb{B}/GF(y) \to \mathbb{B}/y \text{ and } \eta_y\overline{!} \dashv \eta_y^{\Leftarrow} : \mathbb{B}//GF(y) \to \mathbb{B}/y$$

in which the right 2-adjoints are given respectively by the pullback and the comma object along η_y.

Definition 7.4. [Simple, admissible and 2-admissible 2-functors] The 2-functor G is called *simple/*2*-admissible* if $F \dashv G$ is lax idempotent/pre-Kock-Zöberlein, and, for every $y \in \mathbb{B}$,

$$\mathbb{A}//y \underset{\check{G}}{\overset{\varepsilon_y\overline{!}\circ\check{F}}{\leftarrow}} \mathbb{B}//G(y) \qquad (7.0.3)$$

is simple/2-admissible w.r.t. $\eta_y! \dashv \eta_y^{\Leftarrow}$ (see Definitions 4.5 and 4.10).

We say that G is *admissible w.r.t. the basic fibration* if G is fully faithful, and, for every $y \in \mathbb{B}$,

$$\mathbb{A}/y \underset{\check{G}}{\overset{\varepsilon_y!\circ\check{F}}{\leftarrow}} \mathbb{B}/G(y) \qquad (7.0.4)$$

is admissible w.r.t. $\eta_y! \dashv \eta_y^*$.

Remark 7.5. The notion of admissibility w.r.t. the basic fibration is just the direct strict 2-dimensional generalization of the classical notion of admissibility (also called semi-left-exact reflective functor) [7, 4], while the notion of simplicity coincides with that introduced in [9].

In order to establish the direct consequences of the results of Section 4 for the case of 2-admissibility and simplicity, we set some notation below. For

Therefore, by the universal property of the comma object $b \downarrow \mathrm{id}_y$, we get that $\Gamma_{(x,b)} * \mathrm{id}_{\underline{D}_{(x,b)}} = \mathrm{id}_{\mathrm{id}_{\underline{D}_{(x,b)}}}$. This completes the proof that the second triangle identity holds. ∎

7. Admissibility

Throughout this section,

$$\mathbb{A} \underset{G}{\overset{F}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp\,(\varepsilon,\eta)}}}} \mathbb{B}$$

is a given 2-adjunction. By abuse of language, given any 2-functor $H : \mathbb{A} \to \mathbb{B}$, for each object x in $\mathbb{A}$, we denote by the same $\check{H}$ the 2-functors

$$\check{H} : \mathbb{A}/x \to \mathbb{B}/H(x), \quad \check{H} : \mathbb{A}/x \to \mathbb{B}//H(x), \quad \check{H} : \mathbb{A}//x \to \mathbb{B}//H(x)$$

pointwise defined by H. Moreover, given a morphism $f : w \to x$ of $\mathbb{A}$, we denote by

$$f\underline{!} : \mathbb{A}//w \to \mathbb{A}//x$$

the 2-functor defined by the *direct image* between the lax comma 2-categories, whose restriction to $\mathbb{A}/w$ is equal to $f\overline{!}$.

Proposition 7.1. *If G is a locally fully faithful 2-functor then, for each object x of $\mathbb{A}$, both $\check{G} : \mathbb{A}/x \to \mathbb{B}/G(x)$ and $\check{G} : \mathbb{A}//x \to \mathbb{B}//G(x)$ are locally fully faithful.*

Theorem 7.2. *For any object $y \in \mathbb{A}$, we have two 2-adjunctions*

$$\mathbb{A}/y \underset{\check{G}}{\overset{\varepsilon_y!\circ\check{F}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathbb{B}/G(y) \quad \textit{and} \quad \mathbb{A}//y \underset{\check{G}}{\overset{\varepsilon_y\underline{!}\circ\check{F}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathbb{B}//G(y) \qquad (7.0.1)$$

where the counit and the unit of these 2-adjunctions are defined pointwise by the counit and unit of $F \dashv G$.

Corollary 7.3. *For each object $y \in \mathbb{A}$, the 2-adjunctions*

$$\mathbb{A}/y \underset{\check{G}}{\overset{\varepsilon_y!\circ\check{F}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathbb{B}/G(y) \quad \textit{and} \quad \mathbb{A}//y \underset{\check{G}}{\overset{\varepsilon_y\underline{!}\circ\check{F}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathbb{B}//G(y) \qquad (7.0.2)$$

Similarly to the idempotent case, a 2-adjunction induces a lax idempotent 2-monad if and only if it induces a lax idempotent 2-comonad. Furthermore, we give below a lax idempotent analogue of Proposition 3.4.

Theorem 3.12 (Lax idempotent 2-adjunction)**.** *The following statements are equivalent.*

i) $\mathcal{T}$ is lax idempotent;
ii) $G\varepsilon \dashv \eta G$ is a lali adjunction;
iii) $F\eta \dashv \varepsilon F$ is a rali adjunction;
iv) The induced 2-comonad is lax idempotent.

Proof: By Lemma 3.11, it is clear that ii) or iii) implies i). Reciprocally, assuming i), we have by Lemma 3.11 that $\mathrm{id}_{GF} * \eta \dashv \mathrm{id}_G * \varepsilon * \mathrm{id}_F$. By *doctrinal adjunction* (*e.g.* [20]), we conclude that $F(\eta_x) \dashv \varepsilon_{F(x)}$ for every x of $\mathbb{B}$. Finally, again, by doctrinal adjunction, we conclude that $\mathrm{id}_F * \eta \dashv \varepsilon * \mathrm{id}_F$. This proves that i) implies iii).

Analogously, by doctrinal adjunction, we get that i) implies ii). Hence we proved that the first three statements are equivalent.

Since the condition ii) is codual and equivalent to iii), we get that i) is equivalent to its codual – which means iv). ∎

Definition 3.13. [pre-Kock-Zöberlein 2-functor] If the induced 2-monad $\mathcal{T}$ is lax idempotent, the 2-adjunction $F \dashv G$ is *lax idempotent*. In this case if, furthermore, G is 2-premonadic, G is called a *pre-Kock-Zöberlein 2-functor*. Finally, if it is also 2-monadic, G is a *Kock-Zöberlein 2-functor*.

Proposition 3.14. *Assume that $F \dashv G : \mathbb{A} \to \mathbb{B}$ is lax idempotent. The following statements are equivalent.*

i) G is a pre-Kock-Zöberlein 2-functor;
ii) For each object $x \in \mathbb{A}$, ε_x is a regular epimorphism;
iii) For each object $x \in \mathbb{A}$,

$$FGFG(x) \underset{FG(\varepsilon_x)}{\overset{\varepsilon_{FG(x)}}{\rightrightarrows}} FG(x) \xrightarrow{\varepsilon_x} x \qquad (3.3.1)$$

is a coequalizer.

Proof: The result follows directly from the well-known characterization of (2-)premonadic (2-)functors due to Beck (see, for instance, [33, pag. 226]). ∎

Theorem 3.15. *Assume that $F \dashv G : \mathbb{A} \to \mathbb{B}$ is lax idempotent. The following statements are equivalent.*

i) G is a Kock-Zöberlein 2-functor;
ii) G creates absolute coequalizers;
iii) G is a pre-Kock-Zöberlein 2-functor, and, whenever η_y is a rari, there is $x \in \mathbb{A}$ such that $G(x) \cong y$.

Proof: The result follows from Proposition 3.14, and the characterization of algebra structures for lax idempotent 2-monads recalled in Theorem 3.11. ∎

Remark 3.16. [Algebras and free algebras] Corollary 3.10 says that a 2-functor G is Kleisli if and only if it is monadic, whenever $F \dashv G$ induces an idempotent 2-monad. This is not the case when $\mathcal{T}$ is only lax idempotent. The reference [15] provides several counterexamples in this direction. Moreover, in our context, in Section 6, Theorem 6.10 also provides several examples: more precisely, given any 2-category $\mathbb{A}$ and object $z \in \mathbb{A}$, the 2-adjunction between the *lax comma 2-category $\mathbb{A}//z$ (see Definition 5.1)* and the corresponding comma 2-category $\mathbb{A}/z$ usually is a Kleisli 2-adjunction which is not 2-monadic.

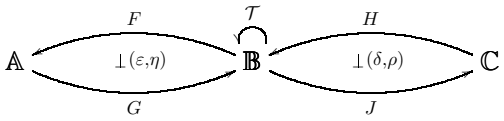
Finally, it should be noted that:

Lemma 3.17. *If $\mathbb{A}$ and $\mathbb{B}$ are locally discrete, we have that $F \dashv G$ is lax idempotent (G is pre-Kock-Zöberlein) if and only if $F \dashv G$ is idempotent (G is full reflective).*

Proof: It is enough to note that a 2-monad defined on a locally discrete 2-category is lax idempotent if and only if it is idempotent. The rest follows from Proposition 3.6. More particularly, it follows from the fact that 2-premonadicity and 2-monadicity are equivalent properties for idempotent 2-adjunctions. ∎

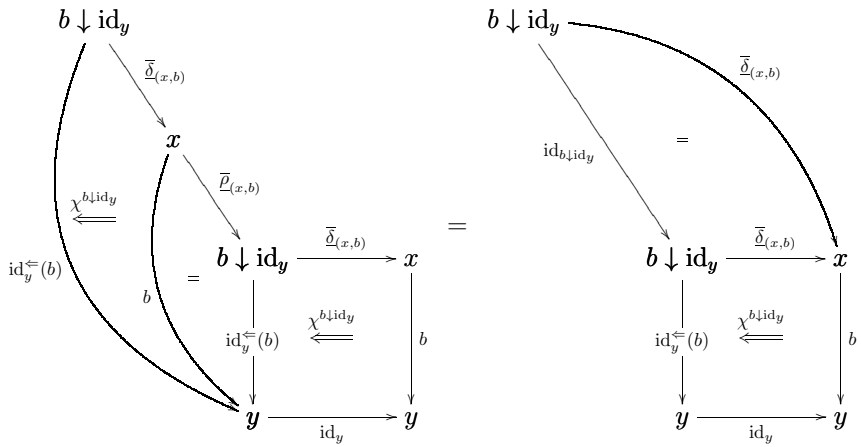
4. Composition of 2-adjunctions

Throughout this section,



(4.0.1)

For each object $(x, b) \in \mathbb{A}/y$, the pair of 2-cells $\left(\chi^{b\downarrow \mathrm{id}_y}, \mathrm{id}_{\overline{\delta}_{(x,b)}}\right)$ satisfies the equation



(6.0.28)

and, hence, by the universal property of the comma object, there is a unique 2-cell $\Gamma_{(x,b)}$ such that the equations

$$\mathrm{id}_{\mathrm{id}_y^{\Leftarrow}(b)} * \Gamma_{(x,b)} = \chi^{b\downarrow \mathrm{id}_y} \quad \text{and} \quad \mathrm{id}_{\overline{\delta}_{(x,b)}} * \Gamma_{(x,b)} = \mathrm{id}_{\overline{\delta}_{(x,b)}}$$

hold. The 2-cells $\Gamma_{(x,b)}$ define a modification

$$\Gamma : \overline{\rho} \cdot \overline{\delta} \Longrightarrow \mathrm{id}_{\mathrm{id}_y \cdot !\mathrm{id}_y^{\Leftarrow} \mathrm{id}_y \cdot !}$$

which we claim to be the counit of the adjunction (6.0.26).

The first triangle identity holds, since, by the definition of Γ above,

$$\mathrm{id}_{\overline{\delta}_{(x,b)}} * \Gamma_{(x,b)} = \mathrm{id}_{\overline{\delta}_{(x,b)}}$$

for every object $(x, b) \in \mathbb{A}/y$.

Finally, for each object $(x, b) \in \mathbb{A}/y$, $\Gamma_{(x,b)} * \mathrm{id}_{\overline{\rho}_{(x,b)}}$ is such that

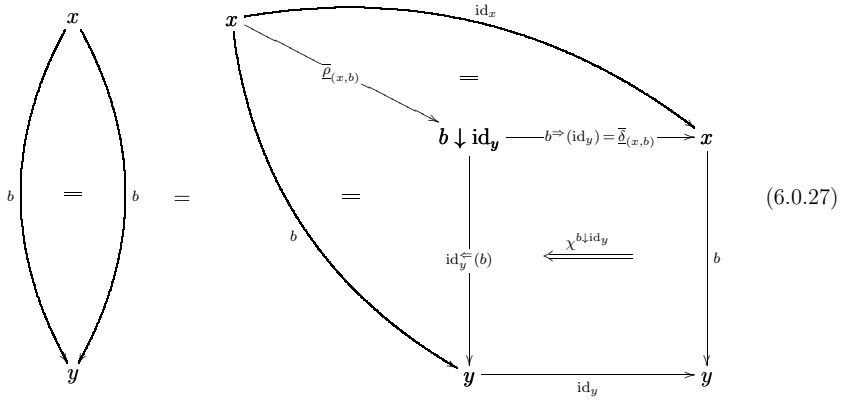
$$\mathrm{id}_{\mathrm{id}_y^{\Leftarrow}(b)} * \Gamma_{(x,b)} * \mathrm{id}_{\overline{\rho}_{(x,b)}} = \chi^{b\downarrow \mathrm{id}_y} * \mathrm{id}_{\overline{\rho}_{(x,b)}} = \mathrm{id}_b$$

by (6.0.27), and, of course,

$$\mathrm{id}_{\overline{\delta}_{(x,b)}} * \Gamma_{(x,b)} * \mathrm{id}_{\overline{\rho}_{(x,b)}} = \mathrm{id}_{\overline{\delta}_{(x,b)} \cdot \overline{\rho}_{(x,b)}}.$$

Moreover, recall that, given an object $(x, b) \in \mathbb{A}/y$, we have that $\overline{\rho}_{(x,b)} = \left(\underline{\rho}_{(x,b)}, \mathrm{id}_b\right)$ in which $\underline{\rho}_{(x,b)}$ is the unique morphism of $\mathbb{A}$ such that the equations

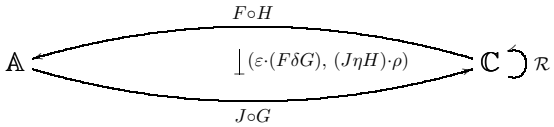
$$\overline{\delta}_{(x,b)} \cdot \overline{\rho}_{(x,b)} = \mathrm{id}_x, \quad \mathrm{id}_y^{\Leftarrow}(b) \cdot \overline{\rho}_{(x,b)} = b, \quad \text{and}$$



(6.0.27)

hold.

are given 2-adjunctions, and $\mathcal{T} = (T, \mu, \eta) = (GF, G\varepsilon F, \eta)$ is the 2-monad induced by the 2-adjunction $F \dashv G$. Recall that we have the composition of 2-adjunctions above given by



(4.0.2)

where $\mathcal{R} = (R, v, \alpha)$ denotes the 2-monad induced by $FH \dashv JG$.

4.1. Idempotent 2-adjunctions. If J and G are full reflective 2-functors, JG is a full reflective 2-functor and, in particular, $FH \dashv JG$ induces an idempotent 2-monad. However, if $F \dashv G$ and $H \dashv J$ are only idempotent 2-adjunctions, we cannot conclude that the composite is idempotent. For instance, consider the 2-adjunctions

(4.1.1)

in which Top is the locally discrete 2-category of topological spaces and continuous functions, CmpHaus is the full sub-2-category of compact Hausdorff spaces, and the right adjoints are the usual forgetful functors. Both 2-adjunctions are idempotent, but the composition induces the ultrafilter (2-)monad which is not idempotent.

Proposition 4.1 characterizes when the composition of the 2-adjunctions is idempotent. It corresponds to the characterization of the simple (reflective) functors in the 1-dimensional case.

Proposition 4.1. *Assume that $F \dashv G$ is idempotent. The following statements are equivalent.*

i) $FH \dashv JG$ is idempotent;
ii) $JGF\delta G$ (or $F\delta GFH$) is a monomorphism;
iii) $FH\alpha$ (or αJG) is an epimorphism.

Proof: Since $F \dashv G$ is idempotent, $G\varepsilon$, εF, $F\eta$ and ηG are invertible.

By Proposition 3.4, the 2-adjunction $FH \dashv JG$ is idempotent if, and only if,

$$JG\left(\varepsilon \cdot (F\delta G)\right) = (JG\varepsilon) \cdot (JT\delta G), \text{ or } \left(\varepsilon \cdot (F\delta G)\right) FH = (\varepsilon FH) \cdot (F\delta TH),$$

is a monomorphism. Therefore, since $JG\varepsilon$ and εFH are invertible, we get that $FH \dashv JG$ is idempotent if, and only if, $JT\delta G$, or $F\delta TH$, is a monomorphism. This proves that i) is equivalent to ii).

Finally, i) is equivalent to iii) by Proposition 3.4. ∎

Corollary 4.2. *If J is full reflective and $F \dashv G$ is idempotent, then the composition is idempotent.*

Proof: In this case, since δ is invertible, we have that $JGF\delta G$ is an isomorphism and, hence, a monomorphism. ∎

Definition 4.3. [Admissible 2-functor] The 2-adjunction $F \dashv G$ is *admissible* w.r.t. $H \dashv J$ if JG is a full reflective 2-functor.

If G is full reflective, and the composition JG is full reflective, we generally cannot conclude that J is full reflective. More precisely, in this case, we have:

Proposition 4.4. *Assuming that G is full reflective, the horizontal composition $F\delta G$ is invertible if and only if the 2-adjunction $F \dashv G$ is admissible w.r.t. $H \dashv J$.*

Proof: Since ε is invertible (by Proposition 3.6), we get that $(F\delta G)$ is invertible if and only if the counit $\varepsilon\,(F\delta G)$ of $FH \dashv JG$ is invertible. By Proposition 3.6, this fact completes the proof. ∎

4.2. Lax idempotent 2-adjunctions. We turn our attention now to analogous results for the lax idempotent case. The main point is to investigate when the composition of the 2-adjunctions is lax idempotent and premonadic.

Definition 4.5. [Simplicity] The 2-adjunction $F \dashv G$ is *simple* w.r.t. $H \dashv J$ if the composition $FH \dashv JG$ is lax idempotent.

As a consequence of the characterization of lax idempotent 2-adjunctions, we get:

Theorem 4.6 (Simplicity)**.** *Assume that G is locally fully faithful. The 2-adjunction $F \dashv G$ is simple w.r.t. $H \dashv J$ if and only if*

$$(\mathrm{id}_{TH} * \alpha) \dashv (\mu * \mathrm{id}_H) \cdot (\mathrm{id}_T * \delta * \mathrm{id}_{TH})$$

is a rali adjunction.

Theorem 6.10. *Let $\mathbb{A}$ be a 2-category, and y an object of $\mathbb{A}$. If $\mathbb{A}$ has comma objects along id_y, then the 2-adjunction*

$$\mathbb{A}//y \underset{\mathrm{id}_y^{\Leftarrow}}{\overset{\mathrm{id}_y\overline{!}}{\overset{\frown}{\underset{\frown}{\perp\,(\delta,\rho)}}}} \mathbb{A}/y \tag{6.0.25}$$

is lax idempotent. Moreover, it is a Kleisli 2-adjunction and, hence, $\mathrm{id}_y^{\Leftarrow}$ is a pre-Kock-Zöberlein 2-functor.

Proof: In order to verify that (6.0.25) is a Kleisli 2-adjunction, it is enough to see that $\mathrm{id}_y\overline{!}$ is bijective on objects. In particular, we conclude that $\mathrm{id}_y^{\Leftarrow}$ is 2-premonadic. Therefore, in order to prove that $\mathrm{id}_y^{\Leftarrow}$ is a pre-Kock-Zöberlein 2-functor, it remains only to prove that the 2-adjunction (6.0.25) is lax idempotent.

We prove below that

$$\mathrm{id}_{\mathrm{id}_y\overline{!}} * \rho \dashv \delta * \mathrm{id}_{\mathrm{id}_y\overline{!}} \tag{6.0.26}$$

is a rari adjunction and, hence, it satisfies the condition iii) of Theorem 3.12, which implies that the 2-adjunction (6.0.25) is lax idempotent.

For short, throughout this proof, we denote $\mathrm{id}_{\mathrm{id}_y!} * \rho$ by $\overline{\rho}$, and $\delta * \mathrm{id}_{\mathrm{id}_y\overline{!}}$ by $\overline{\delta}$.

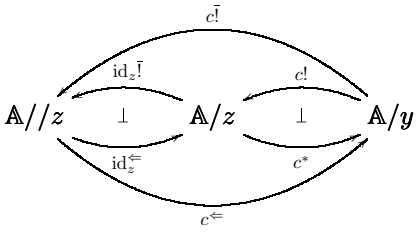
Recall that, given an object $(x, b) \in \mathbb{A}/y$, we have that $\overline{\delta}_{(x,b)}$ is defined by the pair

$$\left(\begin{array}{ccc} b\downarrow \mathrm{id}_y & \xrightarrow{\;b^{\Rightarrow}(\mathrm{id}_y)\;} & x \\ \overline{\underline{\delta}}_{(x,b)},\ \mathrm{id}_y^{\Leftarrow}(b) & \xleftarrow{\;\chi^{b\downarrow \mathrm{id}_y}\;} & b \\ y & \xrightarrow[\mathrm{id}_y]{} & y \end{array}\right)$$

in which, as suggested by the notation, the 2-cell is the comma object in $\mathbb{A}$, and

$$\overline{\underline{\delta}}_{(x,b)} := b^{\Rightarrow}(\mathrm{id}_y).$$

Theorem 6.9. *Let $\mathbb{A}$ be a 2-category, and $c : y \to z$ a morphism of $\mathbb{A}$. If $\mathbb{A}$ has comma objects and pullbacks along c, we have the following commutative diagram of 2-adjunctions*



(6.0.24)

which means that the composition of the 2-adjunction $c! \dashv c^* : \mathbb{A}/z \to \mathbb{A}/y$ with $\mathrm{id}_z! \dashv id_z^{\Leftarrow} : \mathbb{A}//z \to \mathbb{A}/z$ is, up to 2-natural isomorphism, the 2-adjunction

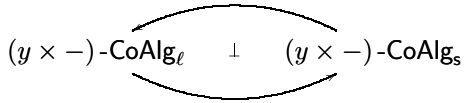
$$c! \dashv c^{\Leftarrow} : \mathbb{A}//z \to \mathbb{A}/y.$$

Given a 2-category $\mathbb{A}$, it is clear that, for any object y of $\mathbb{A}$, the 2-adjunction $\mathrm{id}_y! \dashv \mathrm{id}_y^* : \mathbb{A}/y \to \mathbb{A}/y$ is 2-naturally isomorphic to the identity 2-adjunction $\mathrm{id}_{\mathbb{A}/y} \dashv \mathrm{id}_{\mathbb{A}/y}$ and, in particular, is an idempotent 2-adjunction.

In the setting of Theorem 6.7, that is to say, the comma version of the change of base 2-functor, the 2-adjunction

$$\mathrm{id}_y! \dashv \mathrm{id}_y^{\Leftarrow} : \mathbb{A}//y \to \mathbb{A}/y,$$

is far from being isomorphic to the identity 2-adjunction. It is not even idempotent in most of the cases. Nevertheless, by Lemma 2.5 of [26], if $\mathbb{A}$ has products, the 2-adjunction



as in Remark 6.5 is lax idempotent if the base 2-category $\mathbb{A}$ has suitable comma objects (see, for instance, [27] for opcomma objects). More generally, we have:

Proof: By Theorem 3.12, we conclude that the 2-adjunction $FH \dashv JG$ is lax idempotent if and only if

$$(FH\alpha) \dashv (\varepsilon FH) \cdot (F\delta * TH)$$

is a rali adjunction. Since G is locally fully faithful, we have the rali adjunction above if, and only if, there is a rali adjunction $TH\alpha \dashv (\mu H) \cdot (T\delta TH)$. ∎

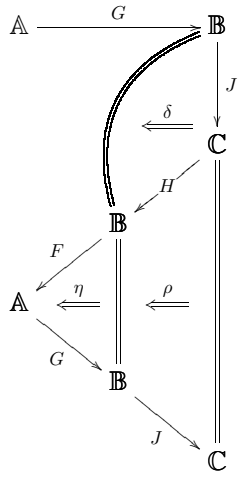
The characterization of Theorem 4.6 turns out to be difficult to apply for most of the examples, since it involves several units and counits of the given 2-adjunctions. Therefore it seems useful to have suitable sufficient conditions to get simplicity.

Theorem 4.7. *a) Assume that $JGF\delta G$ is invertible: $FH \dashv JG$ is lax idempotent if and only if there is a lali adjunction $JG\varepsilon \dashv J\eta G$.*
b) Assume that $F\delta GFH$ is invertible: $FH \dashv JG$ is lax idempotent if and only if there is a rali adjunction $F\eta H \dashv \varepsilon FH$.

Proof: We assume that $JGF\delta G$ is invertible. The other case is entirely analogous and, in fact, dual (3-dimensional codual).

By hypothesis, there is a 2-natural transformation $\vartheta : JGFG \Longrightarrow JGFHJG$ which is the inverse of $JGF\delta G$. Therefore, since

$(JGF\delta G) \cdot (\alpha JG)$ =

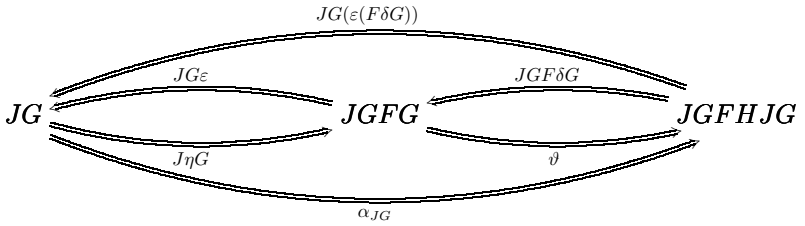


$= J\eta G,$ (4.2.1)

we conclude that

$$\vartheta \cdot (J\eta G) = \vartheta \cdot (JGF\delta G) \cdot (\alpha JG) = \alpha JG. \tag{4.2.2}$$

Therefore we have the following situation



(4.2.3)

in which $\vartheta^{-1} = JGF\delta G$. This is the hypothesis of Corollary 1.3 and, thus, there is a lali adjunction

$$JG\,(\varepsilon \cdot (F\delta G)) \dashv \alpha_{JG}$$

if, and only if, there is a lali adjunction $JG\varepsilon \dashv J\eta G$. By Theorem 3.12, this completes the proof. ∎

Corollary 4.8. *Assume that $F \dashv G$ is lax idempotent.*

a) If $JGF\delta G$ is invertible, then $FH \dashv JG$ is lax idempotent.
b) If $F\delta GFH$ is invertible, then $FH \dashv JG$ is lax idempotent.

Proof: In fact, if $F \dashv G$ is lax idempotent, we have in particular that there are a rali adjunction $F\eta H \dashv \varepsilon FH$ and a lali adjunction $JG\varepsilon \dashv J\eta G$. Therefore the result follows from Theorem 4.7. ∎

It should be noted that the 2-adjunctions in (4.1.1) show in particular that $FH \dashv JG$ might not be lax idempotent, even if $F \dashv G$ and $H \dashv J$ are. However, analogously to the idempotent case (see Corollary 4.2), we have a nicer situation whenever J is full reflective.

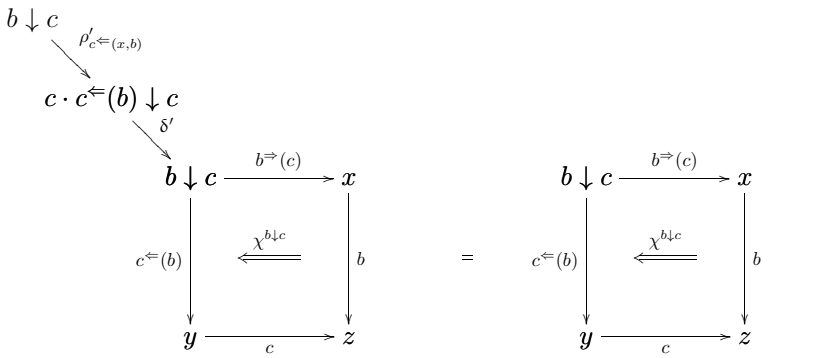
Corollary 4.9. *If J is full reflective, then $F \dashv G$ is lax idempotent if, and only if, $FH \dashv JG$ is lax idempotent.*

Proof: Assuming that J is full reflective, we get that δ is invertible and, thus, $JGF\delta G$ is invertible.

If $F \dashv G$ is lax idempotent, we get that the composite is lax idempotent by Corollary 4.8. Reciprocally, if $FH \dashv JG$ is lax idempotent, by Theorem 4.7, there is a lali adjunction

$$JG\varepsilon \dashv J\eta G.$$

hold, we get that the equations



(6.0.22)

$$c^{\Leftarrow}(b) \cdot \delta' \cdot \rho'_{c^{\Leftarrow}(b)} = c^{\Leftarrow}(b), \qquad b^{\Rightarrow}(c) \cdot \delta' \cdot \rho'_{c^{\Leftarrow}(b)} = b^{\Rightarrow}(c)$$

hold. Since, by the universal property of the comma object of b along c, the morphism satisfying the three equations above is unique, we conclude that $\delta' \cdot \rho'_{c^{\Leftarrow}(x,b)}$ is the identity on $b \downarrow c$. This proves that

$$c^{\Leftarrow}(\delta_{(x,b)}) \cdot \rho_{c^{\Leftarrow}(x,b)} = \mathrm{id}_{c^{\Leftarrow}(x,b)}$$

which proves the second triangle identity. ∎

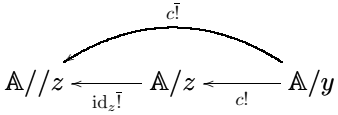
Corollary 6.8. *If $\mathbb{A}$ has comma objects (along identities), then $\mathbb{A}/y \to \mathbb{A}//y$ has a right 2-adjoint which is defined by the comma object along the identity id_y.*

Proof: It follows from Theorem 6.7 and the fact that the inclusion $\mathbb{A}/y \to \mathbb{A}//y$ is actually given by the 2-functor $\mathrm{id}_y\overline{!} : \mathbb{A}/y \to \mathbb{A}//y$ and, hence, it is left 2-adjoint to the 2-functor

$$\mathrm{id}_y^{\Leftarrow} : \mathbb{A}//y \to \mathbb{A}/y.$$

∎

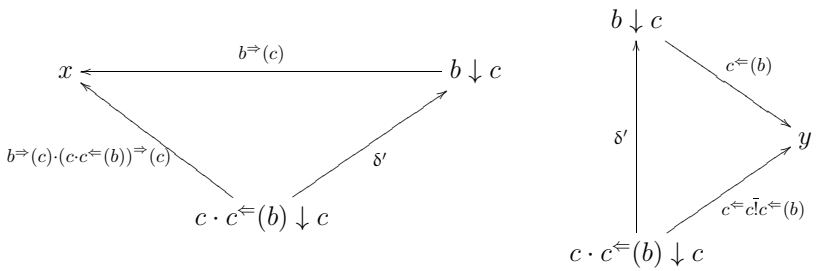
By Theorem 6.7 and the fact that, given a morphism $c : y \to z$ of a 2-category $\mathbb{A}$,



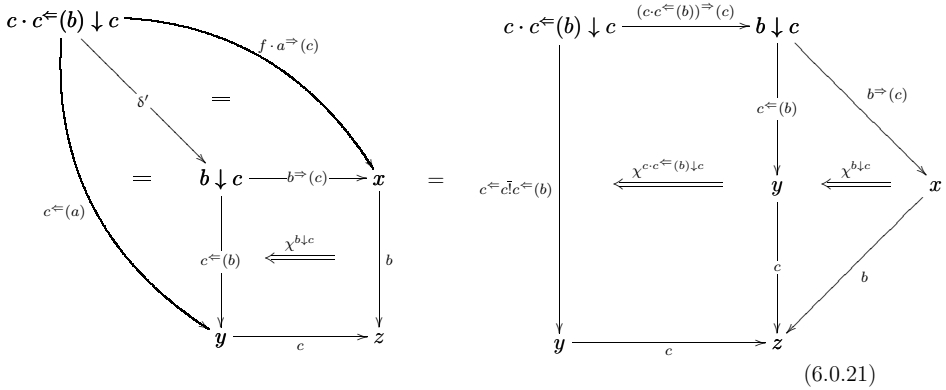
(6.0.23)

commutes, we get that:

in $\mathbb{A}/y$ is defined by the pair $(\delta', \mathrm{id}_{c^{\Leftarrow}c!\bar{c}^{\Leftarrow}(b)})$ in which δ' is the unique morphism in $\mathbb{A}$ making the diagrams



commute, and the equation



(6.0.21)

holds.

Since, by the definition of ρ, the underlying morphism $\rho'_{c^{\Leftarrow}(x,b)}$ of the component of ρ on $c^{\Leftarrow}(x,b)$ is such that the equations

$$\begin{aligned}
\chi^{c\cdot c^{\Leftarrow}(b)\downarrow c} * \mathrm{id}_{\rho'_{c^{\Leftarrow}(x,b)}} &= \mathrm{id}_{c\cdot c^{\Leftarrow}(b)} \\
(c\cdot c^{\Leftarrow}(b))^{\Rightarrow}(c)\cdot \rho'_{c^{\Leftarrow}(b)} &= \mathrm{id}_{b\downarrow c} \\
c^{\Leftarrow}c!\bar{c}^{\Leftarrow}(b)\cdot \rho'_{c^{\Leftarrow}(x,b)} &= c\cdot c^{\Leftarrow}(a)
\end{aligned}$$

Since J is locally an isomorphism, this implies that there is a lali adjunction $G\varepsilon \dashv \eta G$ which proves that $F \dashv G$ is lax idempotent by Theorem 3.12. ∎

Definition 4.10. [2-admissibility] The 2-adjunction $F \dashv G$ is 2-*admissible* w.r.t. $H \dashv J$ if the composition $FH \dashv JG$ is lax idempotent and premonadic (that is to say, JG is pre-Kock-Zöberlein).

As a consequence of Proposition 3.14 and Theorem 4.6, we have:

Theorem 4.11 (2-admissibility)**.** *Assume that G is pre-Kock-Zöberlein. The 2-adjunction $F \dashv G$ is 2-admissible w.r.t. $H \dashv J$ if, and only if, the two conditions below hold.*

- *$TH\alpha \dashv (\mu * \mathrm{id}_H)\cdot(\mathrm{id}_T * \delta * \mathrm{id}_{TH})$ is a lari adjunction (or, equivalently, $F \dashv G$ is simple w.r.t. $H \dashv J$);*
- *For each object $z \in \mathbb{C}$, $(\varepsilon \cdot (F\delta G))_z$ is a regular epimorphism.*

Whenever a (2-)category has coproducts, the composition of a regular epimorphism with a split epimorphism is always a regular epimorphism. Therefore we also have that:

Corollary 4.12. *If $\mathbb{A}$ has coproducts, $F \dashv G$ is simple w.r.t. $H \dashv J$, and $F\delta G$ is a split epimorphism, we conclude that $F \dashv G$ is 2-admissible w.r.t. $H \dashv J$.*

Proof: It follows directly from Theorem 4.11 and the observation above. ∎

Since the composition of a regular epimorphism with an isomorphism is always a regular epimorphism, we get:

Corollary 4.13. *If $F\delta G$ is an isomorphism and G is pre-Kock-Zöberlein, then $F \dashv G$ is 2-admissible w.r.t. $H \dashv J$. In particular, if J is full reflective and G is pre-Kock-Zöberlein, we conclude that JG is pre-Kock-Zöberlein.*

Proof: Since $F\delta G$ is invertible, we get that $JGF\delta G$ is invertible. Therefore, by Corollary 4.8, we get the simplicity. Moreover $\varepsilon \cdot (F\delta G)$ is a regular epimorphism since ε is a regular epimorphism and $(F\delta G)$ is invertible. ∎

5. Lax comma 2-categories

Assuming that $(\mathbb{A}, \otimes, I)$ is a symmetric monoidal 2-category, given an object y of $\mathbb{A}$, there is a bijective correspondence between 2-comonadic structures over the endo-2-functor

$$(y \otimes -) : \mathbb{A} \to \mathbb{A}$$

and the comonoid structures over y. More precisely, each comonoid (structure)

$$(y, m : y \to y \otimes y, \epsilon : y \to I)$$

is associated with the 2-comonad

$$((y \otimes -), \underline{m}, \underline{\epsilon})$$

where, for each object $w \in \mathbb{A}$, the diagrams

$$y \otimes w \xrightarrow{\epsilon \otimes w} I \otimes w \xrightarrow{\cong} w \qquad y \otimes (y \otimes w) \xrightarrow{\cong} (y \otimes y) \otimes w \xrightarrow{\partial^y \otimes w} y \otimes w$$

commute, with isomorphisms given by the symmetric monoidal structure of $(\mathbb{A}, \otimes, I)$.

Therefore, if $(\otimes, I)$ is the cartesian structure $(\times, 1)$ (provided that $\mathbb{A}$ has products), there is a unique 2-comonadic structure over

$$(y \times -) : \mathbb{A} \to \mathbb{A}$$

corresponding to the unique comonoid structure over y which is given by the unique morphism $\iota^y : y \to 1$ playing the role of the counit, and the diagonal morphism $\partial^y : y \to y \times y$ playing the role of the comultiplication. More explicitly, the counit of the (unique) 2-comonadic structure over $(y \times -)$ is pointwise defined by $\mathsf{p}_w : y \times w \to w$, while the comultiplication is pointwise defined by $\partial^y \times \mathrm{id}_w : y \times w \to y \times y \times w$.

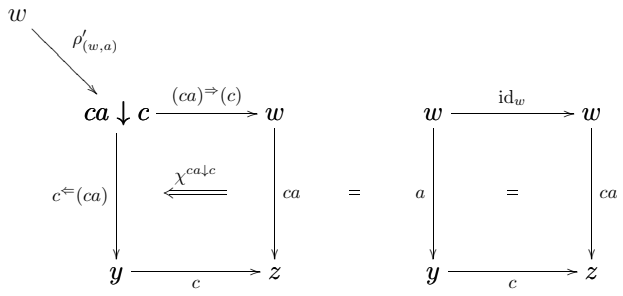
We consider below the locally fully faithful wide inclusion

$$(y \times -)\text{-}\mathsf{CoAlg}_{\mathsf{s}} \longrightarrow (y \times -)\text{-}\mathsf{CoAlg}_{\ell}$$

of the 2-category of $(y \times -)$-coalgebras and $(y \times -)$-morphisms into the 2-category of $(y \times -)$-coalgebras and lax $(y \times -)$-morphisms (see, for instance, [2, 31] for the precise definitions).

A strict $(y \times -)$-coalgebra is a pair (w, a) of an object $w \in \mathbb{A}$ and a morphism $a : w \to y \times w$ of $\mathbb{A}$ such that the usual (co)identity and (co)associativity equations for coalgebras are satisfied. This means, in our case, that the diagrams

in $\mathbb{A}$. By the universal property of the comma object, there is a unique morphism $\rho'_{(w,a)}$ of $\mathbb{A}$ such that the equations



(6.0.20)

$$(ca)^{\Rightarrow}(c) \cdot \rho'_{(w,a)} = \mathrm{id}_w \qquad \text{and} \qquad c^{\Leftarrow}\bar{c!}(a) \cdot \rho'_{(w,a)} = a$$

hold.

By the equation above, the pair $(\rho'_{(w,a)}, \mathrm{id}_a)$ gives a morphism between (w, a) and $(ca \downarrow c, c^{\Leftarrow}\bar{c!}(a))$ in $\mathbb{A}/y$. We claim that the component $\rho_{(w,a)}$ of the unit of $\bar{c!} \dashv c^{\Leftarrow}$ on (w, a) is the morphism defined by the pair $(\rho'_{(w,a)}, \mathrm{id}_a)$.

It is straightforward to see that the definitions above actually give 2-natural transformations $\delta : \bar{c!}c^{\Leftarrow} \longrightarrow \mathrm{id}_{\mathbb{A}//z}$ and $\rho : \mathrm{id}_{\mathbb{A}/y} \longrightarrow c^{\Leftarrow}\bar{c!}$. We prove below that δ and ρ satisfy the triangle identities.

Let (w, a) be an object of $\mathbb{A}/y$.

The image of the morphism $\rho_{(w,a)}$ by the 2-functor $\bar{c!} : \mathbb{A}/y \to \mathbb{A}//z$ is the morphism $(\rho'_{(w,a)}, \mathrm{id}_{ca})$ between $\bar{c!}(w, a) = (w, ca)$ and $(ca \downarrow c, \bar{c!}c^{\Leftarrow}\bar{c!}(a))$ in $\mathbb{A}//z$, while the component $\delta_{\bar{c!}(w,a)} = \delta_{(w,ca)}$ is the morphism $\left((ca)^{\Rightarrow}(c), \chi^{ca\downarrow c}\right)$.

By the definition of $\rho'_{(w,a)}$, we have that $(ca)^{\Rightarrow}(c) \cdot \rho'_{(w,a)} = \mathrm{id}_w$ and $\chi^{ca\downarrow c} * \mathrm{id}_{\rho'_{(w,a)}} = \mathrm{id}_{ca}$. Therefore $\delta_{\bar{c!}(w,a)} \cdot \bar{c!}\left(\rho_{(w,a)}\right)$ is the identity on $\bar{c!}(a)$. This proves the first triangle identity.

Let (x, b) be an object of $\mathbb{A}//z$. Denoting by $(c \cdot c^{\Rightarrow}(b) \downarrow c, \chi^{c \cdot c^{\Leftarrow}(b)\downarrow c})$ the comma object of $c \cdot c^{\Leftarrow}(b)$ along c, we have that the morphism

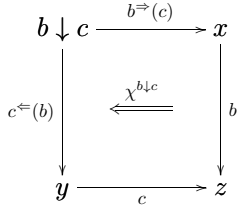
$$c^{\Leftarrow}\left(\delta_{(x,b)}\right) : c^{\Leftarrow}\bar{c!}c^{\Leftarrow}(x, b) \to c^{\Leftarrow}(x, b)$$

Proof: We define below the counit, denoted by δ, and the unit, denoted by ρ, of the 2-adjunction $c^{\overline{!}} \dashv c^{\Leftarrow}$.

For each object

$$\left(x, x \xrightarrow{b} z\right)$$

of $\mathbb{A}//z$, we have the comma object



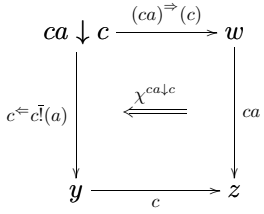
(6.0.18)

as in (6.0.13). We define the counit on (x, b), denoted by $\delta_{(x,b)}$, to be the morphism between $c^{\overline{!}}c^{\Leftarrow}(x, b)$ and (x, b) in $\mathbb{A}//z$ given by the pair $(b^{\Rightarrow}(c), \chi^{b\downarrow c})$.

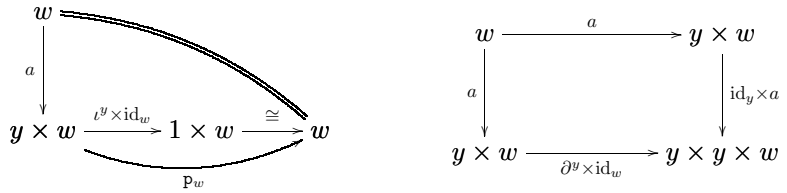
Moreover, for each object

$$\left(w, w \xrightarrow{a} y\right)$$

in $\mathbb{A}/y$, we have the comma object



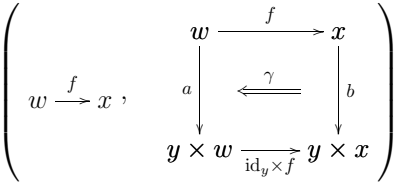
(6.0.19)



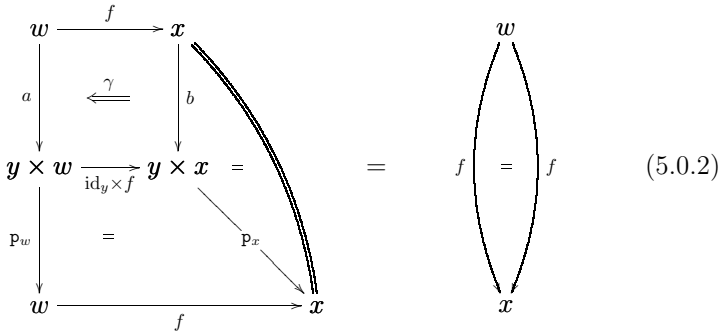
(5.0.1)

commute. It should, however, be observed that the second condition above, called (co)associativity of the coalgebra, is unnecessary for the 2-comonad $(y \times -)$, since, once (w, a) makes the first diagram commutative, the second one trivially commutes.

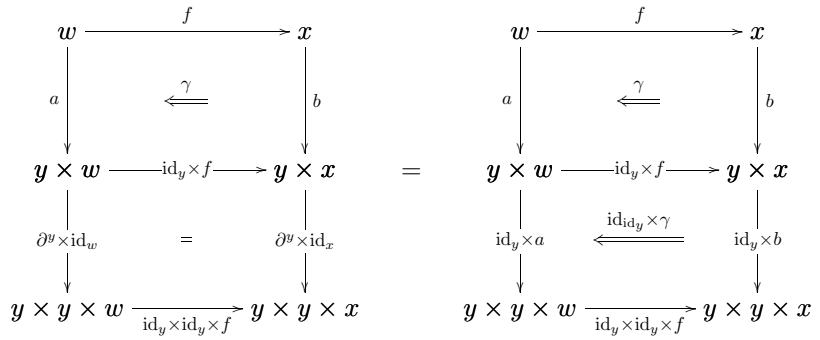
By definition, a *lax* $(y \times -)$*-morphism* between $(y \times -)$-coalgebras $a : w \to y \times w$ and $b : x \to y \times x$ is a pair



of a morphism and a 2-cell of $\mathbb{A}$ such that the equations of (co)associativity and (co)identity are satisfied. In our case, this means that



(5.0.2)



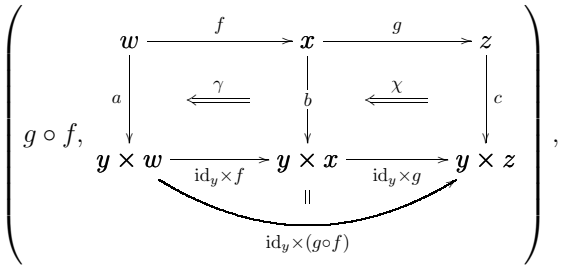
(5.0.3)

hold. Again, it should be observed that Equation (5.0.3), the *(co)associativity*, is redundant. More precisely, if Equation (5.0.2) holds for a pair (f, γ), then (5.0.3) also holds for (f, γ).

Recall moreover that a (strict) $(y \times -)$-*morphism* of $(y \times -)$-coalgebras is a lax $(y \times -)$-morphism

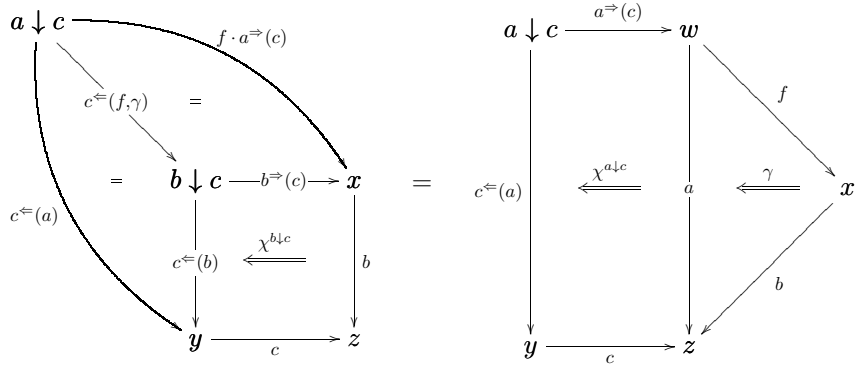
$$(f, \gamma : b \cdot f \Rightarrow (\mathrm{id}_y \times f) \cdot a)$$

such that γ is the identity 2-cell. Furthermore, the composition of two lax $(y \times -)$-morphisms (g, χ) and (f, γ) is given by the pair



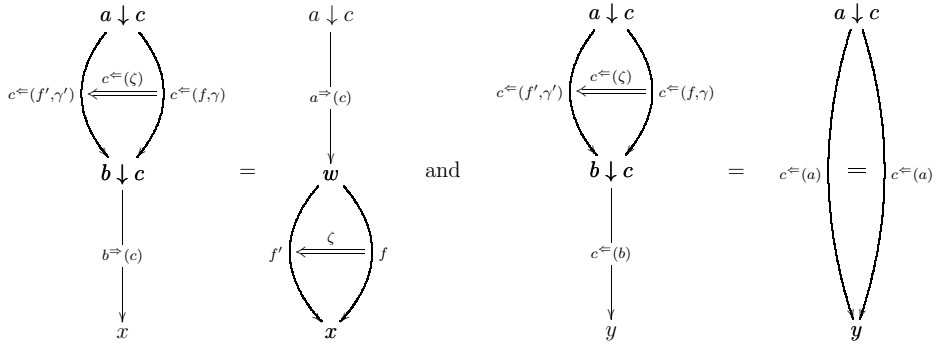
and, hence, the identity on a $(y \times -)$-coalgebra (w, a) is defined by the pair $(\mathrm{id}_w, \mathrm{id}_a)$.

Finally, by definition, a 2-cell $\zeta : (f, \gamma) \Longrightarrow (f', \gamma')$ in $(y \times -)\text{-}\mathsf{CoAlg}_\ell$ between lax $(y \times -)$-morphisms is given by a 2-cell $\zeta : f \Rightarrow f'$ of $\mathbb{A}$ such that



(6.0.15)

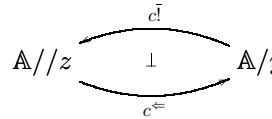
hold. Finally, if $\zeta : f \Rightarrow f' : (w, a) \to (x, b)$ is a 2-cell between morphisms (f, γ) and (f', γ') in $\mathbb{A}//z$, the 2-cell $c^{\Leftarrow}(\zeta)$ is the unique 2-cell such that the equations



(6.0.16)

hold.

Theorem 6.7. *Let $\mathbb{A}$ be any 2-category, and $c : y \to z$ a morphism in $\mathbb{A}$. If $\mathbb{A}$ has comma objects along c, then we have a 2-adjunction*



(6.0.17)

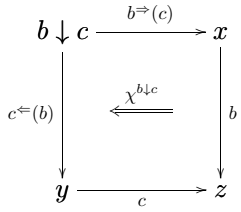
Definition 6.6. [$c^{\Leftarrow}$] Let $\mathbb{A}$ be any 2-category, and $c : y \to z$ a morphism of $\mathbb{A}$. Assume that $\mathbb{A}$ has comma objects along c. We denote by

$$c^{\Leftarrow} : \mathbb{A}//z \to \mathbb{A}/y$$

the 2-functor defined by the comma object along the morphism c. Explicitly, the action on objects of $c^{\Leftarrow}$ is given by

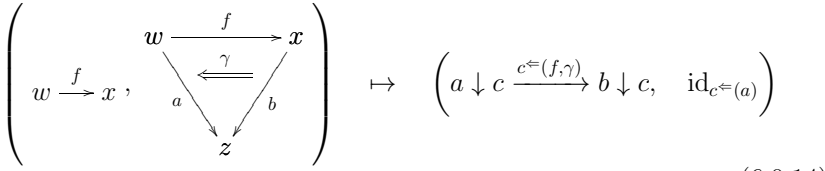
$$(x, b) \mapsto (b \downarrow c, c^{\Leftarrow}(b) : b \downarrow c \to y) \tag{6.0.12}$$

in which



(6.0.13)

is the comma object as in 2.2, while the action on morphisms is given by

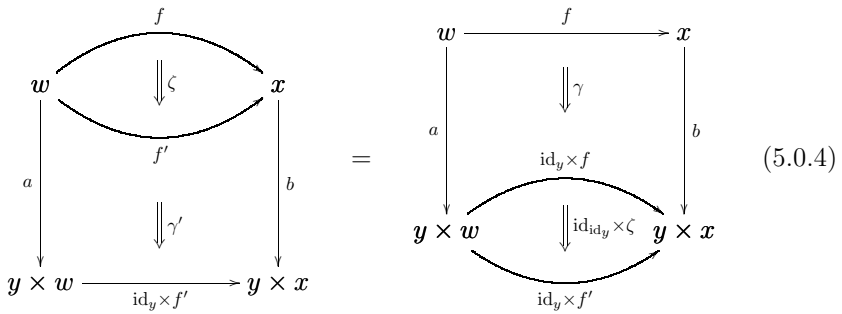


(6.0.14)

in which $c^{\Leftarrow}(f, \gamma)$, sometimes only denoted by $c^{\Leftarrow}(f)$, is the unique morphism of $\mathbb{A}$ such that the equations

$$b^{\Rightarrow}(c) \cdot c^{\Leftarrow}(f, \gamma) = f \cdot a^{\Rightarrow}(c), \qquad c^{\Leftarrow}(b) \cdot c^{\Leftarrow}(f, \gamma) = c^{\Leftarrow}(a),$$

the equation



(5.0.4)

holds. The horizontal and vertical compositions of 2-cells in $(y \times -)$-CoAlg_{ℓ} are defined as in $\mathbb{A}$.

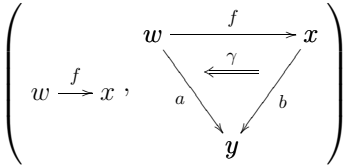
Definition 5.1. [Lax comma 2-category] Given an object y of a 2-category $\mathbb{A}$, we denote by $\mathbb{A}//y$ the 2-category defined by the following.

– The objects are pairs (w, a) in which w is an object of $\mathbb{A}$ and

$$w \xrightarrow{a} y$$

is a morphism of $\mathbb{A}$.

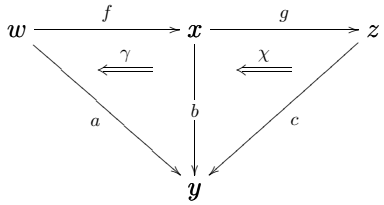
– A morphism in $\mathbb{A}//y$ between objects (w, a) and (x, b) is a pair



in which $f : w \to x$ is a morphism of $\mathbb{A}$ and γ is a 2-cell of $\mathbb{A}$.

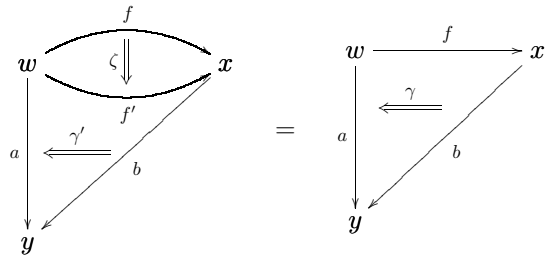
If $(f, \gamma) : (w, a) \to (x, b)$ and $(g, \chi) : (x, b) \to (z, c)$ are morphisms of $\mathbb{A}//y$, the composition is defined by $(g \circ f, \gamma \cdot (\chi * \mathrm{id}_f))$, that is to

say, the composition of the morphisms g and f with the pasting



of the 2-cells χ and γ. Finally, with the definitions above, the identity on the object (w, a) is of course the morphism $(\mathrm{id}_w, \mathrm{id}_a)$.

– A 2-cell between morphisms (f, γ) and (f', γ') is given by a 2-cell $\zeta : f \Rightarrow f'$ such that the equation



holds.

The 2-category $\mathbb{A}//y$ is called the *lax comma* 2*-category* of $\mathbb{A}$ over y, while the 2-category $\mathbb{A}^{\mathrm{co}}//y$ is called the *colax comma* 2*-category* of $\mathbb{A}$ over y.

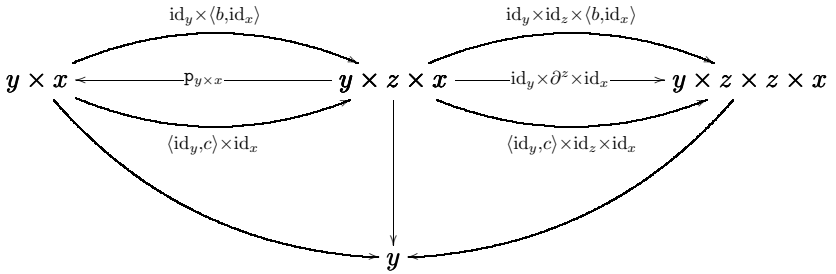
The concept of (co)lax comma 2-category, possibly under other names, has already appeared in the literature. See, for instance, [32, Exercise 5, pag. 115] or [39, pag. 305]. As for our choice of the direction of the 2-cells for the notion of lax comma 2-categories, although we do not follow [39, pag. 305], our choice is compatible with the usual definition of lax natural transformation.

Definition 5.2. [(Strict) comma 2-category] Given an object y of a 2-category $\mathbb{A}$, we denote by $\mathbb{A}/y$ the *comma* 2*-category* over y, defined to be the locally full *wide* sub-2-category of $\mathbb{A}//y$ in which a morphism from (w, a) to (x, b) is a morphism

$$(f, \chi) : (w, a) \to (x, b)$$

such that χ is the identity 2-cell.

More precisely, assuming that $\mathbb{A}$ has products, an object $(x, b : x \to z)$ of $\mathbb{A}/z$ has a right 2-reflection along $c^{\overline{!}}$ if and only if the lax descent object of



(6.0.10)

exists in $\mathbb{A}/y$, and, in this case, the right 2-reflection of $c^{\overline{!}}$ along (x, b) is the lax descent object of (6.0.10).

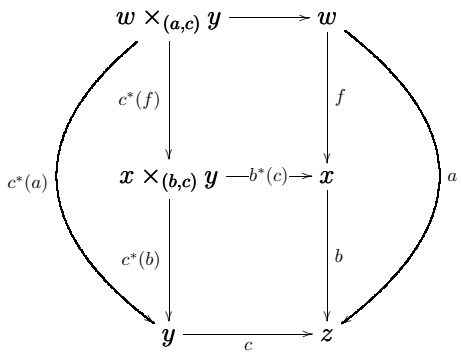
Remark 6.4. By the same theorem on limits of coalgebras mentioned in Remark 6.2, since the endo-2-functor $(y \times -)$ in a 2-category $\mathbb{A}$ (with products) does preserve lax descent objects, we conclude that (6.0.5) creates the existing lax descent objects of $\mathbb{A}$. Moreover, again, it is the case that (6.0.5) creates lax descent objects even when $\mathbb{A}$ does not have products.

Remark 6.5. Still restricting our attention to the case of $\mathbb{A}$ having products, considering the case of $c = \mathrm{id}_y$ in Diagram (6.0.3), in the presence of lax descent objects, we get by the above construction the 2-adjunction

$$\mathbb{A}//y \cong (y \times -)\text{-}\mathsf{CoAlg}_{\ell} \quad \underset{\mathrm{id}_y^{\overline{!}}}{\overset{}{\underset{}{\perp}}} \quad (y \times -)\text{-}\mathsf{CoAlg}_{\mathsf{s}} \cong \mathbb{A}/y \tag{6.0.11}$$

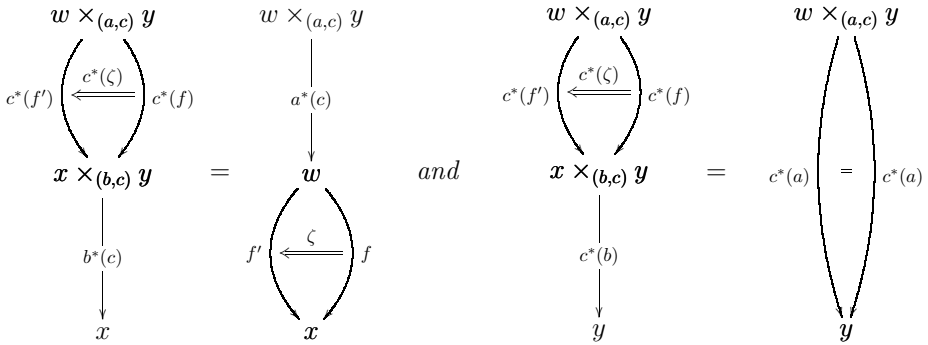
which, in our setting, happens to be a particular case of the general coherence theorems on lax algebras and strict algebras (*e.g.* [25, 29]), since $\mathrm{id}_y^{\overline{!}}$ does coincide with the inclusion $(y \times -)\text{-}\mathsf{CoAlg}_{\mathsf{s}} \to (y \times -)\text{-}\mathsf{CoAlg}_{\ell}$.

We prove that, in our case, we do not need to assume the existence of products to get the 2-adjunction above. More precisely, the only assumption is the existence of comma objects along c in order to define the right 2-adjoint to $c^{\overline{!}}$.



(6.0.8)

are pullbacks. Finally, the image of a 2*-cell* $\zeta : f \Rightarrow f' : (w, a) \to (x, b)$ *is defined by the unique* 2*-cell* $c^*(\zeta)$ *such that the equations*



(6.0.9)

hold.

Analogously to the case of the classical *change of base* 2*-functor* described above, assuming that $\mathbb{A}$ has products and lax descent objects (see [29, 27]), by Diagram (6.0.4) and the adjoint triangle theorem for (lax) (co)algebras (Theorem 5.3 of [29]), we conclude that $c_!$ has a right 2-adjoint.

Remark 5.3. We have an inclusion 2-functor $\mathbb{A}/y \to \mathbb{A}//y$ obviously defined. The morphisms in the image of this inclusion are called *strict* (or *tight*) morphisms of $\mathbb{A}//y$. The 2-category $\mathbb{A}//y$ endowed with this inclusion forms an enhanced 2-category, or, more precisely, an $\mathfrak{F}$-category as defined in [26].

Theorem 5.4. *Assuming that* $\mathbb{A}$ *has products, there is a commutative square*

$$\begin{array}{ccc} (y \times -)\text{-}\mathsf{CoAlg}_{\mathsf{s}} & \xrightarrow{\cong} & \mathbb{A}/y \\ \downarrow & & \downarrow \\ (y \times -)\text{-}\mathsf{CoAlg}_{\ell} & \xrightarrow[\cong]{} & \mathbb{A}//y \end{array} \tag{5.0.5}$$

in which the vertical arrows are the locally full wide inclusions, and the horizontal arrows are invertible 2*-functors.*

Proof: We give the 2-functor

$$(y \times -)\text{-}\mathsf{CoAlg}_{\ell} \xrightarrow{\cong} \mathbb{A}//y \tag{5.0.6}$$

and its inverse explicitly below.

Part A. Our first step is to define the assignment on objects of the 2-functors, and prove that these assignments are inverse of each other.

The 2-functor (5.0.6) is given by

$$\begin{pmatrix} & w & \\ & \downarrow{\scriptstyle a'} & \\ w, & y \times w & \end{pmatrix} \quad \mapsto \quad \begin{pmatrix} & w & \\ & \downarrow{\scriptstyle a'} & \\ w, & y \times w & \\ & \downarrow{\scriptstyle \mathrm{p}_y} & \\ & y & \end{pmatrix} \tag{5.0.7}$$

on the objects, while the inverse assignment is given by

$$\mathbb{A}//y \xrightarrow{\cong} (y \times -)\text{-}\mathsf{CoAlg}_{\ell} \tag{5.0.8}$$

is defined by

$$\begin{pmatrix} & w & \\ & \downarrow{\scriptstyle a} & \\ w, & y & \end{pmatrix} \quad \mapsto \quad \begin{pmatrix} & w & \\ & \downarrow{\scriptstyle \langle a, \mathrm{id}_w \rangle} & \\ w, & y \times w & \end{pmatrix} \tag{5.0.9}$$

where $\langle a, \mathrm{id}_w\rangle$ denotes the morphism induced by the universal property of the product $y \times w$ and the morphisms a and id_w.

We have that (5.0.7) is clearly well defined. In order to verify that (5.0.9) is also well defined, it is enough to see that, for any morphism $a : w \to y$, $\langle a, \mathrm{id}_w\rangle$ makes the diagrams of (5.0.1) commutative.

The assignment on objects of the composition

$$\mathbb{A}//y \longrightarrow (y \times -)\text{-}\mathsf{CoAlg}_\ell \longrightarrow \mathbb{A}//y \tag{5.0.10}$$

is given by

$$\left(\begin{array}{c} w \\ w,\ \downarrow a \\ y \end{array}\right) \mapsto \left(\begin{array}{c} w \\ w,\ \downarrow \langle a,\mathrm{id}_w\rangle \\ y \times w \end{array}\right) \mapsto \left(\begin{array}{ccc} & w & \\ & \downarrow \langle a,\mathrm{id}_w\rangle & \\ w, & y \times w & = \ a \\ & \downarrow \mathtt{p}_y & \\ & y & \end{array}\right) \tag{5.0.11}$$

and, hence, the composition is the identity on objects.

In order to prove that the action on objects of the composition

$$(y \times -)\text{-}\mathsf{CoAlg}_\ell \longrightarrow \mathbb{A}//y \longrightarrow (y \times -)\text{-}\mathsf{CoAlg}_\ell \tag{5.0.12}$$

$$\left(\begin{array}{c} w \\ w,\ \downarrow a' \\ y \times w \end{array}\right) \mapsto \left(\begin{array}{ccc} & w & \\ & \downarrow a' & \\ w, & y \times w & \\ & \downarrow \mathtt{p}_y & \\ & y & \end{array}\right) \mapsto \left(\begin{array}{c} w \\ w,\ \downarrow \langle \mathtt{p}_y \circ a', \mathrm{id}_w\rangle \\ y \times w \end{array}\right) \tag{5.0.13}$$

is also the identity, it is enough to see that, by the identity equation of (5.0.1), we have that

$$a' = \langle \mathtt{p}_y \circ a',\, \mathtt{p}_w \circ a'\rangle = \langle \mathtt{p}_y \circ a',\, \mathrm{id}_w\rangle$$

for any $(y \times -)$-coalgebra $(w, a' : w \to y \times w)$.

Part B. The second step is to give the action of the 2-functors on morphisms.

The action of the morphisms of (5.0.6) is defined by

2-functor

$$(y \times -)\text{-}\mathsf{CoAlg}_{\mathsf{s}} \cong \mathbb{A}/y \longrightarrow \mathbb{A} \tag{6.0.5}$$

creates all the equalizers and pullbacks that $\mathbb{A}$ has, since the endo-2-functor $(y \times -)$ does preserve any equalizer and pullback. Moreover, this result also holds even when $\mathbb{A}$ does not have products. Furthermore, the product of two objects (w, a) and (w', a') of $\mathbb{A}/y$ is given by the pullback of a along a'.

Finally, it is simple to observe that the equalizer of (6.0.4) is actually defined by the pullback of b along c. More precisely, the equalizer of (6.0.4) exists if and only if the pullback $x \times_{(b,c)} y$ exists in $\mathbb{A}$ and, in this case, the equalizer is defined by the pair $\big(x \times_{(b,c)} y, c^*(b)\big)$, following the terminology of (2.1.1).

It is well-known that the 2-adjunction (6.0.1) holds even without assuming the existence of products. We establish this result below.

Proposition 6.3 (Change of base 2-functor)**.** *Let $\mathbb{A}$ be a 2-category with pullbacks. If $c : y \to z$ is any morphism, we get a 2-adjunction*

$$\mathbb{A}/z \underset{c^*}{\overset{c!}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathbb{A}/y \tag{6.0.6}$$

in which c^* is defined by the pullback along c. Explicitly, the assignment of objects of c^* is given by

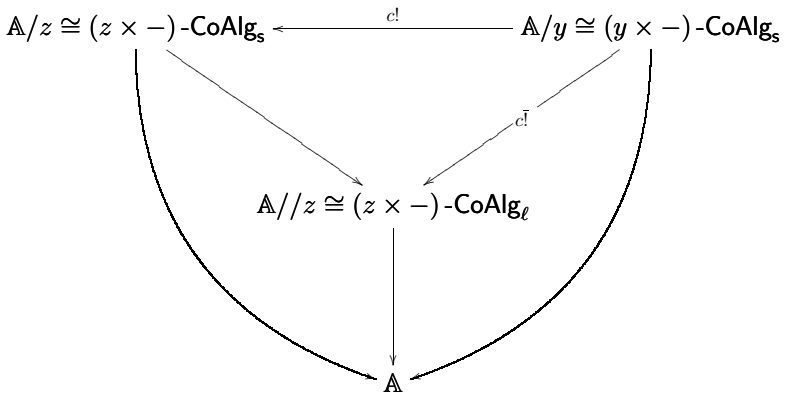
$$(w, a) \mapsto (w \times_{(a,c)} y, c^*(a) : w \times_{(a,c)} y \to y)$$

while the action of c^* on morphisms is given by

$$\Big(w \xrightarrow{f} x,\, \mathrm{id}_a\Big) : (w, a) \to (x, b) \ \mapsto\ \Big(w \times_{(a,c)} y \xrightarrow{c^*(f,\mathrm{id}_a)} x \times_{(b,c)} y,\, \mathrm{id}_{c^*(a)}\Big) : c^*(a) \to c^*(b) \tag{6.0.7}$$

in which all the squares of

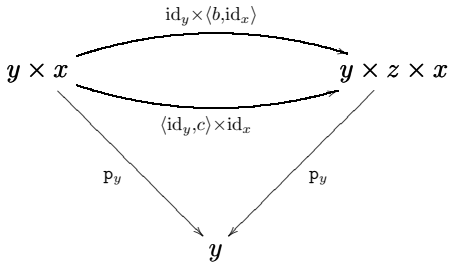
By Theorem 5.4 and Definition 6.1, given a morphism $c : y \to z$ in a 2-category $\mathbb{A}$ with products, we have that the diagram



(6.0.3)

commutes, in which the unlabeled arrows are the obvious forgetful 2-functors.

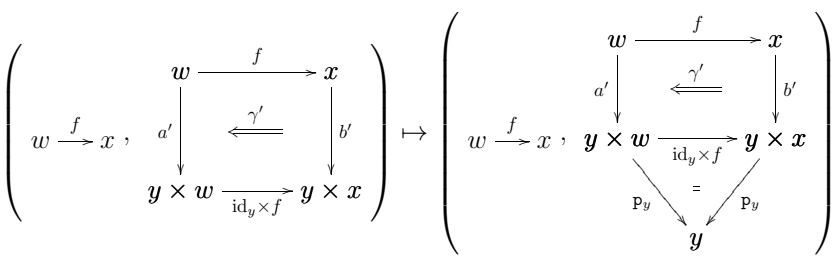
Assuming that $\mathbb{A}$ has products as above, we can get the classical *change of base (*2*-)functor* by the enriched version of Dubuc's adjoint triangle theorem [11, 28]. More precisely, by the enriched adjoint triangle theorem ([28, Proposition 1.1]), we have that $c!$ has a right 2-adjoint c^* if, and only if, for each object $(x, b : x \to z)$ of $\mathbb{A}/z$, the equalizer of



(6.0.4)

exists in $\mathbb{A}/y$. In this case, we have that $c^*(w, b)$ is isomorphic to the equalizer of (6.0.4).

Remark 6.2. It should be noted that, assuming that $\mathbb{A}$ has products, it follows from a well-known result on limits of coalgebras [33] that the forgetful

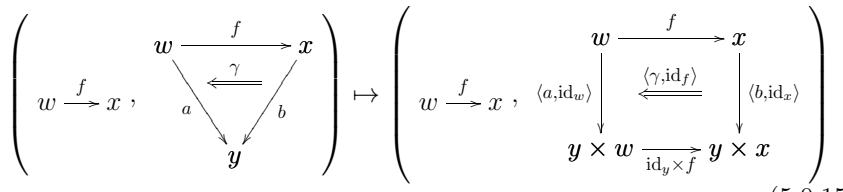


(5.0.14)

which is well defined, since it respects the action on objects previously defined in (5.0.6).

We have that (5.0.14) indeed preserves the identities and compositions. Furthermore, it should be noted that it also preserves *tight* morphisms, that is to say, it takes (strict) $(y \times -)$-morphisms of $(y \times -)$-coalgebras to strict morphisms of $\mathbb{A}//y$.

The action on morphisms of (5.0.8) is given by



(5.0.15)

where $\langle \gamma, \mathrm{id}_f \rangle : \langle b \cdot f, f \rangle \Rightarrow \langle a, f \rangle$ is the 2-cell induced by the universal property of $y \times x$ and the 2-cells $\gamma : b \cdot f \Rightarrow a$ and the identity $\mathrm{id}_f : f \Rightarrow f$.

In order to show that (5.0.15) is well defined, we start by noticing that

$$\mathrm{id}_{\mathrm{p}_x} * \langle \gamma, \mathrm{id}_f \rangle = \mathrm{id}_f$$

holds, by the definition of $\langle \gamma, \mathrm{id}_f \rangle$. Hence, the pair $(f, \langle \gamma, \mathrm{id}_f \rangle)$ satisfies the Equation (5.0.2), which proves that the pair indeed defines a morphism of $(y \times -)$-coalgebras.

Now it is clear that (5.0.15) respects the assignment on objects defined in (5.0.9). Moreover, it is also clear that (5.0.15) preserves composition and identities.

Analogously to the case of the assignment on objects (5.0.11), since $\mathrm{id}_{\mathrm{p}_y} * \langle \gamma, \mathrm{id}_f \rangle = \gamma$, we conclude that the action on morphisms of (5.0.10) is in fact the identity.

Finally, analogously to the case of (5.0.13), by Equation (5.0.2), given any lax $(y \times -)$-morphism of $(y \times -)$-coalgebras (f, γ'), we have that

$$(f, \gamma') = \left(f, \left\langle \mathrm{id}_{\mathrm{p}_y} * \gamma', \mathrm{id}_f \right\rangle\right).$$

By our definitions, this implies that the action on morphisms of the composition (5.0.12) is the identity.

Part C. The actions on the 2-cells are the simplest. In both ways, it takes the 2-cell defined by a 2-cell ζ of $\mathbb{A}$ to the unique 2-cell defined by it. The following facts are straightforward: (1) these actions on the 2-cells are well defined; (2) they preserve compositions and identities and (3) they are inverse of each other.

This completes the definition of the invertible 2-functor (5.0.6).

Finally the fact that both the 2-functors (5.0.6) and (5.0.8) respect strict/*tight* morphisms (see [26]) implies that the restrictions of these 2-functors to $\mathbb{A}/y$ and $(y \times -)$-$\mathsf{CoAlg_s}$ are well defined. Hence, these restrictions define the isomorphism of the top of Diagram (5.0.5). ∎

Remark 5.5. As observed in Remark 5.3, the inclusion $\mathbb{A}/y \to \mathbb{A}//y$ can be seen as a structure of $\mathfrak{F}$-category as introduced in [26]. That being said, the statement of Theorem 5.4 is actually an isomorphism of $\mathfrak{F}$-categories.

Using the theory presented in [26], we could prove a version of Theorem 5.4 using, firstly, Beck's monadicity theorem and, then, Bourke's monadicity theorem [5]. We do not follow this approach for two reasons.

The first reason is that we would need to assume the existence of opcomma objects (along identity) in the base 2-category. The second reason is that our elementary approach above avoids the need of further background material on $\mathfrak{F}$-categories [26].

6. Change of base 2-functors

Assuming that $\mathbb{A}$ has pullbacks, given any morphism $c : y \to z$ of a 2-category $\mathbb{A}$, it is well known that it induces a 2-adjunction

$$\mathbb{A}/z \underset{c^*}{\overset{c!}{\bot}} \mathbb{A}/y \tag{6.0.1}$$

between the (strict) comma 2-categories in which the right 2-adjoint is called the *change of base functor* induced by the morphism c (see, for instance, [18, 19]). Recall that c^* is defined by the pullback along c, and the left adjoint is defined by the composition with c as below.

In the present section, we give the analogue for lax comma 2-categories, that is to say, the *change of base 2-functors for the lax comma 2-categories*. We start by showing how, once we assume the existence of products, by Theorem 5.4 these 2-adjunctions come from general theorems of 2-dimensional monad theory [2, 25, 29]. Then we finish the section by giving an explicit proof of the 2-adjunction in Theorem 6.7, which, besides being an elementary approach, has the advantage of not depending on the existence of products.

Definition 6.1. [Direct image] If $c : y \to z$ is any morphism of a 2-category $\mathbb{A}$, we define the commutative diagram

$$\mathbb{A}//z \xleftarrow{\quad} \mathbb{A}/z \xleftarrow{c!} \mathbb{A}/y \qquad (c! : \mathbb{A}/z \to \mathbb{A}//z) \tag{6.0.2}$$

in which the unlabeled arrow is the obvious inclusion, and

$$c! : \mathbb{A}/y \to \mathbb{A}/z$$

is defined by

$$(x, a) \mapsto (x, ca),\ (f, \mathrm{id}) \mapsto (f, \mathrm{id}_c * \mathrm{id}),\ \zeta \mapsto \zeta,$$

that is to say, the usual direct image 2-functor.