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CUBIC SPLINES IN THE GRASSMANN MANIFOLD GENERATED BY THE DE CASTELJAU ALGORITHM

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ABSTRACT: We present a detailed analysis of the De Casteljau algorithm to generate cubic polynomials satisfying certain boundary conditions in the Grassmann manifold, and extend this approach to produce cubic splines that also solve interpolation problems on that manifold.

KEYWORDS: Cubic splines, Casteljau algorithm, geodesics, Grassmann manifold, interpolating curves.

1. Introduction

Interpolating nonlinear data arises in many different areas, ranging from robotics and computer vision to industrial and medical applications (see, e.g., [4], [12]). In Euclidean spaces, cubic splines, which are $\mathcal{C}^2$-smooth curves obtained by piecing together cubic polynomials, are particularly important since they minimise the average acceleration. A well-known recursive procedure to generate interpolating polynomial curves in Euclidean spaces is the classical De Casteljau algorithm which was introduced, independently, by De Casteljau [6] and Bézier [3]. Generalisations of such curves to non-Euclidean spaces is useful in many engineering applications, with particular emphasis in the case where the data can be represented on a Grassmann manifold. This paper deals with a geometric procedure to generate cubic polynomials and splines in the Grassmann manifold, known as the De Casteljau algorithm. We present a detailed implementation of this algorithm for that manifold, which follows closely the work in [5] concerning the reinterpretation of the De Casteljau algorithm for connected and compact Lie groups.

The main feature of the algorithm is based on recursive geodesic interpolation in order to find a polynomial curve that solves a 2-boundary value problem. The boundary conditions might be of Hermite type, i.e., consisting of initial and final points together with initial and final velocity, or instead consisting of initial and final points, initial velocity and initial intrinsic acceleration. While the first conditions are more natural in applications, they

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pose some difficulties that do not arise in the second type of boundary conditions. We show how these two types of boundary conditions are related to each other.

The organisation of the paper is the following. In Section 2, notations and some preliminary results are introduced. Section 3 includes the geometry of the Grassmann manifold and reveals the importance of that manifold for modelling image-sets. The main results appear in Section 4, where the De Casteljau algorithm to generate geometric cubic polynomials in the Grassmann manifold is explained in detail. In Section 5 the results of the previous section are extended to generate cubic splines. The paper ends with a short conclusion.

2. Notations and auxiliary results

Here we review the geometry of the real Grassmann manifold, hereafter denoted by $G_{k,n}$, with $(0 < k < n)$, which consists of all k-dimensional real subspaces of the Euclidean space $\mathbb{R}^n$, and present some preliminary results that will be important in the main section of the paper. We closely follow the definitions and terminology in [2], but more details concerning these manifolds can be found, for instance, in [1].

In the sequel, $\mathfrak{gl}(n)$ denotes the Lie algebra of $n \times n$ real matrices, equipped with the Lie bracket defined by the commutator, i.e., $[A, B] := AB - BA$, for $A, B \in \mathfrak{gl}(n)$. The adjoint operator in $\mathfrak{gl}(n)$ is defined by $\mathrm{ad}_A B := [A, B]$. The vector space of $n \times n$ symmetric matrices is denoted by $\mathfrak{s}(n)$ and the Lie subalgebra of $\mathfrak{gl}(n)$, consisting of the skewsymmetric $n \times n$ matrices is denoted by $\mathfrak{so}(n)$. The Lie group $SO(n)$, having $\mathfrak{so}(n)$ as its Lie algebra, will also play an important role here.

We now recall some properties of the matrix exponential and of the principal matrix logarithm, that can be found in [8] and [7].

Given $A \in \mathfrak{gl}(n)$, the *matrix exponential of* A, denoted by e^A, is the $n \times n$ real matrix given by the sum of the convergent power series $\sum_{k=0}^{+\infty} \dfrac{A^k}{k!}$, where A^0 is defined to be the identity matrix of order n. So, we write,

$$\mathrm{e}^A = \sum_{k=0}^{+\infty} \frac{A^k}{k!}. \tag{1}$$

The logarithms of an invertible matrix X are the solutions of the matrix equation $\mathrm{e}^A = X$. When X is real and doesn't have eigenvalues in the closed negative real line, i.e., $\sigma(X) \cap \mathbb{R}_0^- = \varnothing$, where $\sigma(X)$ denotes the spectrum of X, there exists a unique real logarithm of X whose spectrum lies in the infinite horizontal strip $\{z \in \mathbb{C} : -\pi < Im(z) < \pi\}$ of the complex plane. This logarithm is usually called the *principal logarithm* of X. This is the only logarithm that we consider, and hereafter will be denoted by $\log X$. When $X \in SO(n)$, $\log X \in \mathfrak{so}(n)$. When $\|X - I\| < 1$, $\log X$ is uniquely defined by the following convergent power series:

$$\log X = \sum_{k=1}^{+\infty} (-1)^{k+1} \frac{(X-I)^k}{k}. \tag{2}$$

This power series define the principal logarithm for matrices which are close to the identity matrix. However, for $\alpha \in [-1, 1]$, $\log(A^\alpha) = \alpha \log A$, so that, making $\alpha = 1/(2^k)$, with $k \in \mathbb{Z}\backslash\{0\}$, one has

$$\log(A^{\frac{1}{2^k}}) = \frac{1}{2^k} \log A.$$

Since $lim_{k\to\infty} A^{\frac{1}{2^k}} = I$, the previous expression allows to compute $\log A$ even for matrices A which are not close to the identity. This procedure, that can be found for instance in [7], is called *inverse scaling and squaring method.*

Lemma 2.1. *Let A, B, C, and X be real square matrices and assume that C is invertible and $\sigma(X) \cap \mathbb{R}_0^- = \varnothing$. Then, the following holds.*

1. $C^{-1} e^A C = e^{C^{-1}AC}$;
2. $e^A B e^{-A} = e^{ad_A}(B) = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \cdots$;
3. $C^{-1}(\log X)C = \log\left(C^{-1}XC\right)$;
4. $\log(X^{-1}) = -\log X$ *and* $\log(X^\top) = (\log X)^\top$;
5. $X^\alpha = e^{\alpha \log X}$, *for* $\alpha \in \mathbb{R}$, *and whenever* $\log X$ *is defined*;
6. $\log(e^A) = A$, *whenever* $\log(e^A)$ *is defined.*

The second of these identities is called the *Campbell-Hausdorff Formula*.

In the sequel we assume the following notations:

$$f(z) = \tfrac{e^z - 1}{z} \text{ stands for the sum of the series } \textstyle\sum_{k=0}^{+\infty} \frac{z^k}{(k+1)!}, \tag{3}$$

and when $|z-1| < 1$,

$$g(z) = \tfrac{\log z}{z-1} \text{ stands for the sum of the series } \textstyle\sum_{k=0}^{+\infty} (-1)^k \dfrac{(z-1)^k}{k+1}. \tag{4}$$

Note that $f(z)g(e^z) = 1$.

Lemma 2.2 ([11]). *Let $t \longmapsto X(t)$ be a differentiable matrix valued function. Then,*

$$\frac{\mathrm{d}}{\mathrm{d}t} e^{X(t)} = \Delta^L_{X(t)}(t)\, e^{X(t)}, \tag{5}$$

where

$$\Delta^L_{X(t)}(t) = \int_0^1 e^{u\, ad_{X(t)}}(\dot{X}(t))du. \tag{6}$$

Or, alternatively,

$$\frac{\mathrm{d}}{\mathrm{d}t} e^{X(t)} = f\left(ad_{X(t)}\right)\left(\dot{X}(t)\right) e^{X(t)}, \tag{7}$$

where f is defined as in (3).

The next three lemmas will play an important role in the main section of the paper.

Lemma 2.3. *Let $t \longmapsto A(t)$ be a differentiable matrix valued function. Then*

$$e^{A(t)} \Delta^L_{-A(t)}(t)\, e^{-A(t)} = -\Delta^L_{A(t)}(t), \tag{8}$$

where $\Delta^L_{A(t)}(t)$ denotes the operator defined on (6).

Proof: It holds that,

$$\begin{aligned}
\mathrm{e}^{A(t)}\Delta^L_{-A(t)}(t)\mathrm{e}^{-A(t)} &= \mathrm{e}^{\mathrm{ad}_{A(t)}}(\Delta^L_{-A(t)}(t)) \\
&= \mathrm{e}^{\mathrm{ad}_{A(t)}}\int_0^1 \mathrm{e}^{u\,\mathrm{ad}_{-A(t)}}(-\dot{A}(t))du \\
&= \mathrm{e}^{\mathrm{ad}_{A(t)}}\int_0^1 \mathrm{e}^{-u\,\mathrm{ad}_{A(t)}}(-\dot{A}(t))du \\
&= -\int_0^1 \mathrm{e}^{\mathrm{ad}_{A(t)}}\mathrm{e}^{-u\,\mathrm{ad}_{A(t)}}(\dot{A}(t))du \\
&= -\int_0^1 \mathrm{e}^{(1-u)\,\mathrm{ad}_{A(t)}}(\dot{A}(t))du.
\end{aligned}$$

Making a change of variable, considering $1-u=z$, we have that $du=-dz$, $u=0$ implies $z=1$ and $u=1$ implies $z=0$. Then,

$$\begin{aligned}
-\int_0^1 \mathrm{e}^{(1-u)\,\mathrm{ad}_{A(t)}}(\dot{A}(t))du &= \int_1^0 \mathrm{e}^{z\,\mathrm{ad}_{A(t)}}(\dot{A}(t))dz \\
&= -\int_0^1 \mathrm{e}^{z\,\mathrm{ad}_{A(t)}}(\dot{A}(t))dz \\
&= -\Delta^L_{A(t)}(t). \quad \blacksquare
\end{aligned}$$

Lemma 2.4. *Let $t \longmapsto A(t)$ be a differentiable matrix valued function. Then, for $k=0,1$,*

$$\Delta^L_{(t-k)A(t)}(t)\Big|_{t=k} = A(k), \text{ and consequently, } \frac{\mathrm{d}}{\mathrm{d}\,t}\Big|_{t=k} \mathrm{e}^{(t-k)A(t)} = A(k). \tag{9}$$

Proof: We present here the proof of the statement for $k=0$, since for $k=1$ the proof is similar. Therefore, for $k=0$, we have that

$$\begin{aligned}
\Delta^L_{tA(t)}(t)\Big|_{t=0} &= \left(\int_0^1 \mathrm{e}^{ut\,\mathrm{ad}_{A(t)}}(A(t)+t\dot{A}(t))du\right)\Big|_{t=0} \\
&= \left(\int_0^1 \mathrm{e}^{ut\,\mathrm{ad}_{A(t)}}(A(t))du\right)\Big|_{t=0} + \left(\int_0^1 \mathrm{e}^{ut\,\mathrm{ad}_{A(t)}}(t\dot{A}(t))du\right)\Big|_{t=0} \\
&= \left(\int_0^1 A(t)du\right)\Big|_{t=0} = A(0).
\end{aligned}$$

Consequently,

$$\left.\frac{\mathrm{d}}{\mathrm{d}\,t}\right|_{t=0} \mathrm{e}^{tA(t)} = \Delta^L_{tA(t)}(t)\mathrm{e}^{tA(t)}\Big|_{t=0} = A(0). \quad \blacksquare$$

Lemma 2.5. *Let $t \longmapsto A(t)$ be a differentiable matrix valued function. Then*

$$\left.\frac{\mathrm{d}}{\mathrm{d}\,t}\right|_{t=0} \Delta^L_{tA(t)}(t) = 2\dot{A}(0) \quad \textit{and} \quad \left.\frac{\mathrm{d}}{\mathrm{d}\,t}\right|_{t=1} \Delta^L_{(t-1)A(t)}(t) = 2\dot{A}(1). \tag{10}$$

Proof: To prove the first identity, we have that

$$\begin{aligned}
\left.\frac{\mathrm{d}}{\mathrm{d}\,t}\right|_{t=0} \Delta^L_{tA(t)}(t) &= \left.\frac{\mathrm{d}}{\mathrm{d}\,t}\right|_{t=0} \left(\int_0^1 \mathrm{e}^{ut\,\mathrm{ad}_{A(t)}}(A(t)+t\dot{A}(t))\,du\right)\\
&= \left.\frac{\mathrm{d}}{\mathrm{d}\,t}\right|_{t=0} \left(\int_0^1 A(t)\,du\right) + \left.\frac{\mathrm{d}}{\mathrm{d}\,t}\right|_{t=0} \left(\int_0^1 \mathrm{e}^{ut\,\mathrm{ad}_{A(t)}}(t\dot{A}(t))\,du\right)\\
&= \int_0^1 \left.\frac{\mathrm{d}}{\mathrm{d}\,t}\right|_{t=0} A(t)\,du + \int_0^1 \left.\frac{\mathrm{d}}{\mathrm{d}\,t}\right|_{t=0} \left(\mathrm{e}^{ut\,\mathrm{ad}_{A(t)}}(t\dot{A}(t))\right)\,du\\
&= \dot{A}(0) + \int_0^1 \Big(\Delta^L_{utA(t)}(t)\mathrm{e}^{utA(t)}(t\dot{A}(t))\mathrm{e}^{-utA(t)}\\
&\qquad + \mathrm{e}^{utA(t)}(\dot{A}(t)+t\ddot{A}(t))\mathrm{e}^{-utA(t)}\\
&\qquad + \mathrm{e}^{utA(t)}(t\dot{A}(t))\Delta^L_{-utA(t)}(t)\mathrm{e}^{-utA(t)}\Big)\Big|_{t=0}\,du\\
&= \dot{A}(0) + \int_0^1 \dot{A}(0)\,du = 2\dot{A}(0).
\end{aligned}$$

The proof of the second identity is now immediate, since it is done with similar computations. $\blacksquare$

3. The Grassmann manifold

3.1. The geometry of the Grassmann manifold. Each k-dimensional subspace of $\mathbb{R}^n$ can be associated to a unique operator of orthogonal projections onto itself, with respect to the Euclidean metric. It is well-known that these operators (or, equivalently, its matrices, called projection matrices) are symmetric, idempotent, and have rank k. Therefore, $G_{k,n}$ can be defined, alternatively, as:

$$G_{k,n} := \big\{P \in \mathfrak{s}(n) \;:\; P^2 = P \text{ and } \mathrm{rank}(P) = k\big\}. \tag{11}$$

It is known that $G_{k,n}$ is a smooth compact connected manifold of real dimension $k(n-k)$, and moreover it is isospectral (each element has the eigenvalues 1 and 0, with multiplicity k and $n-k$, respectively).

For an arbitrary point $P \in G_{k,n}$, define the following sets of matrices

$$\begin{aligned} \mathfrak{gl}_P(n) &:= \{A \in \mathfrak{gl}(n) \,:\, A = PA + AP\}; \\ \mathfrak{s}_P(n) &:= \mathfrak{s}(n) \cap \mathfrak{gl}_P(n); \\ \mathfrak{so}_P(n) &:= \mathfrak{so}(n) \cap \mathfrak{gl}_P(n). \end{aligned} \tag{12}$$

These sets also play an important role in the geometric description of the Grassmann manifold, due to their interesting properties, some of which are listed below.

Lemma 3.1. *Let $P \in G_{k,n}$, $A, B, \in \mathfrak{gl}_P(n)$, and $j \in \mathbb{N}$. Then, the following holds.*

1. $A^{2j-1} = PA^{2j-1} + A^{2j-1}P$;
2. $PA^{2j-1}P = 0$; $\quad PA^{2j} = PA^{2j}P = A^{2j}P$;
3. $(I-2P)A^{2j-1} = -A^{2j-1}(I-2P) = [A^{2j-1}, P]$;
4. $(I-2P)A^{2j} = A^{2j}(I-2P) = -A[A^{2j-1}, P]$;
5. $[A,P] = [B,P] \Longleftrightarrow A = B$.

Proof: The first two properties have been proved, by induction, in [2]. To prove 3., notice that

$$\begin{aligned} (I-2P)A^{2j-1} &= A^{2j-1} - 2PA^{2j-1} \overset{1.}{=} A^{2j-1} - 2\left(A^{2j-1} - A^{2j-1}P\right) \\ &= -A^{2j-1}\left(I-2P\right), \end{aligned}$$

and, on the other hand,

$$\begin{aligned} -A^{2j-1}\left(I-2P\right) &= -A^{2j-1} + 2A^{2j-1}P \overset{1.}{=} -PA^{2j-1} - A^{2j-1}P + 2A^{2j-1}P \\ &= [A^{2j-1}, P]. \end{aligned}$$

This proves 3.. Now,

$$\begin{aligned} (I-2P)A^{2j} &= A^{2j} - 2PA^{2j} \overset{2.}{=} A^{2j} - 2A^{2j}P \\ &= A^{2j}\left(I-2P\right), \end{aligned}$$

and

$$\begin{aligned} A^{2j}\left(I-2P\right) &= AA^{2j-1}\left(I-2P\right) \\ &\overset{3.}{=} -A[A^{2j-1}, P], \end{aligned}$$

which proves 4.. Finally, we prove the last equivalence. For that, we take into consideration that $A, B \in \mathfrak{gl}_P(n)$ and that the matrix $I - 2P$ is orthogonal and symmetric, so its inverse is itself.

$$\begin{aligned}
[A, P] = [B, P] &\iff AP - PA = BP - PB \\
&\iff A - PA - PA = B - PB - PB \\
&\iff A - 2PA = B - 2PB \\
&\iff (I - 2P)A = (I - 2P)B \\
&\iff A = (I - 2P)^{-1}(I - 2P)B \\
&\iff A = B. \quad \blacksquare
\end{aligned}$$

Taking into consideration the previous lemma, we state the next result involving the matrix exponential and the adjoint operator.

Lemma 3.2. *Let $P \in G_{k,n}$, $A \in \mathfrak{gl}_P(n)$, and $t \in \mathbb{R}$. Then,*

$$(I - 2P)e^{tA} = e^{-tA}(I - 2P), \tag{13}$$

and, consequently,

$$e^{2tA}(I - 2P) = e^{ad_{tA}}(I - 2P). \tag{14}$$

Proof: Taking into account the definition of matrix exponential, the first identity (13) is an immediate consequence of properties 3. and 4. of Lemma 3.1. The identity (14) is obtained from the first one just by a few computations and considering the Campbell-Hausdorff Formula in Lemma (2.1). In fact, from (13) we get that

$$\mathrm{e}^{tA}(I - 2P)\mathrm{e}^{tA} = (I - 2P),$$

then, multiplying both terms on the left by e^{tA}, and on the right by e^{-tA}, we have

$$\mathrm{e}^{2tA}(I - 2P) = \mathrm{e}^{tA}(I - 2P)\mathrm{e}^{-tA},$$

which, using the Campbell-Hausdorff Formula, proves the identity (14). $\blacksquare$

The tangent space to the Grassmann manifold at a point $P \in G_{k,n}$ can be defined by

$$T_P G_{k,n} = \big\{[P, \Omega] \;:\; \Omega \in \mathfrak{so}_P(n)\big\}. \tag{15}$$

We consider the Grassmann manifold equipped with a Riemannian metric that is induced by the Euclidean inner product on each tangent space, also known as the Frobenius inner product, defined on the space of square matrices as $\langle A, B\rangle := \operatorname{tr}(A^\top B)$, where tr denotes the trace of a matrix. For the Grassmann manifold, this Riemannian metric can be simplified. Indeed, for $P \in G_{k,n}$, $\Omega_1, \Omega_2 \in \mathfrak{so}_P(n)$, using some properties of the trace, we may write

$$\begin{aligned}
\langle [P,\Omega_1],[P,\Omega_2]\rangle &= \operatorname{tr}\big([P,\Omega_1]^\top [P,\Omega_2]\big)\\
&= \operatorname{tr}\big((P\Omega_1-\Omega_1P)^\top (P\Omega_2-\Omega_2P)\big)\\
&= \operatorname{tr}\big((-\Omega_1P+P\Omega_1)(P\Omega_2-\Omega_2P)\big)\\
&= \operatorname{tr}\big(-\Omega_1P\Omega_2+\Omega_1P\Omega_2P+P\Omega_1P\Omega_2-P\Omega_1\Omega_2P\big)\\
&= \operatorname{tr}\big(-\Omega_1P\Omega_2-P\Omega_1\Omega_2P\big)\\
&= \operatorname{tr}\big(-(\Omega_1-P\Omega_1)\Omega_2-P\Omega_1\Omega_2P\big)\\
&= \operatorname{tr}\big(-\Omega_1\Omega_2+P\Omega_1\Omega_2-P\Omega_1\Omega_2P\big)\\
&= -\operatorname{tr}\big(\Omega_1\Omega_2\big)+\operatorname{tr}\big(P\Omega_1\Omega_2\big)-\operatorname{tr}\big(P^2\Omega_1\Omega_2\big)\\
&= -\operatorname{tr}\big(\Omega_1\Omega_2\big).
\end{aligned}$$

So, the Riemannian metric in $G_{k,n}$ is given by

$$\langle [P,\Omega_1],[P,\Omega_2]\rangle = -\operatorname{tr}\big(\Omega_1\Omega_2\big). \tag{16}$$

Remark 3.1. *Observe that (15) can be rewritten as*

$$T_PG_{k,n} = \big\{ ad_P(\Omega)\ :\ \Omega \in \mathfrak{so}_P(n)\big\}, \tag{17}$$

or, equivalently, as

$$T_PG_{k,n} = \big\{ ad_P^2(S)\ :\ S \in \mathfrak{s}(n)\big\}. \tag{18}$$

Using the last description of the tangent space at P, the normal space at P, with respect to the Riemannian metric (16), can be defined by

$$(T_PG_{k,n})^\perp = \big\{ S - \operatorname{ad}_P^2(S)\ :\ S \in \mathfrak{s}(n)\big\}. \tag{19}$$

We note that the descriptions of the tangent and normal spaces mentioned above are in accordance with the ones that have already appeared in [9].

Lemma 3.3. *Let $P \in G_{k,n}$ and $\Omega \in \mathfrak{so}_P(n)$. Then,*

$$[\Omega,[\Omega,P]] \in (T_PG_{k,n})^\perp. \tag{20}$$

Proof: Let $P \in G_{k,n}$, $\Omega \in \mathfrak{so}_P(n)$ and $[\Omega_1, P]$, with $\Omega_1 \in \mathfrak{so}_P(n)$, be an arbitrary element of $T_PG_{k,n}$. Then, taking into account the identities in 2.

of Lemma 3.1 and some properties of the matrix trace, we have that

$$\begin{aligned}
\langle [\Omega_1, P], [\Omega, [\Omega, P]] \rangle &= \operatorname{tr}\big([\Omega_1, P]^\top [\Omega, [\Omega, P]]\big) \\
&= \operatorname{tr}\big((\Omega_1 P - P\Omega_1)^\top (\Omega(\Omega P - P\Omega) - (\Omega P - P\Omega)\Omega)\big) \\
&= \operatorname{tr}\big((-P\Omega_1 + \Omega_1 P)(\Omega^2 P - 2\Omega P\Omega + P\Omega^2)\big) \\
&= \operatorname{tr}\big((-P\Omega_1 + \Omega_1 P)(2\Omega^2 P - 2\Omega P\Omega)\big) \\
&= 2\operatorname{tr}\big(-P\Omega_1\Omega^2 P + P\Omega_1\Omega P\Omega + \Omega_1 P\Omega^2 P\big) \\
&= 2\operatorname{tr}(-P\Omega_1\Omega^2 P) \\
&= 2\operatorname{tr}\big(P\Omega^2\Omega_1 P\big) = 2\operatorname{tr}\big(\Omega^2 P\Omega_1 P\big) = 0.
\end{aligned}$$

Consequently, $[\Omega, [\Omega, P]] \in (T_P G_{k,n})^\perp$. ■

Now, we present some results about geodesics in the Grassmann manifold with respect to the Riemannian metric in (16).

Lemma 3.4 ([2]). *The unique geodesic $t \mapsto \gamma(t)$ in $G_{k,n}$, satisfying the initial conditions $\gamma(0) = P$ and $\dot\gamma(0) = [\Omega, P]$, where $\Omega \in \mathfrak{so}_P(n)$, is given by*

$$\gamma(t) = e^{t\Omega} P e^{-t\Omega}. \tag{21}$$

The next result gives an explicit formula for the minimising geodesic arc connecting two points in $G_{k,n}$. Although the expression has already appeared in [2], we present bellow an easier alternative proof based on Lemma 3.2.

Proposition 3.5. *Let $P, Q \in G_{k,n}$ be such that the matrix $(I - 2Q)(I - 2P)$ has no negative real eigenvalues. Then, the minimising geodesic arc in $G_{k,n}$ that joins P (at $t = 0$) to Q (at $t = 1$), is parameterised explicitly by*

$$\gamma(t) = e^{t\Omega} P e^{-t\Omega}, \tag{22}$$

with $\Omega = \frac{1}{2}\log((I - 2Q)(I - 2P)) \in \mathfrak{so}_P(n)$.

Proof: Let $P, Q \in G_{k,n}$ and $\gamma(t) = \mathrm{e}^{t\Omega} P \mathrm{e}^{-t\Omega}$, $t \in [0, 1]$ be such that $\gamma(0) = P$. In order to prove the result we need to obtain $\Omega \in \mathfrak{so}_P(n)$, such that $\gamma(1) = Q$, i.e., such that $\mathrm{e}^{\Omega} P \mathrm{e}^{-\Omega} = Q$. According with Lemma 3.2, and since the inverse of the matrix $I-2P$ is itself, the following holds.

$$\begin{aligned} \mathrm{e}^{\Omega}P\mathrm{e}^{-\Omega}=Q &\iff \mathrm{e}^{\Omega}(I-2P)\mathrm{e}^{-\Omega}=I-2Q \\ &\iff \mathrm{e}^{2\Omega}(I-2P)=I-2Q \\ &\iff \mathrm{e}^{2\Omega}=(I-2Q)(I-2P) \\ &\iff \Omega=\frac{1}{2}\log((I-2Q)(I-2P). \end{aligned}$$

We have that $(I-2Q)(I-2P)\in SO(n)$. Then, $\Omega=\frac{1}{2}\log((I-2Q)(I-2P))\in\mathfrak{so}(n)$. Therefore, in order to prove that $\Omega\in\mathfrak{so}_P(n)$, it remains to show that $\Omega P+P\Omega=\Omega$, which is equivalent to prove that

$$2\Omega(I-2P)+(I-2P)2\Omega=0.$$

Taking into account properties in Lemma (2.1) and that $(I-2P)^2=I$, we get,

$$\begin{aligned} &2\Omega(I-2P)+(I-2P)2\Omega \\ &\quad=\left(\log((I-2Q)(I-2P))\right)(I-2P)+(I-2P)\left(\log((I-2Q)(I-2P))\right) \\ &\quad=(I-2P)\left(\log((I-2P)(I-2Q)(I-2P)^2)\right)+(I-2P)\left(\log((I-2Q)(I-2P))\right) \\ &\quad=(I-2P)\left(-\log((I-2Q)(I-2P))\right)+(I-2P)\left(\log((I-2Q)(I-2P))\right) \\ &\quad=0, \end{aligned}$$

which proves the result. ■

Remark 3.2. *Notice that the orthogonal matrix $(I-2Q)(I-2P)$ belongs to $SO(n)$, since the requirement that it has no negative real eigenvalues automatically excludes the orthogonal matrices with determinant equal to -1.*

The geodesic distance between two points P and Q is equal to the length of the geodesic arc that joins P (at $t=0$) to Q (at $t=1$). So, as a consequence of the previous proposition, we can state the following.

Proposition 3.6. *Let $P,Q\in G_{k,n}$ be such that the matrix $(I-2Q)(I-2P)$ has no negative real eigenvalues. Then, the geodesic distance between the points P and Q is given by*

$$d^2(P,Q)=-\frac{1}{4}tr\Big(\log^2((I-2Q)(I-2P))\Big). \tag{23}$$

3.2. Representing Images by Points in a Grassmann Manifold. In this subsection we emphasise the importance of the Grassmann manifold in certain applications dealing with nonlinear data based on images. A popular choice for modelling image-sets is by representing them through linear subspaces. This is done by associating to a set of images a point in the Grassmann manifold $G_{k,n}$, where n is the dimension of the space of features and k is related to the principal features of the images. This enlightens the importance of the Grassmann manifold in many engineering applications, in particular to solve some computer vision problems.

In the context of image processing, a feature vector is a collection of important information that describes an image, differentiating that image from others. Some examples of features are: colour, gray levels, pixel intensities, shapes, edges and gradients.

Given a set of m images of the same object, we associate to that set a point in a Grassmann manifold $G_{k,n}$ as follows:

(1) Each image corresponds to a column matrix in the space of features, so that the m images can be represented by a rectangular matrix $X \in \mathbb{R}^{n\times m}$. We assume that $m < n$.

(2) The matrix X is then decomposed using the Singular Value Decomposition (SVD)

$$X = U\,\Sigma\,V^\top, \tag{24}$$

where $V^\top$ denotes the transpose of the matrix V, the matrices U and V are orthogonal of order n and m respectively ($UU^\top = I_n$, $VV^\top = I_m$) and Σ is a quasi-diagonal matrix containing the singular values $\sigma_1, \cdots, \sigma_m$ of X, in non-increasing order, along the main diagonal. If $\operatorname{rank}(X) = r$ and u_i and v_i denote the column vectors of U and V respectively, the SVD decomposition (24) can be written as

$$X = \sum_{i=1}^{r} \sigma_i\, u_i\, v_i^\top. \tag{25}$$

Since $XX^\top = U(\Sigma\Sigma^\top)U^\top$, the columns of U are the eigenvectors associated to the eigenvalues λ_i of $XX^\top$, which are the non-negative square roots of the singular values and are, by convention, also descendent sorted ($\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_k \geq 0$). The columns of the matrix U are called the eigenvectors of the SVD decomposition and the first columns correspond to the main dominant directions in the image structure.

(3) When a set of images is SVD transformed it is not compressed. Image compression deals with the problem of reducing the amount of computer memory required to represent a digital image. Since the great amount of the image information lies in the first singular values, compression of data can be achieved replacing the matrix X by a good approximation of smaller rank, say of rank $k < r$. The closest matrix of rank k is obtained by truncating the sum in (25) after the first k terms to obtain

$$X \approx \sum_{i=1}^{k} \sigma_i \, u_i \, v_i^\top .$$

As k increases, the image quality increases, but so does the amount of memory needed to store the image. This means that smaller rank SVD approximations are preferable, but the choice of k also depends on the dimensionality of the data. The above truncation corresponds to deleting the last $n-k$ columns of the orthogonal matrix U, to form the sub-matrix $S_{n\times k}$, whose columns form a k-orthonormal frame in $\mathbb{R}^n$, i.e., $S^\top S = I_k$.

(4) From the previous matrix S, we compute a square matrix of order n, $P = SS^\top$, which is symmetric, idempotent ($P^2 = P$) and has rank k. This matrix P gives a representation of the data in the Grassmann manifold $G_{k,n}$.

4. De Casteljau algorithm in the Grassmann manifold

4.1. Revisiting cubic polynomials in Riemannian manifolds. Let M be a m-dimensional connected Riemannian manifold, which is also geodesically complete, so that any pair of points may be joined by a geodesic arc.

Problem 4.1. *Given a set of $\ell+1$ distinct points $p_i \in M$, with $i = 0, 1, \ldots, \ell$, a discrete sequence of $\ell+1$ fixed times, $t_0 < t_1 < \cdots < t_{\ell-1} < t_\ell$, and vectors ξ_0, η_0 tangent to M at p_0, and ξ_ℓ tangent to M at p_ℓ, solve the following problem:*

Find a $\mathcal{C}^2$-smooth curve $\gamma : [t_0, t_\ell] \to M$,
satisfying the interpolation conditions

$$\gamma(t_i) = p_i, \qquad 1 \le i \le \ell - 1, \tag{26}$$

and the boundary conditions (of Hermite type):

$$\begin{aligned} \gamma(t_0) &= p_0, & \gamma(t_\ell) &= p_\ell, \\ \dot\gamma(t_0) &= \xi_0 \in T_{p_0}M, & \dot\gamma(t_\ell) &= \xi_\ell \in T_{p_\ell}M, \end{aligned} \tag{27}$$

or, alternatively, the boundary conditions:

$$\begin{aligned} \gamma(t_0) = p_0, &\quad \gamma(t_\ell) = p_\ell, \\ \dot\gamma(t_0) = \xi_0 \in T_{p_0}M, &\quad \frac{D\dot\gamma}{dt}(t_0) = \eta_0 \in T_{p_0}M. \end{aligned} \tag{28}$$

Without loss of generality, in the sequel we consider $t_0 = 0$ and $t_\ell = 1$, since the reparametrisation $(t \to s)$ defined by $s = t(t_\ell - t_0) + t_0$ maps $[0, 1]$ to $[t_0, t_\ell]$.

In case there are no interpolation conditions, this problem can be solved using the generalised De Casteljau algorithm on manifolds. This is a procedure to generate a smooth curve which is the counterpart, for manifolds, of a cubic polynomial in Euclidean spaces. The classical version of that algorithm was developed by Paul De Casteljau in 1959 [6], and consists in performing successive linear interpolations to generate polynomial curves of arbitrary degree in the Euclidean space $\mathbb{R}^m$. The generalisation of the classical De Casteljau algorithm to Riemannian manifolds, is based on the simple idea of replacing line segments by geodesic arcs, as shown in [10], and later implemented to some concrete manifolds in [5].

In order to implement this algorithm for data on the Grassmann manifold, we review first the three steps of the De Casteljau algorithm to generate cubic polynomials using four ordered distinct points in M, where the first and the forth are the points we want to join and the other two points control the shape of the curve and are related with the boundary conditions. This will become clear later on.

Algorithm 4.2. Generalised De Casteljau Algorithm

Given four distinct points x_0, x_1, x_2 and x_3 in M:

Step 1. *Construct three geodesic arcs $\beta_1(t, x_i, x_{i+1})$, $t \in [0, 1]$ joining, for $i = 0, 1, 2$, the points x_i (at $t = 0$) and x_{i+1} (at $t = 1$).*

Step 2. *Construct two families of geodesic arcs*

$$\begin{aligned} \beta_2(t, x_0, x_1, x_2) &= \beta_1(t, \beta_1(t, x_0, x_1), \beta_1(t, x_1, x_2)), \\ \beta_2(t, x_1, x_2, x_3) &= \beta_1(t, \beta_1(t, x_1, x_2), \beta_1(t, x_2, x_3)), \end{aligned}$$

joining, for $i = 0, 1$ and $t \in [0, 1]$, the point $\beta_1(t, x_i, x_{i+1})$ (at $t = 0$) with the point $\beta_1(t, x_{i+1}, x_{i+2})$ (at $t = 1$).

Step 3. *Construct the family of geodesic arcs*

$$\beta_3(t, x_0, x_1, x_2, x_3) = \beta_1(t, \beta_2(t, x_0, x_1, x_2), \beta_2(t, x_1, x_2, x_3)),$$

joining, for each $t \in [0, 1]$, the points $\beta_2(t, x_0, x_1, x_2)$ (at $t = 0$) and $\beta_2(t, x_1, x_2, x_3)$ (at $t = 1$).

The curve $t \in [0, 1] \mapsto \beta_3(t) := \beta_3(t, x_0, x_1, x_2, x_3)$ obtained in Step 3. of this algorithm generalises cubic polynomials in Euclidean spaces and hereafter will be called *geometric cubic polynomial* in M. We bring to the attention of the reader that this name has been used in the literature for polynomial curves that result from generalising to manifolds other methods to produce cubic polynomials in Euclidean spaces. It is also important to observe that this curve joins the points x_0 (at $t = 0$) and x_3 (at $t = 1$), but does not pass through the other two points x_1 and x_2. These points are usually called *control points*, since they influence the shape of the curve. As will become clear later, the control points can be obtained from the boundary conditions. Although the geometry of a Riemannian manifold possesses enough structure to formulate this construction, it can only be implemented when one can reduce the calculation of these geodesic arcs to a manageable form. Using the result in Proposition 3.5, we can now implement the above algorithm when the manifold M is $G_{k,n}$.

4.2. Implementation of the De Casteljau Algorithm in $G_{k,n}$. Although the Grassmann manifold is geodesically complete, we have seen that an explicit formula for the geodesic that joints two points may be unknown in some particular situations. So, in this case the implementation of the De Casteljau algorithm is restricted to a convex open subset of the manifold where the expression to compute geodesic arcs is known.

When the given points x_0, x_1, x_2 and x_3 belong to $G_{k,n}$, the curves produced at each step of the Algorithm 4.2 are easily derived using the result in Proposition 3.5, and are presented bellow.

Step 1

$$\beta_1(t, x_i, x_{i+1}) = \mathrm{e}^{t\Omega_i^1} x_i \mathrm{e}^{-t\Omega_i^1} = \mathrm{e}^{t\,\mathrm{ad}_{\Omega_i^1}} x_i, \quad i = 0, 1, 2, \tag{29}$$

with

$$\Omega_i^1 = \frac{1}{2} \log((I - 2x_{i+1})(I - 2x_i)) \in \mathfrak{so}_{x_i}(n). \tag{30}$$

Step 2

$$\beta_2(t,x_0,x_1,x_2)=\mathrm{e}^{t\Omega_0^2(t)}\beta_1(t,x_0,x_1)\mathrm{e}^{-t\Omega_0^2(t)}=\mathrm{e}^{t\,\mathrm{ad}_{\Omega_0^2(t)}}\beta_1(t,x_0,x_1), \tag{31}$$

$$\beta_2(t,x_1,x_2,x_3)=\mathrm{e}^{t\Omega_1^2(t)}\beta_1(t,x_1,x_2)\mathrm{e}^{-t\Omega_1^2(t)}=\mathrm{e}^{t\,\mathrm{ad}_{\Omega_1^2(t)}}\beta_1(t,x_1,x_2), \tag{32}$$

with

$$\Omega_0^2(t)=\frac{1}{2}\log((I-2\beta_1(t,x_1,x_2))(I-2\beta_1(t,x_0,x_1)))\in\mathfrak{so}_{\beta_1(t,x_0,x_1)}(n), \tag{33}$$

$$\Omega_1^2(t)=\frac{1}{2}\log((I-2\beta_1(t,x_2,x_3))(I-2\beta_1(t,x_1,x_2)))\in\mathfrak{so}_{\beta_1(t,x_1,x_2)}(n). \tag{34}$$

Step 3

$$\beta_3(t,x_0,x_1,x_2,x_3)=\mathrm{e}^{t\Omega_0^3(t)}\beta_2(t,x_0,x_1,x_2)\mathrm{e}^{-t\Omega_0^3(t)}=\mathrm{e}^{t\,\mathrm{ad}_{\Omega_0^3(t)}}\beta_2(t,x_0,x_1,x_2), \tag{35}$$

with

$$\Omega_0^3(t)=\frac{1}{2}\log((I-2\beta_2(t,x_1,x_2,x_3))(I-2\beta_2(t,x_0,x_1,x_2)))\in\mathfrak{so}_{\beta_2(t,x_0,x_1,x_2)}(n). \tag{36}$$

As a result of applying the De Casteljau algorithm to the given four points, we obtain our definition of a geometric cubic polynomial in the Grassmann manifold.

Definition 4.1. *The curve $t\in[0,1]\mapsto\beta_3(t):=\beta_3(t,x_0,x_1,x_2,x_3)$ in $G_{k,n}$ defined by*

$$\begin{aligned}\beta_3(t) &= e^{t\Omega_0^3(t)}e^{t\Omega_0^2(t)}e^{t\Omega_0^1}x_0e^{-t\Omega_0^1}e^{-t\Omega_0^2(t)}e^{-t\Omega_0^3(t)}\\ &= e^{t\,ad_{\Omega_0^3(t)}}e^{t\,ad_{\Omega_0^2(t)}}e^{t\,ad_{\Omega_0^1}}x_0,\end{aligned} \tag{37}$$

with Ω_0^1, Ω_0^2 and Ω_0^3 given by (30), (33), and (36), is called a geometric cubic polynomial in the Grassmann manifold, associated to the points x_i, $i=0,1,2,3$.

Remark 4.1. *Notice that, as expected, the curve just defined joins the point x_0 (at $t=0$) to x_3 (at $t=1$). It is obvious that $\beta_3(0)=x_0$. To see that $\beta_3(1)=x_3$, we use the following easily derived boundary conditions for the Ω_j^i*

$$\begin{array}{ll}\Omega_0^2(0)=\Omega_0^3(0)=\Omega_0^1, & \Omega_1^2(0)=\Omega_1^1,\\ \Omega_1^2(1)=\Omega_0^3(1)=\Omega_2^1, & \Omega_0^2(1)=\Omega_1^1,\end{array} \tag{38}$$

together with the definition of the geodesic arcs (29), to obtain

$$\beta_3(1)=e^{ad_{\Omega_0^3(1)}}e^{ad_{\Omega_0^2(1)}}e^{ad_{\Omega_0^1}}x_0=e^{ad_{\Omega_2^1}}e^{ad_{\Omega_1^1}}e^{ad_{\Omega_0^1}}x_0=x_3. \tag{39}$$

We now present a few more results for the Ω_j^i that will be used later on.

Lemma 4.3. *Let Ω^i_j, $i=1,2,3$, $j=0,1,2$ be defined as at the beginning of Subsection 4.2. Then, the following identities hold:*

$$\begin{array}{lrcl} (i) & e^{2\Omega^2_0(t)} & = & e^{2t\Omega^1_1}e^{2(1-t)\Omega^1_0} \\ (ii) & e^{2\Omega^2_1(t)} & = & e^{2t\Omega^1_2}e^{2(1-t)\Omega^1_1} \\ (iii) & e^{2\Omega^3_0(t)} & = & e^{2t\Omega^2_1(t)}e^{2(1-t)\Omega^2_0(t)}. \end{array} \tag{40}$$

Proof: The proof of all these identities uses the definition of Ω^i_j and Lemma 3.2. We prove the last one in detail, the others can be proved using similar arguments, but have an even easier proof.
Proof of (iii): From the definition of Ω^3_0, we know that

$$\Omega^3_0(t)=\frac{1}{2}\log((I-2\beta_2(t,x_1,x_2,x_3))(I-2\beta_2(t,x_0,x_1,x_2))).$$

Then, using the relations (31) and (32), we have

$$\begin{aligned} \mathrm{e}^{2\Omega^3_0(t)} &= (I-2\beta_2(t,x_1,x_2,x_3))(I-2\beta_2(t,x_0,x_1,x_2)) \\ &= (I-2\mathrm{e}^{t\mathrm{ad}_{\Omega^2_1(t)}}\beta_1(t,x_1,x_2))(I-2\mathrm{e}^{t\mathrm{ad}_{\Omega^2_0(t)}}\beta_1(t,x_0,x_1)) \\ &= (I-2\mathrm{e}^{t\Omega^2_1(t)}\beta_1(t,x_1,x_2)\mathrm{e}^{-t\Omega^2_1(t)})(I-2\mathrm{e}^{t\Omega^2_0(t)}\beta_1(t,x_0,x_1)\mathrm{e}^{-t\Omega^2_0(t)}). \end{aligned}$$

Therefore, since $\Omega^2_1(t)\in\mathfrak{so}_{\beta_1(t,x_1,x_2)}(n)$ and $\Omega^2_0(t)\in\mathfrak{so}_{\beta_1(t,x_0,x_1)}(n)$, using Lemma 3.2 and the definition of Ω^2_0, we obtain

$$\begin{aligned} \mathrm{e}^{2\Omega^3_0(t)} &= \mathrm{e}^{t\Omega^2_1(t)}(I-2\beta_1(t,x_1,x_2))\mathrm{e}^{-t\Omega^2_1(t)}\mathrm{e}^{t\Omega^2_0(t)}(I-2\beta_1(t,x_0,x_1))\mathrm{e}^{-t\Omega^2_0(t)} \\ &= \mathrm{e}^{2t\Omega^2_1(t)}(I-2\beta_1(t,x_1,x_2))(I-2\beta_1(t,x_0,x_1))\mathrm{e}^{-2t\Omega^2_0(t)} \\ &= \mathrm{e}^{2t\Omega^2_1(t)}\mathrm{e}^{2\Omega^2_0(t)}\mathrm{e}^{-2t\Omega^2_0(t)} \\ &= \mathrm{e}^{2t\Omega^2_1(t)}\mathrm{e}^{2(1-t)\Omega^2_0(t)}. \quad \blacksquare \end{aligned}$$

Lemma 4.4. *Let Ω^i_j, $i=1,2,3$, $j=0,1,2$, be defined as in the steps at the beginning of Subsection 4.2. Then, the following identities hold:*

$$\begin{array}{lrcl} (i) & e^{(t-1)ad_{\Omega^2_0(t)}}e^{t\,ad_{\Omega^1_1}} & = & e^{t\,ad_{\Omega^2_0(t)}}e^{(t-1)ad_{\Omega^1_0}} \\ (ii) & e^{(t-1)ad_{\Omega^2_1(t)}}e^{t\,ad_{\Omega^1_2}} & = & e^{t\,ad_{\Omega^2_1(t)}}e^{(t-1)ad_{\Omega^1_1}} \\ (iii) & e^{(t-1)ad_{\Omega^3_0(t)}}e^{t\,ad_{\Omega^2_1(t)}} & = & e^{t\,ad_{\Omega^3_0(t)}}e^{(t-1)ad_{\Omega^2_0(t)}}. \end{array} \tag{41}$$

Proof: We will prove the identities (i) and (iii). The proof of the identity (ii) is similar to the proof of the identity (i).
Proof of (i): Since $\Omega_0^2(t) \in \mathfrak{so}_{\beta_1(t,x_0,x_1)}(n)$ and $\beta_1(t,x_0,x_1) \in G_{k,n}$, by Lemma 3.2, we have that

$$\mathrm{e}^{2\Omega_0^2(t)}\left(I - 2\beta_1(t,x_0,x_1)\right) = \mathrm{e}^{\mathrm{ad}_{\Omega_0^2(t)}}\left(I - 2\beta_1(t,x_0,x_1)\right).$$

But, from (29), we know that $\beta_1(t,x_0,x_1) = \mathrm{e}^{t\,\mathrm{ad}_{\Omega_0^1}}x_0$, and thus we get

$$\mathrm{e}^{2\Omega_0^2(t)}\mathrm{e}^{t\,\mathrm{ad}_{\Omega_0^1}}\left(I - 2x_0\right) = \mathrm{e}^{\mathrm{ad}_{\Omega_0^2(t)}}\mathrm{e}^{t\,\mathrm{ad}_{\Omega_0^1}}\left(I - 2x_0\right). \tag{42}$$

Then, taking into account the identity (i) of Lemma 4.3 and Lemma 3.2, the left-hand side of (42) can be rewritten as

$$\begin{aligned}
\mathrm{e}^{2\Omega_0^2(t)}\mathrm{e}^{t\,\mathrm{ad}_{\Omega_0^1}}\left(I - 2x_0\right) &= \mathrm{e}^{2\Omega_0^2(t)}\mathrm{e}^{t\,\Omega_0^1}\left(I - 2x_0\right)\mathrm{e}^{-t\,\Omega_0^1} \\
&= \mathrm{e}^{2\Omega_0^2(t)}\mathrm{e}^{2t\,\Omega_0^1}\left(I - 2x_0\right) \\
&= \mathrm{e}^{2t\Omega_1^1}\mathrm{e}^{2(1-t)\Omega_0^1}\mathrm{e}^{2t\,\Omega_0^1}\left(I - 2x_0\right) \\
&= \mathrm{e}^{2t\Omega_1^1}\mathrm{e}^{2\,\Omega_0^1}\left(I - 2x_0\right) \\
&= \mathrm{e}^{2t\Omega_1^1}\mathrm{e}^{\mathrm{ad}_{\Omega_0^1}}\left(I - 2x_0\right) \\
&= \mathrm{e}^{2t\Omega_1^1}\left(I - 2x_1\right) \\
&= \mathrm{e}^{t\mathrm{ad}_{\Omega_1^1}}\mathrm{e}^{\mathrm{ad}_{\Omega_0^1}}\left(I - 2x_0\right).
\end{aligned} \tag{43}$$

Therefore, comparing the last right-hand side of (43) with the right-hand side of (42), we obtain that

$$\mathrm{e}^{t\mathrm{ad}_{\Omega_1^1}}\mathrm{e}^{\mathrm{ad}_{\Omega_0^1}}\left(I - 2x_0\right) = \mathrm{e}^{\mathrm{ad}_{\Omega_0^2(t)}}\mathrm{e}^{t\,\mathrm{ad}_{\Omega_0^1}}\left(I - 2x_0\right).$$

Since $\left(I - 2x_0\right)^{-1} = \left(I - 2x_0\right)$, then $\mathrm{e}^{-\mathrm{ad}_{\Omega_0^2(t)}}\mathrm{e}^{t\mathrm{ad}_{\Omega_1^1}}\mathrm{e}^{\mathrm{ad}_{\Omega_0^1}} = \mathrm{e}^{t\,\mathrm{ad}_{\Omega_0^1}}$, which is equivalent to

$$\mathrm{e}^{-\mathrm{ad}_{\Omega_0^2(t)}}\mathrm{e}^{t\mathrm{ad}_{\Omega_1^1}} = \mathrm{e}^{(t-1)\,\mathrm{ad}_{\Omega_0^1}}. \tag{44}$$

Consequently, multiplying both sides of the last equality by $\mathrm{e}^{t\mathrm{ad}_{\Omega_0^2(t)}}$, we get

$$\mathrm{e}^{(t-1)\mathrm{ad}_{\Omega_0^2(t)}}\mathrm{e}^{t\mathrm{ad}_{\Omega_1^1}} = \mathrm{e}^{t\mathrm{ad}_{\Omega_0^2(t)}}\mathrm{e}^{(t-1)\,\mathrm{ad}_{\Omega_0^1}},$$

which proves the result.

Proof of (iii): We first show that

$$\mathrm{e}^{\mathrm{ad}_{\Omega_0^2(t)}}\beta_1(t,x_0,x_1)=\beta_1(t,x_1,x_2). \tag{45}$$

From the relation (44), it yields that

$$\mathrm{e}^{\mathrm{ad}_{\Omega_0^2(t)}}=\mathrm{e}^{t\mathrm{ad}_{\Omega_1^1}}\mathrm{e}^{(1-t)\,\mathrm{ad}_{\Omega_0^1}}. \tag{46}$$

Then, according with the relations (29) and (46), we have that

$$\begin{aligned}
\mathrm{e}^{\mathrm{ad}_{\Omega_0^2(t)}}\beta_1(t,x_0,x_1) &= \mathrm{e}^{t\mathrm{ad}_{\Omega_1^1}}\mathrm{e}^{(1-t)\,\mathrm{ad}_{\Omega_0^1}}\beta_1(t,x_0,x_1)\\
&= \mathrm{e}^{t\mathrm{ad}_{\Omega_1^1}}\mathrm{e}^{(1-t)\,\mathrm{ad}_{\Omega_0^1}}\mathrm{e}^{t\,\mathrm{ad}_{\Omega_0^1}}x_0\\
&= \mathrm{e}^{t\mathrm{ad}_{\Omega_1^1}}\mathrm{e}^{\mathrm{ad}_{\Omega_0^1}}x_0\\
&= \mathrm{e}^{t\mathrm{ad}_{\Omega_1^1}}x_1\\
&= \beta_1(t,x_1,x_2),
\end{aligned}$$

which proves identity (45). Since $\Omega_0^3(t)\in\mathfrak{so}_{\beta_2(t,x_0,x_1,x_2)}(n)$ and $\beta_2(t,x_0,x_1,x_2)$ is a curve in $G_{k,n}$, by Lemma 3.2 and by (31), we have

$$\begin{aligned}
\mathrm{e}^{2\Omega_0^3(t)}\left(I-2\beta_2(t,x_0,x_1,x_2)\right) &= \mathrm{e}^{\mathrm{ad}_{\Omega_0^3(t)}}\left(I-2\beta_2(t,x_0,x_1,x_2)\right)\\
&= \mathrm{e}^{\mathrm{ad}_{\Omega_0^3(t)}}\mathrm{e}^{t\mathrm{ad}_{\Omega_0^2(t)}}\left(I-2\beta_1(t,x_0,x_1)\right).
\end{aligned} \tag{47}$$

Then, using Lemma 3.2, together with the identity (iii) of Lemma 4.3 and the equality (45), we obtain that the left-hand side of (47) can be rewritten as

$$
\begin{aligned}
e^{2\Omega_0^3(t)}\left(I-2\beta_2(t,x_0,x_1,x_2)\right) &= e^{2\Omega_0^3(t)}e^{2t\Omega_0^2(t)}\left(I-2\beta_1(t,x_0,x_1)\right)\\
&= e^{2t\Omega_1^2(t)}e^{2(1-t)\Omega_0^2(t)}e^{2t\Omega_0^2(t)}\left(I-2\beta_1(t,x_0,x_1)\right)\\
&= e^{2t\Omega_1^2(t)}e^{2\Omega_0^2(t)}\left(I-2\beta_1(t,x_0,x_1)\right)\\
&= e^{2t\Omega_1^2(t)}e^{\mathrm{ad}_{\Omega_0^2(t)}}\left(I-2\beta_1(t,x_0,x_1)\right)\\
&= e^{2t\Omega_1^2(t)}\left(I-2e^{\mathrm{ad}_{\Omega_0^2(t)}}\beta_1(t,x_0,x_1)\right)\\
&= e^{2t\Omega_1^2(t)}\left(I-2\beta_1(t,x_1,x_2)\right)\\
&= e^{t\mathrm{ad}_{\Omega_1^2(t)}}\left(I-2\beta_1(t,x_1,x_2)\right)\\
&= e^{t\mathrm{ad}_{\Omega_1^2(t)}}e^{\mathrm{ad}_{\Omega_0^2(t)}}\left(I-2\beta_1(t,x_0,x_1)\right).
\end{aligned}
\tag{48}
$$

Therefore, from (47) and (48), we get that

$$
e^{t\mathrm{ad}_{\Omega_1^2(t)}}e^{\mathrm{ad}_{\Omega_0^2(t)}}\left(I-2\beta_1(t,x_0,x_1)\right)=e^{\mathrm{ad}_{\Omega_0^3(t)}}e^{t\mathrm{ad}_{\Omega_0^2(t)}}\left(I-2\beta_1(t,x_0,x_1)\right).
$$

Also, since $\beta_1(t,x_0,x_1)\in G_{k,n}$, we know that

$$
\left(I-2\beta_1(t,x_0,x_1)\right)\left(I-2\beta_1(t,x_0,x_1)\right)=I,
$$

and thus

$$
e^{-\mathrm{ad}_{\Omega_0^3(t)}}e^{t\mathrm{ad}_{\Omega_1^2(t)}}e^{\mathrm{ad}_{\Omega_0^2(t)}}=e^{t\mathrm{ad}_{\Omega_0^2(t)}}.
$$

Consequently,

$$
e^{-\mathrm{ad}_{\Omega_0^3(t)}}e^{t\mathrm{ad}_{\Omega_1^2(t)}}=e^{(t-1)\mathrm{ad}_{\Omega_0^2(t)}},
$$

and multiplying both sides of this identity by $e^{t\mathrm{ad}_{\Omega_0^3(t)}}$, the result follows. ■

We are now in conditions to state the following result which contains an alternative way of defining the geometric cubic polynomial β_3 in $G_{k,n}$. The importance of this result lies in the fact that this alternative expression will be particularly useful in the computation of the derivatives of the cubic polynomial at the endpoint ($t=1$).

Theorem 4.5. *Let $t\in[0,1]\mapsto\beta_3(t)$ be the geometric cubic polynomial in $G_{k,n}$ defined in Definition 4.1. Define another curve $t\in[0,1]\mapsto\gamma(t)$ in $G_{k,n}$, by*

$$
\gamma(t)=e^{(t-1)\,ad_{\Omega_0^3(t)}}e^{(t-1)\,ad_{\Omega_1^2(t)}}e^{(t-1)\,ad_{\Omega_2^1}}x_3,
\tag{49}
$$

where Ω_0^3, Ω_1^2 and Ω_2^1 are as defined at the beginning of Subsection 4.2. Then,

$$\beta_3(t) = \gamma(t), \quad t \in [0,1].$$

Proof: Taking into consideration the relation (39) and applying the identities (i), (ii) and (iii) of Lemma 4.4, we may write

$$\begin{aligned}
\gamma(t) &= \mathrm{e}^{(t-1)\,\mathrm{ad}_{\Omega_0^3(t)}}\mathrm{e}^{(t-1)\,\mathrm{ad}_{\Omega_1^2(t)}}\mathrm{e}^{(t-1)\,\mathrm{ad}_{\Omega_2^1}}x_3\\
&\overset{(39)}{=} \mathrm{e}^{(t-1)\,\mathrm{ad}_{\Omega_0^3(t)}}\mathrm{e}^{(t-1)\,\mathrm{ad}_{\Omega_1^2(t)}}\mathrm{e}^{(t-1)\,\mathrm{ad}_{\Omega_2^1}}\mathrm{e}^{\mathrm{ad}_{\Omega_2^1}}\mathrm{e}^{\mathrm{ad}_{\Omega_1^1}}\mathrm{e}^{\mathrm{ad}_{\Omega_0^1}}x_0\\
&= \mathrm{e}^{(t-1)\,\mathrm{ad}_{\Omega_0^3(t)}}\mathrm{e}^{(t-1)\,\mathrm{ad}_{\Omega_1^2(t)}}\mathrm{e}^{t\,\mathrm{ad}_{\Omega_2^1}}\mathrm{e}^{\mathrm{ad}_{\Omega_1^1}}\mathrm{e}^{\mathrm{ad}_{\Omega_0^1}}x_0\\
&\overset{(ii)}{=} \mathrm{e}^{(t-1)\,\mathrm{ad}_{\Omega_0^3(t)}}\mathrm{e}^{t\,\mathrm{ad}_{\Omega_1^2(t)}}\mathrm{e}^{(t-1)\,\mathrm{ad}_{\Omega_1^1}}\mathrm{e}^{\mathrm{ad}_{\Omega_1^1}}\mathrm{e}^{\mathrm{ad}_{\Omega_0^1}}x_0\\
&\overset{(iii)}{=} \mathrm{e}^{t\,\mathrm{ad}_{\Omega_0^3(t)}}\mathrm{e}^{(t-1)\,\mathrm{ad}_{\Omega_0^2(t)}}\mathrm{e}^{t\,\mathrm{ad}_{\Omega_1^1}}\mathrm{e}^{\mathrm{ad}_{\Omega_0^1}}x_0\\
&\overset{(i)}{=} \mathrm{e}^{t\,\mathrm{ad}_{\Omega_0^3(t)}}\mathrm{e}^{t\,\mathrm{ad}_{\Omega_0^2(t)}}\mathrm{e}^{(t-1)\,\mathrm{ad}_{\Omega_0^1}}\mathrm{e}^{\mathrm{ad}_{\Omega_0^1}}x_0\\
&= \mathrm{e}^{t\,\mathrm{ad}_{\Omega_0^3(t)}}\mathrm{e}^{t\,\mathrm{ad}_{\Omega_0^2(t)}}\mathrm{e}^{t\,\mathrm{ad}_{\Omega_0^1}}x_0\\
&= \beta_3(t). \quad \blacksquare
\end{aligned}$$

We now state some results about derivatives of the Ω_j^i that will be necessary to fully understand other important developments.

Lemma 4.6. *For $j = 2, 3$, let $i = 3 - j$. Then, the following holds.*

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\left(e^{2\Omega_0^j(t)}\right)\bigg|_{t=0} = 2\chi_{_0}\left(\dot{\Omega}_0^j(0)\right)e^{2\Omega_0^1}, \quad where \quad \chi_{_0} := \int_0^1 e^{u\,ad_{2\Omega_0^1}}du. \tag{50}$$

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\left(e^{2\Omega_i^j(t)}\right)\bigg|_{t=1} = 2\chi_{_1}\left(\dot{\Omega}_i^j(1)\right)e^{2\Omega_2^1}, \quad where \quad \chi_{_1} := \int_0^1 e^{u\,ad_{2\Omega_2^1}}du. \tag{51}$$

Proof: From Lemma 2.2, we have that, for $j = 2, 3$,

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\left(\mathrm{e}^{2\Omega_0^j(t)}\right) = \int_0^1 \mathrm{e}^{u\,\mathrm{ad}_{2\Omega_0^j(t)}}(2\dot{\Omega}_0^j(t))du\ \mathrm{e}^{2\Omega_0^j(t)}.$$

Then, evaluating at $t=0$, and since $\Omega_0^j(0) = \Omega_0^1$ for $j = 2, 3$, we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\,t}\left(\mathrm{e}^{2\Omega_0^j(t)}\right)\Big|_{t=0} &= 2\int_0^1 \mathrm{e}^{u\,\mathrm{ad}_{2\Omega_0^1}}(\dot{\Omega}_0^j(0))du\ \mathrm{e}^{2\Omega_0^1} \\ &= 2\int_0^1 \mathrm{e}^{u\,\mathrm{ad}_{2\Omega_0^1}}du\ (\dot{\Omega}_0^j(0))\,\mathrm{e}^{2\Omega_0^1} \\ &= 2\chi_0\left(\dot{\Omega}_0^j(0)\right)\mathrm{e}^{2\Omega_0^1}. \end{aligned}$$

The second identity can be proved with similar computations, just taking into account Lemma 2.2, the fact that $\Omega_i^j(1) = \Omega_2^1$ for $j = 2, 3$ and $i = 3 - j$, together with

$$\Delta^L_{2\Omega_i^j(t)}(t)\Big|_{t=1} = 2\chi_1\left(\dot{\Omega}_i^j(1)\right). \tag{52}$$

■

Remark 4.2. *Note that* χ_0 *and* χ_1 *are, alternatively, defined by*

$$\chi_0 := f(ad_{2\Omega_0^1}), \qquad \chi_1 := f(ad_{2\Omega_2^1}), \tag{53}$$

where f *and* g *are as in (3) and (4), respectively.*

Lemma 4.7. *For* $j = 2, 3$, *let* $i = 3 - j$. *Then, from Lemma 2.2 replacing* $X(t)$ *by* $2(1-t)\Omega_0^j(t)$ *and* $2t\Omega_i^j(t)$, *respectively, we have*

$$\begin{aligned} \Delta^L_{2(1-t)\Omega_0^j(t)}(t)\Big|_{t=0} &= -2\Omega_0^1 + 2\chi_0\left(\dot{\Omega}_0^j(0)\right), \\ \Delta^L_{2t\Omega_i^j(t)}(t)\Big|_{t=1} &= 2\Omega_2^1 + 2\chi_1\left(\dot{\Omega}_i^j(1)\right). \end{aligned} \tag{54}$$

Proof: The first identity follows easily from the following computations, when $j = 2, 3$.

$$\begin{aligned}
\Delta^L_{2(1-t)\Omega^j_0(t)}(t)\Big|_{t=0} &= \left(\int_0^1 \mathrm{e}^{2(1-t)u\,\mathrm{ad}_{\Omega^j_0(t)}}(-2\Omega^j_0(t)+2(1-t)\dot{\Omega}^j_0(t))du\right)\Big|_{t=0} \\
&= \left(\int_0^1 \mathrm{e}^{2(1-t)u\,\mathrm{ad}_{\Omega^j_0(t)}}(-2\Omega^j_0(t))du\right)\Big|_{t=0} \\
&\quad + \left(\int_0^1 \mathrm{e}^{2(1-t)u\,\mathrm{ad}_{\Omega^j_0(t)}}(2(1-t)\dot{\Omega}^j_0(t))du\right)\Big|_{t=0} \\
&= -\int_0^1 \mathrm{e}^{2u\,\mathrm{ad}_{\Omega^j_0(0)}}(2\Omega^j_0(0))du + \int_0^1 \mathrm{e}^{2u\,\mathrm{ad}_{\Omega^j_0(0)}}(2\dot{\Omega}^j_0(0))du \\
&= -\int_0^1 \mathrm{e}^{u\,\mathrm{ad}_{2\Omega^1_0}}(2\Omega^1_0)du + 2\int_0^1 \mathrm{e}^{u\,\mathrm{ad}_{2\Omega^1_0}}du(\dot{\Omega}^j_0(0)) \\
&= -2\Omega^1_0 + 2\chi_0\left(\dot{\Omega}^j_0(0)\right).
\end{aligned}$$

The proof of the second identity can be achieved with analogous computations, and taking in consideration that, for $j = 2, 3$ and $i = 3 - j$, $\Omega^j_i(1) = \Omega^1_2$. ■

Remark 4.3. *In what follows, we must guarantee that the operators χ_0 and χ_1 have inverse. From the definition of f and g in (3) and (4) respectively, we know that $f(A)g(e^A) = I$, for $\|e^A - I\| < 1$. So, if this restriction holds for $A = ad_{2\Omega^1_0}$ and for $A = ad_{2\Omega^1_2}$, taking into account the definition of χ_0 and χ_1 in Remark (4.2), we immediately obtain*

$$\chi_0^{-1} := g(e^{ad_{2\Omega^1_0}}) \quad \textit{and} \quad \chi_1^{-1} := g(e^{ad_{2\Omega^1_2}}). \tag{55}$$

Lemma 4.8. *For $j = 2, 3$, let $i = 3 - j$. Then*

$$\dot{\Omega}^j_0(0) = (j-1)\chi_0^{-1}\left(\Omega^1_1 - \Omega^1_0\right), \tag{56}$$

$$\dot{\Omega}^j_i(1) = (j-1)\chi_1^{-1}\left(\Omega^1_2 - e^{2\Omega^1_2}\Omega^1_1 e^{-2\Omega^1_2}\right). \tag{57}$$

Proof: We first show that the identity (56) holds for $j = 2$. Differentiating with respect to t, both sides of the identity (i) of Lemma 4.3, we have that

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\left(\mathrm{e}^{2\Omega^2_0(t)}\right) = 2\Omega^1_1\mathrm{e}^{2\Omega^2_0(t)} + \mathrm{e}^{2t\Omega^1_1}(-2\Omega^1_0)\mathrm{e}^{2(1-t)\Omega^1_0},$$

and, since $\Omega_0^2(0) = \Omega_0^1$, we have

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\left(\mathrm{e}^{2\Omega_0^2(t)}\right)\bigg|_{t=0} = 2\Omega_1^1\mathrm{e}^{2\Omega_0^2(0)} - 2\Omega_0^1\mathrm{e}^{2\Omega_0^1} = 2(\Omega_1^1 - \Omega_0^1)\mathrm{e}^{2\Omega_0^1}. \tag{58}$$

But, from Lemma 4.6, considering $j = 2$, we also have that

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\left(\mathrm{e}^{2\Omega_0^2(t)}\right)\bigg|_{t=0} = 2\chi_{{}_0}\left(\dot{\Omega}_0^2(0)\right)\mathrm{e}^{2\Omega_0^1}, \tag{59}$$

with $\chi_{{}_0} = \displaystyle\int_0^1 \mathrm{e}^{u\,\mathrm{ad}_{2\Omega_0^1}} du$.

Then, comparing the expressions (58) and (59), we get

$$2\chi_{{}_0}\left(\dot{\Omega}_0^2(0)\right)\mathrm{e}^{2\Omega_0^1} = 2(\Omega_1^1 - \Omega_0^1)\mathrm{e}^{2\Omega_0^1} \;\Leftrightarrow\; \dot{\Omega}_0^2(0) = \chi_{{}_0}^{-1}\left(\Omega_1^1 - \Omega_0^1\right), \tag{60}$$

which proves the result, for $j = 2$.

Now, we show that the identity (56) also holds for $j = 3$. Similarly, differentiating with respect to t, both sides of the identity (iii) of Lemma 4.3, and evaluating at $t = 0$, it yields that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\,t}\left(\mathrm{e}^{2\Omega_0^3(t)}\right)\bigg|_{t=0} &= \left(\Delta^L_{2t\Omega_1^2(t)}(t)\mathrm{e}^{2\Omega_0^3(t)} + \mathrm{e}^{2t\Omega_1^2(t)}\Delta^L_{2(1-t)\Omega_0^2(t)}(t)\mathrm{e}^{2(1-t)\Omega_0^2(t)}\right)\bigg|_{t=0} \\ &= \Delta^L_{2t\Omega_1^2(t)}(t)\bigg|_{t=0}\mathrm{e}^{2\Omega_0^3(0)} + \Delta^L_{2(1-t)\Omega_0^2(t)}(t)\bigg|_{t=0}\mathrm{e}^{2\Omega_0^2(0)}.\end{aligned} \tag{61}$$

Since $\Omega_1^2(0) = \Omega_1^1$, by Lemma 2.4, we have that $\Delta^L_{2t\Omega_1^2(t)}(t)\Big|_{t=0} = 2\Omega_1^2(0) = 2\Omega_1^1$. Also, by Lemma 4.7 and, taking into account the relation (60), we have

$$\begin{aligned}\Delta^L_{2(1-t)\Omega_0^2(t)}(t)\bigg|_{t=0} &= -2\Omega_0^1 + 2\chi_{{}_0}\left(\dot{\Omega}_0^2(0)\right) \\ &= -2\Omega_0^1 + 2\chi_{{}_0}\chi_{{}_0}^{-1}\left(\Omega_1^1 - \Omega_0^1\right) \\ &= -4\Omega_0^1 + 2\Omega_1^1.\end{aligned}$$

Consequently, since $\Omega_0^2(0) = \Omega_0^3(0) = \Omega_0^1$, the relation (61) can be rewritten as

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\,t}\left(\mathrm{e}^{2\Omega_0^3(t)}\right)\bigg|_{t=0} &= 2\Omega_1^1\mathrm{e}^{2\Omega_0^1} + \left(-4\Omega_0^1 + 2\Omega_1^1\right)\mathrm{e}^{2\Omega_0^1} \\ &= 4\left(\Omega_1^1 - \Omega_0^1\right)\mathrm{e}^{2\Omega_0^1}.\end{aligned} \tag{62}$$

Therefore, from (62) and Lemma 4.6, with $j = 3$, we get

$$2\chi_{{}_0}\left(\dot{\Omega}_0^3(0)\right)\mathrm{e}^{2\Omega_0^1} = 4\left(\Omega_1^1 - \Omega_0^1\right)\mathrm{e}^{2\Omega_0^1} \;\Leftrightarrow\; \dot{\Omega}_0^3(0) = 2\chi_{{}_0}^{-1}\left(\Omega_1^1 - \Omega_0^1\right), \tag{63}$$

which also proves the result, for $j=3$.

The proof of the second identity uses identical arguments to those applied to show the first, so we only present a sketch of the proof, starting with $j=2$. For that, differentiate, with respect to t, both sides of the identity (ii) of Lemma 4.3 and evaluate them at $t=1$. The result is then achieved, with a few calculations, considering Lemma 4.6, the identity $\Delta^L_{2(1-t)\Omega_1^1}(t)\Big|_{t=1} = -2\Omega_1^1$, the fact that $\Omega_1^2(1)=\Omega_2^1$ and the relation (52), with $j=2$ and $i=1$. For $j=3$, differentiate with respect to t, both sides of the identity (iii) of Lemma 4.3 and evaluate them at $t=1$. Then, with some computations, taking into account Lemma 4.6 and Lemma 4.7, the relation (52), with $j=3$ and $i=0$, the fact that $\Omega_0^3(1)=\Omega_1^2(1)=\Omega_2^1$ and that, by Lemma 2.4, we have $\Delta^L_{2(1-t)\Omega_0^2(t)}(t)\Big|_{t=1} = -2\Omega_0^2(1) = -2\Omega_1^1$, it holds that

$$\dot\Omega_0^3(1) = \Omega_2^1 + \dot\Omega_1^2(1) - \chi_1^{-1}\left(\mathrm{e}^{2\Omega_2^1}\Omega_1^1\mathrm{e}^{-2\Omega_2^1}\right).$$

Therefore, making a few calculations, the result is obtained replacing $\dot\Omega_1^2(1)$ by the identity (57), previously proved for $j=2$, and attending to the fact that $\chi_1\left(\Omega_2^1\right)=\Omega_2^1$. ∎

We are now in conditions to prove the following result and its corollaries.

Theorem 4.9. *The polynomial curve $t\in[0,1]\mapsto\beta_3(t)$ in $G_{k,n}$ defined in (37) satisfies the boundary conditions $\beta_3(0)=x_0$, $\beta_3(1)=x_3$ and*

$$\dot\beta_3(t) = [\Omega(t),\beta_3(t)], \tag{64}$$

with $\Omega(t) = \Delta^L_{t\Omega_0^3(t)}(t) + e^{t\,ad_{\Omega_0^3(t)}}(\Delta^L_{t\Omega_0^2(t)}(t)) + e^{t\,ad_{\Omega_0^3(t)}}e^{t\,ad_{\Omega_0^2(t)}}\Omega_0^1 \in \mathfrak{so}_{\beta_3(t)}(n)$.

Proof: We have already pointed out in Remark 4.1 that $\beta_3(0)=x_0$ and $\beta_3(1)=x_3$. Differentiating (37) with respect to t, and since $\mathrm{e}^{t\Omega_0^1}\Omega_0^1\mathrm{e}^{-t\Omega_0^1} = \mathrm{e}^{t\,\mathrm{ad}_{\Omega_0^1}}\Omega_0^1 = \Omega_0^1$, we obtain

$$\begin{aligned}\dot\beta_3(t) = \; & \Delta^L_{t\Omega_0^3(t)}(t)\beta_3(t) + \mathrm{e}^{t\Omega_0^3(t)}\Delta^L_{t\Omega_0^2(t)}(t)\mathrm{e}^{-t\Omega_0^3(t)}\beta_3(t) \\ & + \left[\mathrm{e}^{t\Omega_0^3(t)}\mathrm{e}^{t\Omega_0^2(t)}\Omega_0^1\mathrm{e}^{-t\Omega_0^2(t)}\mathrm{e}^{-t\Omega_0^3(t)},\beta_3(t)\right] \\ & +\beta_3(t)\mathrm{e}^{t\Omega_0^3(t)}\mathrm{e}^{t\Omega_0^2(t)}\Delta^L_{-t\Omega_0^2(t)}(t)\mathrm{e}^{-t\Omega_0^2(t)}\mathrm{e}^{-t\Omega_0^3(t)} \\ & +\beta_3(t)\mathrm{e}^{t\Omega_0^3(t)}\Delta^L_{-t\Omega_0^3(t)}(t)\mathrm{e}^{-t\Omega_0^3(t)}.\end{aligned} \tag{65}$$

Using Lemma 2.3, with $A(t)$ replaced by $t\Omega_0^j(t)$, for $j = 2, 3$, the fourth and fifth terms in (65) can be rewritten, respectively, as

$$-\beta_3(t)\mathrm{e}^{t\Omega_0^3(t)}\Delta^L_{t\Omega_0^2(t)}(t)\mathrm{e}^{-t\Omega_0^3(t)} \quad \text{and} \quad -\beta_3(t)\Delta^L_{t\Omega_0^3(t)}(t).$$

Then, we get

$$\begin{aligned}\dot{\beta}_3(t) &= \left[\Delta^L_{t\Omega_0^3(t)}(t), \beta_3(t)\right] + \left[\mathrm{e}^{t\Omega_0^3(t)}\Delta^L_{t\Omega_0^2(t)}(t)\mathrm{e}^{-t\Omega_0^3(t)}, \beta_3(t)\right] \\ &\quad + \left[\mathrm{e}^{t\Omega_0^3(t)}\mathrm{e}^{t\Omega_0^2(t)}\Omega_0^1\mathrm{e}^{-t\Omega_0^2(t)}\mathrm{e}^{-t\Omega_0^3(t)}, \beta_3(t)\right] \\ &= \left[\Omega(t), \beta_3(t)\right],\end{aligned}$$

with

$$\begin{aligned}\Omega(t) &= \Delta^L_{t\Omega_0^3(t)}(t) + \mathrm{e}^{t\Omega_0^3(t)}\Delta^L_{t\Omega_0^2(t)}(t)\mathrm{e}^{-t\Omega_0^3(t)} + \mathrm{e}^{t\Omega_0^3(t)}\mathrm{e}^{t\Omega_0^2(t)}\Omega_0^1\mathrm{e}^{-t\Omega_0^2(t)}\mathrm{e}^{-t\Omega_0^3(t)} \\ &= \Delta^L_{t\Omega_0^3(t)}(t) + \mathrm{e}^{t\mathrm{ad}_{\Omega_0^3(t)}}(\Delta^L_{t\Omega_0^2(t)}(t)) + \mathrm{e}^{t\mathrm{ad}_{\Omega_0^3(t)}}\mathrm{e}^{t\mathrm{ad}_{\Omega_0^2(t)}}\Omega_0^1 \in \mathfrak{so}_{\beta_3(t)}(n),\end{aligned}$$

which proves the result. ■

Corollary 4.10. *Let $t \in [0,1] \mapsto \beta_3(t)$ be the geometric cubic polynomial in $G_{k,n}$ defined in (37) and $\Omega(t) \in \mathfrak{so}_{\beta_3(t)}(n)$ as defined in Theorem 4.9. Then,*

$$\frac{D\dot{\beta}_3}{dt}(t) = \left[\dot{\Omega}(t), \beta_3(t)\right].$$

Proof: Differentiating the relation (64) of Theorem 4.9 with respect to t, we get

$$\begin{aligned}\ddot{\beta}_3(t) &= [\dot{\Omega}(t), \beta_3(t)] + [\Omega(t), \dot{\beta}_3(t)] \\ &= [\dot{\Omega}(t), \beta_3(t)] + [\Omega(t), [\Omega(t), \beta_3(t)]].\end{aligned}$$

By Lemma 3.3, we have that $[\Omega(t), [\Omega(t), \beta_3(t)]] \in (T_{\beta_3(t)}G_{k,n})^\perp$. Therefore, since $[\dot{\Omega}(t), \beta_3(t)] \in T_{\beta_3(t)}G_{k,n}$, we obtain that

$$\frac{D\dot{\beta}_3}{dt}(t) = \left[\dot{\Omega}(t), \beta_3(t)\right].$$

■

Corollary 4.11. *Let $t \in [0,1] \mapsto \beta_3(t)$ be the geometric cubic polynomial in $G_{k,n}$ defined in (37) and $\Omega(t) \in \mathfrak{so}_{\beta_3(t)}(n)$ as defined in Theorem 4.9. Then,*

$$\dot{\beta}_3(0) = [3\Omega_0^1, x_0] \quad \textit{and} \quad \frac{D\dot{\beta}_3}{dt}(0) = 6\left[\chi_0^{-1}\left(\Omega_1^1 - \Omega_0^1\right), x_0\right].$$

Proof: From Theorem 4.9, we have that $\beta_3(0) = x_0$ and that

$$\dot{\beta}_3(0) = [\Omega(0), \beta_3(0)],$$

with $\Omega(0) = \left(\Delta^L_{t\Omega^3_0(t)}(t)\right)\Big|_{t=0} + \left(\Delta^L_{t\Omega^2_0(t)}(t)\right)\Big|_{t=0} + \Omega^1_0 \in \mathfrak{so}_{\beta_3(0)}(n)$. But, since $\Omega^2_0(0) = \Omega^3_0(0) = \Omega^1_0$, from Lemma 2.4, we obtain that $\Omega(0) = 3\Omega^1_0$. Therefore, $\dot{\beta}_3(0) = [3\Omega^1_0, x_0]$. From Corollary 4.10, and since $\beta_3(0) = x_0$, we know that

$$\frac{D\dot{\beta}_3}{dt}(0) = \left[\dot{\Omega}(0), x_0\right]. \tag{66}$$

In order to compute $\dot{\Omega}(0)$, let us consider $\omega_1(t) := \mathrm{e}^{t\mathrm{ad}_{\Omega^3_0(t)}}(\Delta^L_{t\Omega^2_0(t)}(t))$ and $\omega_2(t) := \mathrm{e}^{t\mathrm{ad}_{\Omega^3_0(t)}}\mathrm{e}^{t\mathrm{ad}_{\Omega^2_0(t)}}\Omega^1_0$. Then, $\Omega(t) = \Delta^L_{t\Omega^3_0(t)}(t) + \omega_1(t) + \omega_2(t) \in \mathfrak{so}_{\beta_3(t)}(n)$ and, differentiating with respect to t, we have that

$$\dot{\Omega}(t) = \dot{\Delta}^L_{t\Omega^3_0(t)}(t) + \dot{\omega}_1(t) + \dot{\omega}_2(t),$$

with

$$\begin{aligned}\dot{\omega}_1(t) &= \Delta^L_{t\Omega^3_0(t)}(t)\omega_1(t) + \mathrm{e}^{t\mathrm{ad}_{\Omega^3_0(t)}}(\dot{\Delta}^L_{t\Omega^2_0(t)}(t)) + \mathrm{e}^{t\mathrm{ad}_{\Omega^3_0(t)}}\left(\Delta^L_{t\Omega^2_0(t)}(t)\Delta^L_{-t\Omega^3_0(t)}(t)\right)\\ &= \left[\Delta^L_{t\Omega^3_0(t)}(t), \omega_1(t)\right] + \mathrm{e}^{t\mathrm{ad}_{\Omega^3_0(t)}}(\dot{\Delta}^L_{t\Omega^2_0(t)}(t))\end{aligned}$$

and

$$\begin{aligned}\dot{\omega}_2(t) &= \Delta^L_{t\Omega^3_0(t)}(t)\omega_2(t) + \mathrm{e}^{t\mathrm{ad}_{\Omega^3_0(t)}}\left(\Delta^L_{t\Omega^2_0(t)}(t)\mathrm{e}^{t\mathrm{ad}_{\Omega^2_0(t)}}\Omega^1_0\right)\\ &\quad + \mathrm{e}^{t\mathrm{ad}_{\Omega^3_0(t)}}\mathrm{e}^{t\mathrm{ad}_{\Omega^2_0(t)}}\left(\Omega^1_0\Delta^L_{-t\Omega^2_0(t)}(t)\right)\\ &\quad + \mathrm{e}^{t\mathrm{ad}_{\Omega^3_0(t)}}\left(\mathrm{e}^{t\mathrm{ad}_{\Omega^2_0(t)}}(\Omega^1_0)\Delta^L_{-t\Omega^3_0(t)}(t)\right)\\ &= \left[\Delta^L_{t\Omega^3_0(t)}(t) + \mathrm{e}^{t\mathrm{ad}_{\Omega^3_0(t)}}\left(\Delta^L_{t\Omega^2_0(t)}(t)\right), \omega_2(t)\right].\end{aligned}$$

Therefore, evaluating at $t = 0$, and according with Lemma 2.4 and Lemma 2.5, we get

$$\dot{\Omega}(0) = 2\dot{\Omega}^3_0(0) + \dot{\omega}_1(0) + \dot{\omega}_2(0),$$

with

$$\begin{aligned}\dot{\omega}_1(0) &= \Omega^3_0(0)\Omega^2_0(0) + 2\dot{\Omega}^2_0(0) - \Omega^2_0(0)\Omega^3_0(0)\\ &= \left[\Omega^3_0(0), \Omega^2_0(0)\right] + 2\dot{\Omega}^2_0(0)\end{aligned}$$

and

$$\begin{aligned}\dot{\omega}_2(0) &= \Omega_0^3(0)\Omega_0^1 + \Omega_0^2(0)\Omega_0^1 - \Omega_0^1\Omega_0^2(0) - \Omega_0^1\Omega_0^3(0) \\ &= \left[\Omega_0^3(0) + \Omega_0^2(0), \Omega_0^1\right].\end{aligned}$$

Due to the fact that $\Omega_0^2(0) = \Omega_0^3(0) = \Omega_0^1$, it holds that $\dot{\omega}_1(0) = 2\dot{\Omega}_0^2(0)$ and $\dot{\omega}_2(0) = 0$. Then, by Lemma 4.8, we can conclude that

$$\begin{aligned}\dot{\Omega}(0) &= 2\dot{\Omega}_0^3(0) + 2\dot{\Omega}_0^2(0) \\ &= 6\chi_{_0}^{-1}\left(\Omega_1^1 - \Omega_0^1\right).\end{aligned} \tag{67}$$

Consequently, assuming (67), the relation (66) can be rewritten as

$$\frac{D\dot{\beta}_3}{dt}(0) = 6\left[\chi_{_0}^{-1}\left(\Omega_1^1 - \Omega_0^1\right), x_0\right]. \qquad \blacksquare$$

On the next result we derive an expression for the derivative of the geometric cubic polynomial β_3, and for the covariant derivative of the velocity vector field along the curve β_3, at the endpoint $t = 1$. For that, it was fundamental the use of the alternative expression of β_3 established in the Theorem 4.5.

Theorem 4.12. *Let $t \in [0,1] \mapsto \beta_3(t)$ be the geometric cubic polynomial in $G_{k,n}$ given by the alternative formula present in Theorem 4.5. Then,*

$$\dot{\beta}_3(1) = [3\Omega_2^1, x_3] \quad \textit{and} \quad \frac{D\dot{\beta}_3}{dt}(1) = 6\left[\chi_{_1}^{-1}\left(\Omega_2^1 - e^{2\Omega_2^1}\Omega_1^1 e^{-2\Omega_2^1}\right), x_3\right]. \tag{68}$$

Proof: The alternative formula for β_3 present in Theorem 4.5 is

$$\beta_3(t) = \mathrm{e}^{(t-1)\,\mathrm{ad}_{\Omega_0^3(t)}}\mathrm{e}^{(t-1)\,\mathrm{ad}_{\Omega_1^2(t)}}\mathrm{e}^{(t-1)\,\mathrm{ad}_{\Omega_2^1}}x_3. \tag{69}$$

Making a few calculations similar to those that where done in the proof of the Theorem 4.9, it is possible to show that, differentiating with respect to t the expression (69), we obtain

$$\dot{\beta}_3(t) = [\overline{\Omega}(t), \beta_3(t)], \tag{70}$$

with

$$\overline{\Omega}(t) = \Delta^L_{(t-1)\Omega_0^3(t)}(t) + \mathrm{e}^{(t-1)\mathrm{ad}_{\Omega_0^3(t)}}(\Delta^L_{(t-1)\Omega_1^2(t)}(t)) + \mathrm{e}^{(t-1)\mathrm{ad}_{\Omega_0^3(t)}}\mathrm{e}^{(t-1)\mathrm{ad}_{\Omega_1^2(t)}}\Omega_2^1,$$

which belongs to $\mathfrak{so}_{\beta_3(t)}(n)$.

The ingredients to prove the previous identity (70) are, essentially, the formula for the derivative of the exponential map given by Lemma 2.2 and the relation in Lemma 2.3.

Therefore, to obtain the first identity in (68), observe that, for $j = 2, 3$ and $i = 3 - j$, we have that $\left.\mathrm{e}^{(t-1)\mathrm{ad}_{\Omega_i^j(t)}}\right|_{t=1} = I$. Furthermore, consider Lemma 2.4 with $k = 1$, and Remark 4.1, namely, the fact that $\Omega_1^2(1) = \Omega_0^3(1) = \Omega_2^1$. All the rest are simple computations.

In order to prove the second identity in (68), notice that differentiating (70) with respect to t, using similar arguments to those in the proof of the Corollary 4.10, and taking into consideration the Lemma 2.2 and Lemma 2.3, with a few calculations, we get that

$$\frac{D\dot{\beta}_3}{dt}(t) = \left[\dot{\overline{\Omega}}(t), \beta_3(t)\right],$$

with

$$\begin{aligned}\dot{\overline{\Omega}}(t) &= \dot{\Delta}^L_{(t-1)\Omega_0^3(t)}(t) + \left[\Delta^L_{(t-1)\Omega_0^3(t)}(t), \overline{\omega}_1(t)\right] + \mathrm{e}^{(t-1)\mathrm{ad}_{\Omega_0^3(t)}}\left(\dot{\Delta}^L_{(t-1)\Omega_1^2(t)}(t)\right) \\ &\quad + \left[\Delta^L_{(t-1)\Omega_0^3(t)}(t) + \mathrm{e}^{(t-1)\mathrm{ad}_{\Omega_0^3(t)}}\left(\Delta^L_{(t-1)\Omega_1^2(t)}(t)\right), \overline{\omega}_2(t)\right],\end{aligned} \tag{71}$$

where

$$\overline{\omega}_1(t) = \mathrm{e}^{(t-1)\mathrm{ad}_{\Omega_0^3(t)}}(\Delta^L_{(t-1)\Omega_1^2(t)}(t))$$

and $\overline{\omega}_2(t) = \mathrm{e}^{(t-1)\mathrm{ad}_{\Omega_0^3(t)}}\mathrm{e}^{(t-1)\mathrm{ad}_{\Omega_1^2(t)}}\Omega_2^1$.

Consequently, evaluating at $t = 1$ and, essentially, due to Lemma 2.5, Lemma 4.8, Lemma 2.4, with $k = 1$, and Remark 4.1, we obtain

$$\frac{D\dot{\beta}_3}{dt}(1) = 6\left[\chi_1^{-1}\left(\Omega_2^1 - \mathrm{e}^{2\Omega_2^1}\Omega_1^1\mathrm{e}^{-2\Omega_2^1}\right), x_3\right],$$

as required. ■

4.2.1. *Obtaining the Control Points from the Boundary Conditions.* In this subsection we will show how to get the control points from the boundary conditions in order to implement the De Casteljau algorithm to solve interpolation data problems that arise from different areas involving the Grassmann manifold.

• Case 1 - The Boundary Conditions are of Type (27)

When $M = G_{k,n}$, the boundary conditions (27) in Problem 4.1 are:

$$\beta_3(0) = x_0, \quad \beta_3(1) = x_3, \quad \dot{\beta}_3(0) = [V_0, x_0], \quad \dot{\beta}_3(1) = [V_3, x_3], \tag{72}$$

where $x_0, x_3 \in G_{k,n}$, $V_0 \in \mathfrak{so}_{x_0}(n)$, and $V_3 \in \mathfrak{so}_{x_3}(n)$.

According to the implementation of the algorithm, in order to generate the cubic polynomial that satisfies (72), we must be able to choose the control points x_1 and x_2 from those boundary conditions. The following theorem answers this question.

Theorem 4.13. *The control points x_1 and x_2, used in the De Casteljau algorithm to generate the geometric cubic polynomial that satisfies the boundary conditions (72), are given by:*

$$x_1 = \frac{1}{2}\left(I - e^{\frac{2}{3}V_0}(I - 2x_0)\right), \quad x_2 = \frac{1}{2}\left(I - (I - 2x_3)e^{\frac{2}{3}V_3}\right). \tag{73}$$

Proof: From Corollary 4.11 we know that $\dot{\beta}_3(0) = [3\Omega_0^1, x_0]$, with $\Omega_0^1 = \frac{1}{2}\log((I - 2x_1)(I - 2x_0)) \in \mathfrak{so}_{x_0}(n)$. Then, considering the expression of $\dot{\beta}_3(0)$ in (72) and the property 5. in Lemma 3.1, it follows that

$$\Omega_0^1 = \frac{1}{3}V_0. \tag{74}$$

According with the definition of Ω_0^1, we have $\mathrm{e}^{2\Omega_0^1} = (I - 2x_1)(I - 2x_0)$, which is equivalent to $I - 2x_1 = \mathrm{e}^{2\Omega_0^1}(I - 2x_0)$. Therefore, solving the last equation for x_1, and using (74), we obtain $x_1 = \frac{1}{2}\left(I - \mathrm{e}^{\frac{2}{3}V_0}(I - 2x_0)\right)$. Similarly, to obtain the control point x_2 notice that, as proved in Theorem 4.12, $\dot{\beta}_3(1) = [3\Omega_2^1, x_3]$. On the other hand, from (72), $\dot{\beta}_3(1) = [V_3, x_3]$. So, it follows from the property 5. in Lemma 3.1 that $V_3 = 3\Omega_2^1$, that is,

$$\Omega_2^1 = \frac{1}{3}V_3. \tag{75}$$

From the definition of Ω_2^1, we have $\mathrm{e}^{2\Omega_2^1} = (I-2x_3)(I-2x_2)$. Then, solving the last equality for x_2, and using (75), we obtain $x_2 = (I-(I-2x_3)\mathrm{e}^{\frac{2}{3}V_3})/2$. ■

• Case 2 - The Boundary Conditions are of Type (28)

When $M = G_{k,n}$, the boundary conditions (28) in Problem 4.1 are:

$$\beta_3(0) = x_0, \quad \beta_3(1) = x_3, \quad \dot\beta_3(0) = [V_0, x_0], \quad \frac{D\dot\beta_3}{dt}(0) = [W_0, x_0], \tag{76}$$

where $x_0, x_3 \in G_{k,n}$, and V_0, $W_0 \in \mathfrak{so}_{x_0}(n)$.

Theorem 4.14. *The control points x_1 and x_2, used in the De Casteljau algorithm to generate the geometric cubic polynomial that satisfies the boundary conditions (76), are given by:*

$$x_1 = \frac{1}{2}\left(I - e^{\frac{2}{3}V_0}(I-2x_0)\right), \; x_2 = \frac{1}{2}\left(I - e^{\frac{1}{3}\chi_0(W_0)+\frac{2}{3}V_0}\, e^{\frac{2}{3}V_0}(I-2x_0)\right). \tag{77}$$

Proof: It is enough to obtain the control point x_2. Taking into account the expressions for $\dfrac{D\dot\beta_3}{dt}(0)$ in (76) and in the Corollary 4.11, it follows from property 5. in Lemma 3.1 that $6\chi_0^{-1}\left(\Omega_1^1 - \Omega_0^1\right) = W_0$. Thus, also using (74), we obtain

$$\Omega_1^1 = \frac{1}{6}\chi_0(W_0) + \frac{1}{3}V_0. \tag{78}$$

On the other hand, from the definition of Ω_1^1, $\mathrm{e}^{2\Omega_1^1} = (I-2x_2)(I-2x_1)$. So, solving for x_2, one has $x_2 = \dfrac{1}{2}\left(I - \mathrm{e}^{2\Omega_1^1}(I-2x_1)\right)$. Now, using the expression of Ω_1^1 given by (78) and the expression of x_1 in terms of x_0 and V_0, we obtain $x_2 = (I - \mathrm{e}^{\frac{1}{3}\chi_0(W_0)+\frac{2}{3}V_0}\,\mathrm{e}^{\frac{2}{3}V_0}(I-2x_0))/2$. ■

The Case 1, corresponding to the Hermite boundary conditions, can be considered simpler than the Case 2, since it doesn't involves the computation of covariant derivatives. However, the Case 2, where the data is not symmetrically specified, has computational advantages over the Case 1, namely whenever the goal is to generated cubic splines, i.e., piecing together several geometric cubic polynomials so that the overall curve is $\mathcal{C}^2$-smooth.

As a consequence of the last two theorems, we can summarise the relationship between the boundary conditions of types (27) and (28) in the following result.

Corollary 4.15.

$$W_0 = 3\chi_0^{-1}\left(\log\left((I-2x_3)e^{\frac{2}{3}V_3}\, e^{\frac{2}{3}V_0}(I-2x_0)\right) - \frac{2}{3}V_0\right), \tag{79}$$

and

$$V_3 = \frac{3}{2}\log\left((I-2x_3)e^{\frac{1}{3}\chi_0(W_0)+\frac{2}{3}V_0}\,(I-2x_0)e^{-\frac{2}{3}V_0}\right), \tag{80}$$

where $\chi_0 = f(ad_{\frac{2}{3}V_0})$, $\chi_0^{-1} = g(e^{ad_{\frac{2}{3}V_0}})$, *with* f *and* g *as defined in (3) and (4).*

5. Generating Cubic Splines in $G_{k,n}$

We now explain how to solve the interpolation Problem 4.1 for the boundary conditions of type (28). The objective is to generate a geometric cubic spline, i.e., a $\mathcal{C}^2$-smooth curve that satisfies the interpolation and the boundary conditions and such that when restricted to each subinterval is a geometric cubic polynomial. The crucial procedure is the generation of the first cubic polynomial, denoted by γ_1, joining p_0 to p_1 and having prescribed initial velocity $[V_0, p_0]$ and initial covariant acceleration $[W_0, p_0]$. Although this has already been described in the previous section, we summarise the results here for the sake of completeness. We also adapt the notations so that the curve is given in terms of the data. The interpolation curve γ of Problem 4.1 may be generated by piecing together cubic polynomials defined on each subinterval $[t_i, t_{i+1}]$, $i = 0, 1, \ldots, \ell-1$. Without loss of generality, we assume that all spline segments are parameterised in the $[0, 1]$ time interval.

5.1. Generating the First Spline Segment. Apply the De Casteljau algorithm to obtain the first spline segment

$$\gamma_1(t) \;=\; \mathrm{e}^{t\,\mathrm{ad}_{\Omega_0^3(t)}}\mathrm{e}^{t\,\mathrm{ad}_{\Omega_0^2(t)}}\mathrm{e}^{t\,\mathrm{ad}_{\Omega_0^1}}p_0, \tag{81}$$

where

$$\begin{aligned}
\Omega_0^1 &= \frac{1}{2}\log((I-2x_1)(I-2p_0));\\
\Omega_1^1 &= \frac{1}{2}\log((I-2x_2)(I-2x_1));\\
\Omega_2^1 &= \frac{1}{2}\log((I-2p_1)(I-2x_2));\\
\Omega_0^2(t) &= \frac{1}{2}\log((I-2\mathrm{e}^{t\,\mathrm{ad}_{\Omega_1^1}}x_1)(I-2\mathrm{e}^{t\,\mathrm{ad}_{\Omega_0^1}}p_0));\\
\Omega_1^2(t) &= \frac{1}{2}\log((I-2\mathrm{e}^{t\,\mathrm{ad}_{\Omega_2^1}}x_2)(I-2\mathrm{e}^{t\,\mathrm{ad}_{\Omega_1^1}}x_1));\\
\Omega_0^3(t) &= \frac{1}{2}\log((I-2\mathrm{e}^{t\,\mathrm{ad}_{\Omega_1^2(t)}}\mathrm{e}^{t\,\mathrm{ad}_{\Omega_1^1}}x_1)(I-2\mathrm{e}^{t\,\mathrm{ad}_{\Omega_0^2(t)}}\mathrm{e}^{t\,\mathrm{ad}_{\Omega_0^1}}p_0)),
\end{aligned}$$

and the control points are given by

$$\begin{aligned}
x_1 &= \frac{1}{2}(I-\mathrm{e}^{\frac{2}{3}V_0}(I-2p_0));\\
x_2 &= \frac{1}{2}(I-\mathrm{e}^{\frac{1}{3}\chi_0(W_0)+\frac{2}{3}V_0}\,\mathrm{e}^{\frac{2}{3}V_0}(I-2p_0)).
\end{aligned}$$

5.2. Generating Consecutive Spline Segments. After having generated the first spline segment, one continues in a similar way for the second spline segment. Since the cubic spline is required to be $\mathcal{C}^2$-smooth, the initial velocity and initial covariant acceleration for this second spline segment must equal the end velocity and the end covariant acceleration of the previous spline segment, which are given by the formulas in Theorem 4.12. The other $\ell-2$ consecutive segments are generated similarly. The solution of Problem 4.1 is the cubic spline curve resulting from the concatenation of the ℓ consecutive segments.

6. Conclusion

We have presented all the necessary details to implement the De Casteljau algorithm on the Grassmann manifold. This algorithm is a geometric construction, based on successive geodesic interpolation, that generates cubic polynomials and cubic splines. For practical applications, one still needs to rely on computing stable matrix exponentials and matrix logarithms of structured matrices, but this is out of the scope of our work. The problem of computing matrix functions is of growing importance, and efficient numerical methods to solve them have been developed along the years (see, for instance, [7]), and are expanding at a fast rate.

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