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THERE ARE NO NONTRIVIAL TWO-SIDED MULTIPLICATIVE (GENERALIZED)-SKEW DERIVATIONS IN PRIME RINGS

JOSE BROX

ABSTRACT: As originally defined by Mozumder and Dhara ([15]), multiplicative (generalized)-skew derivations must satisfy two identities. In this short note we show that, as a consequence of the simultaneous satisfaction of both identities, a multiplicative (generalized)-skew derivation of a prime ring is either a multiplicative (generalized) derivation (i.e., not skew), or a generalized skew derivation (i.e., additive). Therefore only one of the identities should be taken in the definition of multiplicative (generalized)-skew derivations in order to get a new class of derivations in prime rings.

KEYWORDS: prime rings, derivations, generalized derivations, skew derivations.
MATH. SUBJECT CLASSIFICATION (2010): 16N60, 16W25, 16W20.

1. Introduction

The fundamental concept of derivation of an associative ring R, an additive map $d : R \to R$ such that $d(xy) = d(x)y + xd(y)$, has been progressively generalized in recent literature: by a twisting by an automorphism or a secondary derivation of the ring, by dropping the additivity assumption, by combining both previous ideas, and by repeating the process on the secondary derivation when present ([4],[2],[13],[5],[7],[15]). One of the main purposes of these generalizations is to extend to more sophisticated maps the classic results on derivations in the tradition of Herstein's theory of rings ([16]), in which strong knowledge is gained about the map or the ring through some special (and a priori weaker) property of the map. The main focus is on prime and semiprime rings, or on rings with well-behaved idempotents, which provide a context rich

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enough for the theory to be satisfactorily developed. For example one tries to extend to a more general setting Posner's second theorem for derivations of prime rings ([17, Theorem 2]), which states that a prime ring R is commutative when it has a derivation $d \neq 0$ such that $xd(x) - d(x)x$ is central for every $x \in R$. These efforts have generated literature in abundance (e.g. [1],[3],[6],[8],[9],[10],[12],[14],[18]).

Definitions 1.1. Let R be a ring.

a) A skew derivation ([11, page 170]) is an additive map $d : R \to R$ together with an automorphism $\alpha : R \to R$ such that $d(xy) = d(x)y + \alpha(x)d(y)$.
b) A multiplicative derivation ([4]) is a map $d : R \to R$, not necessarily additive, such that $d(xy) = d(x)y + xd(y)$.
c) A multiplicative skew derivation is a not necessarily additive map $d : R \to R$ together with an automorphism $\alpha : R \to R$ such that $d(xy) = d(x)y + \alpha(x)d(y)$.
d) A generalized derivation ([2]) is an additive map $F : R \to R$ together with a derivation $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$.
e) A generalized skew derivation ([13]) is an additive map $F : R \to R$ together with an automorphism $\alpha : R \to R$ and a skew derivation $d : R \to R$ for α such that $F(xy) = F(x)y + \alpha(x)d(y)$.
f) A multiplicative generalized derivation ([5]) is a map $F : R \to R$, not necessarily additive, together with a derivation $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$.
g) A multiplicative (generalized) derivation ([7]) is a map $F : R \to R$, not necessarily additive, together with a map (not necessarily additive nor a derivation) $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$.

As defined in [15], a multiplicative (generalized)-skew derivation (M(G)S derivation) is a not necessarily additive map $F : R \to R$, together with a not necessarily additive map $d : R \to R$ and an automorphism $\alpha : R \to R$ such that

$$F(xy) = F(x)\alpha(y) + xd(y) \quad \textbf{(Identity 1)}$$
$$F(xy) = F(x)y + \alpha(x)d(y) \quad \textbf{(Identity 2)}$$

Since in this case we get two different identities in the definition, accordingly we will call these two-sided M(G)S derivations. We will say that a map is a type 1 M(G)S derivation (resp. type 2 M(G)S derivation) when it satisfies Identity 1 (resp. Identity 2).

2. Main theorem

In what follows we show that in prime rings there are no nontrivial two-sided M(G)S derivations, since either they are not skew or they actually are generalized skew derivations.

Lemma 2.1. *If R is a semiprime ring and F is a M(G)S derivation of type 1 (resp. type 2) with map $d : R \to R$ and automorphism $\alpha : R \to R$ then d is in fact a multiplicative skew derivation with α as automorphism (resp. satisfies $d(xy) = d(x)\alpha(y) + xd(y)$).*

Proof: For type 1 this is [18, Lemma 2.1]. For type 2 the same proof works. ∎

Theorem 2.2. *Let R be a prime ring and F be a two-sided M(G)S derivation with map $d : R \to R$ and automorphism $\alpha : R \to R$. Then either*

i) $\alpha = \mathrm{id}_R$, *so F is a multiplicative (generalized) derivation, or*
ii) *F and d are additive, so F is a generalized skew derivation.*

Proof: From Identities 1 and 2, $F(x)y+\alpha(x)d(y) = F(xy) = F(x)\alpha(y)+xd(y)$ for every $x, y \in R$, so

$$F(x)(y - \alpha(y)) = (x - \alpha(x))d(y). \qquad \mathbf{(1)}$$

Linearizing in x we get, for every $x, y, z \in R$,

$$F(x + y)(z - \alpha(z)) = (x + y - \alpha(x + y))d(z) = (x + y - \alpha(x) - \alpha(y))d(z) =$$
$$= (x - \alpha(x))d(z) + (y - \alpha(y))d(z) = F(x)(z - \alpha(z)) + F(y)(z - \alpha(z))$$

by **(1)**. So $(F(x + y) - F(x) - F(y))(z - \alpha(z)) = 0$.
Put $G(x, y) := F(x + y) - F(x) - F(y)$. We have, for every $x, y, z \in R$,

$$G(x, y)z = G(x, y)\alpha(z). \qquad \mathbf{(2)}$$

Therefore, for every $w \in R$,

$$G(x, y)wz = G(x, y)\alpha(wz) = (G(x, y)\alpha(w))\alpha(z) = (G(x, y)w)\alpha(z)$$

by **(2)**, hence $G(x, y)w(z - \alpha(z)) = 0$ for every $x, y, z, w \in R$. Since R is prime, either $\alpha(z) = z$ for every $z \in R$ or $G(x, y) = 0$ for every $x, y \in R$. In the first case $\alpha = \mathrm{id}_R$ and F is a multiplicative (generalized) derivation. In the second case we get $\alpha \neq \mathrm{id}_R$ and $F(x + y) = F(x) + F(y)$ for every $x, y \in R$, so F is additive. Now, by Lemma 2.1 above d is another M(G)S derivation associated to $\alpha \neq \mathrm{id}_R$, so analogously d is additive, whence it is a skew derivation and F is a generalized skew derivation. ∎

References

[1] A. Ali, N. Muthana, A. Bano. Multiplicative (generalized)-derivations and left multipliers in semiprime rings. Palest. J. Math. 7, Special Issue I (2018), 170–178.

[2] M. Brešar. On the distance of the composition of two derivations to the generalized derivations. Glasgow Math. J. **33** (1991), 89–93.

[3] J.-C. Chang. Generalized skew derivations with nilpotent values on Lie ideals. Monatsh. Math. **161**(2) (2010), 155–160.

[4] M.N. Daif. When is a multiplicative derivation additive? Int. J. Math. Math. Sci. **14**(3) (1991), 615–618.

[5] M.N. Daif, M.S. Tammam El-Sayiad. Multiplicative generalized derivations which are additive. East-West J. Math. **9**(1) (2007), 31–37.

[6] V. De Filippis, N. Rehman, M.A. Raza. Strong commutativity preserving skew derivations in semiprime rings. Bull. Malays. Math. Sci. Soc. **41**(4) (2018), 1819–1834.

[7] B. Dhara, S. Ali. On multiplicative (generalized)-derivations in prime and semiprime rings. Aequationes Math. **86**(12) (2013), 65—79.

[8] B. Dhara, K.G. Pradhan. A note on multiplicative (generalized) derivations with annihilator conditions. Georgian Math. J., **23**(2) (2016), 191–198.

[9] S. Huang. Commuting multiplicative generalized derivations on Lie ideals of semiprime rings. Gulf J. Math. **6**(3) (2018), 25–32.

[10] B. Hvala. Generalized derivations in rings. Comm. Algebra **26**(4) (1998), 1147–1166.

[11] N. Jacobson. Structure of rings. Amer. Math. Soc., Providence, R.I., 1956, 1964 (revised edition).

[12] E. Koç, Ö. Gölbaşi. Some results on ideals of semiprime rings with multiplicative generalized derivations. Comm. Algebra **46**(11) (2018), 4905–4913.

[13] T.K. Lee. Generalized skew derivations characterized by acting on zero products. Pacific J. Math. **216**(2) (2004), 293–301.

[14] T.K. Lee, W.-K. Shiue. Identities with generalized derivations. Comm. Algebra **29**(10) (2001), 4437–4450.

[15] M.R. Mozumder, B. Dhara. Multiplicative (generalized)-skew derivations on semiprime rings. International conference on semigroups, algebras and applications (2016).

[16] W.S. Martindale III, C.R. Miers. Herstein's Lie theory revisited. J. Algebra **98**(1) (1986), 14–37.

[17] E.C. Posner. Derivations in prime rings. Proc. Amer. Math. Soc. **8** (1957), 1093–1100.

[18] N. Rehman, M.S. Khan. A note on multiplicative (generalized)-skew derivation on semiprime rings. Journal of Taibah University for Science, **12**(4) (2018), 450–454.

Jose Brox
CMUC, Department of Mathematics, University of Coimbra, 3001-501 Coimbra, Portugal

E-mail address: josebrox@mat.uc.pt