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COMMUTATION CLASSES OF THE REDUCED WORDS FOR THE LONGEST ELEMENT OF $\mathfrak{S}_n$

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Abstract: Using the standard Coxeter presentation for the symmetric group $\mathfrak{S}_n$, two reduced expressions for the same group element w are said to be commutative equivalent if one expression can be obtained from the other one by applying a finite sequence of commutations. The commutative classes can be seen as the vertices of a graph $G(w)$, where two classes are connected by an edge if elements of those classes differ by a long braid relation. We compute the radius and diameter of the graph $G(\mathrm{w_0})$, for the longest element $\mathrm{w_0}$ in the symmetric group $\mathfrak{S}_n$, and show that it is not a planar graph for $n \geq 6$. We also describe a family of commutation classes which contains all atoms, that is classes with one single element, and a subfamily of commutation classes whose elements are in bijection with standard Young tableaux of certain moon-polyomino shapes.

1. Introduction

Given a positive integer $n \geq 2$, let $\mathfrak{S}_{n+1}$ denote the symmetric group on the alphabet $[n+1] := \{1, 2, \ldots, n+1\}$. Composition of permutations are read from the left. We represent a permutation $\mathrm{w} \in \mathfrak{S}_{n+1}$ as the word $\mathrm{w} = \mathrm{w}(1)\mathrm{w}(2)\cdots\mathrm{w}(n+1)$ in one-line notation.

The symmetric group $\mathfrak{S}_{n+1}$ is generated by the simple reflections $\{s_1, s_2, \ldots, s_n\}$, where s_i is the transposition $(i\ i{+}1)$. These reflections satisfy the Coxeter relations:

$$s_i^2 = 1, \qquad \text{for all } i, \tag{1.1}$$

$$s_i s_j = s_j s_i, \quad \text{for } |i-j| > 1, \text{ and} \tag{1.2}$$

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \text{ for all } i \leq n-1. \tag{1.3}$$

The relations (1.1) are known as *commutations* or *short braid relations*, and the relations (1.2) are called *long braid relations*. Since $\{s_1, s_2, \ldots, s_n\}$ generates

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$\mathfrak{S}_{n+1}$, any permutation $\mathrm{w} \in \mathfrak{S}_{n+1}$ can be written as a product of adjacent transpositions $\mathrm{w} = s_{i_1} s_{i_2} \cdots s_{i_\ell}$.

Consider $\mathrm{w} \in \mathfrak{S}_{n+1}$ written as a product $\mathrm{w} = s_{i_1} s_{i_2} \cdots s_{i_\ell}$ where $i_j \in [n]$ and ℓ is minimal. The *length* of w is $\ell(\mathrm{w}) := \ell$ and the product $s_{i_1} s_{i_2} \cdots s_{i_{\ell(\mathrm{w})}}$ is a *reduced decomposition* for w. The string of subscripts $w = i_1 i_2 \cdots i_\ell$ is a *reduced word* for w. The content of w is the sequence $cont(w) = (c_1, \ldots, c_n)$, where each c_i is the number of occurrences of the letter i in w. A consecutive substring of w is called a factor, and a word obtained by deleting some of the letters of w is a subword of w.

Reduced decompositions and reduced words are in bijection with each other, and the terms "commutation" and "long braid relation" have natural interpretations in the context of reduced words. The set of all reduced words of w is denoted by $R(\mathrm{w})$.

The length $\ell(\mathrm{w})$ of a permutation w can be computed by counting inversions (see [2]):

$$\ell(\mathrm{w}) = |\{(i,j) : i < j \text{ and } \mathrm{w}(i) > \mathrm{w}(j)\}|. \tag{1.3}$$

As an immediate consequence of (1.3), we have $\ell(\mathrm{w}) = \ell(\mathrm{w}^{-1})$ for any permutation $\mathrm{w} \in \mathfrak{S}_{n+1}$.

The symmetric group $\mathfrak{S}_{n+1}$ has a unique *longest element* $\mathrm{w}_{\mathbf{0}}^n \in \mathfrak{S}_{n+1}$ with length $\ell(\mathrm{w}_{\mathbf{0}}^n) = \binom{n+1}{2}$. In one-line notation, $\mathrm{w}_{\mathbf{0}}^n$ is the permutation $(n+1)n \cdots 321$. The word $w_0 := 1(21)(321) \cdots (n \cdots 21)$ is a reduced word for $\mathrm{w}_{\mathbf{0}}^n$. To simplify notation, we often omit the n superscript in $\mathrm{w}_{\mathbf{0}}^n$, when it is clear from the context.

We define a relation $\sim$ on the set $R(\mathrm{w}_{\mathbf{0}})$ of all reduced words for $\mathrm{w}_{\mathbf{0}}$ by setting $s \sim t$ if and only if s and t differ by a sequence of commutations. This is an equivalence relation and the classes it defines are the *commutation classes* of $\mathrm{w}_{\mathbf{0}}$, denoted by $C(\mathrm{w}_{\mathbf{0}})$. The commutation class of a word $w \in R(\mathrm{w}_{\mathbf{0}})$ is denoted by $[w]$. Although two words in the same commutative class have the same content, this property is not sufficient to characterize the class. The next lemma gives a characterization of the words in a commutative class.

Lemma 1.1. *Let v and w be words over the alphabet $[n]$. Then, $w \sim v$ if and only if for each integer $i \in [n-1]$, we have $w_{|\{i,i+1\}} = v_{|\{i,i+1\}}$.*

Proof: If $w \sim v$, then w can be obtained from v by a sequence of commutative relations that do not change the relative positions of the letters i and $i+1$, and so we must have $w_{|\{i,i+1\}} = v_{|\{i,i+1\}}$ for all $i \in [n-1]$. Reciprocally, if $w_{|\{i,i+1\}} = v_{|\{i,i+1\}}$ for all $i \in [n-1]$, then the relative positions of the letters

cardinality of $[w]$ is again strictly less than the cardinality of $[w_I^-]$. Therefore, we can conclude that $[w_I^-]$ is a local maximum. ∎

Among the cardinalities of all classes in $R(\mathrm{w}_{\mathbf{0}})$, the four atoms we described in Theorem 3.10 are the classes with fewest elements, and the last result shows that each alternating class $[w_I^-]$, with $I \subset [n-1]$ is a local maximum. Computational evidence leads to the following conjecture.

Conjecture. The alternating classes of w_I^- and $\left(w_I^-\right)^{\bullet \mathsf{R}} = w_J^-$ are the ones with maximum cardinality amongst all classes of $R(\mathrm{w}_{\mathbf{0}})$, where I and J are, respectively, the sets of all odd numbers and all even numbers in $[n-1]$.

For instance, when $n = 6$ we have $w_I^- = 1 \cdot 321 \cdot 54321 \cdot 654321 \cdot 6543 \cdot 65$ and $w_J^- = 21 \cdot 4321 \cdot 654321 \cdot 65432 \cdot 654 \cdot 6$. Moreover, when restricted to the alternating classes in $AR(\mathrm{w}_{\mathbf{0}})$, we conjecture that the classes $w_0 = w_{[n-1]}^-$ and $w_\emptyset^-$ are the ones having fewest elements, so that $|[w_0]| \leq |[w]| \leq |w_I^-|$ for any $w \in AR(\mathrm{w}_{\mathbf{0}})$.

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Proof: By Theorem 3.16, the number of commutation classes in the set of all alternating words $AR(\mathrm{w_0})$ is 2^{n-1}, and by Lemma 3.24, each one of these classes has exactly two words from $\mathcal{TSO}(n)$. Consequently, using Lemma 3.6 we conclude that there are at most $3\times 4^{n-2}-2^{n-1}$ distinct commutation classes in the set $\mathcal{TSO}(n)$.

If two $\mathcal{TSO}(n)$ words $w_I^{\mathfrak{b}}$ and $w_{I'}^{\mathfrak{b}'}$ are different and have the same T-value for all triples of integers in $[n+1]$, then by Lemma 3.25, $\mathfrak{b}$ and $\mathfrak{b}'$ have opposite constant signs and $I'=[n-1]\backslash I$. By Lemma 3.24, both words are alternating words. ∎

In the next result, we prove that the cardinality of a commutation class of a alternating word is a local maximum, among the cardinalities of all classes in $R(\mathrm{w_0})$.

Proposition 3.27. *The class $[v]$ is a local maximum, for any alternating reduced word $v\in V$.*

Proof: Let $I=\{i_1,\ldots,i_k\}$ and $w_I^-\in V$, with $\{j_1,\ldots,j_\ell\}=[n-1]\setminus I$ and $j_1<\cdots<j_\ell$. A reduced word $w\notin[w_I^-]$ is connected to some word $v\in[w_I^-]$ only by a long relation $121\underset{\mathsf{L}}{\sim}212$ or $n(n-1)n\underset{\mathsf{L}}{\sim}(n-1)n(n-1)$, since between two consecutive letters i of w_I^-, with $i\neq 1,n-1$, there is always a letter $i-1$ or a letter $i+1$, by the definition of an alternating word. A long relation $121\underset{\mathsf{L}}{\sim}212$ is obtained by using short relations in w_I^- between the letters 1 and 21 of two subwords $\underline{w}_{i_q}^-$ and $\underline{w}_{i_{q+1}}^-$, respectively (assuming $\underline{w}_{i_{q+1}}^-=w_n^-$ if $q=k$), in order to form a factor 121. Similarly, a long relation $n(n-1)n\underset{\mathsf{L}}{\sim}(n-1)n(n-1)$ is obtained by using short relations in w_I^- between the letters $n(n-1)$ and n of two subwords $w_{j_{p+1}}^-$ and $w_{j_p}^-$, respectively (assuming $w_{j_{p+1}}^-=w_n^-$ if $p=\ell$), in order to form a factor $n(n-1)n$.

Let $v=t_1tt_2\sim w_I^-$ be a word in the commutation class of w_I^- with a factor $t=121$, and let $w=t_1\bar{t}t_2\underset{\mathsf{L}}{\sim}v$, with $\bar{t}=212$. Any sequence of short relation on the factor t_1 or t_2 of w can be replicated in v. Moreover, a sequence of short relations which uses one letter 2 in $\bar{t}$ can be replicated with the corresponding letter 1 of t. This defines an injection f from the set $[w]$ into $[v]$. Note also that the sequence of short relations in v that sends the letter 1 of t to the opposite side of the closest letter 3 is not in $f([w])$, showing that the cardinality of $[w]$ is strictly less than that of $[w_I^-]$. An analogous argument shows that if w is connected with $v\sim w_I^-$ by a long relation $n(n-1)n\underset{\mathsf{L}}{\sim}(n-1)n(n-1)$, then the

i and $i+1$ is the same for both words v and w, for all $i\in[n-1]$, and thus v and w can differ only by the positions of integers i and j with $|i-j|>2$. It follows that $w\sim v$. ∎

We write $v\underset{\mathsf{S}}{\sim}w$ (resp. $v\underset{\mathsf{L}}{\sim}w$) when v and w differ by a single short (resp., long) braid relation, and $[v]\underset{\mathsf{L}}{\sim}[w]$ when those classes differ by a long braid relation, *i.e.* if there are $v'\in[v]$ and $w'\in[w]$ such that $v'\underset{\mathsf{L}}{\sim}w'$.

Definition 1.2. The graph $G(\mathrm{w_0})$ of commutation classes of $\mathrm{w_0}\in\mathfrak{S}_{n+1}$ has vertex set $C(\mathrm{w_0})$, and an edge between classes $[u]$ and $[w]$ when $[u]\underset{\mathsf{L}}{\sim}[w]$.

The *distance* $d([u],[w])$ between commutation classes $[u]$ and $[w]$ in $G(\mathrm{w_0})$ is the length of a shortest path joining $[u]$ and $[w]$. The *eccentricity* of $[w]$ is the distance to a farthest commutative class from $[w]$. The *radius* and *diameter* of $G(\mathrm{w_0})$ are the minimum and maximum eccentricities, respectively. Figure 1 depicts the graph $G(\mathrm{w_0})$ for $\mathfrak{S}_4$, which has radius and diameter equal to 4.

In his doctoral thesis and in [5], S. Elnitsky proved that $G(\mathrm{w_0})$ is a connected and bipartite graph, by establishing a bijection between reduced words and rhombic tilings of a certain polygon. Since Elnitsky's work on $G(\mathrm{w_0})$, not much information has been uncover on this graph [4, 11]. In this paper we study various properties of $G(\mathrm{w_0})$, namely we compute its radius and diameter, and show that it is not planar for $n>5$. We also describe a family of commutation classes which contains all atoms, that is classes with one single word, and a subfamily of commutation classes whose elements are in bijection with standard Young tableaux of certain moon-polyomino shapes.

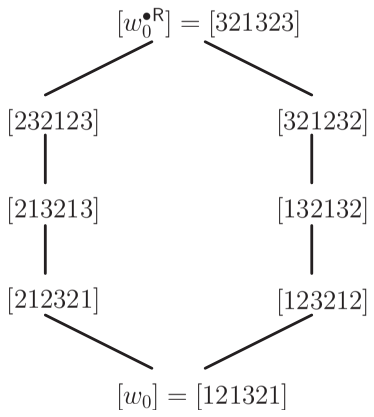


FIGURE 1. The graph $G(\mathrm{w}_{\mathbf{0}}^3)$.

2. Radius, diameter and planarity of $G(\mathbf{w_0})$

Given a reduced word $w = i_1 i_2 \cdots i_\ell$, define the *complement* $w^\bullet$ and *reverse* w^{R} words of w as

$$w^\bullet := (n - i_1 + 1)(n - i_2 + 1) \cdots (n - i_\ell + 1) \quad \text{and} \quad w^{\mathsf{R}} := i_\ell \cdots i_2 i_1.$$

The complement and reverse operations define involutive maps $R(\mathrm{w_0}) \to R(\mathrm{w_0})$ that commute with each other, $(w^\bullet)^{\mathsf{R}} = \left(w^{\mathsf{R}}\right)^\bullet$, and satisfy $cont(w) = cont(w^{\mathsf{R}})$ and $cont(w^\bullet) = cont(w^{\bullet \mathsf{R}})$.

Example 2.1. In $\mathfrak{S}_4$, $w_0 = 121321$, $w_0^\bullet = 323123$, $w_0^{\mathsf{R}} = 123121$ and $w_0^{\bullet \mathsf{R}} = w^{\mathsf{R}\bullet} = 321323$.

Definition 2.1. Given a reduced word $w = i_1 i_2 \cdots i_\ell \in R(\mathrm{w_0})$, let $S(w)$ be the sum of all ℓ letters of w, that is

$$S(w) = \sum_{j=1}^{\ell} i_j = \sum_{j=1}^{n} j \cdot c_j,$$

where $cont(w) = (c_1, \ldots, c_n)$.

Proof: When $\mathfrak{b} = (-)^{n-1}$ the word w_I^- is an element of the set $V \subseteq AR(\mathrm{w_0})$, and if $\mathfrak{b} = (+)^{n-1}$ then, as we have seen above, $w_I^+ \sim w_{I'}^-$, where $I' = \{n - i : i \notin I\}$. Now, suppose $\mathfrak{b}$ is not a constant vector and let $i \in \{2, \ldots, n-1\}$ be the first index such that $\mathfrak{b}_i \neq \mathfrak{b}_{i-1}$. We assume $\mathfrak{b}_i = -$ (the other case is analogous), and consider a two sided ordered word $w_I^{\mathfrak{b}}$. Note that if $a \leq c < d < b$, then the word $t_{ab}^+ \cdot t_{c,d}^-$ is not alternating, since $\left(t_{ab}^+ \cdot t_{c,d}^-\right)_{|\{d-1,d\}} = (d-1)d^2(d-1)$. If $i \notin I$, then $w_I^{\mathfrak{b}}$ has a factor $w_{n-j+1}^{\mathfrak{b}_j} \cdot w_{n-i+1}^{\mathfrak{b}_i} = t_{ab}^+ \cdot t_{cd}^-$, for some $j < i$ and $a \leq c < d < b$. Similarly, if $i \in I$, then $w_I^{\mathfrak{b}}$ has a factor $\underline{w}_{n-i+1}^{\mathfrak{b}_i} \cdot \underline{w}_{n-j+1}^{\mathfrak{b}_j} = t_{cd}^- \cdot t_{ab}^+$, for some $j < i$ and $a \leq c < d < b$, where we let $\underline{w}_{n-j+1}^{\mathfrak{b}_j} = w_n^{\mathfrak{b}_1}$ if $j = 1$. It follows that $w_I^{\mathfrak{b}}$ is not an alternating word. ∎

We are now ready to compute the number of commutation classes having an element of the set $\mathcal{TSO}(n)$ as a representative. Given $w_I^{\mathfrak{b}} \in \mathcal{TSO}(n)$ and $i \in [n]$, define the integers

$$x_i := 1 + \#\{j < i : \mathfrak{b}_j = +\} \quad \text{and} \quad y_i := n + 1 - \#\{j < i : \mathfrak{b}_j = -\}.$$

Note that the permutation associated with $w_{n+1-i}^{\mathfrak{b}_i}$ sends x_i to y_i if $\mathfrak{b}_i = +$, and sends y_i to x_i if $\mathfrak{b}_i = -$.

Lemma 3.25. *Let $w_I^{\mathfrak{b}} \in \mathcal{TSO}(n)$. For $i \in [n]$, define the sets*

$$\Delta^i = \begin{cases} \{(x_i, z) : x_i < z \leq y_i\}, & \text{if } \mathfrak{b}_i = + \\ \{(z, y_i) : x_i \leq z < y_i\}, & \text{if } \mathfrak{b}_i = - \end{cases} \quad \text{and} \quad \Delta_I^i = \bigcup_{\substack{j > i \\ n+1-j \in I}} \Delta^j.$$

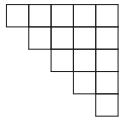
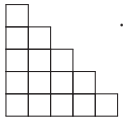
Then, we have

$$T(w_I^{\mathfrak{b}}, x_i z y_i) = \begin{cases} +1, & \text{if } (\mathfrak{b}_i = + \text{ and } (z, y_i) \notin \Delta_I^i) \text{ or } (\mathfrak{b}_i = - \text{ and } (x_i, z) \in \Delta_I^i) \\ -1, & \text{if } (\mathfrak{b}_i = - \text{ and } (x_i, z) \notin \Delta_I^i) \text{ or } (\mathfrak{b}_i = + \text{ and } (z, y_i) \in \Delta_I^i) \end{cases}.$$

Proof: Let $i \in [n]$ and assume $\mathfrak{b}_i = +$ (the other case is analogous). The set Δ_I^i stores all pairs of integers (a, b) with $a < b$ such that $\mathrm{w}(a) > \mathrm{w}(b)$ for w the permutation associated with one of the factors $\underline{w}_{n+1-j}^{\mathfrak{b}_j}$, with $j > i$. Then the permutation associated with $w_I^{\mathfrak{b}}$ sends x_i to y_i. If the pair (z, y_i) was not inverted by now (that is, $(x_i, z) \in \Delta_I^i$), then the pair (x_i, z) is inverted before the pair (z, y_i) and, consequently, $T(w_I^{\mathfrak{b}}, x_i z y_i) = +1$. ∎

Proposition 3.26. *The elements of the set $\mathcal{TSO}(n)$ belong to $3 \times 4^{n-2} - 2^{n-1}$ distinct commutation classes.*

of $w_0 = w^-_{[n-1]}$ and $w^-_\emptyset$ are, respectively:

 and  .

Note that $P(w_0)$ is a shifted standard Young tableau, *i.e.* a standard Young tableau of shifted shape given by the strict partition $(n, n-1, \dots, 1)$. The bijection P extends the bijection between reduced words in the class of w_0 and standard Young tableau of shifted shape given by $(n, n-1, \dots, 1)$ constructed in [8]. A formula for the number of these standard Young tableau of shifted shape can be found in [9].

Proposition 3.23 (See [8])**.** *The commutation classes of $w^-_\emptyset$ and of $w^-_{[n-1]}$ have*

$$\binom{n}{2}! \prod_{i=0}^{n-2} \frac{i!}{(2i+1)!} \tag{3.3}$$

elements each.

Proof: It is proven in [8] that the number of reduced words in the commutation class of $w_0 = w^-_{[n-1]}$ is given by (3.3). This is also the number of elements in the commutation class of the word $w^-_\emptyset$, since $w_0^{\bullet \mathsf{R}} = w^-_\emptyset$ and $\bullet\mathsf{R}$ establishes a bijection between the commutation classes $[w^-_{\{1,\cdots,n-1\}}]$ and $[w^-_\emptyset]$. ∎

If $I \cup J$ is a partition of $[n-1]$, then we have $\left(w^-_I\right)^{\mathsf{R}} = w^+_J$, and the shape of the tableau of $\left(w^-_I\right)^{\mathsf{R}}$ is the reflection under the diagonal line $(i,j) \mapsto (j,i)$ of the tableau of w^-_I. Moreover, if the entry of position (i,j) of $P\left(w^-_I\right)$ is a, then the entry in position (j,i) of $P(\left(w^-_I\right)^{\mathsf{R}})$ is $n-a+1$. It follows that

$$\left(w^-_I\right)^{\mathsf{R}} = w^+_J \sim w^-_{I'},$$

where $I' = \{n - i : i \in I\}$. Therefore, the operation R establishes a bijection between the commutation classes of $[w^-_I]$ and $[w^-_{I'}]$, for each $I \subseteq [n-1]$. Since the operations $\bullet$ and R are involutions that commute with each other, it follows that the operation $\bullet$ establishes a bijection between the commutation classes of $[w^-_I]$ and $[w^-_{[n-1]\setminus I'}]$, for each $I \subseteq [n-1]$.

Lemma 3.24. *Let $w^{\mathfrak{b}}_I \in \mathcal{TSO}(n)$. Then $w^{\mathfrak{b}}_I$ is an alternating word if and only if $\mathfrak{b}$ has constant sign.*

The number $S(w)$ is invariant for any word in the equivalence class of $[w]$, and thus defines a map $C(\mathrm{w_0}) \to \mathbb{N}$. For instance, we have

$$S(w_0) = S\left(w_0^{\mathsf{R}}\right) = \sum_{j=1}^{n} j(n-j+1) = \frac{n(n+1)(n+2)}{6}$$

and

$$S(w_0^\bullet) = S\left(w_0^{\bullet\mathsf{R}}\right) = \sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}.$$

The next lemma follows directly from the definition of a long braid relation.

Lemma 2.2. *If $v \underset{\mathsf{L}}{\sim} w$, then $|S(v) - S(w)| = 1$.*

Proposition 2.3. *Let $v \in R(\mathrm{w_0})$. If $[v] \neq [w_0]$ then there is $u \in R(\mathrm{w_0})$ such that $[u] \underset{\mathsf{L}}{\sim} [v]$ and $S(u) < S(v)$.*

Proof: We prove the contrapositive assertion. Let $v \in R(\mathrm{w_0})$, and assume that for any class $[u] \underset{\mathsf{L}}{\sim} [v]$ we have $S(u) \geq S(v)$. Then, by Lemma 2.2, $S(u) = S(v) + 1$. This implies that a factor $i(i-1)i$ cannot appear in any word of the class $[v]$, for any $i = 2, \dots, n$. Since between two letters n in a reduced word for $\mathrm{w_0}$ there must be a letter $n-1$, then a word in the class $[v]$ can only have one letter n.

Between two consecutive letters $n-1$ in a reduced word for $\mathrm{w_0}$, there must be a letter n or a letter $n-2$. Since we have established that a word in the class $[v]$ cannot have a factor $i(i-1)i$, for any i, it follows that there must exist a word in $[v]$ having the factor $(n-1)n(n-1)$ or $(n-1)n(n-2)(n-1)$. In any case, in between two consecutive letters $n-1$ it has to appear the letter n. This implies that any word in $[v]$ has at most two letters $n-1$.

Repeating this argument, we conclude that any word in $[v]$ has at most i letters $n-i+1$, for $i = 1, \dots, n$, and that between two consecutive letters i, there is exactly one letter $i+1$, for all i. On the other hand, being reduced, the length of the words in $[v]$ is $\binom{n+1}{2}$. Thus, any word in $[v]$ has exactly i letters $n-i+1$, for $i = 1, \dots, n$, and applying commutations, it follows that the first word of $[v]$, in lexicographic order, is w_0. ∎

The previous result shows that $S(w_0)$ is the minimum value for the map S, and that $S(w) = S(w_0)$ if and only if $w \in [w_0]$. Moreover, an analogous argument shows that $S(w_0^\bullet)$ is the maximum value for the map S and $S(w) =$

$S(w_0^\bullet)$ if and only if $w \in [w_0^\bullet]$. That is, $S(w_0) \leq S(w) \leq S(w_0^\bullet)$ for any $w \in R(\mathrm{w}_\mathbf{0})$. We can also conclude that $w_0 \sim w_0^{\mathsf{R}}$ and $w_0^\bullet \sim w_0^{\bullet\mathsf{R}}$.

It also follows from Proposition 2.3 that any commutation class distinct from $[w_0]$ is linked to a class with a smaller S-value, which means that any class is linked to w_0. Thus, we recover the following result from S. Elnitsky [5], which is also a consequence of Matsumoto's Theorem [7].

Corollary 2.4. *The graph $G(\mathrm{w}_\mathbf{0})$ is connected.*

Additionally, by Lemma 2.2 it follows that $G(\mathrm{w})$ is a bipartite graph, with the partition of $C(\mathrm{w})$ given by the parity of the S-values of its vertices.

Corollary 2.5. *The graph $G(\mathrm{w}_\mathbf{0})$ is bipartite.*

The map S may also be used to compute the diameter of $G(\mathrm{w}_\mathbf{0})$.

Theorem 2.6. *The diameter of $G(\mathrm{w}_\mathbf{0})$ is $S(w_0^\bullet) - S(w_0) = \binom{n+1}{3}$.*

Proof: By Proposition 2.3, there is $w \in R(\mathrm{w}_\mathbf{0})$ such that $[w] \underset{\mathsf{L}}{\sim} [w_0^\bullet]$, and moreover $S(w) = S(w_0^\bullet) - 1$ by Lemma 2.2. Repeating this process $S(w_0^\bullet) - S(w_0)$ times, we arrive at the class $[w_0]$, proving that

$$d([w_0],[w_0^\bullet]) = S(w_0^\bullet) - S(w_0) = \frac{(n-1)n(n+1)}{6} = \binom{n+1}{3}.$$

Similarly, we can see that for any class $[w]$ we have $d([w_0],[w]) = S(w) - S(w_0)$ and $d([w],[w_0^\bullet]) = S(w_0^\bullet) - S(w)$.

To prove that the diameter is $\binom{n+1}{3}$, it remains to show that this number is the maximal distance between any two classes in the graph. Consider two commutation classes $[w]$ and $[w']$. Since

$$d([w_0],[w]) + d([w],[w_0^\bullet]) + d([w_0],[w']) + d([w'],[w_0^\bullet]) = 2\left(S(w_0^\bullet) - S(w_0)\right),$$

using the triangle inequality, we conclude that

$$\begin{aligned} d([w],[w']) &\leq \min\{d([w_0],[w]) + d([w_0],[w']),\, d([w],[w_0^\bullet]) + d([w'],[w_0^\bullet])\} \\ &\leq S(w_0^\bullet) - S(w_0), \end{aligned}$$

proving that the distance between any two commutation classes $[w]$ and $[w']$ is at most $S(w_0^\bullet) - S(w_0)$. It follows that the maximum eccentricity of a commutation class in the graph $G(\mathrm{w}_\mathbf{0})$ is $S(w_0^\bullet) - S(w_0)$. ∎

Theorem 3.21. *The map $P : AR(\mathrm{w}_\mathbf{0}) \to PV$, that sends w into $P(w)$, is a bijection. Moreover, $w \sim v$ if and only if $P(v)$ and $P(w)$ have the same shape.*

Proof: Lemma 3.19 shows that P is well defined. By Algorithm 1, the shape of the tableau $P(w)$, with $w \in AR(\mathrm{w}_\mathbf{0})$, is completely characterized by the subwords $w_{|\{i,i+1\}}$ for $i \in [n-1]$. It follows by Lemma 1.1 that given $w, v \in AR(\mathrm{w}_\mathbf{0})$, we have $w \sim v$ if and only if $P(v)$ and $P(w)$ have the same shape.

Note also that the map P is invertible. If $Q \in PV$, has the shape of $P(w_I^-)$, we can construct a word w over the alphabet $\{1,2,\ldots,n\}$ by setting the letter i in position k whenever the tableau Q has the integer k in a box of the D_{i-1} diagonal, for all $i \in [n]$. Since any two consecutive diagonals of Q satisfy the Young tableau condition, it follows that each subword $w_{|\{i,i+1\}}$ is alternating and satisfy $w_{|\{i,i+1\}} = \left(w_I^-\right)_{|\{i,i+1\}}$ for all $i \in [n-1]$. By Lemma 1.1, we have $w \sim w_I^-$, and thus $w \in AR(\mathrm{w}_\mathbf{0})$, with $P(w) = Q$. This proves that P is a bijection and that $w \sim v$ if and only if $P(v)$ and $P(w)$ have the same shape. ∎

Example 3.4. The alternating word $w = 253145213425312$ in Example 3.3 is in the class of $w_{\{2,3\}}^-$ since the shapes of $P(w_{\{2,3\}}^-)$ and $P(w)$ are the same, with

$$P\left(w_{\{2,3\}}^-\right) = \begin{array}{|c|c|c|c|c|}\hline & & 6 & & \\ \hline 1 & 3 & 7 & 11 & \\ \hline 2 & 4 & 8 & 12 & 15 \\ \hline & 5 & 9 & 13 & \\ \hline & & 10 & 14 & \\ \hline \end{array}\ .$$

Note that the filling of $P\left(w_{\{2,3\}}^-\right)$ is obtained by writing the integers from 1 to 15 down columns, starting from the leftmost one. The tableaux of the words in V are obtained in the same manner.

Corollary 3.22. *The map $s : V \to s(V)$ is a bijection that sends each word $v \in V$ to the shape of $P(v)$.*

Note that $(w_n^-)^{\bullet\mathsf{R}} = w_n$ and for $i \neq n$, $\left(\underline{w_i^-}\right)^{\bullet\mathsf{R}} = w_i^-$ and $\left(w_i^-\right)^{\bullet\mathsf{R}} = \underline{w_i^-}$. That is, $\left(w_I^-\right)^{\bullet\mathsf{R}} = w_{[n-1]\setminus I}^-$, and thus the shape of $P(w_I^-)$ is the 180 degree rotation of the shape of $P(w_{[n-1]\setminus I}^-)$. Therefore, the operation $\bullet\mathsf{R}$ establishes a bijection between the commutation classes $[w_I^-]$ and $[w_{[n-1]\setminus I}^-]$. In particular, the commutation classes of $w_0 = w_{[n-1]}^-$ and $w_0^{\bullet\mathsf{R}} = w_\emptyset^-$ have the same number of elements. For $n = 5$, the shapes corresponding to the words in the classes

i. Add c_i cells at positions $(a+j, b+1+j)$, $j = 0, \ldots, c_i - 1$.
ii. Fill these c_i cells with the positions of the letters i, from left to right, in w. Let $P_i(w)$ be the resulting tableau.

Example 3.3. The successive steps in the construction of the tableau $P(w)$ corresponding to the reduced alternating word $w = 253145213425312 \in AR(\mathrm{w_0})$ are

$$P_1(w) = \begin{array}{|c|}\hline 4 \\ \hline\end{array}\begin{array}{|c|}\hline 8 \\ \hline\end{array}\begin{array}{|c|}\hline 14 \\ \hline\end{array}, \quad P_2(w) = \begin{array}{|c|}\hline 1 \\ \hline\end{array}\begin{array}{|c|}\hline 4 \\ \hline\end{array}\begin{array}{|c|}\hline 7 \\ \hline\end{array}\begin{array}{|c|}\hline 8 \\ \hline\end{array}\begin{array}{|c|}\hline 11 \\ \hline\end{array}\begin{array}{|c|}\hline 14 \\ \hline\end{array}\begin{array}{|c|}\hline 15 \\ \hline\end{array}, \quad P_3(w) = \begin{array}{|c|}\hline 1 \\ \hline\end{array}\begin{array}{|c|}\hline 3 \\ \hline\end{array}\begin{array}{|c|}\hline 4 \\ \hline\end{array}\begin{array}{|c|}\hline 7 \\ \hline\end{array}\begin{array}{|c|}\hline 9 \\ \hline\end{array}\begin{array}{|c|}\hline 8 \\ \hline\end{array}\begin{array}{|c|}\hline 11 \\ \hline\end{array}\begin{array}{|c|}\hline 13 \\ \hline\end{array}\begin{array}{|c|}\hline 14 \\ \hline\end{array}\begin{array}{|c|}\hline 15 \\ \hline\end{array},$$

$$P_4(w) = \begin{array}{|c|}\hline 1 \\ \hline\end{array}\begin{array}{|c|}\hline 3 \\ \hline\end{array}\begin{array}{|c|}\hline 5 \\ \hline\end{array}\begin{array}{|c|}\hline 4 \\ \hline\end{array}\begin{array}{|c|}\hline 7 \\ \hline\end{array}\begin{array}{|c|}\hline 9 \\ \hline\end{array}\begin{array}{|c|}\hline 10 \\ \hline\end{array}\begin{array}{|c|}\hline 8 \\ \hline\end{array}\begin{array}{|c|}\hline 11 \\ \hline\end{array}\begin{array}{|c|}\hline 13 \\ \hline\end{array}\begin{array}{|c|}\hline 14 \\ \hline\end{array}\begin{array}{|c|}\hline 15 \\ \hline\end{array}, \quad P_5(w) = P(w) = \begin{array}{|c|}\hline 2 \\ \hline\end{array}\begin{array}{|c|}\hline 1 \\ \hline\end{array}\begin{array}{|c|}\hline 3 \\ \hline\end{array}\begin{array}{|c|}\hline 5 \\ \hline\end{array}\begin{array}{|c|}\hline 6 \\ \hline\end{array}\begin{array}{|c|}\hline 4 \\ \hline\end{array}\begin{array}{|c|}\hline 7 \\ \hline\end{array}\begin{array}{|c|}\hline 9 \\ \hline\end{array}\begin{array}{|c|}\hline 10 \\ \hline\end{array}\begin{array}{|c|}\hline 12 \\ \hline\end{array}\begin{array}{|c|}\hline 8 \\ \hline\end{array}\begin{array}{|c|}\hline 11 \\ \hline\end{array}\begin{array}{|c|}\hline 13 \\ \hline\end{array}\begin{array}{|c|}\hline 14 \\ \hline\end{array}\begin{array}{|c|}\hline 15 \\ \hline\end{array}.$$

Lemma 3.19. *If w is an alternating word, then $P(w)$ is a Young tableau.*

Proof: The alternating property of the subword $w_{|\{i,i+1\}}$, for $i \in [n-1]$, shows that any two consecutive diagonals of $P(w)$ satisfy the tableau condition, that is the entries increase along rows from left to right, and along columns, from top to bottom. It follows that $P(w)$ is a Young tableau. ∎

Lemma 3.20. *Let $I = \{i_1 < \cdots < i_k\} \subseteq [n-1]$ and $[n] \setminus I = \{j_1 < \cdots < j_\ell < n\}$. Then the shape of $P(w_I^-)$ is the n-diagonal moon-polyomino $(i_1, \ldots, i_k, n, j_\ell, \ldots, j_1)$.*

Proof: The word w_I^- can be written as a product of factors $w_I^- = \underline{w}_{i_1}^- \cdots \underline{w}_{i_k}^- \cdot w_n^- \cdot w_{j_\ell}^- \cdots w_{j_1}^-$. By construction, each of these factors corresponds to a column of $P(w_I^-)$, with the number of boxes equal to the length of the corresponding factor. Since all factors $\underline{w}_{i_q}^-$ end with the letter 1, then all the columns $1, 2, \cdots, k+1$ end in the diagonal D_0 associated with the letter 1, and similarly, since all factor $w_{j_p}^-$ start with the letter n, then the last $\ell+1$ columns start in the same diagonal D_{n-1}.

Since the indices in I appear in w_I^- in increasing order, the set of row coordinates of column q is contained in the set of row coordinates of column $q+1$, for $q = 1, \ldots, k$, and the same happens with the last $\ell+1$ columns. This shows that the shape of $P(w_I^-)$ is the n-diagonal moon-polyomino $(i_1, \ldots, i_k, n, j_\ell, \ldots, j_1)$. ∎

Denote by $s(V)$ the set of shapes of the tableaux $P(v)$, with $v \in V$, and let PV be the set of all Young tableaux with shapes in $s(V)$.

We will now prove that the eccentricity of any commutation class is, in fact, $\binom{n+1}{3}$ and therefore prove that the radius of $R(\mathrm{w_0})$ is equal to the diameter. To this end, we will define an auxiliary function T.

Let T_n be the set of all triples $(a, b, c) \in [n+1]^3$ such that $a < b < c$. For any $w \in R(\mathrm{w_0})$ and any $(a, b, c) \in T_n$ define $T(w, abc) = 1$ if, by the action of the generators of w on $[n+1]$, the inversion of the pair (a, b) occurs before the inversion of (b, c); and define $T(w, abc) = -1$ otherwise. The number $T(w, abc)$ can be easily read off of the line diagram of a permutation. The line diagram of $w = s_{i_1} s_{i_2} \cdots s_{i_\ell}$ [6] is the array $[n+1] \times [\ell]$ in the Cartesian coordinates, which describes the trajectories of the numbers $1, 2, \ldots, n+1$ as they are arranged into the permutation w by the successive simple transpositions s_{i_j}. Note that since w is a reduced word for $\mathrm{w_0}$, any two integers $a < b$ in $[n+1]$ will invert positions in the line diagram of w, thus showing that T is well defined.

Figure 2 shows the line diagram of the word $w = 212321 \in G(\mathrm{w_0^4})$, and it follows that $T(w, 123) = -1$ and $T(w, abc) = 1$ for all remaining triples (a, b, c), with $a < b < c$.

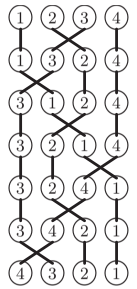


FIGURE 2. Line diagram of the word 212321.

Lemma 2.7. *Two reduced words $w, w' \in R(\mathrm{w_0})$ are in the same commutation class if and only if $T(w, abc) = T(w', abc)$ for all triples $(a, b, c) \in T_n$.*

Proof: The operator T is invariant for words in the same commutation class, since any possible change of order of the generators is done between generators acting on disjoint pairs of numbers.

Reciprocally, suppose $T(w, abc) = T(w', abc)$ and let i be the leftmost letter of w and suppose $w' = u\, i\, v$, where u has no letter i. If u has a letter $i-1$, then the first occurrence of $i-1$ acts over a pair of integers (a, b) with $a < b = i$. On

the other hand, the generator i in the word w, acts over the pair (b, c) where $b = i < c = i + 1$. Thus, $T(w, abc) = -1$ and $T(w', abc) = 1$. It follows that u cannot have the letter $i - 1$, and similar reasoning shows that it cannot have $i+1$. By commutation relations we can write $w' \sim w''$, where i is the leftmost letter of w''. By induction on the first letter where w and w' differ, we conclude that $w \sim w'$. ∎

In view of the result above, we will write $T([w], abc)$ to represent the common number $T(v, abc)$ for all $v \in [w]$.

Lemma 2.8. *If* $[w] \underset{\text{L}}{\sim} [w']$*, then* $T(w, abc) = T(w', abc)$ *for all triples in* T_n*, except for one.*

Proof: Without loss of generality, let $w = u\, i(i+1)i\, v \in R(\mathrm{w_0})$ and $w' = u\,(i+1)i(i+1)\, v$, where u and v are, respectively, the initial and final factors of w and w'. If a, b, c are such that $\mathrm{u}(a) = i$, $\mathrm{u}(b) = i+1$ and $\mathrm{u}(c) = i+2$, where u is the permutation corresponding to u, then $a < b < c$ since otherwise w would not be reduced. Moreover, the permutation corresponding to the left factor $u\, i(i+1)i$ is equal to the one corresponding to $u\,(i+1)i(i+1)$. The transposition corresponding to the generator i, applied to the permutation u, inverts the integers a and b, while the transposition corresponding to the $i + 1$ applied to u inverts the integers b and c. Therefore, $T(w, abc) = -T(w', abc)$. Finally, since the permutation corresponding to $i(i+1)i = (i+1)i(i+1)$ only acts over a, b and c, for any other triple $x < y < z$, we have $T(w, xyz) = T(w', xyz)$. ∎

Proposition 2.9. *For any* $w \in R(\mathrm{w_0})$ *and any triple* $(a, b, c) \in T_n$*, we have*

(a) $T([w], abc) = -T\left([w^\bullet], (n+2-c)(n+2-b)(n+2-a)\right)$;
(b) $T([w], abc) = T\left([w^R], (n+2-c)(n+2-b)(n+2-a)\right)$;
(c) $T([w], abc) = -T\left([w^{\bullet R}], abc\right)$.

Proof: (a) The line diagram of $w^\bullet$ corresponds to the horizontal reflection of the line diagram of w. Since the word $w^\bullet$ consists in replacing each generator i, in the word w, by the generator $n + 1 - i$, the inversion of the integers a and b is achieved by the action of w in the same order that the inversion of the integers $n + 2 - a$ and $n + 2 - b$ is achieved by $w^\bullet$. And so, if the action of w inverts the integers a and b before it inverts b and c, then the action of $w^\bullet$, inverts the integers $n+2-a$ and $n+2-b$ before it inverts $n+2-b$ and $n + 2 - c$. Thus the result follows.

(b) Note that the line diagram of w^R corresponds to the 180 degrees rotation of the line diagram of w, and the image of a by $\mathrm{w_0}$ is $n + 2 - a$. Reading

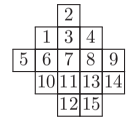


FIGURE 6. SYT of 5-diagonal moon-polyomino shape

The number of SYT of shape δ is invariant under reflection in a diagonal line $(i, j) \mapsto (j, i)$ or $(i, j) \mapsto (-j, -i)$, and reflection in the origin $(i, j) \mapsto (-i, -j)$, that is rotation by 180° [1], which is the composition of the other two reflections.

Let $w = w_1 w_2 \cdots w_\ell \in AR(\mathrm{w_0})$ be an alternating word with content $c(w) = (c_1, \ldots, c_n)$. We assign to w a tableau $P(w)$ that we will show is a SYT of n-diagonal moon-polyomino shape. This tableau is constructed by the overlapping of n diagonals D_{i-1}, where each diagonal D_{i-1} contains the positions of the letters i in w. The first box of D_i is placed over or in front of the first box of D_{i-1}, according to whether the first occurrence of the letter $i + 1$ in w is before or after the first occurrence of the letter i.

The following algorithm encodes this procedure, by constructing a sequence of tableaux $\emptyset = P_0(w), P_1(w), \ldots, P_n(w) = P(w)$, where $P_i(w)$ is obtained from $P_{i-1}(w)$ by overlapping it with the diagonal D_i.

Algorithm 1.

1. Start with the empty subset $P_0(w)$ of $\mathbb{Z} \times \mathbb{Z}$.
2. Add c_1 cells at positions (j, j), $j = 1, \ldots, c_1$.
3. Fill these c_1 cells with the positions of the letters 1, from left to right, in w. Let $P_1(w)$ be the resulting tableau.
4. For $i = 2, \ldots, n$,
   a. Let (a, b) be the coordinates of the cell in $P_{i-1}(w)$ corresponding to the leftmost letter $i - 1$ of w, and let k be its label.
   b. If the first occurrence of the letter i in w is before the kth place, then
      i. Add c_i cells at positions $(a - 1 + j, b + j)$, $j = 0, \ldots, c_i - 1$.
      ii. Fill these c_i cells with the positions of the letters i, from left to right, in w. Let $P_i(w)$ be the resulting tableau.

   Else, do

The shape of an n-diagonal moon-polyomino is completely determined by the sequence of its column lengths, and thus it is identified by that sequence. For example, the shape of the last 5-diagonal moon-polyomino in Figure 5 is $(1, 5, 4, 3, 2)$.

Proposition 3.18. *There are exactly 2^{n-1} n-diagonal moon-polyominoes.*

Proof: An n-diagonal moon-polyomino is completely characterized by choosing on which side of the moon-polyomino the column of length i will be, for each $i \in [n-1]$. ∎

Figure 5 shows the 2^4 5-diagonal moon-polyominoes.

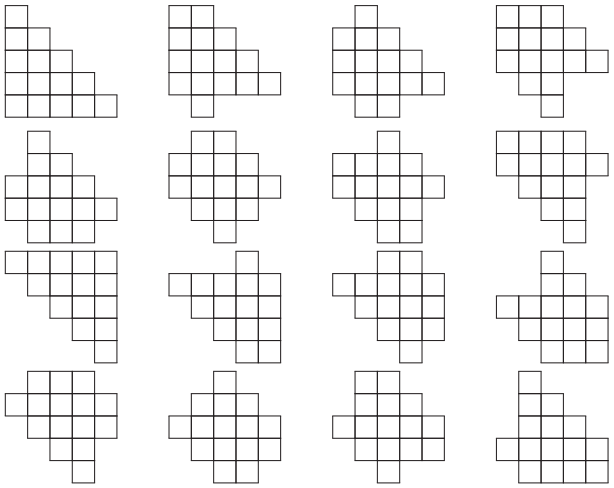


FIGURE 5. The 5-diagonal moon-polyominoes.

A *tableau* P of shape δ is an assignment of integers to the cells of δ. If the entries of the cells of P are the integers in $[|\delta|] = \{1, 2, \ldots, |\delta|\}$, used exactly once, the tableau is called *standard*. A *Young tableau* is a tableau in which the entries are increasing down columns, and across rows, from left to right. A *standard Young tableau* (SYT) is a Young tableau which is also a standard tableau. Figure 6 shows a SYT of the 5-diagonal moon-polyomino of shape $(1, 3, 5, 4, 2)$. Stanley [10] proved that the cardinality of $R(\mathrm{w_0})$ is given by the number of all SYT with partition shape $(n, n-1, \ldots, 1)$.

the word w backwards, any inversion of the integers a and b corresponds to a inversion of the integers $n+2-a$ and $n+2-b$ in the reverse order. Thus, if a and b are inverted before b and c by the action of w, then $n+2-b$ and $n+2-c$ are inverted before $n+2-b$ and $n+2-a$ are inverted by the action of w^{R}. It follows that $T([w], abc) = T\left([w^{\mathsf{R}}], n+2-c, n+2-b, n+2-a\right)$.

(c) Follows from the previous two cases, noticing that the line diagram of $w^{\bullet R}$ corresponds to the vertical reflection of the line diagram of w. ∎

Corollary 2.10. *For any $w \in R(\mathrm{w_0})$,*

$$d\left([w], [w^{\bullet R}]\right) = \binom{n+1}{3}.$$

Proof: By Theorem 2.6, the distance between any two words is at most $\binom{n+1}{3}$. By Proposition 2.9, for any triple (a, b, c), $T([w], abc) = -T\left([w^{\bullet \mathsf{R}}], abc\right)$ and, by Lemma 2.8, this means that there are necessary at least $|T_n|$ long relations in a path linking $[w]$ and $\left[w^{\bullet \mathsf{R}}\right]$. Thus $d\left([w], [w^{\bullet \mathsf{R}}]\right) \geq |T_n| = \binom{n+1}{3}$, which concludes the proof. ∎

The following result is a consequence of the previous corollary and Theorem 2.6.

Theorem 2.11. *The eccentricity of any class $[w]$ is $\binom{n+1}{3}$, and therefore the radius of $G(\mathrm{w_0})$ is $\binom{n+1}{3}$.*

As can be seen in Figures 1 and 3, the graphs $G(\mathrm{w}_0^n)$ for $n \leq 4$ are planar graphs. In Figure 3, the vertices A and B correspond to the commutation classes of w_0 and $w_0^{\bullet \mathsf{R}}$, and the vertices in each dashed circle have the same S-value, with one unit of increment (resp., decrement) for each circle starting from $S(w_0) = 20$ (resp., $S(w_0^{\bullet \mathsf{R}}) = 30$) in the center, having the external circles the same S-value.

We prove next that for $n > 4$ the graph $G(\mathrm{w_0})$ is not planar, using Wagner's Theorem [3]. An *edge contraction* of an edge $e = \{u, v\}$ in a graph is the graph obtained by combining the vertices u and v into a single vertex, which is adjacent to every vertex that was adjacent to u and v in the original graph. A *graph minor* of a graph is a new graph obtained by deleting vertices, deleting edges, and/or contracting edges of the original graph. Wagner's Theorem states that a graph is planar if and only if it does not contain K_5 or $K_{3,3}$ as a graph minor.

Lemma 2.12. *Given $n \geq 2$, the graph $G(\mathrm{w}_0^{n-1})$ is a subgraph of $G(\mathrm{w}_0^n)$.*

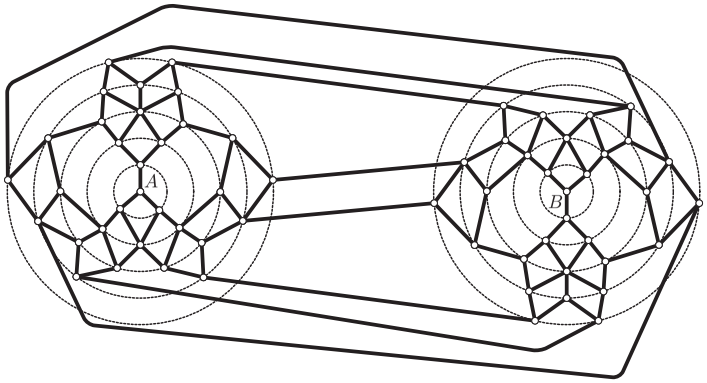


FIGURE 3. The graph $G(\mathrm{w}_{\mathbf{0}}^4)$.

Proof: Notice that for any word w in $G(\mathrm{w}_{\mathbf{0}}^{n-1})$, the word $w' = w \cdot w_n^-$ is a word in $G(\mathrm{w}_{\mathbf{0}}^n)$, where we set $w_n^- = n(n-1)\cdots 21$ (see Definition 3.1). The subgraph of $G(\mathrm{w}_{\mathbf{0}}^n)$ formed by the classes of the words $w \cdot w_n^-$, with $w \in R(\mathrm{w}_{\mathbf{0}}^{n-1})$, is isomorphic to $G(\mathrm{w}_{\mathbf{0}}^{n-1})$. ∎

Theorem 2.13. *For $n > 4$ the graph $G(\mathrm{w}_{\mathbf{0}}^n)$ is not planar.*

Proof: Using Lemma 2.12, it is enough to prove that $G(\mathrm{w}_{\mathbf{0}}^5)$ is not planar. We will do so by proving that it has $K_{3,3}$ as minor.

The minor of $G(\mathrm{w}_{\mathbf{0}}^5)$ having as vertices the sets:
$A = \{[132132432154321]\}$,
$B = \{[123214321354321]\}$,
$C = \{[123212432543212]\}$,
$D = \{[123212432154321]\}$,
$E = \{[132134321354321], [132134323543212], [123214323543212]\}$,
$F = \{[123214321543214], [132134321543214], [321234321543214],$
$[321234325432124], [321234323543212], [321232432543212],$
$[132132432543212]\}$,
where E and F are the edge contractions of their vertices, is isomorphic to $K_{3,3}$, since each of the vertices A, B, C is connected to all the vertices D, E, F. ∎

3.3. Complete moon-polyominoes. Next we give an interpretation of alternating words as standard fillings of certain moon-polyomino Young tableaux.

A *diagram* δ is a finite subset of the two-dimensional integer lattice $\mathbb{Z}^2$, which we identify with a set of cells in the plane, using the English convention for the coordinates of each cell, *i.e.* matrix-like coordinates. The number $|\delta|$ of cells in the diagram is the *size* of δ. A *column* of δ is the set of cells along a vertical line, and a *row* is the set of cells along a horizontal line. A *diagonal* D_k of a diagram δ, with $k \in \mathbb{Z}$, is the set $D_k = \{(i,j) \in \delta : i - j = k\}$.

A diagram δ is *convex* if for any two cells in a either column or row, the elements of $\mathbb{Z}^2$ in between are also cells of the diagram. It is *intersection-free* if any two columns are *comparable*, *i.e.* the set of row coordinates of cells in one column is contained in the set of row coordinates of cells in the other. For example, the first diagram in Figure 4 is convex but not intersection-free, the second is neither convex nor intersection-free, while the third is a convex intersection-free diagram.

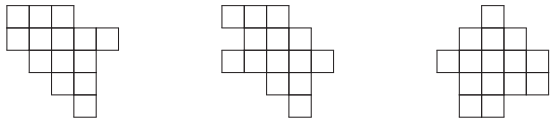


FIGURE 4. Example of diagrams.

Definition 3.17. A *moon-polyomino* is a convex intersection-free diagram. A moon-polyomino with exactly n columns is said to be a *n-diagonal moon-polyomino* if it has a column of length i, for all $i \in [n]$, and exactly n diagonals.

For instance, the third diagram in Figure 4 is a moon-polyomino but not a 5-diagonal moon-polyomino. Note that since an n-diagonal moon-polyomino is convex intersection-free, all columns on the right side of the column of length n are by decreasing order. Moreover, they all start at the same diagonal. We call this set of columns, including the column of length n, the *right side* of the moon-polyomino. Similarly, all columns on the left side of the column of length n are by increasing order, and end at the same diagonal. We call these set of columns, excluding the column of length n, the *left side* of the moon-polyomino. Analogously we define the *up side* and *down side* of a moon-polyomino as the subdiagram formed by all rows including and below the row of length n, and as the diagram formed by all rows above the row of length n, respectively.

According to Definition 3.1, we have

$$\underline{w_i^-} = i(i-1)\cdots 21 \qquad \text{and} \qquad w_j^- = n(n-1)\cdots(n-j+2)(n-j+1),$$

for $i \in I$ and $j \in J$, and thus $w_I^- \in AR(\mathrm{w_0})$.

Lemma 3.14. *The set V is a subset of $AR(\mathrm{w_0})$ with cardinality 2^{n-1}. Moreover, the commutation classes of any two distinct words of V are distinct.*

Proof: By construction, each word w_I^-, with $I \subseteq [n-1]$, is the minimum element, in lexicographic order, of its class, since they are formed by the concatenation of strictly decreasing subwords, with the rightmost letter of a subword strictly smaller than the leftmost letter of the subword sitting on its right. Therefore, each word in V is in a different commutation class, and the number of words in V is the number of subsets of $[n-1]$. ∎

Example 3.2. For $n+1 = 6$, the set V is formed by the 2^4 words below:

$$\begin{aligned}
w_{\emptyset}^- &= 54321\cdot 5432\cdot 543\cdot 54\cdot 5, & w_{\{1\}}^- &= 1\cdot 54321\cdot 5432\cdot 543\cdot 54,\\
w_{\{1,2\}}^- &= 1\cdot 21\cdot 54321\cdot 5432\cdot 543, & w_{\{1,3\}}^- &= 1\cdot 321\cdot 54321\cdot 5432\cdot 54,\\
w_{\{1,4\}}^- &= 1\cdot 4321\cdot 54321\cdot 543\cdot 54, & w_{\{2\}}^- &= 21\cdot 54321\cdot 5432\cdot 543\cdot 5,\\
w_{\{2,3\}}^- &= 21\cdot 321\cdot 54321\cdot 5432\cdot 5, & w_{\{2,4\}}^- &= 21\cdot 4321\cdot 54321\cdot 543\cdot 5,\\
w_{\{3\}}^- &= 321\cdot 54321\cdot 5432\cdot 54\cdot 5, & w_{\{3,4\}}^- &= 321\cdot 4321\cdot 54321\cdot 54\cdot 5,\\
w_{\{4\}}^- &= 4321\cdot 54321\cdot 543\cdot 54\cdot 5, & w_{\{1,2,3\}}^- &= 1\cdot 21\cdot 321\cdot 54321\cdot 5432,\\
w_{\{1,2,4\}}^- &= 1\cdot 21\cdot 4321\cdot 54321\cdot 543, & w_{\{1,3,4\}}^- &= 1\cdot 321\cdot 4321\cdot 54321\cdot 54,\\
w_{\{2,3,4\}}^- &= 21\cdot 321\cdot 4321\cdot 54321\cdot 5, & w_{\{1,2,3,4\}}^- &= 1\cdot 21\cdot 321\cdot 4321\cdot 54321.
\end{aligned}$$

Notice that the word $w_0 = w_{[n-1]}^- \in V$. As a particular case of Proposition 3.12, we get the following result.

Corollary 3.15. *Let $w_I^- \in V$. Then, the value of $T(w_I^-, xiy)$ is 1 (respectively -1) if $i-1 \in I$ (respectively $i-1 \notin I$), for all integer $x < i < y$.*

We will show next that each alternating reduced word in $AR(\mathrm{w_0})$ is in a class of an element of V.

Theorem 3.16. *There are exactly 2^{n-1} commutation classes in $AR(\mathrm{w_0})$.*

Proof: By Lemma 2.7, the commutation class of $w \in R(\mathrm{w_0})$ is characterized by the values of $T(w, abc)$, for all triples $a < b < c$. Proposition 3.12 shows that it is enough to know the values of $T(w, k(k+1)(k+2))$, for $k = 1, \ldots, n-1$. Thus, there are at most 2^{n-1} commutation classes in $AR(\mathrm{w_0})$. Lemma 3.14 shows that there are exactly 2^{n-1} such classes. ∎

3. Commutative Classes

In this section, we define a family of reduced words for $\mathrm{w_0}$ generated by a concatenation of monotone words of lengths $n, \ldots, 2, 1$, and indexed by binary vectors of length $n-1$. This family contains, as a particular case, the reduced word $w_0 = 1(21)(321)\cdots(n\cdots 21)$.

Definition 3.1. Given a binary vector $\mathfrak{b} = (\mathfrak{b}_1, \mathfrak{b}_2, \ldots, \mathfrak{b}_{n-1})$, with $\mathfrak{b}_i \in \{+,-\}$ for all $i = 1, 2, \ldots, n-1$, we construct the word

$$w^{\mathfrak{b}} = w_n^{\mathfrak{b}_1} w_{n-1}^{\mathfrak{b}_2} \cdots w_2^{\mathfrak{b}_{n-1}} w_1,$$

where each $w_{n-\ell+1}^{\mathfrak{b}_\ell}$ is a monotone subword of length $n-\ell+1$ defined recursively as follows (we set $w_1 = w_1^{\mathfrak{b}_n}$, with $\mathfrak{b}_n = +$):

- $w_n^{\mathfrak{b}_1} = \begin{cases} 12\cdots n, & \text{if } \mathfrak{b}_1 = +\\ n\cdots 21, & \text{if } \mathfrak{b}_1 = - \end{cases}$.
- For $\ell = 1, \ldots, n-1$,
  - if $\mathfrak{b}_\ell = +$ and $w_{n-\ell+1}^{\mathfrak{b}_\ell} = i(i+1)\cdots(j-1)j$ with $i < j$, then set

$$w_{n-\ell}^{\mathfrak{b}_{\ell+1}} = \begin{cases} i(i+1)\cdots(j-1), & \text{if } \mathfrak{b}_{\ell+1} = +\\ (j-1)\cdots(i+1)i, & \text{if } \mathfrak{b}_{\ell+1} = - \end{cases},$$

  - if $\mathfrak{b}_\ell = -$ and $w_{n-\ell+1}^{\mathfrak{b}_\ell} = j(j-1)\cdots(i+1)i$ with $i < j$, then set

$$w_{n-\ell}^{\mathfrak{b}_{\ell+1}} = \begin{cases} (i+1)\cdots(j-1)j, & \text{if } \mathfrak{b}_{\ell+1} = +\\ j(j-1)\cdots(i+1), & \text{if } \mathfrak{b}_{\ell+1} = - \end{cases}.$$

Note that each word $w_{n-\ell}^{\mathfrak{b}_{\ell+1}}$ is obtained by removing the rightmost letter from $w_{n-\ell-1}^{\mathfrak{b}_\ell}$, and sorting the remaining letters by increasing or decreasing order according to the sign of $\mathfrak{b}_{\ell+1}$. Also, note that in Definition 3.1, the sign $\mathfrak{b}_n$ is irrelevant for the construction $w_1 = w_1^{\mathfrak{b}_n}$, that is, the letter $w_1^{\mathfrak{b}_n}$ is completely determined by the previous subword $w_2^{\mathfrak{b}_{n-1}}$.

Example 3.1. Consider the binary vector $\mathfrak{b} = (+,+,-,+,-)$ of length 5. We then construct the subwords

$$\begin{aligned} w_6^{\mathfrak{b}_1} &= w_n^+ = 123456, \\ w_5^{\mathfrak{b}_2} &= w_5^+ = 12345, \\ w_4^{\mathfrak{b}_3} &= w_4^- = 4321, \\ w_3^{\mathfrak{b}_4} &= w_3^+ = 234, \\ w_2^{\mathfrak{b}_5} &= w_2^- = 32, \\ w_1 &= 3. \end{aligned}$$

Thus, the word indexed by $\mathfrak{b}$ is $w^{\mathfrak{b}} = 123456 \cdot 12345 \cdot 4321 \cdot 234 \cdot 32 \cdot 3$.

Let $\mathcal{O}(n) = \{w^{\mathfrak{b}} : \mathfrak{b} \in \{+,-\}^{n-1}\}$ denote the set of all words indexed by binary vectors of length $n-1$, constructed in Definition 3.1. The words of this set will be designated *ordered words*, and are formed by subwords $w_{n-\ell}^{\mathfrak{b}_{\ell+1}}$ in decreasing order of their lengths. They will serve as a basis for the construction of a larger class of commutation classes in $G(\mathrm{w}_{\mathbf{0}})$.

Proposition 3.2. *Any ordered word $w \in \mathcal{O}(n)$ is a reduced word for $\mathrm{w}_{\mathbf{0}}$.*

Proof: Let $w^{\mathfrak{b}} \in \mathcal{O}(n)$, with $\mathfrak{b}$ a binary vector of length $n-1$. Then, the length of $w^{\mathfrak{b}}$ is the sum of the lengths of the monotone subwords $w_{n-\ell+1}^{\mathfrak{b}_\ell}$, for $\ell = 1, 2 \ldots, n$. That is

$$\ell(w^{\mathfrak{b}}) = \sum_{\ell=1}^{n} (n-\ell+1) = \frac{(n+1)n}{2} = \ell(\mathrm{w}_{\mathbf{0}}).$$

Let us now prove that the permutation associated with $w^{\mathfrak{b}}$ is the longest permutation $\mathrm{w}_{\mathbf{0}} \in \mathfrak{S}_{n+1}$. Assume that $\mathfrak{b}_1 = -$. The permutation associated with $w_n^- = n \cdots 21$ has one-line notation $23 \cdots (n+1)1$. On the other hand, the permutation associated with $w_{n-1}^{\mathfrak{b}_2} \cdots w_2^{\mathfrak{b}_{n-1}} w_1$ leaves the letter 1 invariant, and only acts on the set $\{2, \ldots, n+1\}$. By induction, the permutation associated with $w_{n-1}^{\mathfrak{b}_2} \cdots w_2^{\mathfrak{b}_{n-1}} w_1^+$ is the longest permutation on the set $\{2, \ldots, n+1\}$, and it follows that the permutation associated with $w^{\mathfrak{b}}$ is the longest permutation of $\mathfrak{S}_{n+1}$.

The proof is similar if $\mathfrak{b}_1 = +$. Therefore $w^{\mathfrak{b}}$ is a reduced word for $\mathrm{w}_{\mathbf{0}}$. ∎

Using Lemma 3.4, it is easy to see that $(w^{\mathfrak{b}})^{\mathsf{R}} = w_{[n-1]}^{-\mathfrak{b}}$ and $(w^{-\mathfrak{b}})^{\mathsf{R}} = w_{[n-1]}^{\mathfrak{b}}$, showing that the four atoms $w^{\mathfrak{b}}, w^{-\mathfrak{b}}, (w^{\mathfrak{b}})^{\mathsf{R}}$ and $(w^{-\mathfrak{b}})^{\mathsf{R}}$ are words in $\mathcal{TSO}(n)$.

3.2. Alternating classes.

Definition 3.11. A word w over the alphabet $\{a, b\}$ is *alternating* if it is of the form $(ab)^k$ or $(ab)^k a$ for some integer $k \geq 0$. A word w over the alphabet $[n]$ is *alternating* if each subword $w_{|\{i,i+1\}}$ of w, formed only by the letters i and $i+1$, is alternating, for $i = 1, 2, \ldots, n-1$.

It follows that the difference between the number of letters i and $i+1$ in an alternating word w is at most one. Moreover, between two consecutive letters i in w there is exactly one letter $i-1$ and exactly one letter $i+1$. Denote by $AR(\mathrm{w}_{\mathbf{0}})$ the set of all alternating reduced words in $R(\mathrm{w}_{\mathbf{0}})$. We will characterize all commutation classes in $AR(\mathrm{w}_{\mathbf{0}})$.

Proposition 3.12. *Let $w \in AR(\mathrm{w}_{\mathbf{0}})$ be an alternating word, and $i \in \{2, \ldots, n\}$. Then, $T(w, xiy)$ is constant for all triples $x < i < y$.*

Proof: Suppose the letter i occurs in w to the left of the first letter $i-1$ (the other case is analogous), that is $w_{|\{i-1,i\}} = (i\,(i{-}1))^k$ or $w_{|\{i-1,i\}} = (i\,(i{-}1))^k i$, for some integer k. Then, by the action of the generators of w in the set $[n{+}1]$, the first inversion of i occurs with a pair (i, y) with $i < y$. Thus, the integer i, after this inversion, is in position $i+1$, and only the generators i and $i+1$ can act on it. Since w is an alternating word, there is a letter $i+1$ before the next letter i in w, which means that the next inversion of the integer i is again with an integer y' such that $i < y'$. This process is repeated until all pairs of integers (i, t) had invert position, for $i < t$. It follows that $T(w, xiy) = -1$. ∎

We now define a subset V of $\mathcal{TSO}(n)$, whose elements are alternating words. Moreover, we will prove that the minimum element, in lexicographic order, of any alternating class is an element of the set V.

Definition 3.13. Let V be the set of words

$$w_I^- = \prod_{i \in I} \underline{w_i^-} \cdot w_n^- \cdot \prod_{j \in J} w_j^- \in \mathcal{TSO}(n),$$

where $-$ in w_I^- stands for the binary vector $(-)^{n-1}$ of length $n-1$, and $I \cup J$ is any partition of $[n-1]$.

By Proposition 3.2, $w^{\mathfrak{b}}$ and $w^{-\mathfrak{b}}$ are reduced words for $\mathrm{w}_{\mathbf{0}}$, and each factor of length 2 is formed by consecutive letters. It follows that $w^{\mathfrak{b}}$ and $w^{-\mathfrak{b}}$ are atoms of $G(\mathrm{w}_{\mathbf{0}})$. Similarly, each factor of length 2 of the reduced words $(w^{\mathfrak{b}})^{\mathsf{R}}$ and $(w^{-\mathfrak{b}})^{\mathsf{R}}$, is formed by consecutive letters, showing that they are also atoms of $G(\mathrm{w}_{\mathbf{0}})$. For $n \geq 3$, these four atoms are all distinct.

We will prove next that these four words are the only atoms in $G(\mathrm{w}_{\mathbf{0}})$. Let $w \in R(\mathrm{w}_{\mathbf{0}})$ be an atom. Since the sets of letters appearing in any two reduced words for $\mathrm{w}_{\mathbf{0}}$ are the same (see [2]), a reduced word for $\mathrm{w}_{\mathbf{0}}$ must have the letters 1 and n. By Lemma 3.9, w must have as a factor $t^{+}_{1,n} = 12\cdots n$ or $t^{-}_{1,n} = n\cdots 21$, since any factor of length 2 of w consists of consecutive letters. Suppose the first case happens (the other is analogous).

If w has two 1s and two ns then, again by Lemma 3.9, we would have $t^{+}_{1,n}t^{-}_{1,n-1}t^{+}_{2,n}$ as a factor of w, which by Lemma 3.8 is not reduced. Thus, w has at least two 1s and one n, or one 1 and at least two ns. Notice that one of these cases must occur, since otherwise we would have

$$w = w' \cdot (12\cdots n) \cdot w'',$$

with w' and w'' words over the alphabet $\{2,\ldots,n-1\}$. In this case, the permutation associated with w' leaves invariant the positions 1 and $n+1$, and then the permutation associated with $w' \cdot (12\cdots n)$ sends 1 to $n+1$ and $n+1$ to n. Therefore, w cannot be a word for $\mathrm{w}_{\mathbf{0}}$ since it does not send $n+1$ to 1, contradicting the definition of w. Then, w must have a factor

$$t^{+}_{1,n} \cdot t^{-}_{1,n-1} \cdot t^{+}_{2,k} \quad \text{or} \quad t^{-}_{1,k} \cdot t^{+}_{2,n} \cdot t^{-}_{1,n-1},$$

for some $k < n$. Assume the former case (the other is analogous). Then, this must be the leftmost factor of w, since otherwise w would have the factor $2 \cdot t^{+}_{1,n} \cdot t^{-}_{1,n-1} \cdot 2$, which by Lemma 3.8 is not reduced.

Since $k < n$, the next factor in w must be a decreasing sequence $t^{-}_{i,k-1}$ for some integer i. For the same reason as before i is must be bigger than 1, otherwise w would have the factor $t^{-}_{1,k} \cdot t^{+}_{2,k} \cdot t^{-}_{1,k-1}$, which is not reduced. Consequently, the new factor should be shorter (in length) than the previous one. Repeating the same reasoning, w is formed by a sequence of factors in decreasing order of lengths, alternating between increasing and decreasing factors. So it has, at most, n factors and, to be reduced, these lengths add up to $\frac{(n+1)n}{2}$. Therefore, these lengths must be, respectively, $n, (n-1), \ldots, 1$.

Therefore, w is the atom $w^{\mathfrak{b}}$ that we have constructed above. The different choices we can made in the proof gives the other three atoms. ∎

Proposition 3.3. *The set $\mathcal{O}(n)$ contains 2^{n-1} distinct words, each belonging to a different commutation class.*

Proof: By construction, there are a total of 2^{n-1} words $w^{\mathfrak{b}}$, with $\mathfrak{b}$ a binary vector of length $n-1$, and they are all distinct. We prove by induction on $n \geq 2$ that any two of these words are in distinct commutation classes.

When $n = 2$ there are only two words $w^{(+)} = 12 \cdot 1$ and $w^{(-)} = 21 \cdot 2$, which clearly are not in the same commutation class. Suppose now $n > 2$, and consider words $w^{\mathfrak{b}}$ and $w^{\mathfrak{b}'}$ with $\mathfrak{b}_1 = +$ and $\mathfrak{b}'_1 = -$. Then, $T(w^{\mathfrak{b}}, xyz) = 1$ for any integers $y < z$ with $x = 1$, while $T(w^{\mathfrak{b}'}, xyz) = -1$ for any integers $x < y$ and $z = n+1$. Thus, $T(w^{\mathfrak{b}}, 1y(n+1)) \neq T(w^{\mathfrak{b}'}, 1y(n+1))$, for any $2 \leq y \leq n$, showing, by Lemma 2.7, that the commutation classes of $w^{\mathfrak{b}}$ and $w^{\mathfrak{b}'}$ are distinct. Consider next the case $\mathfrak{b}_1 = \mathfrak{b}'_1$. Then, we can write $w^{\mathfrak{b}} = w_n^{\mathfrak{b}_1} \cdot u^{\mathfrak{d}}$ and $w^{\mathfrak{b}'} = w_n^{\mathfrak{b}_1} \cdot u^{\mathfrak{d}'}$, where $\mathfrak{d} = (\mathfrak{b}_2, \ldots, \mathfrak{b}_{n-1})$ and $\mathfrak{d}' = (\mathfrak{b}'_2, \ldots, \mathfrak{b}'_{n-1})$. By the inductive step, we find that the words $u^{\mathfrak{d}}$ and $u^{\mathfrak{d}'}$ are in distinct commutation classes, showing that $w^{\mathfrak{b}}$ and $w^{\mathfrak{b}'}$ are in distinct commutation classes. ∎

We can use short and long braid relations in an ordered word $w^{\mathfrak{b}}$ to "move" a factor $w^{\mathfrak{b}_\ell}_{n-\ell+1}$ within the other factors of $w^{\mathfrak{b}}$, according to the rules of the following lemma. These rules will be used to construct two classes of words: atoms, that is reduced words with no short braid relations, and alternating words, which can be interpreted as fillings of certain moon-polyomino shapes.

Lemma 3.4. *Given positive integers $a < k \leq b$, consider the increasing word $w^{+} = a(a+1)\cdots(b-1)b$ and the decreasing word $w^{-} = b(b-1)\cdots(a+1)a$. Then,*

(1) $[k \cdot w^{+}] \underset{\mathsf{L}}{\sim} [w^{+} \cdot (k-1)]$.
(2) $[(k-1) \cdot w^{-}] \underset{\mathsf{L}}{\sim} [w^{-} \cdot k]$.

For instance, the following list of reduced words for $\mathrm{w}_{\mathbf{0}}$ shows how we can use short and long braid relations to "move" the factor $w^{+}_3 = 234$ of $w^{\mathfrak{b}}$ in Example 3.1, to the leftmost position:

$$\begin{aligned} &123456 \cdot 12345 \cdot 4321 \cdot \mathbf{234} \cdot 32 \cdot 3 = w^{\mathfrak{b}} \\ &123456 \cdot 12345 \cdot \mathbf{123} \cdot 4321 \cdot 32 \cdot 3 \\ &123456 \cdot \mathbf{234} \cdot 12345 \cdot 4321 \cdot 32 \cdot 3 \\ &\mathbf{345} \cdot 123456 \cdot 12345 \cdot 4321 \cdot 32 \cdot 3 \end{aligned} \tag{3.1}$$

Given an ordered word $w^{\mathfrak{b}}$, and a partition $I \cup J$ of $[n-1]$, we construct the *two side ordered word* $w_I^{\mathfrak{b}}$ as the concatenation

$$\prod_{i\in I} \underline{w}_i^{\mathfrak{b}_{n-i+1}} \cdot w_n^{\mathfrak{b}_1} \cdot \prod_{j\in J} w_j^{\mathfrak{b}_{n-j+1}}, \tag{3.2}$$

where each letter of $\underline{w}_i^{\mathfrak{b}_{n-i+1}}$ is obtained from the corresponding letter of $w_i^{\mathfrak{b}_{n-i+1}}$ by adding the value $\sum_{k=i+1}^{n} \mathfrak{b}_k$, considering the signals $+$ and $-$ as 1 and -1, respectively. The factor $\underline{w}_i^{\mathfrak{b}_{n-i+1}}$ can also be seen as obtained from the factor $w_i^{\mathfrak{b}_{n-i+1}}$ of $w^{\mathfrak{b}}$ by the application of short and long braid relations, according to the rules of Lemma 3.4. Moreover, in $\prod_{i\in I} \underline{w}_i^{\mathfrak{b}_{n-i+1}}$ the factors are written from left to right in increasing order of their lengths, and in $\prod_{j\in J} w_j^{\mathfrak{b}_{n-j+1}}$ the factors are written from left to right in decreasing order of their lengths.

For example, the word (3.1) can be written as $w_I^{\mathfrak{b}}$, where $b = (+,+,-,+,-)$ and $I = \{3\}$, where $\underline{w}_3^{+} = 345$.

Definition 3.5. Let $\mathcal{TSO}(n) = \{w_I^{\mathfrak{b}} : w^{\mathfrak{b}} \in \mathcal{O}(n) \text{ and } I \subseteq [n-1]\}$ denote the set of all two side ordered words.

Note that each word in $\mathcal{TSO}(n)$ is a reduced word for $\mathrm{w_0}$, since it is obtained from some ordered word in $\mathcal{O}(n)$, which by Proposition 3.2 is reduced, by applying short and long braid relations.

Lemma 3.6. *The set $\mathcal{TSO}(n)$ contains $3 \times 4^{n-2}$ distinct words.*

Proof: To construct a two side ordered word, there are two possibilities for the sign $\mathfrak{b}_1$ of $w_n^{\mathfrak{b}_1}$, and two possibilities for whether w_1 belongs to the set I or not. For all other factors there are four possible choices, two for the signal and two for whether it belongs to I or not. This amounts to 4^{n-1} possibilities. Note however, that if $\mathfrak{b}_1$ and $\mathfrak{b}_2$ have distinct signs, then $w_n^{\mathfrak{b}_1} w_{n-1}^{\mathfrak{b}_2} = \underline{w}_{n-1}^{\mathfrak{b}_1} w_n^{\mathfrak{b}_2}$. Thus, there are a total of $4^{n-1} - 4^{n-2} = 3 \times 4^{n-2}$ distinct two side ordered words in $\mathcal{TSO}(n)$. ∎

3.1. Atoms in $G(\mathbf{w_0})$.

Definition 3.7. A reduced word $w \in R(\mathrm{w_0})$ whose commutation class contains only itself is called an *atom* of $G(\mathrm{w_0})$.

Clearly, a reduced word $w \in R(\mathrm{w_0})$ is an atom if and only if each factor ij of length 2 of w is formed by consecutive letters, *i.e.* $|i-j| = 1$. We will show that there are exactly four atoms in $G(\mathrm{w_0})$, for $n \geq 3$, namely the words $w^{\mathfrak{b}}, w^{-\mathfrak{b}}, \left(w^{\mathfrak{b}}\right)^{\mathsf{R}}$ and $\left(w^{-\mathfrak{b}}\right)^{\mathsf{R}}$, with $\mathfrak{b} = (+,-,+,-,\ldots)$.

To simplify notation, whenever $1 \leq a < b \leq n$ define the words

$$t_{a,b}^{+} := a(a+1)\cdots(b-1)b, \qquad \text{and}$$
$$t_{a,b}^{-} := b(b-1)\cdots(a+1)a.$$

The word $t_{a,b}^{+}$ is reduced since by (1.3) its associated permutation has length $b-a+1$. Also, $t_{a,b}^{-} = \left(t_{a,b}^{+}\right)^{\mathsf{R}}$ is the reverse word of $t_{a,b}^{+}$, and thus it is reduced as well. The same argument proves the following result.

Lemma 3.8. *Let a, b be integers such that $a < b-1$. Then, the words*

(1) $(a+1)\cdot t_{a,b}^{+} \cdot t_{a,b-1}^{-} \cdot (a+1)$, *and*
(2) $t_{a,b}^{+} \cdot t_{a,b-1}^{-} \cdot t_{a+1,b}^{+}$.

are not reduced.

Lemma 3.9. *The increasing word $12\cdots n$ is the only reduced word amongst the set of all words of length $\geq n$ over the alphabet $[n]$, having leftmost letter 1 and rightmost letter n, and where each factor of length 2 is formed by consecutive letters.*

Proof: Let k be the length of a word u in the conditions of the lemma. We start by noticing that if $k = n$, then the increasing word $u = 12\cdots n$ is reduced, since by (1.3) the corresponding permutation $(n+1)12\cdots n$ has length n. Assuming now that $k > n$, the word u is a concatenation of increasing factors with decreasing factors. Since $u_1 = 1$ and $u_k = n$, u must have a factor of the form

$$t_{a,b}^{+} \cdot t_{a,b-1}^{-} \cdot t_{a+1,b}^{+},$$

for some integers $a < b$, which by Lemma 3.8 is not reduced. It follows that u is not reduced. ∎

Theorem 3.10. *For $n \geq 3$, there are exactly 4 atoms in the graph $G(\mathrm{w_0})$.*

Proof: We start by exhibiting the four atoms of $G(\mathrm{w_0})$. Let $\mathfrak{b} = (+,-,+,-,\ldots)$ and $-\mathfrak{b} = (-,+,-,+,\ldots)$ be the binary vectors of length $n-1$ having alternating signs, and consider the corresponding ordered words $w^{\mathfrak{b}}$ and $w^{-\mathfrak{b}}$ in $\mathcal{O}(n)$:

$$w^{\mathfrak{b}} = w_n^{+} w_{n-1}^{-} w_{n-2}^{+} \cdots w_1 \text{ and } w^{-\mathfrak{b}} = w_n^{-} w_{n-1}^{+} w_{n-2}^{-} \cdots w_1 = \left(w^{\mathfrak{b}}\right)^{\bullet}.$$