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THE KAROUBI ENVELOPE OF THE MIRAGE OF A SUBSHIFT

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ABSTRACT: We study a correspondence associating to each subshift $\mathcal{X}$ of $A^{\mathbb{Z}}$ a subcategory of the Karoubi envelope of the free profinite semigroup generated by A. The objects of this category are the idempotents in the mirage of $\mathcal{X}$, that is, in the set of pseudowords whose finite factors are blocks of $\mathcal{X}$. The natural equivalence class of the category is shown to be invariant under flow equivalence. As a corollary of our proof, we deduce the flow invariance of the profinite group that Almeida associated to each irreducible subshift. We also show, in a functorial manner, that the isomorphism class of the category is invariant under conjugacy. Finally, we see that the zeta function of $\mathcal{X}$ is naturally encoded in the category. These results hold, with obvious translations, for relatively free profinite semigroups over many pseudovarieties, including all of the form $\overline{\mathsf{H}}$, with H a pseudovariety of groups.

KEYWORDS: Subshift, symbolic dynamics, free profinite semigroup, Karoubi envelope, zeta function, pseudovariety.
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1. Introduction

Relatively free profinite semigroups and their elements, pseudowords, play an important role in finite semigroup theory. Around 2003, Almeida established the following connection between them and symbolic dynamics [Alm03]: in the A-generated relatively free profinite semigroup $\widehat{F}_{\mathsf{V}}(A)$, where V is a semigroup pseudovariety containing $\mathcal{L}\mathsf{Sl}$, associate to each subshift $\mathcal{X}$ of $A^{\mathbb{Z}}$ the topological closure in $\widehat{F}_{\mathsf{V}}(A)$ of the set $L(\mathcal{X})$ of finite blocks of $\mathcal{X}$. This connection proved to be very useful for a better understanding of structural aspects of $\widehat{F}_{\mathsf{V}}(A)$, even in the most difficult case where V is the pseudovariety S of all finite semigroups. One of the most relevant aspects of this line of research concerned the case of irreducible subshifts. When $\mathcal{X}$ is irreducible, the union of the $\mathcal{J}$-classes intersecting the topological closure $\overline{L(\mathcal{X})} \subseteq \widehat{F}_{\mathsf{V}}(A)$

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contains a minimum $\mathcal{J}$-class $J_{\mathsf{V}}(\mathcal{X})$, which is a regular $\mathcal{J}$-class of $\widehat{F}_{\mathsf{V}}(A)$. If $\mathsf{V} = \mathsf{V} * \mathsf{D}$, the corresponding Schützenberger group $G_{\mathsf{V}}(\mathcal{X})$, the profinite group isomorphic to all maximal subgroups of $J_{\mathsf{V}}(\mathcal{X})$, is invariant under *conjugacy* [Cos06], the name given to the isomorphism relation between topological dynamical systems. The conjugacy invariance of $G_{\overline{\mathsf{H}}}(\mathcal{X})$ was crucial to the proof in [CS11] that if H is an extension-closed pseudovariety of groups containing infinitely many groups of prime order, and if $L(\mathcal{X})$ is recognized by a semigroup of $\overline{\mathsf{H}}$, then the maximal subgroups of $J_{\overline{\mathsf{H}}}(\mathcal{X})$ are free pro-H groups of countable rank, unless $\mathcal{X}$ is periodic, in which case $G_{\overline{\mathsf{H}}}(\mathcal{X})$ is free pro-aperiodic. The profinite group $G_{\mathsf{S}}(\mathcal{X})$ was also identified in many instances where $\mathcal{X}$ is minimal [Alm05a, AC13, AC16], in the process being shown to sometimes not be free, although it is always projective accordingly to [RS08]. In this paper, we add information about the dynamical meaning of $G_{\mathsf{V}}(\mathcal{X})$, as briefly contextualized in the following paragraphs.

Some techniques used in [Cos06] to prove the conjugacy invariance of $G_{\mathsf{V}}(\mathcal{X})$ were adapted in the same paper in order to obtain conjugacy invariants encoded in the syntactic semigroup $S(\mathcal{X})$ of the language $L(\mathcal{X})$, when $\mathcal{X}$ is sofic (that is, when $L(\mathcal{X})$ is rational). These syntactic invariants where shown in [CS16] to be invariants with respect to another relation of significant importance in symbolic dynamics, *flow equivalence* (cf. [LM95, Section 13.6]), the relation, coarser than conjugacy, identifying subshifts with suitably equivalent suspension flows (or mapping tori). This was done by showing that those invariants are encoded in the Karoubi envelope of $S(\mathcal{X})$, a small category whose equivalence class was shown in [CS16] to be a flow invariant (even if $\mathcal{X}$ is not sofic), and in fact, as also proved there, the best possible syntactic flow equivalence invariant for sofic systems.

The importance for semigroup theory of the Karoubi envelope $\mathbb{K}(S)$ of a semigroup S became clear with Tilson's seminal paper [Til87] (there, it is denoted S_E). Inspired by [CS16], we now consider the Karoubi envelope of $\widehat{F}_{\mathsf{V}}(A)$ in relation with the subshift $\mathcal{X}$. In fact, we view $\mathbb{K}(\widehat{F}_{\mathsf{V}}(A))$ as a compact topological category. In the exploration of the connections between symbolic dynamics and free profinite semigroups, the convenience of considering the set of pseudowords of $\widehat{F}_{\mathsf{V}}(A)$ whose finite factors are elements of $L(\mathcal{X})$ soon became apparent (here, as before, $\mathsf{V} \supseteq \mathcal{L}\mathsf{Sl}$) [Cos06, AC09]. This set, the *mirage* of $\mathcal{X}$ in $\widehat{F}_{\mathsf{V}}(A)$, denoted $\mathcal{M}_{\mathsf{V}}(\mathcal{X})$, always contains $\overline{L(\mathcal{X})}$, but it may contain elements not in $\overline{L(\mathcal{X})}$. The arrows (e, u, f) in $\mathbb{K}(\widehat{F}_{\mathsf{V}}(A))$ such that $u \in \mathcal{M}_{\mathsf{V}}(\mathcal{X})$ form a compact subcategory of $\mathbb{K}(\widehat{F}_{\mathsf{V}}(A))$, which we call the Karoubi envelope of the mirage of $\mathcal{X}$ (with respect to V), and denote by $\mathbb{K}(\mathcal{M}_{\mathsf{V}}(\mathcal{X}))$.

In this paper, we show that the correspondence $\mathcal{X} \mapsto \mathbb{K}(\mathcal{M}_{\mathsf{V}}(\mathcal{X}))$ establishes a functor from the category of symbolic dynamical systems to that of compact zero-dimensional categories, whenever $\mathsf{V} = \mathsf{V} * \mathsf{D}$ and $\mathsf{V} \supseteq \mathcal{L}\mathsf{SI}$. From this functor, we get for free a new proof that the profinite group $G_{\mathsf{V}}(\mathcal{X})$ is a conjugacy invariant, when $\mathcal{X}$ is irreducible, $\mathsf{V} = \mathsf{V} * \mathsf{D}$ and $\mathsf{V} \supseteq \mathcal{L}\mathsf{SI}$. Under the additional mild assumption that V is monoidal, we show that the natural equivalence class of $\mathbb{K}_{\mathsf{V}}(\mathcal{X})$ is actually invariant under flow equivalence, deducing from that, in the irreducible case, the invariance under flow equivalence of the profinite group $G_{\mathsf{V}}(\mathcal{X})$.

When $\mathcal{X}$ is irreducible, the mirage $\mathcal{M}_{\mathsf{V}}(\mathcal{X})$ contains a minimum $\mathcal{J}$-class $\widetilde{J}_{\mathsf{V}}(\mathcal{X})$, which is regular and therefore possesses a profinite Schützenberger group $\widetilde{G}_{\mathsf{V}}(\mathcal{X})$ isomorphic to its maximal subgroups. We deduce, just as for $G_{\mathsf{V}}(\mathcal{X})$, that $\widetilde{G}_{\mathsf{V}}(\mathcal{X})$ is a conjugacy invariant when $\mathcal{L}\mathsf{SI} \subseteq \mathsf{V} = \mathsf{V} * \mathsf{D}$, and that $\widetilde{G}_{\mathsf{V}}(\mathcal{X})$ is a flow equivalence invariant when V is also monoidal. It may be interesting and challenging to investigate the group $\widetilde{G}_{\mathsf{V}}(\mathcal{X})$, which remains largely unknown when $\mathcal{X}$ is not minimal (in the minimal case the equality $G_{\mathsf{V}}(\mathcal{X}) = \widetilde{G}_{\mathsf{V}}(\mathcal{X})$ holds).

The structure of $\mathbb{K}(\mathcal{M}_{\mathsf{V}}(\mathcal{X}))$ says more about $\mathcal{X}$ than $G_{\mathsf{V}}(\mathcal{X})$ or $\widetilde{G}_{\mathsf{V}}(\mathcal{X})$. Indeed, we show that the zeta function of $\mathcal{X}$ is encoded in $\mathbb{K}(\mathcal{M}_{\mathsf{V}}(\mathcal{X}))$, by showing that the periodic points correspond to the objects of $\mathbb{K}(\mathcal{M}_{\mathsf{V}}(\mathcal{X}))$ with a finite isomorphism class. This is done by identifying the regular $\mathcal{J}$-classes of $\widehat{F}_{\mathsf{V}}(A)$ containing a finite number of $\mathcal{H}$-classes.

This introduction is followed by two sections of preliminaries, about symbolic dynamics and free profinite semigroups. Section 4 gives tools for the establishment, in Section 5, of the above mentioned functor between subshifts and compact categories. The results about flow equivalence are treated in Section 6, (with an appendix at the end of the paper, concerning one technical consequence). Finally, Section 7 deals with the connections with the zeta function.

2. Symbolic dynamics

In this section we provide a brief introduction to symbolic dynamics. For a very developed introduction, see the book [LM95]. In the context of this paper, the short text [Cos18] might also be useful.

It is helpful to begin by recalling some terminology and notation about free semigroups. In this paper, an *alphabet* will always be a finite nonempty set. The elements of the alphabet A are the *letters* of A. A *word* over A is a finite nonempty sequence of letters of A. The words over A form the semigroup A^+, for the operation of concatenation of words. The free monoid A^* is obtained from A^+ by adjoining the empty sequence (the *empty word*, here denoted by the symbol ε), which is the neutral element of A^* for the concatenation operation. As it is usual in the literature, the length of a word u is denoted by $|u|$.

2.1. The category of symbolic dynamical systems. Let A be an alphabet. Endow A with the discrete topology, and $A^{\mathbb{Z}}$ with the corresponding product topology. Note that, by Tychonoff's theorem and our convention that all alphabets are finite sets, the space $A^{\mathbb{Z}}$ is compact. We assume that compact topological spaces are Hausdorff. The *shift map* $\sigma_A\colon A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is the mapping defined by

$$\sigma_A((x_i)_{i\in\mathbb{Z}}) = (x_{i+1})_{i\in\mathbb{Z}}.$$

A *symbolic dynamical system*, also called a *subshift*, or just a *shift*, of $A^{\mathbb{Z}}$ is a nonempty closed subset $\mathcal{X}$ of $A^{\mathbb{Z}}$ such that $\sigma_A(\mathcal{X}) = \mathcal{X}$. The subshifts are the objects of the category of symbolic dynamical systems. In this category, a morphism between a subshift of $A^{\mathbb{Z}}$ and a subshift of $B^{\mathbb{Z}}$ is a continuous mapping $\varphi\colon \mathcal{X} \to \mathcal{Y}$ such that the diagram

$$\begin{tikzcd} \mathcal{X} \arrow[r, "\varphi"] \arrow[d, "\sigma_A"'] & \mathcal{Y} \arrow[d, "\sigma_B"] \\ \mathcal{X} \arrow[r, "\varphi"] & \mathcal{Y}. \end{tikzcd}$$

commutes. In the category of symbolic dynamical systems, an isomorphism is usually called a *conjugacy*, and two isomorphic subshifts are said to be *conjugate*.

A *block* of a subshift $\mathcal{X}$ is a nonempty word u appearing in some element of $\mathcal{X}$, that is, a word u such that for some $x \in \mathcal{X}$ and some integers $i \leq j$, the equality $u = x_i x_{i+1} \ldots x_{j-1} x_j$ holds. The word u may then be denoted by $x_{[i,j]}$. We denote the set of blocks of $\mathcal{X}$ by $L(\mathcal{X})$. One has $\mathcal{X} \subseteq \mathcal{Y}$ if and only if $L(\mathcal{X}) \subseteq L(\mathcal{Y})$, for all subshifts $\mathcal{X}$ and $\mathcal{Y}$ of $A^{\mathbb{Z}}$.

The notion of block is the groundwork for a form of producing morphisms between subshifts, which we next describe. For alphabets A and B, and a positive integer N, take a map $\Phi\colon A^N \to B$, where N is some positive integer. Let m and n be nonnegative integers such that $N = m + n + 1$. In the context of this paper, such a map is called a *block map*. The integer N is the *window size* of the block map. Consider the mapping $\varphi\colon A^{\mathbb{Z}} \to B^{\mathbb{Z}}$ defined by the correspondence

$$\varphi((x_i)_{i\in\mathbb{Z}}) = (\Phi(x_{[i-m,i+n]}))_{i\in\mathbb{Z}}.$$

We say that φ is the *sliding block code* from $A^{\mathbb{Z}}$ to $B^{\mathbb{Z}}$ with block map Φ, *memory* m and *anticipation* n. More generally, if the subshifts $\mathcal{X} \subseteq A^{\mathbb{Z}}$ and $\mathcal{Y} \subseteq B^{\mathbb{Z}}$ are such that $\varphi(\mathcal{X}) \subseteq \mathcal{Y}$, then the induced restriction $\varphi\colon \mathcal{X} \to \mathcal{Y}$ is also called a sliding block code, from $\mathcal{X}$ to $\mathcal{Y}$, with memory m and anticipation n. Note that $\varphi\colon \mathcal{X} \to \mathcal{Y}$ is determined by the restriction of Φ to the set of words of $L(\mathcal{X})$ with length N.

We are now ready to state a fundamental result of symbolic dynamics, the Curtis–Hedlund–Lyndon theorem [Hed69], fully characterizing the morphisms of subshifts (cf. [LM95, Theorem 6.2.9]).

Theorem 2.1. *The morphisms between subshifts are precisely the sliding block codes.*

Let us say that a block map $\Psi\colon A^N \to B$ is a *central block map* if N is odd. If $N = 2k+1$, then we say that k is the *wing* of Ψ. Given a sliding block code $\psi\colon \mathcal{X} \to \mathcal{Y}$, a *central block map of* ψ is a central block map $\Psi\colon A^{2k+1} \to B$ for which ψ has Ψ as a block map with both memory and anticipation equal to k.

Fact 2.2. Every sliding block code $\psi\colon \mathcal{X} \to \mathcal{Y}$ has a central block map.

Proof: If $\psi\colon \mathcal{X} \to \mathcal{Y}$ is a sliding block code with block map $\Phi\colon A^{m+n+1} \to B$, memory m and anticipation n, and letting $k = \max\{m, n\}$, then the map $\Psi\colon A^{2k+1} \to B$ defined by

$$\Psi(a_{-k}a_{-k+1}\cdots a_{-1}a_0a_1\cdots a_{k-1}a_k) = \Phi(a_{-m}a_{-m+1}\cdots a_{-1}a_0a_1\cdots a_{n-1}a_n),$$

where $a_i \in A$ for all $i \in \{-k, -k+1, \dots, k-1, k\}$, is such that ψ has Ψ as block map with memory k and anticipation k. ∎

A 1-*code* is a block map having a central block map of window size 1 (that is, wing 0). A 1-conjugacy is a 1-code that is a conjugacy.

Remark 2.3. The composition of two 1-codes is a 1-code.

With the help of Theorem 2.1, one gets the next useful result. In the diagram included, the double arrow represents an isomorphism, a convention reprised throughout the paper.

Proposition 2.4 (cf. [LM95, Proposition 1.5.12]). *If $\varphi\colon \mathcal{X} \to \mathcal{Y}$ is a morphism of subshifts, then there are* 1*-codes $\alpha\colon \mathcal{Z} \to \mathcal{X}$ and $\beta\colon \mathcal{Z} \to \mathcal{Y}$, for some subshift $\mathcal{Z}$, such that α is a conjugacy and the diagram*

$$\begin{tikzcd} & \mathcal{Z} \arrow[dl, Rightarrow, "\alpha"'] \arrow[dr, "\beta"] & \\ \mathcal{X} \arrow[rr, "\varphi"] & & \mathcal{Y} \end{tikzcd}$$

commutes, that is, $\varphi = \beta \circ \alpha^{-1}$.

2.2. Classification of subshifts. We review some important classes of subshifts.

A subshift $\mathcal{X}$ of $A^{\mathbb{Z}}$ is *irreducible* if there is $x \in A^{\mathbb{Z}}$ with positive dense orbit, that is, such that $\{\sigma_A^n(x) \mid n \geq 1\}$ is dense in $\mathcal{X}$. Clearly, being irreducible is a property invariant under conjugacy. Next is a convenient characterization in terms of words. Say that a subset K of a semigroup S is *irreducible* if, for every $u, v \in K$, there is $w \in S$ such that $uwv \in S$. It turns out that a subshift $\mathcal{X}$ of $A^{\mathbb{Z}}$ is irreducible if and only if $L(\mathcal{X})$ is irreducible in the semigroup A^+.

A subshift $\mathcal{X}$ of $A^{\mathbb{Z}}$ is *sofic* when its elements are the labels of the bi-infinite paths in a fixed graph with edges labeled by letters of A. A sofic subshift is irreducible if and only such a graph can be chosen to be strongly connected.

Example 2.5. The *even subshift* is the irreducible sofic subshift $\mathcal{X}$ of $\{a, b\}^{\mathbb{Z}}$ with presentation given by the labeled graph in Figure 1. That is, when u is

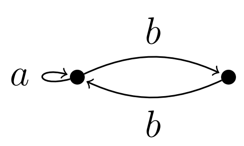

FIGURE 1. The even subshift.

a word over $\{a, b\}$, one has $u \in L(\mathcal{X})$ if and only if $ab^n a$ is not a factor of u for some odd n.

A subshift conjugate to a sofic subshift is also sofic. Within sofic shifts, the more salient class closed under conjugacy is that of *finite type shifts*. These are the subshifts conjugate with *edge shifts*, the latter being the subshifts presented by a labeled graph where distinct edges have distinct labels. The most famous open problem of symbolic dynamics is to know if we can always decide if two given edge shifts are conjugate or not.

A conjugacy-closed class quite distinct from sofic shifts, that has received a lot of attention in the literature (see [Fog02, Lot02]), is that of *minimal subshifts*: the subshift $\mathcal{X}$ is minimal if, whenever $\mathcal{Y}$ is a subshift, the inclusion $\mathcal{Y} \subseteq \mathcal{X}$ implies $\mathcal{Y} = \mathcal{X}$. All minimal subshifts are irreducible.

3. Free profinite semigroups

We assume knowledge about basic features of semigroups, like Green's relations (a short introduction may be found in [RS09, Appendix A]). In this section we quickly review some aspects of profinite semigroup theory. One of our purposes is to fix notation. For a more paused but short introduction to the subject, see for example [Alm05b]. The book [RS09] is also an updated guiding reference. We finish this section reviewing some connections with symbolic dynamics.

3.1. Languages and pseudovarieties. A subset of the free semigroup A^+ is called a *language* of A^+. A language L of A^+ is said to be *recognized* by a finite semigroup S if there is a homomorphism $\varphi\colon A^+ \to S$ such that $L = \varphi^{-1}(\varphi(L))$. Without giving details, we recall the well-known fact, not difficult to prove, that a language L is recognizable in this algebraic sense if and only if it is recognized by some finite automaton. Hence, a subshift $\mathcal{X}$ is sofic if and only if $L(\mathcal{X})$ is recognizable.

A *pseudovariety of semigroups* is a class of finite semigroups closed under taking subsemigroups, homomorphic images, and finitary products. The intersection of pseudovarieties is clearly a pseudovariety, and so we may talk of the pseudovariety generated by a class of semigroups. In Section 6 we shall have to restrict ourselves to monoidal pseudovarieties, the semigroup pseudovarieties generated by a class of finite monoids. Here are some pseudovarieties of semigroups, relevant for this paper, with only the last three examples not being monoidal:

- The pseudovariety S of all finite semigroups.
- The pseudovariety I of one-element semigroups.

- The pseudovariety G of all finite groups.
- The pseudovariety A of finite *aperiodic* semigroups, that is, semigroups all of whose subgroups (i.e., subsemigroups with group structure) are trivial.
- The pseudovariety $\overline{\mathsf{H}}$ of finite semigroups whose maximal subgroups belong to the pseudovariety of groups H.
- The pseudovariety Sl of *semilattices*, that is, commutative semigroups all of whose elements are idempotent.
- Given a pseudovariety V, the pseudovariety $\mathcal{L}\mathsf{V}$ of semigroups S such that, for every idempotent e of S, the subsemigroup eSe belongs to V.
- The pseudovariety N of finite *nilpotent* semigroups, which are the finite semigroups with a zero element 0 such that $S^n = \{0\}$ for some $n \geq 1$.
- The pseudovariety D of finite semigroups such that $Se = \{e\}$ for every idempotent e if S.

One of the main interests of semigroup pseudovarieties is that quite often one decides if a recognizable language L satisfies a certain combinatorial property by deciding if L is recognized by a semigroup from a certain pseudovariety V. Sometimes, these pseudovarieties are expressed as the result of operations on other pseudovarieties. An important example is the *semidirect product* $\mathsf{V} * \mathsf{W}$ of two pseudovarieties V and W, the least semigroup pseudovariety containing the semidirect products of elements of V with elements of W. This is an associative operation on the lattice of pseudovarieties. Another important operation, non-associative, is the *Mal'cev product* $\mathsf{V} \textcircled{m} \mathsf{W}$, briefly mentioned in one example later on, and which is the pseudovariety generated by finite semigroups S for which there is a homomorphism $\varphi : S \to T$ with $T \in \mathsf{W}$ and $\varphi^{-1}(e) \in \mathsf{V}$ for every idempotent e of T. The interested reader is referred to [RS09] for more information on these operations.

Example 3.1. A language L of A^+ is said to be *locally testable* if it is a finite Boolean combination of languages of the form uA^*, A^*u and A^*uA^*, where u denotes a (non-fixed) word of A^+. One of the first successes of finite semigroup theory was the proof that being locally testable is a decidable property by showing that a language is locally testable if and only if it is recognized by a semigroup in $\mathcal{L}\mathsf{Sl}$ [BS73, McN74, Zal73, Zal72]. In terms of pseudovarieties, this amounts to the equality $\mathcal{L}\mathsf{Sl} = \mathsf{Sl} * \mathsf{D}$. If $\mathcal{X}$ is a subshift of $A^{\mathbb{Z}}$ of finite type, then $L(\mathcal{X})$ is locally testable: indeed, it is of the form

$L(\mathcal{X}) = A^+ \setminus A^*WA^*$ for some finite set W of words. Conversely, if $\mathcal{X}$ is irreducible and $L(\mathcal{X})$ is locally testable, then $\mathcal{X}$ is of finite type (see [Cos07a] for a proof).

3.2. Relatively free profinite semigroups. A *compact semigroup* is a semigroup endowed with a topology for which the semigroup operation is continuous. We view finite semigroups as compact semigroups with the discrete topology.

In general, a pseudovariety of semigroups V is too small to contain free objects. An approach commonly followed is to find room for free objects by considering the inverse limits of semigroups of V, viewed as compact semigroups. These semigroups are the *pro-*V *semigroups*. Note that the semigroups from V are pro-V. Conversely, finite pro-V semigroups must belong to V. When dealing with the pseudovariety S of all finite semigroups, one uses the terminology *profinite* instead of pro-S.

If A is an alphabet, then the natural inverse limit defined by the finite quotients of A^+ that belong to V is a pro-V semigroup, denoted by $\widehat{F}_{\mathsf{V}}(A)$. Our assumption that all alphabets are finite guarantees that the topology of $\widehat{F}_{\mathsf{V}}(A)$ is metrizable.

The least closed subsemigroup of $\widehat{F}_{\mathsf{V}}(A)$ containing the image of the generating map $\iota\colon A \to \widehat{F}_{\mathsf{V}}(A)$ is $\widehat{F}_{\mathsf{V}}(A)$. The pro-$\mathsf{V}$ semigroup $\widehat{F}_{\mathsf{V}}(A)$ is the free object generated by A in the category of pro-V semigroups, as the map $\iota\colon A \to \widehat{F}_{\mathsf{V}}(A)$ satisfies the following universal property: for every map $\varphi\colon A \to S$ into a pro-V semigroup, there is a unique continuous semigroup homomorphism $\hat{\varphi}\colon \widehat{F}_{\mathsf{V}}(A) \to S$ such that $\hat{\varphi} \circ \iota = \varphi$. Hence, we say that $\widehat{F}_{\mathsf{V}}(A)$ is the *free pro-*V *semigroup generated by* A, or that it is the *free profinite semigroup relative to* V *generated by* A.

Let V be a pseudovariety of semigroups containing the pseudovariety N of finite nilpotent semigroups. Then the unique extension of $\iota\colon A \to \widehat{F}_{\mathsf{V}}(A)$ to a semigroup homomorphism $A^+ \to \widehat{F}_{\mathsf{V}}(A)$ is an injective map, and it is from this viewpoint that we consider ι as the inclusion and A^+ as a subsemigroup of $\widehat{F}_{\mathsf{V}}(A)$. One should bear in mind that A^+ is dense in $\widehat{F}_{\mathsf{V}}(A)$. Moreover, the hypothesis $\mathsf{N} \subseteq \mathsf{V}$ guarantees that the elements of A^+ are isolated in $\widehat{F}_{\mathsf{V}}(A)$. Hence, one may view the elements of $\widehat{F}_{\mathsf{V}}(A)$ as generalizations of finite words, for which reason we call them *pseudowords*, and we are justified to say that the elements of A^+ are the *finite* pseudowords of $\widehat{F}_{\mathsf{V}}(A)$, while those of $\widehat{F}_{\mathsf{V}}(A) \setminus A^+$ are the *infinite* pseudowords of $\widehat{F}_{\mathsf{V}}(A)$.

The following theorem gives us a glimpse of the reasons why relatively free profinite semigroups and pseudowords are useful. It essentially says that $\widehat{F}_{\mathsf{V}}(A)$ is the Stone dual of the Boolean algebra of languages recognized by semigroups of $\mathsf{V} \supseteq \mathsf{N}$.

Theorem 3.2 (cf. [Alm95, Theorem 3.6.1])**.** *Let V be a pseudovariety of semigroups containing N. Then a language $L \subseteq A^+$ is recognized by a semigroup of V if and only if its topological closure $\overline{L}$ in $\widehat{F}_{\mathsf{V}}(A)$ is open, if and only if $L = K \cap A^+$ for some clopen subset K of $\widehat{F}_{\mathsf{V}}(A)$.*

Given a semigroup S, we denote by S^I the monoid $S \uplus \{I\}$ extending the semigroup operation of S by adjoining an identity I. For example, A^* is (isomorphic to) the monoid $(A^+)^I$. If S is a compact semigroup, then we view S^I as a compact monoid extending S, by letting I be an isolated point. If $\varphi\colon S \to T$ is a function between semigroups, then its extension $S^I \to T^I$ that maps I to I, may still be denoted by φ, in the absence of confusion.

3.3. Pseudowords defined by subshifts. We briefly review some data relating relatively free profinite semigroups with symbolic dynamics, in part already met in Section 1, most of which is explained in [Cos06, Section 3.2] or [AC09]. Fix a semigroup pseudovariety V containing $\mathcal{L}\mathsf{Sl}$. The *mirage* of a subshift $\mathcal{X} \subseteq A^{\mathbb{Z}}$ is the set $\mathcal{M}_{\mathsf{V}}(\mathcal{X})$ of elements of $\widehat{F}_{\mathsf{V}}(A)$ whose finite factors are in $L(\mathcal{X})$. It helps to also consider the set $\mathcal{M}_{\mathsf{V},k}(\mathcal{X})$ of elements of $\widehat{F}_{\mathsf{V}}(A)$ whose finite factors of length at most k belong to $L(\mathcal{X})$. One clearly has $\mathcal{M}_{\mathsf{V}}(\mathcal{X}) = \bigcap_{k\geq 1} \mathcal{M}_{\mathsf{V},k}(\mathcal{X})$.

Remark 3.3. The set $\mathcal{M}_{\mathsf{V},k}(\mathcal{X})$ is the finite intersection of subsets of $\widehat{F}_{\mathsf{V}}(A)$ of the form $\widehat{F}_{\mathsf{V}}(A) \setminus \overline{A^*uA^*}$, with $u \in A^+ \setminus L(\mathcal{X})$ having length at most k. Hence, $\mathcal{M}_{\mathsf{V},k}(\mathcal{X})$ is clopen, in view of Theorem 3.2, as the locally testable language A^*uA^* is recognized by a semigroup of $\mathcal{L}\mathsf{Sl}$.

A subset K of a semigroup S is said to be *factorial* if every factor of an element of K belongs to K, and is said to be *prolongable* with respect to a subset A of S if $uA \cap K \neq \emptyset$ and $Au \cap K \neq \emptyset$ for each $u \in K$. The languages of the form $L(\mathcal{X})$, with $\mathcal{X}$ a subshift of $A^{\mathbb{Z}}$, are precisely the nonempty languages of A^+ that are factorial and prolongable with respect to A. With routine topological arguments, one easily deduces that $\overline{L(\mathcal{X})}$, $\mathcal{M}_{\mathsf{V},k}(\mathcal{X})$ and $\mathcal{M}_{\mathsf{V}}(\mathcal{X})$ are prolongable subsets of $\widehat{F}_{\mathsf{V}}(A)$, with respect to A. Note also that each of these sets contains infinite pseudowords, for example, every accumulation point of a sequence of words in $L(\mathcal{X})$ with increasing length.

Again applying standard topological arguments, one sees that the inclusion $\overline{L(\mathcal{X})} \subseteq \mathcal{M}_{\mathsf{V}}(\mathcal{X})$ holds. This inclusion may be strict. In fact, it is clear that $\mathcal{M}_{\mathsf{V},k}(\mathcal{X})$ and $\mathcal{M}_{\mathsf{V}}(\mathcal{X})$ are factorial, but the next example shows that $\overline{L(\mathcal{X})}$ may not be factorial, as seen in Example 3.4, taken from [Cos07b]. In that example, we use the notation s^ω, standard in (pro)finite semigroup theory, for the unique idempotent in the closed subsemigroup of S generated by s, where s is an element in a compact semigroup S. If S is profinite, one has $s = \lim s^{n!}$.

Example 3.4. Let $A = \{a, b, c, d\}$ and consider the sofic subshift $\mathcal{X}$ of $A^{\mathbb{Z}}$ presented in Figure 2. In $\widehat{F}_{\mathcal{L}\mathsf{Sl}}(A)$, the pseudoword $v = a^\omega b a^\omega c a^\omega$ belongs to

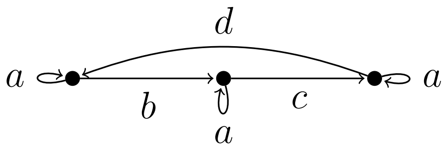

FIGURE 2. An irreducible sofic subshift.

$\overline{L(\mathcal{X})}$, since one clearly has $a^*ba^*ca^* \subseteq L(\mathcal{X})$. Moreover, in $\widehat{F}_{\mathcal{L}\mathsf{Sl}}(A)$ we have $a^\omega cv = v$ and so cv is a factor of v. The topological closure of the locally testable language $K = cA^* \cap A^*bA^* \cap A^+ \setminus A^*dA^*$ is a clopen neighborhood of cv (cf. Theorem 3.2). Therefore, if we had $cv \in \overline{L(\mathcal{X})}$, then we would have $L(\mathcal{X}) \cap K \neq \emptyset$, which is false.

On the other hand, if $\mathsf{V} = \mathsf{A} \circledast \mathsf{V}$ (for example, if $\mathsf{V} = \overline{\mathsf{H}}$), then $\overline{L(\mathcal{X})}$ is factorial [AC09]. For arbitrary V, consider another set, the *shadow of* $\mathcal{X}$, denoted by $\mathrm{Sha}_{\mathsf{V}}(\mathcal{X})$, defined as the union of the $\mathcal{J}$-classes of $\widehat{F}_{\mathsf{V}}(A)$ intersecting $\overline{L(\mathcal{X})}$. Note that $\mathrm{Sha}_{\mathsf{V}}(\mathcal{X}) = \overline{L(\mathcal{X})}$ if $L(\mathcal{X})$ is factorial. One has $\mathrm{Sha}_{\mathsf{V}}(\mathcal{X}) \subseteq \mathcal{M}_{\mathsf{V}}(\mathcal{X})$, with equality if $\mathcal{X}$ is of finite type. The equality also holds if $\mathcal{X}$ is minimal, a fact recorded in Theorem 3.6 below.

We already mentioned that $\mathcal{X}$ is irreducible if and only if $L(\mathcal{X})$ is an irreducible subset of A^+. From that, again with routine topological arguments, one deduces that if $\mathcal{X}$ is irreducible then $\overline{L(\mathcal{X})}$, $\mathrm{Sha}_{\mathsf{V}}(\mathcal{X})$ and $\mathcal{M}_{\mathsf{V}}(\mathcal{X})$ are irreducible. If K is a nonempty closed irreducible factorial subset of a compact semigroup, then it contains a $\mathcal{J}$-minimum $\mathcal{J}$-class, which is regular, as seen in [CS11]. All elements of K are then factors of all elements of such $\mathcal{J}$-class. Therefore, if $\mathcal{X}$ is irreducible, $\mathrm{Sha}_{\mathsf{V}}(\mathcal{X})$ contains a $\mathcal{J}$-minimum $\mathcal{J}$-class $J_{\mathsf{V}}(\mathcal{X})$ and $\mathcal{M}_{\mathsf{V}}(\mathcal{X})$ contains a $\mathcal{J}$-minimum $\mathcal{J}$-class $\widetilde{J}_{\mathsf{V}}(\mathcal{X})$, both regular $\mathcal{J}$-classes.

A *$\mathcal{J}$-maximal infinite element* of $\widehat{F}_{\mathsf{V}}(A)$ is an element u of $\widehat{F}_{\mathsf{V}}(A)$ such that $u \leq_{\mathcal{J}} v$ implies $v \in A^+$.

Remark 3.5. Every infinite pseudoword has some infinite idempotent as a factor [Alm95, Corollary 5.6.2], and so every $\mathcal{J}$-maximal infinite element of $\widehat{F}_{\mathsf{V}}(A)$ is regular. Moreover, every infinite pseudoword w has some $\mathcal{J}$-maximal infinite element as a factor, by Zorn's Lemma, because, by compactness, every $\leq_{\mathcal{J}}$-chain of infinite pseudowords that are factors of w clusters to an infinite pseudoword which is also a factor of w.

A *$\mathcal{J}$-maximal infinite $\mathcal{J}$-class* of $\widehat{F}_{\mathsf{V}}(A)$ is a $\mathcal{J}$-class consisting of $\mathcal{J}$-maximal infinite elements of $\widehat{F}_{\mathsf{V}}(A)$.

Theorem 3.6. *Let V be a pseudovariety of semigroups containing $\mathcal{L}\mathsf{SI}$. The correspondence $\mathcal{X} \mapsto J_{\mathsf{V}}(\mathcal{X})$ is a bijection from the set of minimal subshifts of $A^{\mathbb{Z}}$ to the set of $\mathcal{J}$-maximal infinite classes of $\widehat{F}_{\mathsf{V}}(A)$. Moreover, for every minimal subshift $\mathcal{X}$, the equalities* $\mathrm{Sha}_{\mathsf{V}}(\mathcal{X}) = L(\mathcal{X}) \cup J_{\mathsf{V}}(\mathcal{X}) = \mathcal{M}_{\mathsf{V}}(\mathcal{X})$ *hold.*

Theorem 3.6 is from [Alm05a]. Another proof, substantially different, is given in [AC09]. The following related proposition will be used in Section 7.

Proposition 3.7. *Let $\mathcal{X}$ be a subshift of $A^{\mathbb{Z}}$. Consider a pseudovariety of semigroups containing $\mathcal{L}\mathsf{SI}$. The $\mathcal{J}$-maximal infinite elements of $\widehat{F}_{\mathsf{V}}(A)$ contained in $\mathcal{M}_{\mathsf{V}}(\mathcal{X})$ are the $\mathcal{J}$-maximal infinite elements of $\widehat{F}_{\mathsf{V}}(A)$ contained in* $\mathrm{Sha}_{\mathsf{V}}(\mathcal{X})$.

Proof: Since the factorial set $\mathcal{M}_{\mathsf{V}}(\mathcal{X})$ contains infinite pseudowords, we may take some $\mathcal{J}$-maximal infinite element w of $\widehat{F}_{\mathsf{V}}(A)$ belonging to $\mathcal{M}_{\mathsf{V}}(\mathcal{X})$ (cf. Remark 3.5,). By Theorem 3.6, there is a minimal subshift $\mathcal{Y}$ such that $w \in \mathcal{M}_{\mathsf{V}}(\mathcal{Y})$ and all elements of $L(\mathcal{Y})$ are finite factors of w. By the definition of $\mathcal{M}_{\mathsf{V}}(\mathcal{X})$, we then have $L(\mathcal{Y}) \subseteq L(\mathcal{X})$, whence $\mathrm{Sha}_{\mathsf{V}}(\mathcal{Y}) \subseteq \mathrm{Sha}_{\mathsf{V}}(\mathcal{X})$. Looking again at Theorem 3.6, one sees that $\mathcal{M}_{\mathsf{V}}(\mathcal{Y}) = \mathrm{Sha}_{\mathsf{V}}(\mathcal{Y})$. Therefore, we have $w \in \mathrm{Sha}_{\mathsf{V}}(\mathcal{X})$. ■

4. Pseudoword block codes

In this section we present a technique emulating for pseudowords the sliding block code process used for bi-infinite sequences. This will permit to build in Section 5 the functors mentioned in Section 1. This technique was applied in [Cos06], explicitly for free profinite semigroups over S, implicitly for free profinite semigroups over pseudovarieties V such that $\mathsf{V} = \mathsf{V} * \mathsf{D}$ and $\mathsf{V} \supseteq \mathcal{L}\mathsf{SI}$. In Theorem 4.2 we see that these pseudovarieties give the exact scope of validity of this technique. While the facts in Theorem 4.2 are not original, they are dispersed in the literature and may not be easily accessible (for example, that all pseudovarieties V for which the technique holds satisfy $\mathsf{V} = \mathsf{V} * \mathsf{D}$ is, as far as we know, only explicitly mentioned, *en passant*, in the thesis [Cos07b], written in Portuguese).

4.1. Word and pseudoword block codes. We use the following convenient notation: given a word u of length $n \geq 1$, over the alphabet A, if $u = a_1 a_2 \cdots a_n$, with $a_i \in A$ for each $i \in \{1, \ldots, n\}$, we represent by $u_{[p,q]}$ the word $a_p a_{p+1} \cdots a_{q-1} a_q$, whenever $1 \leq p \leq q \leq n$. If $1 \leq k \leq n$, then we define $\mathsf{i}_k(u) = u_{[1,k]}$ and $\mathsf{t}_k(u) = u_{[n-k+1,n]}$, that is, $\mathsf{i}_k(u)$ and $\mathsf{t}_k(u)$ are respectively the unique prefix and the unique suffix of u with length k. If $k > n$, then we let $\mathsf{i}_k(u) = u = \mathsf{t}_k(u)$. Moreover, for $k = 0$, we make $\mathsf{i}_0(u) = \varepsilon = \mathsf{t}_0(u)$.

If V contains $\mathcal{L}\mathsf{I}$, then the maps $u \mapsto \mathsf{t}_k(u)$ and $u \mapsto \mathsf{i}_k(u)$, with $u \in A^*$, admit a unique continuous extension to maps $\mathsf{i}_k \colon \widehat{F}_{\mathsf{V}}(A)^I \to A^*$ and $\mathsf{t}_k \colon \widehat{F}_{\mathsf{V}}(A)^I \to A^*$, respectively, where we consider the discrete topology on A^* (take [Alm95, Sections 3.7 and 5.2] as reference, with [AC09, Section 2.5] as a possible helpful text). Hence, for every pseudoword $u \in \widehat{F}_{\mathsf{V}}(A) \setminus A^+$, the word $\mathsf{i}_k(u)$ (respectively, $\mathsf{t}_k(u)$) is the unique prefix (respectively, suffix) of u which is a word of length k.

Given a block map $\Psi \colon A^N \to B$, we are interested in the map $\overline{\Psi} \colon A^* \to B^*$ defined as follows: if u is a word of A^* of length at most $N-1$ then $\overline{\Psi}(u) = 1$, and if $u = a_1 \cdots a_M$ is a word of length $M \geq N$, with $a_i \in A$ for all $i \in \{1, \ldots, M\}$, then we have

$$\overline{\Psi}(u) = \Psi(u_{[1,N]}) \cdot \Psi(u_{[2,N+1]}) \cdot \Psi(u_{[3,N+2]}) \cdots \Psi(u_{[M-N+1,M]}). \tag{4.1}$$

Example 4.1. Let Ψ be a central block map $A^{2k+1} \to B$. Consider the sliding block code $\psi \colon A^{\mathbb{Z}} \to B^{\mathbb{Z}}$ having Ψ as a central block map. Let $x \in A^{\mathbb{Z}}$, and $y = \psi(x)$. Then, for all $i \in \mathbb{Z}$, we have

$$\Psi(x_{[i-k,i+k]}) = y_i$$

and so, applying formula (4.1), we obtain

$$\overline{\Psi}(x_{[i-k,j+k]}) = y_{[i,j]}$$

whenever $i, j \in \mathbb{Z}$ are such that $i \leq j$.

Intuitively, what $\overline{\Psi}$ does is to "encode" the word u into a new word $\overline{\Psi}(u)$ of B^*, by "reading" the consecutive factors on length N and assigning the corresponding letters from B. Loosely speaking, we are coding words as we code elements of a subshift via block maps, for which reason we say that $\overline{\Psi}$ is a *word block code*. Theorem 4.2 below characterizes the pseudovarieties for which we can extend this process in the most natural way, to what we shall call *pseudoword block codes*.

In preparation for Theorem 4.2, we introduce some notation. For each alphabet A and positive integer N, we denote by $A^{(<N)}$ the set of elements of A^* with length at most $N-1$. We will sometimes view the set A^N, of words of A^+ with length N, as an alphabet of its own. Viewed as an alphabet, A^N may be denoted A_N, to facilitate the understanding of the context in which the elements of A^N are being seen.

For the special case where Ψ is the identity map $A^N \to A_N$, we use the notation Υ_N for the corresponding word code $\overline{\Psi}$. In the literature (eg. [Alm95, AK20, PW02]), the map Υ_N is sometimes denoted by Φ_{N-1} or σ_{N-1}. These two notations are somewhat unfortunate in the context of this paper, the latter because of the standard notation for the shift map, the former because it is also usual, in the symbolic dynamics literature, to use the letter Φ to denote arbitrary block maps (see eg. [LM95]).

In this section we work with pseudovarieties V satisfying $\mathsf{V} = \mathsf{V} * \mathsf{D}$ and $\mathcal{L}\mathsf{I} \subseteq \mathsf{V}$. After the next theorem, we deal with them using their characterization in the theorem, without needing the original definition in terms of semidirect products.

Theorem 4.2. *Let* V *be a pseudovariety of semigroups containing* $\mathcal{L}\mathsf{I}$. *The following conditions are equivalent:*

(1) $\mathsf{V} = \mathsf{V} * \mathsf{D}$*;*
(2) for every alphabet A *and every positive integer* N*, the word block code* $\Upsilon_N \colon A^* \to (A_N)^*$ *admits a unique extension to a continuous mapping* $\Upsilon_N^{\mathsf{V}} \colon \widehat{F}_{\mathsf{V}}(A)^I \to \widehat{F}_{\mathsf{V}}(A_N)^I$*;*
(3) for every alphabet A*, positive integer* N*, and block map* $\Psi \colon A^N \to B$*, the word block code* $\overline{\Psi} \colon A^* \to B^*$ *admits a unique extension to a continuous mapping* $\overline{\Psi}_{\mathsf{V}} \colon \widehat{F}_{\mathsf{V}}(A)^I \to \widehat{F}_{\mathsf{V}}(B)^I$.

Moreover, assuming the equivalent conditions (1)-(3)*, and denoting by* λ_{V} *the unique continuous homomorphism* $\widehat{F}_{\mathsf{V}}(A_N) \to \widehat{F}_{\mathsf{V}}(A)$ *such that* $\lambda_{\mathsf{V}}(u) = \mathsf{i}_1(u)$ *for each* $u \in A^N$*:*

(4) we have the equality

$$\lambda_{\mathsf{V}}(\Upsilon^{\mathsf{V}}_N(uv)) = u \tag{4.2}$$

for every $u \in \widehat{F}_{\mathsf{V}}(A)$ *and* $v \in A^{N-1}$*, so that in particular* Υ^{V}_N *is injective on* $\widehat{F}_{\mathsf{V}}(A) \setminus A^{(<N)}$.

Theorem 4.2 derives from [Alm95, Chapter 10], and some parts are more or less explicitly stated there. In the paper [AK20] and in the thesis [Cos07b] (written in Portuguese) more details are given for other parts. The following proof is for the reader's convenience, so that the proof can be found in one location.

Proof of Theorem 4.2: Throughout the proof, we refer to the pseudovariety D_k of finite semigroups such that $St = \{t\}$ whenever $t \in S^k$, where k is a positive integer. By convention, one has $\mathsf{D}_0 = \mathsf{I}$. In fact, the equality $\mathsf{D} = \bigcup_{k\geq 1} \mathsf{D}_k$ holds (cf. [Alm95, Sections 10.4 and 10.6]), and $\mathsf{V} * \mathsf{D} = \bigcup_{k\geq 1} \mathsf{V} * \mathsf{D}_k$. We proceed in several steps.

(1) $\Rightarrow$ (2): The validity of this implication when $\mathsf{V} = \mathsf{S}$ is Lemma 10.6.11 from [Alm95].

For each alphabet X, denote by p^{V}_X the unique continuous homomorphism from $\widehat{F}_{\mathsf{S}}(X)$ onto the X-generated profinite semigroup $\widehat{F}_{\mathsf{V}}(X)$ that extends the identity on X. As before, we also use the notation p^{V}_X for the extension $\widehat{F}_{\mathsf{S}}(X)^I \to \widehat{F}_{\mathsf{V}}(X)^I$ mapping I to I. In a somewhat different language, Theorem 10.6.12 from [Alm95] affirms in particular that if W is a pseudovariety strictly containing $\mathcal{L}\mathsf{I}$ and such that $\mathsf{W} = \mathsf{W} * \mathsf{D}_{N-1}$, then one has

$$p^{\mathsf{W}*\mathsf{D}_{N-1}}_A(u) = p^{\mathsf{W}*\mathsf{D}_{N-1}}_A(v) \Leftrightarrow \begin{cases} p^{\mathsf{W}}_{A^N}(\Upsilon^{\mathsf{S}}_N(u)) = p^{\mathsf{W}}_{A^N}(\Upsilon^{\mathsf{S}}_N(v)) \\ \mathsf{i}_{N-1}(u) = \mathsf{i}_{N-1}(v) \\ \mathsf{t}_{N-1}(u) = \mathsf{t}_{N-1}(v) \end{cases} \tag{4.3}$$

for all $u, v \in \widehat{F}_{\mathsf{S}}(A)$. But this equivalence is also valid when $\mathsf{V} = \mathcal{L}\mathsf{I}$, because, in what is a well-know property of pseudowords (see [Cos01, Section 2.3] for example*), when $u, v \in \widehat{F}_{\mathsf{V}}(A)$ one has $p^{\mathcal{L}\mathsf{I}}_A(u) = p^{\mathcal{L}\mathsf{I}}_A(v)$ if and only if

*We give [Cos01, Section 2.3] as a reference for this property of $\mathcal{L}\mathsf{I}$ for the sake of better readability, but the property was known before: in the language of pseudowords, it is implicit in [Alm95,

$i_k(u) = i_k(v)$ and $t_k(u) = t_k(v)$ for every positive integer k (in particular, either $u = v$ or u, v are infinite pseudowords with the same finite prefixes and the same finite suffixes). So, we may in fact suppose that $\mathsf{V} \supseteq \mathcal{L}\mathsf{I}$. Taking $\mathsf{W} = \mathsf{V}$, and since $\mathsf{V} = \mathsf{V} * \mathsf{D} = \mathsf{V} * \mathsf{D} * \mathsf{D}_{N-1} = \mathsf{V} * \mathsf{D}_{N-1}$ (as $\mathsf{D} = \mathsf{D} * \mathsf{D}_{N-1}$, cf. [Alm95, Sections 10.4 and 10.6]), we obtain the implication

$$p^{\mathsf{V}}_A(u) = p^{\mathsf{V}}_A(v) \Rightarrow p^{\mathsf{V}}_{A^N}(\Upsilon^{\mathsf{S}}_N(u)) = p^{\mathsf{V}}_{A^N}(\Upsilon^{\mathsf{S}}_N(v))$$

and so we may define a (unique) map $\Upsilon^{\mathsf{V}}_N \colon \widehat{F}_{\mathsf{V}}(A) \to \widehat{F}_{\mathsf{V}}(A^N)^I$ for which the diagram

$$\begin{tikzcd}
\widehat{F}_{\mathsf{S}}(A) \arrow[r, "\Upsilon^{\mathsf{S}}_N"] \arrow[d, "p^{\mathsf{V}}_A"'] & \widehat{F}_{\mathsf{S}}(A_N)^I \arrow[d, "p^{\mathsf{V}}_{A_N}"] \\
\widehat{F}_{\mathsf{V}}(A) \arrow[r, "\Upsilon^{\mathsf{V}}_N"] & \widehat{F}_{\mathsf{V}}(A_N)^I
\end{tikzcd} \tag{4.4}$$

commutes. Finally, because the other maps in the diagram are continuous maps between compact spaces, one sees that Υ^{V}_N is also continuous[†].

(2) $\Rightarrow$ (4): Consider words $v, w \in A^+$ with length $N-1$, and letters $a, b \in A$ such that $av = wb$. By definition, we have $\lambda_{\mathsf{V}}(\Upsilon^{\mathsf{V}}_N(av)) = a$. This provides the base step for the following inductive argument to show the equality (4.2) for words, inducting on the length of words. If $z \in A^+$ has length $M \geq N$, then, according to formula (4.1) applied to the case where $\overline{\Psi}$ acts in A^N as the identity, we have

$$\Upsilon^{\mathsf{V}}_N(z) = \Upsilon^{\mathsf{V}}_N(z_{[1,M-1]}) \cdot \Upsilon^{\mathsf{V}}_N(z_{[M-N+1,M]}). \tag{4.5}$$

Therefore, for every $u \in A^+$, by putting $z = uav = uwb$ in (4.5), one has the equality $\Upsilon^{\mathsf{V}}_N(uav) = \Upsilon^{\mathsf{V}}_N(uw) \cdot \Upsilon^{\mathsf{V}}_N(av)$, so that

$$\lambda_{\mathsf{V}}(\Upsilon^{\mathsf{V}}_N(uav)) = \lambda_{\mathsf{V}}(\Upsilon^{\mathsf{V}}_N(uw)) \cdot \lambda_{\mathsf{V}}(\Upsilon^{\mathsf{V}}_N(av)) = \lambda_{\mathsf{V}}(\Upsilon^{\mathsf{V}}_N(uw)) \cdot a = ua,$$

where in the last equality we use the induction hypothesis. Since A^+ is dense in $\widehat{F}_{\mathsf{V}}(A)$, and $\lambda_{\mathsf{V}} \circ \Upsilon^{\mathsf{V}}_N$ is continuous on $A^+ \setminus A^{(<N)}$, we immediately extend the scope of equality (4.2) to every $u \in \widehat{F}_{\mathsf{V}}(A)$ and $v \in A^{N-1}$.

Section 3.7], and in fact it amounts to the fact that $\mathcal{L}\mathsf{I}$ is the join of D and its dual K, a fact already appearing in [Eil76].

[†]The arguments used in the proof of this implication are basically the same that were used in the proof of [AK20, Lemma 2.2], but there one finds the assumption that V contains Sl to guarantee that V does contain nontrivial monoids and therefore is according to the statement in [Alm95, Theorem 10.6.12]. As seen in our recapitulation of those arguments, such assumption is unnecessary.

Let $w \in \widehat{F}_{\mathsf{V}}(A) \setminus A^{(<N)}$. Then $w = uv$, for some $u \in \widehat{F}_{\mathsf{V}}(A)^I$ and $v \in A^N$. Let $(w_k)_k$ be a sequence of words of length at least N converging to w. By the continuity of t_N, we have $v = \mathsf{t}_N(w_k) = \mathsf{t}_N(w)$ for all sufficiently large k. If we see v as letter of A_N, then, by the continuity of Υ_N^{V}, for all sufficiently large k we have $v = \mathsf{t}_1(\Upsilon_N^{\mathsf{V}}(w_k)) = \mathsf{t}_1(\Upsilon_N^{\mathsf{V}}(w))$. Therefore, if w and w' are elements of $\widehat{F}_{\mathsf{V}}(A) \setminus A^{(<N)}$ such that $\Upsilon_N^{\mathsf{V}}(w) = \Upsilon_N^{\mathsf{V}}(w')$, then $\mathsf{t}_N(w)$ and $\mathsf{t}_N(w')$ are the same word v. In particular, $\mathsf{t}_{N-1}(w)$ and $\mathsf{t}_{N-1}(w')$ are both equal to the word $\tilde{v} = \mathsf{t}_{N-1}(v)$. On the other hand, we have factorizations $w = u\tilde{v}$ and $w' = u'\tilde{v}$, for some $u, u' \in \widehat{F}_{\mathsf{V}}(A)$. Since $u = \lambda_{\mathsf{V}}(\Upsilon_N^{\mathsf{V}}(u\tilde{v})) = \lambda_{\mathsf{V}}(\Upsilon_N^{\mathsf{V}}(u'\tilde{v})) = u'$, we conclude that $w = w'$ and that Υ_N^{V} is injective.

(2) $\Rightarrow$ (3): Let $\widehat{\Psi}$ be the unique continuous homomorphism $\widehat{F}_{\mathsf{V}}(A_N) \to \widehat{F}_{\mathsf{V}}(B)$ such that $\widehat{\Psi}(u) = \Psi(u)$ for every $u \in A_N$. We also work with the extension $\widehat{F}_{\mathsf{V}}(A_N)^I \to \widehat{F}_{\mathsf{V}}(B)^I$, still denoted $\widehat{\Psi}$, mapping I to I. By the hypothesis that (2) holds, the composition $\overline{\Psi}_{\mathsf{V}} = \widehat{\Psi}_{\mathsf{V}} \circ \Upsilon_N^{\mathsf{V}}$ is a continuous mapping from $\widehat{F}_{\mathsf{V}}(A)^I$ into $\widehat{F}_{\mathsf{V}}(B)^I$. One sees straightforwardly by induction on the length of $u \in A^+$ that $\overline{\Psi}_{\mathsf{V}}(u) = \overline{\Psi}(u)$ for every $u \in A^+$. Since A^* in dense in $\widehat{F}_{\mathsf{V}}(A)^I$, the mapping $\overline{\Psi}_{\mathsf{V}}$ is the unique continuous extension of $\overline{\Psi}\colon A^* \to B^*$ to a mapping $\widehat{F}_{\mathsf{V}}(A)^I \to \widehat{F}_{\mathsf{V}}(B)^I$.

Observing that (3) $\Rightarrow$ (2) is trivial, it remains to check (2) $\Rightarrow$ (1). We shall use the following facts, valid for all pseudovarieties V and W:

- if $\mathsf{W} \supseteq \mathsf{V}$, then the kernel of W is contained in the kernel of V (this is because $\widehat{F}_{\mathsf{V}}(A)$ is a pro-W semigroup when $\mathsf{V} \subseteq \mathsf{W}$);
- p_X^{W} and p_X^{V} have the same kernel if and only if $\mathsf{V} = \mathsf{W}$ (this is just a reformulation of the fact that V and W are equal if and only if they satisfy the same "pseudoidentities", see for example [Alm05b] for details if necessary).

Observe Diagram (4.4), which, under our assumption that (2) holds, is commutative: indeed, the restrictions to A^+ of the continuous mappings $p_{A_N}^{\mathsf{V}} \circ \Upsilon_N^{\mathsf{S}}$ and $\Upsilon_N^{\mathsf{V}} \circ p_A^{\mathsf{V}}$ clearly coincide, and A^+ is dense in $\widehat{F}_{\mathsf{V}}(A)$. As already seen in the proof of the implication (1) $\Rightarrow$ (2), if W is a pseudovariety containing $\mathcal{L}\mathsf{I}$, then the equivalence (4.3) holds for all $u, v \in \widehat{F}_{\mathsf{V}}(A)$. Taking $\mathsf{W} = \mathsf{V}$, and using the commutativity of Diagram (4.4), we then get the equivalence

$$p_A^{\mathsf{V}*\mathsf{D}_{N-1}}(u) = p_A^{\mathsf{V}*\mathsf{D}_{N-1}}(v) \Leftrightarrow \begin{cases} \Upsilon_N^{\mathsf{V}}[p_A^{\mathsf{V}}(u)] = \Upsilon_N^{\mathsf{V}}[p_A^{\mathsf{V}}(v)] \\ \mathsf{i}_{N-1}(u) = \mathsf{i}_{N-1}(v) \\ \mathsf{i}_{N-1}(u) = \mathsf{i}_{N-1}(v) \end{cases} \tag{4.6}$$

for all $u, v \in \widehat{F}_{\mathsf{V}}(A)$. We claim that in fact we have

$$p_A^{\mathsf{V}*\mathsf{D}_{N-1}}(u) = p_A^{\mathsf{V}*\mathsf{D}_{N-1}}(v) \Leftrightarrow p_A^{\mathsf{V}}(u) = p_A^{\mathsf{V}}(v).$$

Since $\mathsf{V} * \mathsf{D}_{N-1}$ contains V, the direct implication is immediate. Conversely, suppose that $u, v \in \widehat{F}_{\mathsf{V}}(A)$ are such that $p_A^{\mathsf{V}}(u) = p_A^{\mathsf{V}}(v)$. Since V contains $\mathcal{L}\mathsf{I}$, this implies $p_A^{\mathcal{L}\mathsf{I}}(u) = p_A^{\mathcal{L}\mathsf{I}}(v)$, which is the same as having $\mathsf{i}_k(u) = \mathsf{i}_k(v)$ and $\mathsf{t}_k(u) = \mathsf{t}_k(v)$ for every positive integer k. It then follows from (4.6) that $p_A^{\mathsf{V}*\mathsf{D}_{N-1}}(u) = p_A^{\mathsf{V}*\mathsf{D}_{N-1}}(v)$. Therefore, p_A^{V} and $p_A^{\mathsf{V}*\mathsf{D}_{N-1}}$ have the same kernel, whence $\mathsf{V} = \mathsf{V} * \mathsf{D}_{N-1}$. As this is true for all $N \geq 1$, we conclude that $\mathsf{V} = \mathsf{V} * \mathsf{D}$. ■

Given a block map $\Psi\colon A^N \to B$, we say that the map $\overline{\Psi}_{\mathsf{V}}\colon \widehat{F}_{\mathsf{V}}(A)^I \to \widehat{F}_{\mathsf{V}}(B)^I$, introduced in Theorem 4.2, is a *pseudoword block code.*

The next corollary is in [AK20], with the additional hypothesis that V is monoidal.

Corollary 4.3. *Let* V *be a pseudovariety of semigroups such that* $\mathsf{V} = \mathsf{V} * \mathsf{D}$ *and* $\mathcal{L}\mathsf{I} \subseteq \mathsf{V}$. *If* u, v *are words of* A^* *with the same length, and* $\pi, \rho \in \widehat{F}_{\mathsf{V}}(A)$ *are such that* $\pi u = \rho v$ *or that* $u\pi = v\rho$, *then* $u = v$ *and* $\pi = \rho$.

Proof: Let n be the length of u and v. If $\pi u = \rho v$, then $u = \mathsf{t}_n(\pi u) = \mathsf{t}_n(\rho v) = v$, and the equality $\pi = \rho$ follows from equality (4.2) in Theorem 4.2 (with $N = n + 1$). The case $u\pi = v\rho$ is treated similarly, using the dual of equality (4.2). ■

If we are in the conditions of Corollary 4.3, then, for each positive integer N, and pseudoword $u \in \widehat{F}_{\mathsf{V}}(A) \setminus A^+$, we denote by $\mathsf{i}_N(u)^{-1} \cdot u$ the unique pseudoword u' such that $u = \mathsf{i}_N(u) \cdot u'$. Similarly, we denote by $u \cdot \mathsf{t}_N(u)^{-1}$ the unique pseudoword u'' such that $u = u'' \cdot \mathsf{t}_N(u)$.

Lemma 4.4. *The maps* $u \mapsto \mathsf{i}_N(u)^{-1} \cdot u$ *and* $u \mapsto u \cdot \mathsf{t}_N(u)^{-1}$ *are continuous on the space* $\widehat{F}_{\mathsf{V}}(A) \setminus A^+$.

Proof: Suppose that $(u_n)_n$ converges to u in $\widehat{F}_{\mathsf{V}}(A) \setminus A^+$. As i_N is continuous, we have $\mathsf{i}_N(u_n) = \mathsf{i}_N(u)$ for all large enough n. Therefore, we have

$u_n = \mathsf{i}_N(u)v_n$ for all large enough n, where $v_n = \mathsf{i}_N(u_n)^{-1} \cdot u_n$. Every accumulation point v of $(v_n)_n$ is such that $u = \mathsf{i}_N(u)v$, that is, $v = \mathsf{i}_N(u)u^{-1}$. Since we are dealing with a compact space, this means that $\mathsf{i}_N(u_n)^{-1} \cdot u_n \to \mathsf{i}_N(u)^{-1} \cdot u$. Similarly, we have $u_n \cdot \mathsf{t}_N(u_n)^{-1} \to u \cdot \mathsf{t}_N(u)^{-1}$. ■

4.2. Some properties of pseudoword block codes. We now introduce some useful properties of pseudoword block codes. Until the end of this section, we work with a fixed pseudovariety of semigroups such that $\mathsf{V} = \mathsf{V} * \mathsf{D}$ and $\mathcal{L}\mathsf{I} \subseteq \mathsf{V}$. In the absence of confusion, we may denote a pseudoword block code $\overline{\Psi}_{\mathsf{V}}$, from $\widehat{F}_{\mathsf{V}}(A)^I$ to $\widehat{F}_{\mathsf{V}}(B)^I$, simply by $\overline{\Psi}$, dropping the subscript V.

Lemma 4.5. *Consider a pseudovariety of semigroups* V *such that* $\mathsf{V} = \mathsf{V} * \mathsf{D}$ *and* $\mathcal{L}\mathsf{I} \subseteq \mathsf{V}$. *Let* $\varphi\colon \mathcal{X} \to \mathcal{Y}$ *be a morphism of subshifts with central block map* $\Phi\colon A^{2k+1} \to B$. *Take a morphism of subshifts* $\psi\colon \mathcal{Y} \to \mathcal{Z}$ *with central block map* $\Psi\colon B^{2l+1} \to C$. *Then the map* $\Lambda\colon A^{2k+2l+1} \to C$ *defined by* $\Lambda(u) = \Psi \circ \overline{\Phi}(u)$ *is a central block map for* $\psi \circ \varphi$. *Moreover, the equality*

$$\overline{\Lambda}(u) = \overline{\Psi} \circ \overline{\Phi}(u) \tag{4.7}$$

holds for every $u \in \widehat{F}_{\mathsf{V}}(A)$.

Proof: Let $x \in \mathcal{X}$, $y = \varphi(x)$ and $z = \psi(y)$. Note that (cf. Example 4.1) for each $i \in \mathbb{Z}$, we have

$$z_i = \Psi([y_{[i-l,i+l]}]) = \Psi\left(\prod_{j\in[i-l,i+l]} \Phi(x_{[j-k,j+k]})\right) = \Psi(\overline{\Phi}(x_{[i-l-k,i+l+k]})),$$

and so Λ is indeed a central block map for $\psi \circ \varphi$.

We may in particular suppose that $\mathcal{X} = A^{\mathbb{Z}}$, $\mathcal{Y} = B^{\mathbb{Z}}$ and $\mathcal{Z} = C^{\mathbb{Z}}$. Let u be a word of A^+ of length $n \geq 2k + 2l + 1$. Then $u = x_{[1,n]}$ for some $x \in \mathcal{X}$. As we already checked that Λ is a central block map for $\varphi \circ \psi$, we know that, for $y = \varphi(x)$ and $z = \psi(y)$, the following chain of equalities holds:

$$\overline{\Lambda}(x_{[1,n]}) = z_{[1+k+l,n-k-l]} = \overline{\Psi}(y_{[1+k,n-k]}) = \overline{\Psi}(\overline{\Phi}(x_{[1,n]})).$$

Hence, equality (4.7) holds for every $u \in A^+$ of length at least $2k+2l+1$. It also holds if u has smaller length: in that case, we have $\overline{\Lambda}(u) = \varepsilon = \overline{\Psi}(\overline{\Phi}(u))$ (for the latter equality, note that the length of $\overline{\Phi}(u)$ will be smaller than $2l + 1$). As $\overline{\Lambda}$, $\overline{\Psi}$ and $\overline{\Phi}$ are continuous in $\widehat{F}_{\mathsf{V}}(A)^I = \overline{A^*}$, it follows that (4.7) holds for every $u \in \widehat{F}_{\mathsf{V}}(A)^I$. ■

Proposition 4.6. *Take a pseudovariety of semigroups* V *such that* $\mathsf{V} = \mathsf{V} * \mathsf{D}$ *and* $\mathcal{L}\mathsf{I} \subseteq \mathsf{V}$. *Consider a block map* $\Psi\colon A^N \to B$. *For all pseudowords* u *and* v *of* $\widehat{F}_{\mathsf{V}}(A)$, *we have*

$$\overline{\Psi}(uv) = \overline{\Psi}(u\,\mathsf{i}_{N-1}(v)) \cdot \overline{\Psi}(v) = \overline{\Psi}(u) \cdot \overline{\Psi}(\mathsf{t}_{N-1}(u)v). \tag{4.8}$$

If, moreover, $N = 2k + 1$, *then*

$$\overline{\Psi}(uv) = \overline{\Psi}(u\,\mathsf{i}_k(v)) \cdot \overline{\Phi}(\mathsf{t}_k(u)\,v) \tag{4.9}$$

holds.

For $\overline{\Psi} = \Upsilon_N$, the property in (4.8) is [Alm95, Exercise 10.6.6]. In its entirety, Proposition 4.6 is proved in the thesis [Cos07b]. For the reader's convenience, a short proof is given here, which seems more transparent than that in [Cos07b].

Proof: By the continuity of $\overline{\Psi}$, and since A^+ is dense in $\widehat{F}_{\mathsf{V}}(A)$, it suffices to check (4.8) and (4.9) for elements of A^+. We only do it for (4.9), as (4.8) may be treated similarly. Let $u, v \in A^+$ be words of lengths n and m, respectively. If $n \le k$, then $\mathsf{t}_k(u) = u$ and $|u\,\mathsf{i}_k(v)| < 2k + 1$, whence $\overline{\Psi}(u\,\mathsf{i}_k(v)) = \varepsilon$ and the equality holds trivially. Similarly for the case $m \le k$. Finally, suppose that $n, m > k$. Consider the sliding block code $\psi\colon A^{\mathbb{Z}} \to B^{\mathbb{Z}}$ having Ψ as a central block map. Then, we may choose some $x \in A^{\mathbb{Z}}$ such that $u = x_{[1,n]}$ and $v = x_{[n+1,n+m]}$. Let $y = \psi(x)$. Since $n + m > 2k + 1$, we have

$$\begin{aligned}\overline{\Psi}(uv) = \overline{\Psi}(x_{[1,n+m]}) = y_{[1+k,n+m-k]} &= y_{[1+k,n]} \cdot y_{[n+1,n+m-k]} \\ &= \overline{\Psi}(x_{[1,n+k]}) \cdot \overline{\Psi}(x_{[n+1-k,n+m]}) \\ &= \overline{\Psi}(u\,\mathsf{i}_k(v)) \cdot \overline{\Psi}(\mathsf{t}_k(u)\,v),\end{aligned}$$

establishing the equality (4.9). ■

Pseudoword block codes behave well with respect to the sets $\overline{L(\mathcal{X})}$ and $\mathcal{M}_{\mathsf{V}}(\mathcal{X})$, in the sense of the next proposition. Note the assumption that V contains $\mathcal{L}\mathsf{Sl}$ is necessary because in the proof we need to guarantee that $\mathcal{M}_{\mathsf{V},k}(\mathcal{X})$ is clopen, for every positive integer k. Recall that, under the hypothesis $\mathsf{V} = \mathsf{V} * \mathsf{D}$, the inclusion $\mathcal{L}\mathsf{Sl} \subseteq \mathsf{V}$ is equivalent to $\mathsf{Sl} \subseteq \mathsf{V}$, since $\mathcal{L}\mathsf{Sl} = \mathcal{L}\mathsf{Sl} * \mathsf{D}$.

Proposition 4.7. *Consider a pseudovariety of semigroups* V *such that* $\mathsf{V} = \mathsf{V} * \mathsf{D}$ *and* $\mathcal{L}\mathsf{Sl} \subseteq \mathsf{V}$. *Let* $\varphi\colon \mathcal{X} \to \mathcal{Y}$ *be a sliding block code of subshifts, with* *central block map* Φ. *The inclusions* $\overline{\Phi}\Big(\overline{L(\mathcal{X})}\Big) \subseteq \overline{L(\mathcal{Y})} \cup \{\varepsilon\}$, $\overline{\Phi}\big(\mathcal{M}_{\mathsf{V}}(\mathcal{X})\big) \subseteq \mathcal{M}_{\mathsf{V}}(\mathcal{Y}) \cup \{\varepsilon\}$ *and* $\overline{\Phi}\big(\mathrm{Sha}_{\mathsf{V}}(\mathcal{X})\big) \subseteq \mathrm{Sha}_{\mathsf{V}}(\mathcal{Y}) \cup \{\varepsilon\}$, *hold.*

We omit the routine proof of Proposition 4.7, appearing in [Cos06, Lemma 3.2] under the assumption $\mathsf{V} = \mathsf{S}$, irrelevant for the proof given there. Proposition 4.7 is also proved in the thesis [Cos07b] (with exactly the same hypothesis as here). We just underline that the last of the three inclusions is a direct consequence of the first inclusion and of the implication $u \leq_{\mathcal{J}} v \Rightarrow \overline{\Phi}(u) \leq_{\mathcal{J}} \overline{\Phi}(v)$, justified by Proposition 4.6. More precisely, if Φ has window size N, and $u = xvy$, then

$$\overline{\Phi}(u) = \overline{\Phi}(x\,\mathsf{i}_{N-1}(vy)) \cdot \overline{\Phi}(vy) = \overline{\Phi}(x\,\mathsf{i}_{N-1}(vy)) \cdot \overline{\Phi}(v) \cdot \overline{\Phi}(\mathsf{t}_{N-1}(v)y) \leq_{\mathcal{J}} \Phi(v).$$

5. A functorial correspondence from subshifts to categories

By a *compact category* we mean a small category C such that:

(1) the set $\mathrm{Obj}(C)$ of objects of C and the set $\mathrm{Mor}(C)$ of arrows (i.e., morphisms) of C are both compact topological spaces;
(2) both incidence maps, respectively assigning the domain $d(e) = x$ and the co-domain $r(e) = y$ to each arrow $e\colon x \to y$, are continuous maps from the space of arrows to the space of objects;
(3) the map $x \mapsto 1_x$ is continuous, where 1_x denotes the identity at x;
(4) the map $(s,t) \mapsto st$ defined on the set of composable arrows is continuous.

The morphisms between compact categories are the functors that restrict to continuous mappings between the corresponding spaces of objects and arrows.

For each semigroup S, we denote by $E(S)$ the set of idempotents of S. The *Karoubi envelope* of a semigroup S is a small category $\mathbb{K}(S)$, whose objects are the idempotents of S, and whose morphisms $f \to e$ are triples $(e,s,f) \in E(S) \times S \times E(S)$ such that $s = esf$, with composition $(e,s,f)(f,t,g) = (e,st,g)$. The identity morphism 1_e at object e is the triple (e,e,e). If S is a compact semigroup, then $E(S)$ is a nonempty compact subspace [CHK83, Theorem 3.5], and $\mathbb{K}(S)$ becomes a compact category, if we consider the space of morphisms endowed with the topology induced from the product space $E(S) \times S \times E(S)$.

Remark 5.1. Each continuous homomorphism $\varphi\colon T \to R$ of compact semigroups induces a continuous functor $\mathbb{K}(\varphi)\colon \mathbb{K}(T) \to \mathbb{K}(R)$, with $\mathbb{K}(\varphi)(e) = \varphi(e)$ when $e \in E(T)$, and $\mathbb{K}(\varphi)(e,s,f) = (\varphi(e),\varphi(s),\varphi(f))$ when (e,s,f) is an arrow of $\mathbb{K}(T)$. Moreover, an inverse limit $S = \varprojlim S_i$ of finite semigroups induces the equality $\mathbb{K}(S) = \varprojlim \mathbb{K}(S_i)$. A compact category is *profinite* when it is the inverse limit of an inverse system of finite categories. Hence, the Karoubi envelope of a profinite semigroup is a profinite category. We shall not need this fact, but one should have it in mind, as we will work with Karoubi envelopes of (free) profinite semigroups.

For later reference, we collect a couple of simple facts about the Karoubi envelope of a semigroup. For each idempotent e of a semigroup S, let G_e be the $\mathcal{H}$-class of e, that is, G_e is the group of units of the monoid eSe. Recall that G_e is a compact/profinite group if S is compact/profinite.

Proposition 5.2. *If e is an idempotent of the compact semigroup S, then the group of automorphisms of e in $\mathbb{K}(S)$ is a compact group isomorphic to G_e.*

Proof: The map $(e,s,e) \mapsto s$ is an isomorphism from the group of automorphisms of e in $\mathbb{K}(S)$ onto G_e (see for example [CS15]). This map is clearly continuous. ■

In a category, an object c is a retract of an object d, denoted $c \prec d$, if there are arrows $\varphi\colon c \to d$ and $\psi\colon d \to c$ with $\psi \circ \varphi = 1_c$. The relation $\prec$ is a partial order.

Proposition 5.3. *Let e, f be idempotents of the semigroup S. Then $e \leq_{\mathcal{J}} f$ if and only if $e \prec f$.*

Proof: If $e = xfy$, with $x, y \in S^I$, then $1_e = (e, exf, f)(f, fye, e)$, establishing the "only if" part. Conversely, if $1_e = (e,s,f)(f,t,e)$, then $e = st = sft$. ■

If F is a closed factorial subset of S, we denote by $\mathbb{K}(F)$ the subgraph of $\mathbb{K}(S)$ whose edges are the morphisms (e,s,t) such that $s \in F$, and whose objects are the idempotents of S belonging to F. The graph $\mathbb{K}(F)$ may not be a subcategory.

Example 5.4. Let $\mathcal{X}$ be the even subshift from Example 2.5. Then $s = (a^\omega, a^\omega b^\omega, b^\omega)$ and $t = (b^\omega, b^{\omega+1}a^\omega, a^\omega)$ belong to $\mathbb{K}(\mathrm{Sha}_{\mathsf{V}}(\mathcal{X}))$, but not $st =$

$(a^\omega, a^\omega b^{\omega+1}a^\omega, a^\omega)$. Indeed, $a^\omega b^{\omega+1}a^\omega$ belongs to $K = \overline{a^+(b^2)^*ba^+}$, and so, since $K \cap L(\mathcal{X}) = \emptyset$ and K is open, one has $a^\omega b^{\omega+1}a^\omega \notin \overline{L(\mathcal{X})}$.

Let $\mathcal{X}$ be a subshift of $A^{\mathbb{Z}}$ and let V be a pseudovariety containing $\mathcal{L}\mathsf{Sl}$. Suppose that (e, u, f) and (f, v, g) are arrows of $\mathbb{K}(\widehat{F}_{\mathsf{V}}(A))$ with $u, v \in \mathcal{M}_{\mathsf{V}}(\mathcal{X})$. Then, we have $uv = ufv \in \mathcal{M}_{\mathsf{V}}(\mathcal{X})$: indeed, in what has some similarity with properties of ordinary words, a finite factor in a product $w_1 \cdots w_n$ of infinite pseudowords over $\mathsf{V} \supseteq \mathcal{L}\mathsf{Sl}$ is either a factor of some w_i, or a product of a suffix of w_i and a prefix of w_{i+1}, for some i (see [AV06, Lemma 8.2] for a formal statement and proof), so that a finite factor of ufv is either a factor of $uf = u$ or of $v = fv$. Hence $\mathbb{K}(\mathcal{M}_{\mathsf{V}}(\mathcal{X}))$ is a compact subcategory of $\mathbb{K}(\widehat{F}_{\mathsf{V}}(A))$.

In this section, V is always a pseudovariety of semigroups containing $\mathcal{L}\mathsf{Sl}$ and such that $\mathsf{V} = \mathsf{V} * \mathsf{D}$.

Lemma 5.5. *Consider a central block map $\Phi\colon A^{2k+1} \to B$. For every idempotent e of $\widehat{F}_{\mathsf{V}}(A)$, the pseudoword $\overline{\Phi}(\mathsf{t}_k(e) \cdot e \cdot \mathsf{i}_k(e))$ is an idempotent of $\widehat{F}_{\mathsf{V}}(B)$.*

Proof: Applying the property expressed in equality (4.9), from Proposition 4.6, to the pseudowords $u = \mathsf{t}_k(e) \cdot e$ and $v = e \cdot \mathsf{i}_k(e)$, we obtain

$$\overline{\Phi}(\mathsf{t}_k(e) \cdot e \cdot \mathsf{i}_k(e)) \cdot \overline{\Phi}(\mathsf{t}_k(e) \cdot e \cdot \mathsf{i}_k(e)) = \overline{\Phi}(\mathsf{t}_k(e) \cdot e \cdot e \cdot \mathsf{i}_k(e)) = \overline{\Phi}(\mathsf{t}_k(e) \cdot e \cdot \mathsf{i}_k(e)),$$

showing that $\overline{\Phi}(\mathsf{t}_k(e) \cdot e \cdot \mathsf{i}_k(e))$ is an idempotent. ∎

In the setting of Lemma 5.5, we denote the idempotent $\Phi(\mathsf{t}_k(e) \cdot e \cdot \mathsf{i}_k(e))$ by $\Phi_{\mathbb{K}}(e)$.

Proposition 5.6. *Consider a central block map $\Phi\colon A^{2k+1} \to B$. The following data defines a functor:*

$$\begin{array}{rrcl} \Phi_{\mathbb{K}_{\mathsf{V}}}\colon & \mathbb{K}(\widehat{F}_{\mathsf{V}}(A)) & \to & \mathbb{K}(\widehat{F}_{\mathsf{V}}(B)) \\ & e & \mapsto & \Phi_{\mathbb{K}_{\mathsf{V}}}(e) \\ & (e,u,f) & \mapsto & \Big(\Phi_{\mathbb{K}_{\mathsf{V}}}(e), \overline{\Phi}(\mathsf{t}_k(e)\, u\, \mathsf{i}_k(f)), \Phi_{\mathbb{K}_{\mathsf{V}}}(f)\Big) \end{array}$$

Proof: Thanks to Lemma 5.5, we already know that this correspondence is correctly defined on objects. Since $u = euf$, we have $\mathsf{i}_k(u) = \mathsf{i}_k(e)$, $\mathsf{t}_k(u) =$

$\mathsf{t}_k(f)$, and therefore, applying (4.9), we get the following chain of equalities:

$$\begin{aligned}\overline{\Phi}(\mathsf{t}_k(e)\,u\,\mathsf{i}_k(f)) &= \overline{\Phi}(\mathsf{t}_k(e)\,e\cdot u\,\mathsf{i}_k(f))\\ &= \overline{\Phi}(\mathsf{t}_k(e)\,e\,\mathsf{i}_k(e))\cdot\overline{\Phi}(\mathsf{t}_k(e)\,u\,\mathsf{i}_k(f))\\ &= \overline{\Phi}(\mathsf{t}_k(e)\,e\,\mathsf{i}_k(e))\cdot\overline{\Phi}(\mathsf{t}_k(e)\,u\cdot f\,\mathsf{i}_k(f))\\ &= \overline{\Phi}(\mathsf{t}_k(e)\,e\,\mathsf{i}_k(e))\cdot\overline{\Phi}(\mathsf{t}_k(e)\,u\,\mathsf{i}_k(f))\cdot\overline{\Phi}(\mathsf{t}_k(f)\,f\,\mathsf{i}_k(f)).\end{aligned}$$

This shows that $\Phi_{\mathbb{K}_\mathsf{V}}$ is a morphism of graphs. Similarly, if (e,u,f) and (f,v,g) are two composable arrows of $\mathbb{K}(\widehat{F}_\mathsf{V}(A))$, by applying again (4.9) we get

$$\overline{\Phi}(\mathsf{t}_k(e)\,u\,\mathsf{i}_k(f))\cdot\overline{\Phi}(\mathsf{t}_k(f)\,v\,\mathsf{i}_k(g)) = \overline{\Phi}(\mathsf{t}_k(e)\,uv\,\mathsf{i}_k(g)),$$

thus establishing that $\Phi_{\mathbb{K}_\mathsf{V}}$ is a functor. ■

For $u\in\widehat{F}_\mathsf{V}(A)$ and idempotents e,f with $u=euf$, one has $u\mathrel{\mathcal{J}}\mathsf{t}_k(e)\,u\,\mathsf{i}_k(f)$, thus $u\in\mathcal{M}_\mathsf{V}(\mathcal{X})$ if and only if $\mathsf{t}_k(e)\,u\,\mathsf{i}_k(f)\in\mathcal{M}_\mathsf{V}(\mathcal{X})$. Therefore, by Proposition 4.7, the functor $\Phi_{\mathbb{K}_\mathsf{V}}$ restricts to a functor $\mathbb{K}(\mathcal{M}_\mathsf{V}(\mathcal{X}))\to\mathbb{K}(\mathcal{M}_\mathsf{V}(\mathcal{Y}))$ whenever Φ is a central block map of a sliding block code $\varphi\colon\mathcal{X}\to\mathcal{Y}$. We proceed to show that this restriction depends on φ only.

Lemma 5.7. *Consider a sliding block code $\varphi\colon\mathcal{X}\to\mathcal{Y}$, where $\mathcal{X}$ is a subshift of $A^\mathbb{Z}$ and $\mathcal{Y}$ is a subshift of $B^\mathbb{Z}$. Suppose that Φ and Ψ are central block maps of φ, with wings l and k, respectively, and suppose that $k\geq l$. Take $v\in\widehat{F}_\mathsf{V}(A)$ and words u,w of A^* with length k and such that every factor of uvw with length $2k+1$ belongs to $L(\mathcal{X})$. Then the equality*

$$\overline{\Psi}(uvw) = \overline{\Phi}(\mathsf{t}_l(u)\,v\,\mathsf{i}_l(w))$$

holds.

Proof: As reasoned in previous proofs, it suffices to consider the case $v\in A^+$. Suppose first that $v\in A$. Then, uvw is a word of length $2k+1$, and so $uvw = x_{[-k,k]}$ for some $x\in\mathcal{X}$. Then (cf. Example 4.1), we have

$$\overline{\Psi}(uvw) = (\varphi(x))_0 = \overline{\Phi}(x_{[-l,l]}) = \overline{\Phi}(\mathsf{t}_l(u)\,v\,\mathsf{i}_l(w)),$$

settling the case where v is a letter. Let φ' (respectively, φ'') be the sliding block code $A^\mathbb{Z}\to B^\mathbb{Z}$ having Φ (respectively, Ψ) as a central block map. Let $x\in A^\mathbb{Z}$ and $n\geq 2k+1$ be such that $x_{[1,n]}=uvw$. Take $y=\varphi'(x)$ and $z=\varphi''(x)$. Assuming $1+k\leq i\leq n-k$, the word $x_{[i-k,i+k]}$ belongs to $L(\mathcal{X})$, as it has length $2k+1$. By the already settled case, we know that

$$y_i = \Psi(x_{[i-k,i+k]}) = \overline{\Phi}(x_{[i-l,i+l]}) = z_i,$$

whenever $1 + k \leq i \leq n - k$. Finally, we have

$$\begin{aligned}\overline{\Phi}(\mathsf{t}_l(u)\, v\, \mathsf{i}_l(w)) = \overline{\Phi}(x_{[1+(k-l),n-(k-l)]}) &= z_{[1+k,n-k]} \\ &= y_{[1+k,n-k]} = \overline{\Psi}(x_{[1,n]}) = \overline{\Psi}(uvw),\end{aligned}$$

establishing the result. ■

Corollary 5.8. *Consider a sliding block code $\varphi\colon \mathcal{X} \to \mathcal{Y}$, with $\mathcal{X} \subseteq A^{\mathbb{Z}}$ and $\mathcal{Y} \subseteq B^{\mathbb{Z}}$. If Φ and Ψ are central block maps of φ, then the restrictions of $\Phi_{\mathbb{K}_{\mathsf{V}}}$ and $\Psi_{\mathbb{K}_{\mathsf{V}}}$ to $\mathbb{K}(\mathcal{M}_{\mathsf{V}}(\mathcal{X}))$ are equal.*

Proof: Suppose that the wings of Φ and Ψ are respectively l and k. Let u be an element of $\mathcal{M}_{\mathsf{V}}(\mathcal{X})$ such that $u = euf$ for some idempotents e and f of $\widehat{F}_{\mathsf{V}}(A)$. Then we have $\overline{\Psi}(\mathsf{t}_k(e)\, u\, \mathsf{i}_k(f)) = \overline{\Phi}(\mathsf{t}_l(e)\, u\, \mathsf{i}_l(f))$ by Lemma 5.7. ■

Definition 5.9. Let $\varphi\colon \mathcal{X} \to \mathcal{Y}$ be a sliding block code, with $\mathcal{X} \subseteq A^{\mathbb{Z}}$ and $\mathcal{Y} \subseteq B^{\mathbb{Z}}$. We define the functor $\varphi_{\mathbb{K}_{\mathsf{V}}}\colon \mathbb{K}(\mathcal{M}_{\mathsf{V}}(\mathcal{X})) \to \mathbb{K}(\mathcal{M}_{\mathsf{V}}(\mathcal{Y}))$ as being the restriction to $\mathbb{K}(\mathcal{M}_{\mathsf{V}}(\mathcal{X}))$ of the functor $\Phi_{\mathbb{K}_{\mathsf{V}}}\colon \mathbb{K}(\widehat{F}_{\mathsf{V}}(A)) \to \mathbb{K}(\widehat{F}_{\mathsf{V}}(B))$, whenever Φ is a central block map of φ (remember that $\Phi_{\mathbb{K}_{\mathsf{V}}}\big(\mathbb{K}(\mathcal{M}_{\mathsf{V}}(\mathcal{X}))\big)$ is indeed contained in $\mathbb{K}(\mathcal{M}_{\mathsf{V}}(\mathcal{Y}))$, as observed in the paragraph before Lemma 5.7). By Corollary 5.8, the map $\varphi_{\mathbb{K}_{\mathsf{V}}}$ does not depend on the choice of Φ.

Remark 5.10. Let $\mathcal{X}$ be a subshift of $A^{\mathbb{Z}}$. The identity $1_A\colon A \to A$ is a central block map of the identity $1_{\mathcal{X}}\colon \mathcal{X} \to \mathcal{X}$, and so the formula $(1_{\mathcal{X}})_{\mathbb{K}_{\mathsf{V}}} = 1_{\mathcal{M}_{\mathsf{V}}(\mathcal{X})}$ holds.

Remark 5.11. By Proposition 4.7, $\varphi_{\mathbb{K}_{\mathsf{V}}}(\mathrm{Sha}_{\mathsf{V}}(\mathcal{X})) \subseteq \mathrm{Sha}_{\mathsf{V}}(\mathcal{Y})$ for every morphism $\varphi\colon \mathcal{X} \to \mathcal{Y}$.

The next proposition is the first step to show that the correspondence $\varphi \mapsto \varphi_{\mathbb{K}_{\mathsf{V}}}$ defines a functor.

Proposition 5.12. *Let $\varphi\colon \mathcal{X} \to \mathcal{Z}$ and $\psi\colon \mathcal{Z} \to \mathcal{Y}$ be sliding block codes such that either φ or ψ is a 1-code. Then $\psi_{\mathbb{K}_{\mathsf{V}}} \circ \varphi_{\mathbb{K}_{\mathsf{V}}} = (\psi \circ \varphi)_{\mathbb{K}_{\mathsf{V}}}$.*

Proof: Note that it suffices to prove that the restrictions of $\psi_{\mathbb{K}_{\mathsf{V}}} \circ \varphi_{\mathbb{K}_{\mathsf{V}}}$ and $(\psi \circ \varphi)_{\mathbb{K}_{\mathsf{V}}}$ to the set of morphisms of $\mathbb{K}(\mathcal{M}_{\mathsf{V}}(\mathcal{X}))$ are equal. Let (e, u, f) be a morphism of $\mathbb{K}(\mathcal{M}_{\mathsf{V}}(\mathcal{X}))$. Suppose that Φ and Ψ are central block maps of φ and ψ, respectively.

Suppose first that Ψ has wing 0, and let k be the wing of Φ. Then $\psi_{\mathbb{K}_{\mathsf{V}}} \circ \varphi_{\mathbb{K}_{\mathsf{V}}}(e, u, f)$ is equal to the triple

$$\Big(\overline{\Psi} \circ \bar{\Phi}[\mathsf{t}_k(e)\, e\, \mathsf{i}_k(e)], \overline{\Psi} \circ \overline{\Phi}[\mathsf{t}_k(e)\, euf\, \mathsf{i}_k(f)], \overline{\Psi} \circ \overline{\Phi}[\mathsf{t}_k(f)\, f\, \mathsf{i}_k(f)]\Big).$$

By Lemma 4.5, the latter is equal to $(\psi\circ\varphi)_{\mathbb{K}_{\mathsf{V}}}(e,u,f)$.

It remains to consider the case in which Φ has wing 0. Let k be the wing of Ψ. Then

$$\begin{aligned}
&\psi_{\mathbb{K}_{\mathsf{V}}}\circ\varphi_{\mathbb{K}_{\mathsf{V}}}(e,u,f) = \\
&= \Psi_{\mathbb{K}_{\mathsf{V}}}\Big(\overline{\Phi}(e),\overline{\Phi}(u),\overline{\Phi}(f)\Big)\\
&= \Big(\overline{\Psi}[\mathsf{t}_k(\overline{\Phi}(e))\cdot\overline{\Phi}(e)\cdot\mathsf{i}_k(\overline{\Phi}(e))],\,\overline{\Psi}[\mathsf{t}_k(\overline{\Phi}(e))\cdot\overline{\Phi}(u)\cdot\mathsf{i}_k(\overline{\Phi}(f))],\,\overline{\Psi}[\mathsf{t}_k(\overline{\Phi}(f))\cdot\overline{\Phi}(f)\cdot\mathsf{i}_k(\overline{\Phi}(f))]\Big)\\
&= \Big(\overline{\Psi}\circ\overline{\Phi}[\mathsf{t}_k(e)\cdot e\cdot\mathsf{i}_k(e)],\,\overline{\Psi}\circ\overline{\Phi}[\mathsf{t}_k(e)\cdot u\cdot\mathsf{i}_k(f)],\,\overline{\Psi}\circ\overline{\Phi}[\mathsf{t}_k(f)\cdot f\cdot\mathsf{i}_k(f)]\Big),
\end{aligned}$$

where the last equality holds because $\overline{\Phi}$ is a homomorphism. Again by Lemma 4.5, we conclude that $\psi_{\mathbb{K}_{\mathsf{V}}}\circ\varphi_{\mathbb{K}_{\mathsf{V}}}(e,u,f) = (\psi\circ\varphi)_{\mathbb{K}_{\mathsf{V}}}(e,u,f)$. ∎

Proposition 5.13. *If the sliding block code $\varphi\colon\mathcal{X}\to\mathcal{Y}$ is a conjugacy of subshifts, then the functor $\varphi_{\mathbb{K}_{\mathsf{V}}}\colon\mathbb{K}(\mathcal{M}_{\mathsf{V}}(\mathcal{X}))\to\mathbb{K}(\mathcal{M}_{\mathsf{V}}(\mathcal{Y}))$ is an isomorphism of compact categories, and the equality*

$$(\varphi^{-1})_{\mathbb{K}_{\mathsf{V}}} = (\varphi_{\mathbb{K}_{\mathsf{V}}})^{-1}$$

holds.

Proof: By Proposition 2.4, if the sliding block code φ is a conjugacy, then there are 1-conjugacies α and β such that the following diagram commutes:

$$\begin{tikzcd}
& \mathcal{Z} \arrow[dl, Rightarrow, "\alpha"'] \arrow[dr, "\beta"] & \\
\mathcal{X} \arrow[rr, "\varphi"] & & \mathcal{Y}
\end{tikzcd}$$

Using several times Proposition 5.12, we deduce the following chain of equalities:

$$
\begin{aligned}
1_{\mathbb{K}(\mathcal{M}_{\mathsf{V}}(\mathcal{X}))} &= (1_{\mathcal{X}})_{\mathbb{K}_{\mathsf{V}}} \\
&= (\alpha \circ \alpha^{-1})_{\mathbb{K}_{\mathsf{V}}} \\
&= \alpha_{\mathbb{K}_{\mathsf{V}}} \circ (\alpha^{-1})_{\mathbb{K}_{\mathsf{V}}} \qquad \text{(since } \alpha \text{ is a 1-code)} \\
&= \alpha_{\mathbb{K}_{\mathsf{V}}} \circ 1_{\mathbb{K}(\mathcal{M}_{\mathsf{V}}(\mathcal{Z}))} \circ (\alpha^{-1})_{\mathbb{K}_{\mathsf{V}}} \\
&= \alpha_{\mathbb{K}_{\mathsf{V}}} \circ (1_{\mathcal{Z}})_{\mathbb{K}_{\mathsf{V}}} \circ (\alpha^{-1})_{\mathbb{K}_{\mathsf{V}}} \\
&= \alpha_{\mathbb{K}_{\mathsf{V}}} \circ (\beta^{-1} \circ \beta)_{\mathbb{K}_{\mathsf{V}}} \circ (\alpha^{-1})_{\mathbb{K}_{\mathsf{V}}} \\
&= \alpha_{\mathbb{K}_{\mathsf{V}}} \circ (\beta^{-1})_{\mathbb{K}_{\mathsf{V}}} \circ \beta_{\mathbb{K}_{\mathsf{V}}} \circ (\alpha^{-1})_{\mathbb{K}_{\mathsf{V}}} \qquad \text{(since } \beta \text{ is a 1-code)} \\
&= (\alpha \circ \beta^{-1})_{\mathbb{K}_{\mathsf{V}}} \circ (\beta \circ \alpha^{-1})_{\mathbb{K}_{\mathsf{V}}} \qquad \text{(because } \beta \text{ and } \alpha \text{ are 1-codes)} \\
&= (\varphi^{-1})_{\mathbb{K}_{\mathsf{V}}} \circ \varphi_{\mathbb{K}_{\mathsf{V}}}.
\end{aligned}
$$

Similarly, the next chain of equalities is valid:

$$
\begin{aligned}
1_{\mathbb{K}(\mathcal{M}_{\mathsf{V}}(\mathcal{Y}))} &= (1_{\mathcal{Y}})_{\mathbb{K}_{\mathsf{V}}} \\
&= (\beta \circ \beta^{-1})_{\mathbb{K}_{\mathsf{V}}} \\
&= \beta_{\mathbb{K}_{\mathsf{V}}} \circ (\beta^{-1})_{\mathbb{K}_{\mathsf{V}}} \qquad \text{(since } \beta \text{ is a 1-code)} \\
&= \beta_{\mathbb{K}_{\mathsf{V}}} \circ 1_{\mathbb{K}(\mathcal{M}_{\mathsf{V}}(\mathcal{Z}))} \circ (\beta^{-1})_{\mathbb{K}_{\mathsf{V}}} \\
&= \beta_{\mathbb{K}_{\mathsf{V}}} \circ (1_{\mathcal{Z}})_{\mathbb{K}_{\mathsf{V}}} \circ (\beta^{-1})_{\mathbb{K}_{\mathsf{V}}} \\
&= \beta_{\mathbb{K}_{\mathsf{V}}} \circ (\alpha^{-1} \circ \alpha)_{\mathbb{K}_{\mathsf{V}}} \circ (\beta^{-1})_{\mathbb{K}_{\mathsf{V}}} \\
&= \beta_{\mathbb{K}_{\mathsf{V}}} \circ (\alpha^{-1})_{\mathbb{K}_{\mathsf{V}}} \circ \alpha_{\mathbb{K}_{\mathsf{V}}} \circ (\beta^{-1})_{\mathbb{K}_{\mathsf{V}}} \qquad \text{(since } \alpha \text{ is a 1-code)} \\
&= (\beta \circ \alpha^{-1})_{\mathbb{K}_{\mathsf{V}}} \circ (\alpha \circ \beta^{-1})_{\mathbb{K}_{\mathsf{V}}} \qquad \text{(because } \alpha \text{ and } \beta \text{ are 1-codes)} \\
&= \varphi_{\mathbb{K}_{\mathsf{V}}} \circ (\varphi^{-1})_{\mathbb{K}_{\mathsf{V}}}.
\end{aligned}
$$

Therefore, the proposition holds. ∎

The following statement is now obvious.

Corollary 5.14. *Let* V *be a pseudovariety of semigroups containing* $\mathcal{L}\mathsf{Sl}$ *and such that* $\mathsf{V} = \mathsf{V} * \mathsf{D}$. *The compact category* $\mathbb{K}(\mathcal{M}_{\mathsf{V}}(\mathcal{X}))$ *is a conjugacy invariant, up to isomorphism of compact categories.*

Assuming $\mathcal{X}$ is irreducible, we may consider the $\mathcal{J}$-minimum $\mathcal{J}$-class $J_{\mathsf{V}}(\mathcal{X})$ of $\mathrm{Sha}_{\mathsf{V}}(\mathcal{X})$ and the $\mathcal{J}$-minimum $\mathcal{J}$-class $\widetilde{J}_{\mathsf{V}}(\mathcal{X})$ of $\mathcal{M}_{\mathsf{V}}(\mathcal{X})$. To such $\mathcal{X}$ we associate two profinite groups: the Schützenberger group $G_{\mathsf{V}}(\mathcal{X})$ of $J_{\mathsf{V}}(\mathcal{X})$ and the Schützenberger group $\widetilde{G}_{\mathsf{V}}(\mathcal{X})$ of $\widetilde{J}_{\mathsf{V}}(\mathcal{X})$, respectively isomorphic to the maximal subgroups of $J_{\mathsf{V}}(\mathcal{X})$ and to the maximal subgroups of $\widetilde{J}_{\mathsf{V}}(\mathcal{X})$. The following straightforward consequence of Proposition 5.13 was first established in [Cos06].

Corollary 5.15. *Let* V *be a pseudovariety of semigroups containing* $\mathcal{L}\mathsf{SI}$ *and such that* $\mathsf{V} = \mathsf{V} * \mathsf{D}$*. Suppose that the subshift* $\mathcal{X}$ *is irreducible. Then the profinite groups* $G_{\mathsf{V}}(\mathcal{X})$ *and* $\widetilde{G}_{\mathsf{V}}(\mathcal{X})$ *are conjugacy invariants, up to isomorphism of profinite groups.*

Proof: The idempotents of $\widetilde{J}_{\mathsf{V}}(\mathcal{X})$ are the minimal objects of $\mathbb{K}(\mathcal{M}_{\mathsf{V}}(\mathcal{X}))$ with respect to the retraction order $\prec$, by Proposition 5.3. The partial order $\prec$ is clearly preserved by isomorphisms of categories. Hence, in view of Proposition 5.13, if $\varphi\colon \mathcal{X} \to \mathcal{Y}$ is a conjugacy, then each idempotent e in $\widetilde{J}_{\mathsf{V}}(\mathcal{X})$ is mapped via $\varphi_{\mathbb{K}_{\mathsf{V}}}$ to an idempotent $f = \varphi_{\mathbb{K}_{\mathsf{V}}}(e)$ in $\widetilde{J}_{\mathsf{V}}(\mathcal{Y})$. In particular, the profinite groups G_e and G_f are isomorphic, by Proposition 5.2, establishing the conjugacy invariance of $\widetilde{G}_{\mathsf{V}}(\mathcal{X})$.

As $\varphi_{\mathbb{K}_{\mathsf{V}}}(\mathrm{Sha}_{\mathsf{V}}(\mathcal{X})) \subseteq \mathrm{Sha}_{\mathsf{V}}(\mathcal{Y})$ and $\varphi_{\mathbb{K}_{\mathsf{V}}}^{-1}(\mathrm{Sha}_{\mathsf{V}}(\mathcal{Y})) \subseteq \mathrm{Sha}_{\mathsf{V}}(\mathcal{X})$ (Remark 5.11), the arguments in the previous paragraph also yeld the conjugacy invariance of the profinite group $G_{\mathsf{V}}(\mathcal{X})$. ■

We now establish the functoriality of the correspondence $\mathcal{X} \mapsto \mathbb{K}(\mathcal{M}_{\mathsf{V}}(\mathcal{X}))$.

Theorem 5.16. *Let* V *be a pseudovariety of semigroups containing* $\mathcal{L}\mathsf{SI}$ *and such that* $\mathsf{V} = \mathsf{V} * \mathsf{D}$*. The following data defines a functor from the category of shifts to the category of compact categories.*

$$\begin{tikzcd} \mathcal{X} \arrow[r] \arrow[d, "\varphi"'] & \mathbb{K}(\mathcal{M}_{\mathsf{V}}(\mathcal{X})) \arrow[d, "\varphi_{\mathbb{K}_{\mathsf{V}}}"] \\ \mathcal{Y} \arrow[r] & \mathbb{K}(\mathcal{M}_{\mathsf{V}}(\mathcal{Y})) \end{tikzcd}$$

Proof: Consider composable sliding block codes $\varphi\colon \mathcal{X}_1 \to \mathcal{X}_2$ and $\psi\colon \mathcal{X}_2 \to \mathcal{X}_3$. Then, by Proposition 2.4, we may build a commutative diagram of sliding block codes, displayed in Figure 3, such that all the maps not in the base of the outer triangle (the maps α, β, γ, δ, μ and ν) are 1-codes, with α, γ and μ being conjugacies.

$$
\begin{tikzcd}
& & \mathcal{Z} \arrow[dl, Rightarrow, "\mu"'] \arrow[dr, "\nu"] & & \\
& \mathcal{Y}_1 \arrow[dl, Rightarrow, "\alpha"'] \arrow[dr, "\beta"] & & \mathcal{Y}_2 \arrow[dl, Rightarrow, "\gamma"'] \arrow[dr, "\delta"] & \\
\mathcal{X}_1 \arrow[rr, "\varphi"] & & \mathcal{X}_2 \arrow[rr, "\psi"] & & \mathcal{X}_3
\end{tikzcd}
$$

FIGURE 3. Triangle with base $\psi \circ \varphi$.

Applying several times Proposition 5.12 and 5.13, we deduce the following chain of equalities:

$$
\begin{aligned}
(\psi \circ \varphi)_{\mathbb{K}_{\mathsf{V}}} &= ((\delta \circ \nu) \circ (\alpha \circ \mu)^{-1})_{\mathbb{K}_{\mathsf{V}}} \\
&= (\delta \circ \nu)_{\mathbb{K}_{\mathsf{V}}} \circ ((\alpha \circ \mu)^{-1})_{\mathbb{K}_{\mathsf{V}}} && \text{(by Proposition 5.12, since } \delta \circ \nu \text{ is a 1-code)} \\
&= (\delta \circ \nu)_{\mathbb{K}_{\mathsf{V}}} \circ ((\alpha \circ \mu)_{\mathbb{K}_{\mathsf{V}}})^{-1} && \text{(by Proposition 5.13)} \\
&= \delta_{\mathbb{K}_{\mathsf{V}}} \circ \nu_{\mathbb{K}_{\mathsf{V}}} \circ (\alpha_{\mathbb{K}_{\mathsf{V}}} \circ \mu_{\mathbb{K}_{\mathsf{V}}})^{-1} && \text{(by Proposition 5.12, as } \delta,\ \nu,\ \alpha \text{ and } \mu \text{ are 1-codes)} \\
&= \delta_{\mathbb{K}_{\mathsf{V}}} \circ \nu_{\mathbb{K}_{\mathsf{V}}} \circ (\mu_{\mathbb{K}_{\mathsf{V}}})^{-1} \circ (\alpha_{\mathbb{K}_{\mathsf{V}}})^{-1} \\
&= \delta_{\mathbb{K}_{\mathsf{V}}} \circ \nu_{\mathbb{K}_{\mathsf{V}}} \circ (\mu^{-1})_{\mathbb{K}_{\mathsf{V}}} \circ (\alpha^{-1})_{\mathbb{K}_{\mathsf{V}}} && \text{(by Proposition 5.13)} \\
&= \delta_{\mathbb{K}_{\mathsf{V}}} \circ (\nu \circ \mu^{-1})_{\mathbb{K}_{\mathsf{V}}} \circ (\alpha^{-1})_{\mathbb{K}_{\mathsf{V}}} && \text{(by Proposition 5.12, as } \nu \text{ is a 1-code)} \\
&= \delta_{\mathbb{K}_{\mathsf{V}}} \circ (\gamma^{-1} \circ \beta)_{\mathbb{K}_{\mathsf{V}}} \circ (\alpha^{-1})_{\mathbb{K}_{\mathsf{V}}} \\
&= \delta_{\mathbb{K}_{\mathsf{V}}} \circ (\gamma^{-1})_{\mathbb{K}_{\mathsf{V}}} \circ \beta_{\mathbb{K}_{\mathsf{V}}} \circ (\alpha^{-1})_{\mathbb{K}_{\mathsf{V}}} && \text{(by Proposition 5.12, as } \beta \text{ is a 1-code)} \\
&= (\delta \circ \gamma^{-1})_{\mathbb{K}_{\mathsf{V}}} \circ (\beta \circ \alpha^{-1})_{\mathbb{K}_{\mathsf{V}}} && \text{(by Proposition 5.12, as } \delta \text{ and } \beta \text{ are 1-codes)} \\
&= \psi_{\mathbb{K}_{\mathsf{V}}} \circ \varphi_{\mathbb{K}_{\mathsf{V}}},
\end{aligned}
$$

thus establishing the result. ■

Our proof of Theorem 5.16 depends on the Curtis–Hedlund–Lyndon theorem (Theorem 2.1). It may be interesting to obtain a more direct proof, not depending on the use of block maps. We leave that as an open problem.

6. Flow equivalence

We turn our attention to flow equivalence, having [LM95, Section 13.7] and [BBEP10] as guiding references. Two discrete-time dynamical systems are *flow equivalent* if their suspension flows (or mapping tori) are conjugate modulo a time change. Parry and Sullivan showed that within the class of subshifts, flow equivalence is the equivalence relation between subshifts generated by conjugacy and *symbol expansion* [PS75], described next. Fix an alphabet A and a letter α of A. Let $\diamond$ be a letter not in A, and let $B = A \cup \{\diamond\}$. The *symbol expansion of A associated to α* is the homomorphism $\mathcal{E}\colon A^+ \to B^+$ such that $\mathcal{E}(\alpha) = \alpha\diamond$ and $\mathcal{E}(a) = a$ for all $a \in A \setminus \{\alpha\}$. The *symbol expansion of a subshift $\mathcal{X}$ of $A^{\mathbb{Z}}$ relative to α* is the least subshift $\mathcal{X}'$ of $B^{\mathbb{Z}}$ such that $L(\mathcal{X}')$ contains $\mathcal{E}(L(\mathcal{X}))$. A *symbol expansion of $\mathcal{X}$* is a symbol expansion of $\mathcal{X}$ relative to some letter.

Remark 6.1. Using induction on the length of words, one verifies that

$$\mathcal{E}(A^+) = B^+ \setminus \Big(\diamond B^* \cup B^*\alpha \cup \bigcup_{x\in A} B^*\alpha x B^* \cup \bigcup_{x\in B\setminus\{\alpha\}} B^* x \diamond B^*\Big).$$

In particular, one sees that $\mathcal{E}(A^+)$ is a locally testable language.

Throughout this section, as in Section 5, V will always be a pseudovariety of semigroups containing $\mathcal{L}\mathsf{SI}$ such that $\mathsf{V} = \mathsf{V} * \mathsf{D}$, but (unlike Section 5) with the additional requirement that V is monoidal. It is folklore that if V is monoidal and contains SI, then $S \in \mathsf{V}$ if and only if $S^I \in \mathsf{V}$ [Eil76]. From that it follows that $\widehat{F}_{\mathsf{V}}(A)^I$ is pro-V whenever V is monoidal and contains SI, a property that we shall need.

Let us return to the symbol expansion homomorphism $\mathcal{E}\colon A^+ \to B^+$ introduced in the first paragraph of this section. The unique extension of $\mathcal{E}$ to a continuous homomorphism $\widehat{F}_{\mathsf{V}}(A) \to \widehat{F}_{\mathsf{V}}(B)$ will also be denoted by $\mathcal{E}$. We let $\mathcal{E}(I) = I = \varepsilon$. Because $\widehat{F}_{\mathsf{V}}(A)^I$ is a pro-V semigroup, as we are assuming V to be monoidal, we may consider the unique continuous semigroup homomorphism $\mathcal{C}\colon \widehat{F}_{\mathsf{V}}(B) \to \widehat{F}_{\mathsf{V}}(A)^I$ such that $\mathcal{C}(\diamond) = \varepsilon = I$ and $\mathcal{C}(a) = a$ for all $a \in A$. The notation $\mathcal{C}$ is used because its restriction to B^+ is said to be a *symbol contraction.* Note that $\mathcal{C} \circ \mathcal{E}(u) = u$ for all $u \in \widehat{F}_{\mathsf{V}}(A)$, since this is clearly true for finite words and $\mathcal{C} \circ \mathcal{E}$ is continuous. In particular, $\mathcal{E}$ is injective, and we may use the notation $\mathcal{E}^{-1}$ for the restriction of $\mathcal{C}$ to $\mathcal{E}(\widehat{F}_{\mathsf{V}}(A))$. Observe that $\mathcal{E}(\widehat{F}_{\mathsf{V}}(A))$ is clopen by Remark 6.1.

Lemma 6.2. *Let $v \in \widehat{F}_{\mathsf{V}}(A)$. The following properties hold:*

1. *For $x, y, u \in \widehat{F}_{\mathsf{V}}(B)^I$, if $x \cdot \mathcal{E}(v) \cdot y = \mathcal{E}(u)$ then $x, y \in \mathcal{E}(\widehat{F}_{\mathsf{V}}(A)^I)$ and $u = \mathcal{E}^{-1}(x) v \mathcal{E}^{-1}(y)$.*
2. *If $\mathcal{E}(v) \in \overline{L(\mathcal{X}')}$, then we have $v \in \overline{L(\mathcal{X})}$.*

Proof: The case where x, y, v, u are finite words is Lemma 12.3 in [CS16], following very easily from Remark 6.1.

For the general case, suppose that $x \cdot \mathcal{E}(v) \cdot y = \mathcal{E}(u)$ and let $(x_n)_n$, $(v_n)_n$, $(y_n)_n$ be sequences of words respectively converging to x, v and y. Since $\mathcal{E}(\widehat{F}_{\mathsf{V}}(A))$ is open, for all large enough n there is a word u_n such that $x_n \cdot \mathcal{E}(v_n) \cdot y_n = \mathcal{E}(u_n)$. Taking subsequences, we may as well suppose that $u_n \to u$. By the special case for words, we have $x_n, y_n \in \mathcal{E}(A^*)$ and $u_n = \mathcal{E}^{-1}(x_n)v_n\mathcal{E}^{-1}(y_n)$ (bear in mind that $\mathcal{E}$ is injective). Because the mapping $\mathcal{E}^{-1}$, from the closed space $\mathcal{E}(\widehat{F}_{\mathsf{V}}(A)^I)$ to $\widehat{F}_{\mathsf{V}}(A)^I$, is continuous, we deduce that $u = \mathcal{E}^{-1}(x)v\mathcal{E}^{-1}(y)$.

Suppose now that $\mathcal{E}(v) \in \overline{L(\mathcal{X}')}$. There is a sequence $(u_n)_n$ of elements of $L(\mathcal{X}')$ converging to $\mathcal{E}(v)$. Because $\mathcal{E}(\widehat{F}_{\mathsf{V}}(A))$ is clopen, and by compactness, by taking subsequences we may in fact suppose that $u_n = \mathcal{E}(w_n) \in L(\mathcal{X}')$ for a sequence $(w_n)_n$ converging to some $w \in \widehat{F}_{\mathsf{V}}(A)$. Again by the case for words, we get $w_n \in L(\mathcal{X})$, thus $w \in \overline{L(\mathcal{X})}$. Since $\mathcal{E}$ is continuous, we have $\mathcal{E}(v) = \lim u_n = \mathcal{E}(w)$, whence $v = w \in \overline{L(\mathcal{X})}$, as $\mathcal{E}$ is injective. ■

In what follows, $\mathcal{X}$ is a subshift of $A^{\mathbb{Z}}$. We begin to record that the shadow of $\mathcal{X}$ is preserved by the symbol expansion.

Lemma 6.3. *The inclusion* $\mathcal{E}(\mathrm{Sha}_{\mathsf{V}}(\mathcal{X})) \subseteq \mathrm{Sha}_{\mathsf{V}}(\mathcal{X}')$ *holds.*

Proof: This is immediate, since $\mathcal{E}(L(\mathcal{X})) \subseteq L(\mathcal{X}')$ by the definition of $\mathcal{X}'$ and because $\mathcal{E}\colon \widehat{F}_{\mathsf{V}}(A) \to \widehat{F}_{\mathsf{V}}(B)$ is a continuous homomorphism. ■

We next prove that the mirage is also preserved by the symbol expansion.

Lemma 6.4. *The inclusion*

$$\mathcal{E}(\mathcal{M}_{\mathsf{V}}(\mathcal{X})) \subseteq \mathcal{M}_{\mathsf{V}}(\mathcal{X}')$$

holds. More precisely, one has

$$\mathcal{E}(\mathcal{M}_{\mathsf{V},k}(\mathcal{X})) \subseteq \mathcal{M}_{\mathsf{V},k}(\mathcal{X}')$$

for every positive integer k.

Proof: Clearly, it suffices to show the second inclusion, as $\mathcal{M}_{\mathsf{V}}(\mathcal{Z}) = \bigcap_{k\geq 1} \mathcal{M}_{\mathsf{V},k}(\mathcal{Z})$ for every subshift $\mathcal{Z}$.

Let u be an element of $\mathcal{M}_{\mathsf{V},k}(\mathcal{X})$. Suppose that $w \in B^+$ is a finite factor of $\mathcal{E}(u)$ with length at most k. Let $(u_n)_n$ be a sequence of elements of A^+ converging to u. Then $u_n \in \mathcal{M}_{\mathsf{V},k}(\mathcal{X})$ for all sufficiently large n, as $\mathcal{M}_{\mathsf{V},k}(\mathcal{X})$ is a (clopen) neighborhood of u (cf. Remark 3.3). On the other hand, $\overline{B^*wB^*}$ is a clopen of $\widehat{F}_{\mathsf{V}}(B)$ containing $\mathcal{E}(u)$. Since $\mathcal{E}$ is continuous, we also have

$\mathcal{E}(u_n) \in B^*wB^*$ for all sufficiently large n. Therefore, we may take some word u_m in the intersection $\mathcal{M}_{\mathsf{V},k}(\mathcal{X}) \cap \mathcal{E}^{-1}(B^*wB^*)$. Since w is a factor of $\mathcal{E}(u_m)$, in view of the equality in Remark 6.1, we see that there are words $p, q \in B^*$, $x \in \{\alpha, \varepsilon\}$ and $y \in \{\diamond, \varepsilon\}$ such that $\mathcal{E}(u_m) = pxwyq$ and xwy belongs to the image $\mathrm{Im}\mathcal{E}$, the possibilities for x and y depending on whether w starts with $\diamond$ or not, and whether w ends with α or not. By Lemma 6.2, the words p and q also belong to $\mathrm{Im}\mathcal{E}$ and

$$u_m = \mathcal{E}^{-1}(p) \cdot \mathcal{E}^{-1}(xwy) \cdot \mathcal{E}^{-1}(q). \tag{6.1}$$

Moreover, if $x = \alpha$, then w starts with the letter $\diamond$. Hence, if $x = \alpha$ or $y = \diamond$, then xwy has at least one occurrence of the letter $\diamond$, and it has at least two occurrences if $x = \alpha$ and $y = \diamond$. Therefore, for whatever possibility for $x \in \{\alpha, \varepsilon\}$ and $y \in \{\diamond, \varepsilon\}$, it follows from the definition of the symbol contraction $\mathcal{C}$ that $|\mathcal{E}^{-1}(xwy)| \leq |w| \leq k$. Since $u_m \in \mathcal{M}_{\mathsf{V},k}(\mathcal{X})$ and (6.1) holds, it follows that $\mathcal{E}^{-1}(xwy) \in L(\mathcal{X})$. Therefore, $xwy \in L(\mathcal{X}')$, and so $w \in L(\mathcal{X}')$. This proves that $\mathcal{E}(u) \in \mathcal{M}_{\mathsf{V},k}(\mathcal{X}')$. ∎

Concerning the contraction homomorphism, we first note the following fact.

Lemma 6.5. *The inclusion* $\mathcal{C}(L(\mathcal{X}') \setminus \{\diamond\}) \subseteq L(\mathcal{X})$ *holds, and so does the inclusion* $\mathcal{C}(\mathrm{Sha}_{\mathsf{V}}(\mathcal{X}') \setminus \{\diamond\}) \subseteq \mathrm{Sha}_{\mathsf{V}}(\mathcal{X})$.

Proof: If $u \in L(\mathcal{X}')$, then u is a factor of $\mathcal{E}(v)$ for some $v \in L(\mathcal{X})$, whence $\mathcal{C}(u)$ is a factor of $\mathcal{C} \circ \mathcal{E}(v) = v$, showing that $\mathcal{C}(L(\mathcal{X}')) \subseteq L(\mathcal{X}) \cup \{\varepsilon\}$. Moreover, if $u \in L(\mathcal{X}') \setminus \{\diamond\}$, then at least one letter appearing in u is not $\diamond$ (as $\diamond\diamond \notin L(\mathcal{X}')$), thus $\mathcal{C}(u) \neq \varepsilon$. Since $\mathcal{C}$ is a continuous homomorphism from $\widehat{F}_{\mathsf{V}}(B)$ to $\widehat{F}_{\mathsf{V}}(A)^I$, we immediately obtain $\mathcal{C}(\mathrm{Sha}_{\mathsf{V}}(\mathcal{X}') \setminus \{\diamond\}) \subseteq \mathrm{Sha}_{\mathsf{V}}(\mathcal{X})$. ∎

We will need the following lemma.

Lemma 6.6. *Every pseudoword u in $\mathcal{M}_{\mathsf{V},2}(\mathcal{X}')$ is of one, and only one, of the following four types:*

(1) $u \in \{\alpha, \diamond\}$
(2) $u \in \mathcal{E}(\widehat{F}_{\mathsf{V}}(A))$
(3) $u = \diamond v$ *for some* $v \in \mathcal{E}(\widehat{F}_{\mathsf{V}}(A))$
(4) $u = v\alpha$ *for some* $v \in \mathcal{E}(\widehat{F}_{\mathsf{V}}(A))$
(5) $u = \diamond v\alpha$ *for some* $v \in \mathcal{E}(\widehat{F}_{\mathsf{V}}(A)) \cup \{\varepsilon\}$

Proof: We assume $u \in B^+$ first. We prove, by induction on the length of the word u, that if u belongs to $\mathcal{M}_{\mathsf{V},2}(\mathcal{X}') \cap B^+$, then u is of one of five types (1)-(5). The base step is immediate: if the length of u is one, then u is of type (1) or (2).

Suppose that u is a word with length at least two and that the lemma holds for words of smaller length. Consider first the case in which u starts with the letter $\diamond$, and take a factorization $u = \diamond w$. Since w also belongs to the factorial set $\mathcal{M}_{\mathsf{V},2}(\mathcal{X}')$, we may apply the induction hypothesis to w. Let us see what happens in each case:

- If w is of type (1), then from $u = \diamond w \in L(\mathcal{X}')$ we get $w = \alpha$, and so u is of type (5).
- If w falls into type (2), then u is of type (3).
- It is impossible that w falls into types (3) or (5), otherwise $\diamond\diamond$ would be a prefix of u, contradicting that every factor of length two of u is in $L(\mathcal{X}')$.
- If w falls into type (4), then u is of type (5).

Therefore, in all possible cases, u is of one of the listed types, whenever $u \in \diamond B^*$.

Suppose now that u starts with the letter α. Since the factors of length two of u belong to $L(\mathcal{X}')$, we must have $u = \alpha \diamond w = \mathcal{E}(\alpha) \cdot w$ for some $w \in B^* \setminus \diamond B^*$. Applying the induction hypothesis to w, one sees that u must be either of type (2) or (4). A similar reasoning is valid if u starts with a letter $a \in A \setminus \{\alpha\}$, as we then have $u = aw = \mathcal{E}(a) \cdot w$ for some $w \in B^* \setminus \diamond B^*$. We have thus concluded that the inductive step holds, and that the lemma is valid for every $u \in \mathcal{M}_{\mathsf{V},2}(\mathcal{X}') \cap B^+$.

Now, let u be a pseudoword belonging to $\mathcal{M}_{\mathsf{V},2}(\mathcal{X}')$. Since $\mathcal{M}_{\mathsf{V},2}(\mathcal{X}')$ is clopen, there is a sequence $(u_n)_n$ of elements of $\mathcal{M}_{\mathsf{V},2}(\mathcal{X}') \cap B^+$ converging to u. As the number of possible types is finite, taking subsequences, we may as well suppose that all elements of $(u_n)_n$ are of the same type, among the five possible types (1)-(5). Since $\mathcal{E}(\widehat{F}_{\mathsf{V}}(A))$ is a closed set and the multiplication is continuous, it follows that u is of the same type as that of the terms u_n.

We end by observing that no pseudoword can be of more than one of the five types (1)-(5), since no element of $\mathcal{E}(\widehat{F}_{\mathsf{V}}(A))$ starts with $\diamond$ or ends with α. ■

Next is a sort of weak converse of Lemma 6.4.

Lemma 6.7. *The inclusion*

$$\mathcal{C}(\mathcal{M}_{\mathsf{V}}(\mathcal{X}') \setminus \{\diamond\}) \subseteq \mathcal{M}_{\mathsf{V}}(\mathcal{X})$$

holds. More precisely, one has

$$\mathcal{C}(\mathcal{M}_{\mathsf{V},2k}(\mathcal{X}') \setminus \{\diamond\}) \subseteq \mathcal{M}_{\mathsf{V},k}(\mathcal{X})$$

for every positive integer k.

Proof: Because $\mathcal{M}_{\mathsf{V}}(\mathcal{Z}) = \bigcap_{k\geq 1} \mathcal{M}_{\mathsf{V},2k}(\mathcal{Z})$ for every subshift $\mathcal{Z}$, we are reduced to showing the second inclusion.

Let $u \in \mathcal{M}_{\mathsf{V},2k}(\mathcal{X}') \setminus \{\diamond\}$. Let w be a finite factor of $\mathcal{C}(u)$ of length at most k. By Lemma 6.6, there are $x \in \{\diamond, \varepsilon\}$, $y \in \{\alpha, \varepsilon\}$ and $v \in \widehat{F}_{\mathsf{V}}(A)^I$ such that

$$u = x\mathcal{E}(v)y.$$

Since $\mathcal{C} \circ \mathcal{E}$ is the identity, we have

$$\mathcal{C}(u) = vy.$$

Hence $\mathcal{E}(w)$ is a finite factor of $\mathcal{E}(v)\cdot\mathcal{E}(y)$. Observe that $|\mathcal{E}(w)| \leq 2|w| \leq 2k$.

Suppose that $y = \varepsilon$. Then $\mathcal{E}(w)$ is a factor of $\mathcal{E}(v)$, and so it is a factor of u. Since $u \in \mathcal{M}_{\mathsf{V},2k}(\mathcal{X}')$, it follows that $\mathcal{E}(w) \in L(\mathcal{X}')$. Applying Lemma 6.2, we then get $w \in L(\mathcal{X})$.

Finally, suppose that $y = \alpha$. Then we have $u\diamond = x\mathcal{E}(v)\alpha\diamond = x\mathcal{E}(v)\mathcal{E}(\alpha)$, and $\mathcal{E}(w)$ is a factor of $u\diamond$. As discussed in Section 3.3, the set $\mathcal{M}_{\mathsf{V},2k}(\mathcal{X}')$ is prolongable, whence $ub \in \mathcal{M}_{\mathsf{V},2k}(\mathcal{X}')$ for some letter b. But $y = \alpha$ is a suffix of u, and so αb is a finite suffix of $u\diamond$. In particular, $\alpha b \in L(\mathcal{X}')$, implying $b = \diamond$. Therefore, we have $u\diamond \in \mathcal{M}_{\mathsf{V},2k}(\mathcal{X}')$. Since $\mathcal{E}(w)$ is a finite factor of $u\diamond$ of length at most $2k$, we must have $\mathcal{E}(w) \in L(\mathcal{X}')$. Again by Lemma 6.2, we conclude that $w \in L(\mathcal{X})$. ■

The following improvement of Lemma 6.6 is not necessary for the sequel, but it may be worthwhile to have it in mind.

Corollary 6.8. *The equality*

$$\mathcal{M}_{\mathsf{V}}(\mathcal{X}') \cap \mathcal{E}(\widehat{F}_{\mathsf{V}}(A)) = \mathcal{E}(\mathcal{M}_{\mathsf{V}}(\mathcal{X})). \tag{6.2}$$

holds. Consequently, every pseudoword u in $\mathcal{M}_{\mathsf{V}}(\mathcal{X}')$ is of one, and only one, of the following four types:

(1) $u \in \{\alpha, \diamond\}$
(2) $u \in \mathcal{E}(\mathcal{M}_{\mathsf{V}}(\mathcal{X}))$
(3) $u = \diamond v$ for some $v \in \mathcal{E}(\mathcal{M}_{\mathsf{V}}(\mathcal{X}))$
(4) $u = v\alpha$ for some $v \in \mathcal{E}(\mathcal{M}_{\mathsf{V}}(\mathcal{X}))$
(5) $u = \diamond v\alpha$ for some $v \in \mathcal{E}(\mathcal{M}_{\mathsf{V}}(\mathcal{X})) \cup \{\varepsilon\}$

Proof: The inclusion $\mathcal{E}(\mathcal{M}_{\mathsf{V}}(\mathcal{X})) \subseteq \mathcal{M}_{\mathsf{V}}(\mathcal{X}') \cap \mathcal{E}(\widehat{F}_{\mathsf{V}}(A))$ is in Lemma 6.4. Conversely, if $v \in \mathcal{M}_{\mathsf{V}}(\mathcal{X}') \cap \mathcal{E}(\widehat{F}_{\mathsf{V}}(A))$, then, by Lemma 6.7, we have $\mathcal{E}^{-1}(v) = \mathcal{C}(v) \in \mathcal{M}_{\mathsf{V}}(\mathcal{X})$, and so (6.2) holds.

Let $u \in \mathcal{M}_{\mathsf{V}}(\mathcal{X}')$. Then u is in one of the situations of Lemma 6.6. Since $\mathcal{M}_{\mathsf{V}}(\mathcal{X}')$ is factorial and the equality (6.2) is valid, we conclude that in the list given for such u by Lemma 6.6, we may replace $\mathcal{E}(\widehat{F}_{\mathsf{V}}(A))$ by $\mathcal{E}(\mathcal{M}_{\mathsf{V}}(\mathcal{X}))$. ■

We adapt to compact categories the notions of isomorphism of functors and of equivalence of categories. For that purpose, the following simple fact is needed.

Lemma 6.9. *In a compact category C, the set of isomorphisms is a closed subspace of* $\mathrm{Mor}(C)$*, and the mapping $\varphi \mapsto \varphi^{-1}$ is continuous on this subspace.*

Proof: Observe first that the set of identities $\{1_c \mid c \in \mathrm{Obj}(C)\}$ is a closed subspace of $\mathrm{Mor}(C)$, since the map $c \in \mathrm{Obj}(C) \mapsto 1_c$ is continuous and $\mathrm{Obj}(C)$ is compact. Therefore, if the net $(\varphi_i)_{i\in I}$ of isomorphisms of C converges to φ then, by the continuity of the composition, every convergent subnet of $(\varphi_i^{-1})_{i\in I}$ converges to an inverse of φ. As $\mathrm{Mor}(C)$ is compact, we deduce that $(\varphi_i^{-1})_{i\in I}$ converges to φ^{-1}. ■

Two continuous functors $F, G\colon C \to D$ between compact categories are *continuously isomorphic*, written $F \cong G$, when there is a *continuous* natural isomorphism $\eta\colon F \Rightarrow G$, which we define as natural isomorphism $\eta\colon F \Rightarrow G$ such that the function $\mathrm{Obj}(C) \to \mathrm{Mor}(D)$ mapping each object c of C to the morphism $\eta_c\colon F(c) \to G(c)$ is continuous. By Lemma 6.9, the inverse of a continuous natural isomorphism is a continuous natural isomorphism, and so the relation $\cong$ is symmetric. Moreover, it is straightforward that for all continuous functors $F, G\colon C \to D$ and $H, K\colon D \to E$ of compact categories, if $F \cong G$ and $H \cong K$ then $H \circ F \cong K \circ G$.

A functor $F\colon C \to D$ between compact categories C and D is a *continuous equivalence* if there is a continuous functor $G\colon D \to C$, such that $F \circ G \cong 1_D$ and $G \circ F \cong 1_C$. Such G is a *continuous pseudo-inverse* of F. We say that C and D are *continuously equivalent* if there is a continuous equivalence $F\colon C \to D$. Note that the continuous equivalence of compact categories is an equivalence relation.

We are now ready to state the next theorem. We mention that it applies in particular when $\mathsf{V} = \overline{\mathsf{H}}$, for a pseudovariety of groups H, as the equality $\overline{\mathsf{H}} = \overline{\mathsf{H}} * \mathsf{D}$ holds, and $\overline{\mathsf{H}} \supseteq \mathsf{A} \supseteq \mathcal{L}\mathsf{Sl}$ [Eil76, RS09].

Theorem 6.10. *Let* V *be a monoidal pseudovariety of semigroups containing* $\mathcal{L}\mathsf{Sl}$ *and such that* $\mathsf{V} = \mathsf{V} * \mathsf{D}$*. With respect to the continuous equivalence of compact categories, the equivalence class of the compact category* $\mathbb{K}(\mathcal{M}_{\mathsf{V}}(\mathcal{X}))$ *is a flow equivalence invariant.*

Proof: Thanks to Corollary 5.14, to show the flow invariance of the continuous equivalence class of $\mathbb{K}(\mathcal{M}_{\mathsf{V}}(\mathcal{X}))$, it only remains to show that it is invariant under symbol expansion.

Lemmas 6.4 and 6.7 guarantee the correctness of the choice of the codomains in the definition of both of the continuous functors $F\colon \mathbb{K}(\mathcal{M}_{\mathsf{V}}(\mathcal{X})) \to \mathbb{K}(\mathcal{M}_{\mathsf{V}}(\mathcal{X}'))$ and $G\colon \mathbb{K}(\mathcal{M}_{\mathsf{V}}(\mathcal{X}')) \to \mathbb{K}(\mathcal{M}_{\mathsf{V}}(\mathcal{X}))$ given by the rules

$$F(e,u,f) = (\mathcal{E}(e),\mathcal{E}(u),\mathcal{E}(f)) \quad \text{and} \quad G(e,u,f) = (\mathcal{C}(e),\mathcal{C}(u),\mathcal{C}(f)).$$

We prove the theorem by showing that F and G are continuous pseudo-inverses. Clearly, $1_{\mathbb{K}(\mathcal{M}_{\mathsf{V}}(\mathcal{X}))} = G \circ F$.

In the next lines, we use the notation u' for the pseudoword $(\mathsf{i}_1(u)^{-1}u) \cdot \mathsf{t}_1(u)^{-1}$, where u is an infinite pseudoword. Note that the map $u' \mapsto u$ is continuous, by Lemma 4.4. Suppose that e is an idempotent of $\mathcal{M}_{\mathsf{V}}(\mathcal{X}')$ not belonging to the image of $\mathcal{E}$. Then, by Lemma 6.6, either the first letter of e is $\diamond$, or the last letter of e is α. Since every finite factor of e belongs to $L(\mathcal{X}')$ and $e = e \cdot e$, if follows that in fact both situations happen, entailing $e = \diamond e'\alpha$. Note that $e' \in \mathrm{Im}\mathcal{E}$ according to Lemma 6.6 and the definition of the pseudoword e'. Since $\mathcal{C} \circ \mathcal{E}$ is the identity, and $e' \in \mathrm{Im}\mathcal{E}$, we have $e' = \mathcal{E} \circ \mathcal{C}(e')$, and so

$$F(G(e)) = \mathcal{E}(\mathcal{C}(\diamond e'\alpha)) = \mathcal{E}(\mathcal{C}(e') \cdot \alpha) = \mathcal{E}(\mathcal{C}(e')) \cdot \mathcal{E}(\alpha) = e'\alpha\diamond\,. \tag{6.3}$$

If the idempotent e of $\mathcal{M}_{\mathsf{V}}(\mathcal{X}')$ belongs to the image of $\mathcal{E}$, define $\eta_e = (e,e,e)$; if $e \notin \mathrm{Im}\mathcal{E}$, then we define $\eta_e = (e, e\diamond, e'\alpha\diamond)$. Note that in both cases η_e is an isomorphism of $\mathbb{K}(\mathcal{M}_{\mathsf{V}}(\mathcal{X}'))$, in the second case the inverse being $(e'\alpha\diamond, e'\alpha e, e)$.

Let $(e_n)_{n\in\mathbb{N}}$ be a sequence of idempotents of $\mathcal{M}_{\mathsf{V}}(\mathcal{X})$, converging to the idempotent e. Since $\mathcal{E}(A^+)$ is a locally testable set (cf. Remark 6.1), we know that $\mathcal{E}(\widehat{F}_{\mathsf{V}}(A)) = \overline{\mathcal{E}(A^+)}$ is clopen. Therefore, there is $p \in \mathbb{N}$ such that either $e_n \in \mathrm{Im}\mathcal{E}$ for all $n \geq p$, or $e_n \notin \mathrm{Im}\mathcal{E}$ for all $n \geq p$. Since $\lim(e_n)' = e'$, by continuity of the operator $u \mapsto u'$, we conclude that the mapping $e \mapsto \eta_e$ is continuous, viewing η_e as an element of the space $\widehat{F}_{\mathsf{V}}(B) \times \widehat{F}_{\mathsf{V}}(B) \times \widehat{F}_{\mathsf{V}}(B)$.

Let (e, u, f) be a morphism of $\mathbb{K}_{\mathsf{V}}(\mathcal{M}(\mathcal{X}))$. The proof of the theorem is now reduced to showing that Diagram 6.4 commutes.

$$\begin{tikzcd} e & F \circ G(e) \arrow[l, "\eta_e"'] \\ f \arrow[u, "{(e,u,f)}"] & F \circ G(f) \arrow[l, "\eta_f"] \arrow[u, "{F\circ G(e,u,f)}"'] \end{tikzcd} \tag{6.4}$$

We have several cases to consider:

(i) Suppose first that $e \in \mathrm{Im}\mathcal{E}$. If b is the first letter of $\mathcal{E}^{-1}(e)$, then the first letter of $\mathcal{E}(b)$ is the first letter of $u = eu$. Hence, the first letter of u is not $\diamond$. We have two subcases to consider:

(a) If $f \in \mathrm{Im}\mathcal{E}$, then just as we reasoned for the first letter of u, we see that the last letter of $u = uf$ is not α. Since $u \in \mathcal{M}_{\mathsf{V}}(\mathcal{X}')$, it follows from Lemma 6.6 that $u \in \mathrm{Im}\mathcal{E}$. Therefore, as $\mathcal{E} \circ \mathcal{C}$ restricts to the identity on $\mathrm{Im}\mathcal{E}$, we have $F \circ G(e, u, f) = (\mathcal{E}(\mathcal{C}(e)), \mathcal{E}(\mathcal{C}(u)), \mathcal{E}(\mathcal{C}(f))) = (e, u, f)$. And since in this case $\eta_e = 1_e$ and $\eta_f = 1_f$, the commutativity of Diagram 6.4 is immediate.

(b) If $f \notin \mathrm{Im}\mathcal{E}$, then we have the factorization $f = \diamond f'\alpha$, entailing $F(G(f)) = f'\alpha\diamond$ (cf. (6.3)). The last letter of $u = uf$ is α, while first letter is not $\diamond$, and so by Lemma 6.6 we have $u = \mathcal{E}(w)\alpha$ for some $w \in \widehat{F}_{\mathsf{V}}(A)$. It follows that $\mathcal{E} \circ \mathcal{C}(u) = \mathcal{E} \circ \mathcal{C}(\mathcal{E}(w)\alpha) = \mathcal{E}(w)\alpha\diamond = u\diamond$ and $F \circ G(e, u, f) = (e, u\diamond, f'\alpha\diamond)$. Then, by the definition of η_f when $f \notin \mathrm{Im}\mathcal{E}$, we have

$$\begin{aligned} \eta_e \circ (F \circ G)(e, u, f) &= 1_e \circ (e, u\diamond, f'\alpha\diamond) = (e, uf\diamond, f'\alpha\diamond) \\ &= (e, u, f) \circ (f, f\diamond, f'\alpha\diamond) = (e, u, f) \circ \eta_f, \end{aligned}$$

establishing that Diagram 6.4 is commutative in this case.

(ii) Suppose now that $e \notin \mathrm{Im}\mathcal{E}$, so that $e = \diamond e'\alpha$. As $u = eu \in \mathcal{M}_{\mathsf{V}}(\mathcal{X})$, the first letter of u must be $\diamond$. Recall also that $F(G(e)) = e'\alpha\diamond$ (cf. (6.3)). Again, we have two subcases to consider:

(a) If $f \in \mathrm{Im}\mathcal{E}$, then, as seen in case (i), the last letter of $u = uf$ is not α. It follows from Lemma 6.6 that $u = \diamond\mathcal{E}(w)$ for some $w \in \widehat{F}_{\mathsf{V}}(A)$. Then we have $\mathcal{E}(\mathcal{C}(u)) = \mathcal{E}(\mathcal{C}(\diamond\mathcal{E}(w))) = \mathcal{E}(\mathcal{C}(\mathcal{E}(w))) =$

$\mathcal{E}(w)$. On the other hand, $\diamond\mathcal{E}(w) = u = eu = \diamond e'\alpha u$, thus $\mathcal{E}(w) = e'\alpha u$ (cf. Corollary 4.3). We conclude that $F\circ G(e,u,f) = (e'\alpha\diamond, e'\alpha u, f)$, thus

$$\begin{aligned}\eta_e \circ (F\circ G)(e,u,f) &= (e, e\diamond, e'\alpha\diamond)(e'\alpha\diamond, e'\alpha u, f)\\ &= (e, e\cdot \underbrace{(\diamond e'\alpha)}_{=e}\cdot u, f) = (e,u,f) = (e,u,f)\circ \eta_f,\end{aligned}$$

proving that Diagram 6.4 commutes in this case also.

(b) If $f \notin \mathrm{Im}\mathcal{E}$, then we have the factorization $f = \diamond f'\alpha$. Since the first and last letters of $u = euf$ are respectively $\diamond$ and α, applying Lemma 6.6 we conclude that $u = \diamond\mathcal{E}(w)\alpha$ for some $w \in \widehat{F}_{\mathsf{V}}(A)$. Therefore, $\mathcal{E}\circ\mathcal{C}(u) = \mathcal{E}(w\alpha) = \mathcal{E}(w)\alpha\diamond$. On the other hand, because $\diamond\mathcal{E}(w)\alpha = u = eu = \diamond e'\alpha u$, we have $\mathcal{E}(w)\alpha = e'\alpha u$ by Corollary 4.3, and so we get $F\circ G(e,u,f) = (e'\alpha\diamond, e'\alpha u\diamond, f'\alpha\diamond)$. Finally, we have

$$\begin{aligned}&\eta_e \circ (F\circ G)(e,u,f)\\ &= (e, e\diamond, e'\alpha\diamond)(e'\alpha\diamond, e'\alpha u\diamond, f'\alpha\diamond) = (e, e(\diamond e'\alpha)u\diamond, f'\alpha\diamond)\\ &= (e, u\diamond, f'\alpha\diamond) = (e, u(f\diamond), f'\alpha\diamond) = (e,u,f)\circ(f, f\diamond, f'\alpha\diamond)\\ &= (e,u,f)\circ\eta_f.\end{aligned}$$

With all cases having been exhausted, the proof is concluded. ∎

Remark 6.11. By Lemmas 6.3 and 6.5, the functors F and G in the proof of Theorem 6.10 restrict to graph homomorphisms $\mathbb{K}(\mathrm{Sha}_{\mathsf{V}}(\mathcal{X})) \to \mathbb{K}(\mathrm{Sha}_{\mathsf{V}}(\mathcal{X}'))$ and $\mathbb{K}(\mathrm{Sha}_{\mathsf{V}}(\mathcal{X}')) \to \mathbb{K}(\mathrm{Sha}_{\mathsf{V}}(\mathcal{X}))$, respectively.

In the appendix section at the end of this paper we describe a labeled poset considered in [Cos06], and check that it is encapsulated in $\mathbb{K}(\mathcal{M}_{\mathsf{V}}(\mathcal{X}))$. The invariance under flow equivalence of such labeled poset then follows from Theorem 6.10. A direct proof of the conjugacy invariance was given in [Cos06]. The description of the labeled poset and the proof of its invariance are somewhat technical. The most interesting information associated to that labeled poset is the following more palatable result, which we next easily deduce directly from the proof of Theorem 6.10.

Corollary 6.12. *Suppose that $\mathcal{X}$ is an irreducible subshift. Let V be a monoidal pseudovariety of semigroups containing $\mathcal{L}\mathsf{Sl}$ and such that $\mathsf{V} =$*

$\mathsf{V} * \mathsf{D}$. *The profinite groups* $G_{\mathsf{V}}(\mathcal{X})$ *and* $\widetilde{G}_{\mathsf{V}}(\mathcal{X})$ *are flow equivalence invariants.*

Proof: By Corollary 5.15, to show the flow invariance of $\widetilde{G}_{\mathsf{V}}(\mathcal{X})$ we only need to check that $\widetilde{G}_{\mathsf{V}}(\mathcal{X})$ and $\widetilde{G}_{\mathsf{V}}(\mathcal{X}')$ are isomorphic profinite groups. By Theorem 6.10, there is a continuous equivalence $F\colon \mathbb{K}(\mathcal{M}_{\mathsf{V}}(\mathcal{X})) \to \mathbb{K}(\mathcal{M}_{\mathsf{V}}(\mathcal{X}'))$. In every category, the retraction order $\prec$ is preserved by every equivalence functor, and so if e is an idempotent in $\widetilde{J}_{\mathsf{V}}(\mathcal{X})$, then $F(e)$ is an idempotent in $\widetilde{J}_{\mathsf{V}}(\mathcal{X}')$, by Proposition 5.3. Also, every continuous equivalence functor of compact categories preserves the compact group of automorphisms in each object, so that G_e and $G_{F(e)}$ are isomorphic compact groups, establishing the flow invariance of $\widetilde{G}_{\mathsf{V}}(\mathcal{X})$.

For what follows we use the specific functor $F\colon \mathbb{K}(\mathcal{M}_{\mathsf{V}}(\mathcal{X})) \to \mathbb{K}(\mathcal{M}_{\mathsf{V}}(\mathcal{X}'))$ given by $F(e,u,f) = (\mathcal{E}(e), \mathcal{E}(u), \mathcal{E}(f))$, already met in the proof of Theorem 6.10, where we saw that it is indeed a continuous equivalence. By Lemma 6.3, if e is an idempotent in $J_{\mathsf{V}}(\mathcal{X})$, then $F(e)$ is an idempotent in $J_{\mathsf{V}}(\mathcal{X}')$, also because of the preservation of the retraction order by equivalence functors. And as G_e will then be isomorphic to $G_{F(e)}$, we get the flow invariance of $G_{\mathsf{V}}(\mathcal{X})$. ■

Remark 6.13. In the paper [AC16] a sort of geometric interpretation was given to $G_{\mathsf{S}}(\mathcal{X})$ when $\mathcal{X}$ is minimal, in which case $G_{\mathsf{S}}(\mathcal{X}) = \widetilde{G}_{\mathsf{S}}(\mathcal{X})$: there it was shown that the profinite group $G_{\mathsf{S}}(\mathcal{X})$ is an inverse limit of profinite completions of fundamental groups in an inverse system of the so called *Rauzy graphs* of $\mathcal{X}$. A geometric interpretation of this sort is yet to be obtained in the general case in which $\mathcal{X}$ is irreducible but may be non-minimal. The approach followed in [AC16] was based on exploring a profinite semigroupoid (a semigroupoid is a "category possibly without identities"), there denoted $\widehat{\Sigma}_\infty(\mathcal{X})$, and already considered in [AC09], which is determined by the infinite paths in the free profinite semigroupoid generated by the inverse limit of the Rauzy graphs of $\mathcal{X}$. The proof for the geometric interpretation made in [AC16] included the proof that if $\mathcal{X}$ is minimal then $\mathbb{K}(\mathcal{M}_{\mathsf{S}}(\mathcal{X}))$ and $\widehat{\Sigma}_\infty(\mathcal{X})$ are isomorphic compact categories. But that no longer holds if $\mathcal{X}$ is not minimal, as then $\widehat{\Sigma}_\infty(\mathcal{X})$ is not a category.

7. Relationship with the zeta function

The *orbit* of an element x of $A^{\mathbb{Z}}$ is the set $\mathcal{O}(x) = \{\sigma^n(x) \mid n \in \mathbb{Z}\}$. An element x of $A^{\mathbb{Z}}$ is said to be a *periodic* point, if $\sigma^n(x) = x$ for some positive integer n, equivalently, if $\mathcal{O}(x)$ is finite. A positive integer n such that $\sigma^n(x) = x$ is a *period* of x. The *least period* of a periodic point x is the smallest positive integer n such that $\sigma^n(x) = x$, that is, the least period of such x is the cardinal of $\mathcal{O}(x)$. A subshift $\mathcal{X}$ of $A^{\mathbb{Z}}$ is said to be a *periodic subshift* if $\mathcal{X} = \mathcal{O}(x)$ for some periodic point x of $A^{\mathbb{Z}}$. Every periodic subshift is both minimal and of finite type.

Given a subshift $\mathcal{X}$ of $A^{\mathbb{Z}}$, we denote by $p_{\mathcal{X}}(n)$ the number of periodic points with period n (i.e., with least period dividing n), and by $q_{\mathcal{X}}(n)$ the number of periodic points with least period n. The sequences $(p_{\mathcal{X}}(n))_{n\geq 1}$ and $(q_{\mathcal{X}}(n))_{n\geq 1}$ determine each other [LM95, Exercise 6.3.1]. The *zeta function* of $\mathcal{X}$, defined by

$$\zeta_{\mathcal{X}}(t) = \exp\Bigl(\sum_{n=1}^{+\infty} \frac{p_{\mathcal{X}}(n)}{n} t^n\Bigr)$$

encodes the sequence $(p_{\mathcal{X}}(n))_{n\geq 1}$ enumerating the number of periods, and so it also encodes the sequence $(q_{\mathcal{X}}(n))_{n\geq 1}$ enumerating the number of least periods. The zeta function is an important conjugacy invariant, namely of sofic subshifts (cf. [LM95]). In this section, we show that the zeta function of $\mathcal{X}$ is encoded in $\mathbb{K}(\mathcal{M}_{\mathsf{V}}(\mathcal{X}))$ as an invariant of isomorphism of compact categories (Corollary 7.10).

Two elements u and v in a semigroup S are said to be *conjugate*, and we write $u \sim_c v$, if there are elements $x, y \in S^I$ such that $u = xy$ and $v = yx$. For each $u \in S$, the elements v such that $v \sim_c u$ are the *conjugates* of u. In the next few lines, we focus on $S = A^+$, in which case $\sim_c$ is an equivalence relation. Indeed, the words conjugate to $u \in A^+$ are those of the form $v = p^{-1}up$, for some prefix p of u. A word $v \in A^+$ is *primitive* if $v = w^k$ implies that $v = w$. Every conjugate of a primitive word is primitive, and the number of conjugates of a primitive word v is the length of v. The latter fact may be seen as a consequence of one of the most basic properties of combinatorics of words (cf. [Lot83, Proposition 1.3.2]): if $x, y \in A^*$ are words such that $xy = yx$, then there is $z \in A^*$ such that $x, y \in z^*$.

Given a word v of length n of A^+, we denote by v^∞ the unique element x of $A^{\mathbb{Z}}$ such that $x_{[0,n-1]} = v$ and $\sigma^n(x) = x$. Likewise, we shall also use the notation $v^{+\infty}$ for the right infinite sequence $x \in A^{\mathbb{N}}$ such that $x_{[kn,(k+1)n-1]} = v$ for every $k \geq 0$, and $v^{-\infty}$ will be the left infinite sequence $x \in A^{\mathbb{Z}^-}$ such that $x_{[kn,(k+1)n-1]} = v$ for every $k \leq -1$. When $y \in A^{\mathbb{N}}$ and $x \in A^{\mathbb{Z}^-}$, we use the notation $z = x.y$ for $z \in A^{\mathbb{Z}}$ such that $z_i = x_i$ and $z_j = y_j$ for every $i \in \mathbb{Z}^-$ and $j \in \mathbb{N}$. Hence, $v^\infty = v^{-\infty}.v^{+\infty}$ if v is a word.

The notion of primitive word is useful for dealing with periodic points, because of the following simple fact.

Fact 7.1. Let x be a periodic element of $A^{\mathbb{Z}}$. Then, there is a unique primitive word $v \in A^+$ such that $x = v^\infty$. Moreover, we have the equality $\mathcal{O}(x) = \{u^\infty \mid u \sim_c v\}$, and the least period of x is the length of v.

We collect some more properties of primitive words.

Lemma 7.2. *If v is a primitive word of A^+, then the language v^+ is locally testable.*

Proof: Let $\mathcal{X} = \mathcal{O}(v^\infty)$. Then, denoting by $[v]_{\sim_c}$ the $\sim_c$-class of v, we have the equality

$$v^+ = (L(\mathcal{X}) \setminus A^{(<|v|)}) \setminus \bigcup_{u,w \in [v]_{\sim_c} \setminus \{v\}} (uA^* \cup A^*w).$$

Since $\mathcal{X}$ is of finite type, the language $L(\mathcal{X})$ is locally testable. As $A^{(<|v|)}$, uA^* and A^*w are also locally testable, we conclude that v^+ is locally testable. ∎

Lemma 7.2 may be seen as an application of the main result of [Res74], a more general result about very pure codes (see also [BPR10, Proposition 7.1.1]).

Lemma 7.3. *If v is a primitive word of A^+ with length n, then the inclusion $v^* \cdot v^2 \cdot A^{(<n)} \cap A^* \cdot v^2 \subseteq v^+$ holds.*

Proof: Let $w \in v^* \cdot v^2 \cdot A^{(<n)} \cap A^* \cdot v^2$. Then $w = v^k q$ for some $k \geq 2$ and some (possibly empty) word q of length at most $n-1$, and v^2 is a suffix of $v^2 q$. We are reduced to showing that $q = \varepsilon$. Take the word p such that $v^2 q = pv^2$. Then we have $|p| = |q| < |v|$, and $v = pv' = v''q$ for some $v', v'' \in A^+$ such that $|v'| = |v''|$. As the following chain of equalities

$$p \cdot v'' \cdot qv = p \cdot (v''q) \cdot v = pv^2 = v^2 q = p \cdot v' \cdot vq$$

holds, comparing the extremities of the chain, we deduce from $|v''| = |v'|$ that $qv = vq$. Therefore, by the aforemention property of commuting words, one concludes that $v, q \in w^*$ for some word w. But as v is primitive and $|q| < |v|$, one must have $w = v$ and $q = \varepsilon$. ∎

We remark, *en passant*, that the word v^2 is really relevant in Lemma 7.3. More precisely, the inclusion $v^* \cdot v \cdot A^{(<|v|)} \cap A^* \cdot v \subseteq v^+$ fails, for example, for $A = \{a, b\}$ and the primitive word $v = bab$, since $(bab)ab = ba(bab)$ is not a power of bab.

We turn now our attention to pseudowords. Let V be a pseudovariety of semigroups containing $\mathcal{L}\mathsf{I}$. Suppose that $u \in \widehat{F}_{\mathsf{V}}(A) \setminus A^+$. We denote by $\overrightarrow{u}$ the unique element $x = (x_i)_{i\in\mathbb{N}}$ of $A^{\mathbb{N}}$ such that $x_{[0,n-1]}$ is the prefix of length n of u, whenever n is a positive integer. We say that $\overrightarrow{u}$ is the *positive ray* of u. Symmetrically, the *negative ray* of u, denoted $\overleftarrow{u}$, is the unique element $x = (x_i)_{i\in\mathbb{Z}^-}$ of $A^{\mathbb{Z}^-}$ such that $x_{[-n,-1]}$ is the suffix of length n of u, whenever n is a positive integer. Let u and v be elements of $\widehat{F}_{\mathsf{V}}(A) \setminus A^+$. Note that if $u = vw$ for some $w \in \widehat{F}_{\mathsf{V}}(A)^I$, then $\overrightarrow{u} = \overrightarrow{v}$, but the converse is not true: $u = a^\omega b$ and $v = a^\omega c$ are such that $\overrightarrow{u} = \overrightarrow{v}$, but neither $u \leq_{\mathcal{R}} v$ nor $v \leq_{\mathcal{R}} u$. In contrast, we have the following proposition.

Proposition 7.4 ([AC09, Lemma 6.6] and [AC12, Lemma 5.3])**.** *Consider a pseudovariety of semigroups V containing $\mathcal{L}\mathsf{SI}$. Let $\mathcal{X}$ be a minimal subshift. For every $u, v \in J_{\mathsf{V}}(\mathcal{X})$, the equivalences*

$$u \,\mathcal{R}\, v \Leftrightarrow \overrightarrow{u} = \overrightarrow{v} \qquad \textit{and} \qquad u \,\mathcal{L}\, v \Leftrightarrow \overleftarrow{u} = \overleftarrow{v}$$

hold, and therefore so does the equivalence

$$u \,\mathcal{H}\, v \Leftrightarrow \overleftarrow{u}\,.\,\overrightarrow{u} = \overleftarrow{v}\,.\,\overrightarrow{v}.$$

Moreover, the $\mathcal{H}$-class of $u \in J_{\mathsf{V}}(\mathcal{X})$ is a maximal subgroup of $J_{\mathsf{V}}(\mathcal{X})$ if and only if $\overleftarrow{u}\,.\,\overrightarrow{u} \in \mathcal{X}$.

In other words, Proposition 7.4 states in particular that if $\mathcal{X}$ is minimal then the $\mathcal{R}$-classes and the $\mathcal{L}$-classes of $J_{\mathsf{V}}(\mathcal{X})$ are respectively parameterized by the positive rays of $\mathcal{X}$ and the negative rays of $\mathcal{X}$, provided V contains $\mathcal{L}\mathsf{SI}$.

Corollary 7.5. *Consider a pseudovariety of semigroups V containing $\mathcal{L}\mathsf{SI}$. Let $\mathcal{X}$ be a minimal subshift. If $\mathcal{X}$ is not a periodic subshift, then $J_{\mathsf{V}}(\mathcal{X})$ contains $2^{\aleph_0}$ many $\mathcal{R}$-classes and $2^{\aleph_0}$ many $\mathcal{L}$-classes. If $\mathcal{X}$ is a periodic subshift of least period n, then $\mathcal{X}$ contains precisely n $\mathcal{R}$-classes, n $\mathcal{L}$-classes, n^2 $\mathcal{H}$-classes and n idempotents, and these idempotents are the pseudowords of the form u^ω with u a conjugate word of v, where v is a primitive word of length n such that $\mathcal{X} = \mathcal{O}(v^\infty)$.*

Proof: It suffices to combine Proposition 7.4 with the following facts that we recall. First, it is known that a nonperiodic minimal subshift has $2^{\aleph_0}$ many negative rays, and $2^{\aleph_0}$ many positive rays (cf. [Lot02, Chapter 2]). Second, if we assume that $\mathcal{X}$ is a periodic subshift of period n, with $\mathcal{X} = \mathcal{O}(v^\infty)$ for some primitive word v of length n, then it is clear that $\mathcal{X}$ has n positive rays, namely those of the form $u^{-\infty}$ with u a conjugate of v. And whenever u and w are conjugates of v, one has $u^{-\infty}.w^{+\infty} \in \mathcal{X}$ if and only if $u = w$, since periodic shifts are minimal and hence Proposition 7.4 applies. Finally, if u is conjugate with the primitive word v, then u^ω is an idempotent in $J_{\mathsf{V}}(\mathcal{X})$, the one in the unique maximal subgroup of $J_{\mathsf{V}}(\mathcal{X})$ whose elements have negative ray $u^{-\infty}$ and positive ray $u^{+\infty}$. ■

For later reference, we state the next well known and easy to prove lemma.

Lemma 7.6. *Suppose that xy is an idempotent in a semigroup S, and consider the conjugate yx. Then $(yx)^2$ is an idempotent of S which is $\mathcal{J}$-equivalent to xy.*

Next is another well known fact (cf. [RS09, Propositions A.1.15 and 3.1.10]) that we shall use.

Lemma 7.7. *In a compact semigroup, every two $\mathcal{J}$-equivalent idempotents are conjugate.*

In what follows, J_e denotes the $\mathcal{J}$-class of e.

Proposition 7.8. *Let V be a pseudovariety of semigroups containing $\mathcal{L}\mathsf{Sl}$. Let e be an idempotent of $\widehat{F}_{\mathsf{V}}(A)$. The following conditions are equivalent:*

(1) $e = u^\omega$ for some $u \in A^+$;
(2) J_e contains a finite number of $\mathcal{H}$-classes;
(3) J_e contains a finite number of $\mathcal{R}$-classes;
(4) J_e contains a finite number of $\mathcal{L}$-classes;
(5) J_e contains a finite number of idempotents.

In the following proof of Proposition 7.8 we use profinite powers u^ν, with ν belonging to the profinite completion $\widehat{\mathbb{N}}$ of $\mathbb{N}$ (details may be found in [AV06, Section 2]). The power u^ω is an example of such powers, with $\omega = \lim n!$ in $\widehat{\mathbb{N}}$. What is most relevant for the proof is that, for every $u \in A^+$, the power u^ν belongs to the maximal subgroup of $\widehat{F}_{\mathsf{V}}(A)$ containing u^ω if and only if $\nu \in \widehat{\mathbb{N}} \setminus \mathbb{N}$.

Proof of Proposition 7.8: The implication (1) $\Rightarrow$ (2) is encapsulated in Corollary 7.5. The implications (2) $\Rightarrow$ (3) and (2) $\Rightarrow$ (4) follow immediately from each $\mathcal{R}$-class and each $\mathcal{L}$-class being a union of $\mathcal{H}$-classes.

(3) $\Rightarrow$ (1): Suppose there is no $u \in A^+$ such that $e = u^\omega$. Let f be a $\mathcal{J}$-maximal idempotent such that $e \leq_{\mathcal{J}} f$. Such an idempotent f exists, as mentioned in Remark 3.5. Let $x, y \in \widehat{F}_{\mathsf{V}}(A)$ be such that $e = xfy$. Since $fy \cdot xf$ is a conjugate of $xf \cdot fy = e$, the pseudoword $h = (fyxf)^2$ is an idempotent in J_e, by Lemma 7.6. Let f' be an idempotent in J_f. Then $f = zt$ and $f' = tz$ for some $z, t \in \widehat{F}_{\mathsf{V}}(A)$ (Lemma 7.7). Since $e = xfzf'ty$, we know that $h' = (f'tyxfz)^2$ is an idempotent in J_e (Lemma 7.6). By Proposition 7.4, if f and f' are not $\mathcal{R}$-equivalent then $\overrightarrow{f} \neq \overrightarrow{f}$. Since $h = fh$ and $h' = f'h'$, the inequality $\overrightarrow{f} \neq \overrightarrow{f}$ in turn implies the inequality $\overrightarrow{h} \neq \overrightarrow{h}'$. This shows that if f and f' are not $\mathcal{R}$-equivalent, then h and h' are not $\mathcal{R}$-equivalent, and so J_h has at least as many $\mathcal{R}$-classes as J_f has. By Corollary 7.5, if f is not of the form v^ω, then J_f has $2^{\aleph_0}$ $\mathcal{R}$-classes, and so $J_e = J_h$ has at least $2^{\aleph_0}$ $\mathcal{R}$-classes.

From hereon, we suppose that $f = v^\omega$ for some word $v \in A^+$, which we may as well assume to be primitive. Since $h = v^\omega h v^\omega$, one has

$$h \in \overline{v^2 \cdot v^+ \cdot A^* \cap A^* \cdot v^+ \cdot v^2}, \tag{7.1}$$

a fact which is the base of the reasoning that follows. Let $n = |v|$. For each $z \in A^n$, consider the language $K_z = v^+ \cdot z \cdot A^*$. Note that,

$$v^2 \cdot v^+ \cdot A^* \cap A^* \cdot v^+ \cdot v^2 \subseteq \left[\bigcup_{z \in A^n \setminus \{v\}} K_z \right] \cup (v^+ \cdot v^2 \cdot A^{(<n)} \cap A^* \cdot v^2). \tag{7.2}$$

Since v is primitive, we know by Lemma 7.3 that the inclusion

$$v^+ \cdot v^2 \cdot A^{(<n)} \cap A^* \cdot v^2 \subseteq v^+ \tag{7.3}$$

holds. Combining (7.1), (7.2) and (7.3), and noticing that the family $(K_z)_z$ is finite, we conclude that

$$h \in \left[\bigcup_{z \in A^n \setminus \{v\}} \overline{K_z} \right] \cup \overline{v^+}.$$

If $h \in \overline{v^+}$, then h is the unique idempotent v^ω in $\overline{v^+}$, thus $e \in J_{v^\omega}$. By Corollary 7.5, this contradicts our assumption that e is not of the form u^ω with $u \in A^+$.

Therefore, we may take $z \in A^n \setminus \{v\}$ such that $h \in \overline{K_z}$. Take a sequence $(h_k)_k = (v^{r_k} z w_k)_k$ of words of K_z converging to h, with $r_k \geq 1$. By taking subsequences, we may as well suppose that $(v^{r_k})_k$ and $(w_k)_k$ respectively converge to some pseudowords v^α and w of $\widehat{F}_{\mathsf{V}}(A)^I$, with $\alpha \in \widehat{\mathbb{N}}$, thanks to the compactness of $\widehat{F}_{\mathsf{V}}(A)$ and $\widehat{\mathbb{N}}$. Note that $h = v^\alpha \cdot z \cdot w$. If $\alpha \in \mathbb{N}$, then $v^\alpha \cdot z$ is the prefix of length $(\alpha+1)\cdot n$ of h. But since $h = v^\omega h$, the prefix of length $(\alpha+1)\cdot n$ of h is actually $v^{\alpha+1}$, and so we reached a contradiction with $v \neq z$. To avoid the contradiction, we must have $\alpha \in \widehat{\mathbb{N}} \setminus \mathbb{N}$. Therefore, for each positive integer k, we may consider the pseudoword $g_k = (v^k \cdot z \cdot w \cdot v^{\alpha-k})^2$, which is an idempotent $\mathcal{J}$-equivalent to h (Lemma 7.6). If $k < \ell$, then the prefix of length $(k+1)n$ of g_ℓ is v^{k+1}, while the prefix of the same length of g_k is $v^k z \neq v^{k+1}$. Hence, we conclude that g_k and g_ℓ are not $\mathcal{R}$-equivalent whenever $k \neq \ell$, thus showing that $J_e = J_h$ has at least $\aleph_0$ $\mathcal{R}$-classes.

(4) $\Rightarrow$ (1): This implication holds with a proof entirely symmetric to the proof of the implication (3) $\Rightarrow$ (1).

At this point, we have established the equivalences (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4). The implication (1) $\Rightarrow$ (5) is also encapsulated in Corollary 7.5. Finally, the implication (5) $\Rightarrow$ (1) follows from the well-know fact that, in a stable semigroup, every $\mathcal{R}$-class contained in a regular $\mathcal{J}$-class contains at least one idempotent. ■

Corollary 7.9. *Let $\mathcal{X}$ be a subshift of $A^{\mathbb{Z}}$. Suppose that V is a pseudovariety of semigroups containing $\mathcal{L}\mathsf{SI}$. Then $q_{\mathcal{X}}(n)$ is the number of objects of the category $\mathbb{K}(\mathcal{M}_{\mathsf{V}}(\mathcal{X}))$ whose isomorphism class is a set of cardinal n.*

Proof: Let P be the set of primitive words with length n belonging to $L(\mathcal{X})$. Then the mapping $u \mapsto u^\infty$ is a bijection between P and the set of periodic points of $\mathcal{X}$ with least period n. Moreover, the mapping $\psi\colon u^\infty \mapsto u^\omega$, with $u \in P$, is injective, and for every $u \in P$ the idempotent u^ω is an object of $\mathbb{K}(\mathcal{M}_{\mathsf{V}}(\mathcal{X}))$ whose isomorphism class is a set with n elements (cf. Proposition 7.4 and Corollary 7.5).

On the other hand, by Proposition 7.8, if e is an object of $\mathbb{K}(\mathcal{M}_{\mathsf{V}}(\mathcal{X}))$ whose isomorphism class has n elements, then $e = u^\omega$ for some primitive word $u \in L(\mathcal{X})$, which, by Corollary 7.5, has length n. Therefore, the image of the injective map ψ is the set of objects of the category $\mathbb{K}(\mathcal{M}_{\mathsf{V}}(\mathcal{X}))$ whose isomorphism class is a set of cardinal n. ■

The following perspective about zeta functions is immediate from Corollary 7.10.

Corollary 7.10. *Let $\mathcal{X}$ and $\mathcal{Y}$ be subshifts such that $\mathbb{K}(\mathcal{M}_{\mathsf{V}}(\mathcal{X}))$ and $\mathbb{K}(\mathcal{M}_{\mathsf{V}}(\mathcal{Y}))$ are isomorphic, where V is a pseudovariety of semigroups containing $\mathcal{L}\mathsf{Sl}$. Then we have $\zeta_{\mathcal{X}} = \zeta_{\mathcal{Y}}$.*

Appendix A. A labeled topological poset

Here a *topological poset* T is a partially ordered set T such that T is a topological space and the partial order $\leq$ of T is a closed subset of $T \times T$. A *labeled* topological poset is a topological poset T together with a labeling map λ, of domain T, assigning to each element t of T its *label*, denoted $\lambda(t)$.

Consider labeled topological posets T and R, respectively with partial orders $\leq_T$ and $\leq_R$, and labeling maps λ_T and λ_R. An isomorphism of labeled topological posets between T and R is a homeomorphism $\varphi\colon T \to R$ that preserves orders (that is, $t_1 \leq_T t_2 \Leftrightarrow \varphi(t_1) \leq_R \varphi(t_2)$ for every $t_1, t_2 \in T$) and labels (that is, $\lambda_R(\varphi(t)) = \lambda_T(t)$ for every $t \in T$). Naturally, T and R are said to be isomorphic labeled topological posets when such an isomorphism exists.

An element s of a semigroup S is said to have *local units* in S if $s = esf$ for some idempotents e, f of S.

Suppose that $s \in S$ has local units and let $t \in S$ be $\mathcal{J}$-equivalent to s. Since $s \in SsS$, there are $x, y \in S$ such that $t = xty$, whence $t = x^k t y^k$ for every $k \geq 1$. Since in a compact semigroup the closure of a monogenic semigroup contains an idempotent [CHK83, Theorem 3.5], we conclude that t also has local units. Therefore, in a compact semigroup, the set of local units is a union of $\mathcal{J}$-classes.

For each subset K of a semigroup S, we denote by $LU(K)$ the set of elements of K which have local units in S. Suppose that S is a compact semigroup and that K is a closed subset of S which is factorial. We associate to $LU(K)$ a labeled topological poset, denoted by $LU(K)^\dagger$, as follows:

(1) The underlying space of $LU(K)^\dagger$ is the quotient of the space $LU(K)$ by the restriction to $LU(K)$ of the relation $\mathcal{J}$. In other words, the underlying space is the space of $\mathcal{J}$-classes contained in $LU(K)$.
(2) One has $J_1 \leq J_2$ in $LU(K)^\dagger$ if and only if $u \leq_{\mathcal{J}} v$ for some (equivalently, for all) elements $u \in J_1$ and $v \in J_2$, whenever J_1 and J_2 are $\mathcal{J}$-classes contained in $LU(K)$.

(3) The label of each regular $\mathcal{J}$-class contained in K is the pair $(\varepsilon, \Gamma(J))$ such that $\varepsilon = 1$ if J is regular and $\varepsilon = 0$ if J is not regular, and $\Gamma(J)$ is the isomorphism class of the Schützenberger group of J, as a compact group.

Proposition A.1. *Let $\mathcal{X}, \mathcal{Y}$ be subshifts for which there is a continuous equivalence functor $F\colon \mathbb{K}(\mathcal{M}_{\mathsf{V}}(\mathcal{X})) \to \mathbb{K}(\mathcal{M}_{\mathsf{V}}(\mathcal{Y}))$. Then the labeled topological posets $LU(\mathcal{M}_{\mathsf{V}}(\mathcal{X}))^\dagger$ and $LU(\mathcal{M}_{\mathsf{V}}(\mathcal{Y}))^\dagger$ are isomorphic. If, moreover, F is such that the inclusion $F(\mathbb{K}(\mathrm{Sha}_{\mathsf{V}}(\mathcal{X}))) \subseteq \mathbb{K}(\mathrm{Sha}_{\mathsf{V}}(\mathcal{Y}))$ holds, and for some continuous pseudo-inverse G of F, the inclusion $G(\mathbb{K}(\mathrm{Sha}_{\mathsf{V}}(\mathcal{Y}))) \subseteq \mathbb{K}(\mathrm{Sha}_{\mathsf{V}}(\mathcal{X}))$ also holds, then the labeled topologically posets $LU(\mathrm{Sha}_{\mathsf{V}}(\mathcal{X}))^\dagger$ and $LU(\mathrm{Sha}_{\mathsf{V}}(\mathcal{Y}))^\dagger$ are isomorphic.*

The proof of Proposition A.1 will be later deduced as a consequence of some intermediate technical results.

Remark A.2. A reader familiar with the paper [CS15] will note the similarity of Proposition A.1 with the Theorem 6.3 from [CS15], which concerns labeled posets, without topology, with the Schützenberger groups in the labels not being viewed as topological groups. But the techniques of [CS15] are not suitable for the topological ingredients which we add here. Indeed, one crucial step of the approach made in [CS15] consists of the following: for each element s with local units in a semigroup S, choose idempotents e_s, f_s such that $s = e_s s f_s$. There is no reason to expect continuous choices $s \mapsto e_s$ and $s \mapsto f_s$.

Combining Proposition A.1 with Theorems 5.14 and 6.10, one immediately gets the following consequence, which is the reason for this appendix.

Corollary A.3. *Let V be a pseudovariety of semigroups containing $\mathcal{L}\mathsf{Sl}$ and such that $\mathsf{V} = \mathsf{V} * \mathsf{D}$. The labeled topological posets $LU(\mathcal{M}_{\mathsf{V}}(\mathcal{X}))^\dagger$ and $LU(\mathrm{Sha}_{\mathsf{V}}(\mathcal{X}))^\dagger$ are conjugacy invariants, and they are invariants of flow equivalence if V is monoidal.*

Remark A.4. Note that, when $\mathcal{X}$ is irreducible, the conjugacy/flow invariance of $\widetilde{G}_{\mathsf{V}}(\mathcal{X})$ and of $G_{\mathsf{V}}(\mathcal{X})$, stated in Corollaries 5.15 and 6.12, may also be derived from Corollary A.3, because $\widetilde{J}(\mathcal{X})$ is the minimum element of $LU(\mathcal{M}_{\mathsf{V}}(\mathcal{X}))^\dagger$ and $J(\mathcal{X})$ is the minimum element of $LU(\mathrm{Sha}_{\mathsf{V}}(\mathcal{X}))^\dagger$.

In what follows, when we refer to a "category", we mean a "small category". The Green relations on the set of morphisms of a category C may be defined by adapting in a direct and natural manner the usual definitions of the Green relations on a monoid. Alternatively, one may use a classical construction, the semigroup C_{cd}, which is called the *consolidation* of C. The elements of C_{cd} are the morphisms of C together with an extra element 0, which is as zero of C_{cd}. For any morphisms φ, ψ of C, the product $\varphi\psi$ in C_{cd} equals the composition $\varphi \circ \psi$ when $d(\varphi) = r(\psi)$, and equals 0 when $d(\varphi) \neq r(\psi)$. Then, for each $\mathcal{K} \in \{\mathcal{R}, \mathcal{L}, \mathcal{D}, \mathcal{H}, \mathcal{J}, \leq_{\mathcal{R}}, \leq_{\mathcal{L}}, \leq_{\mathcal{J}}\}$, one has $\varphi \mathrel{\mathcal{K}} \psi$ in C if and only if $\varphi \mathrel{\mathcal{K}} \psi$ in C_{cd}, for all morphisms φ, ψ of C.

Lemma A.5. *In a compact category C, each relation $\mathcal{R}, \mathcal{L}, \mathcal{D}, \mathcal{H}, \mathcal{J}, \leq_{\mathcal{R}}, \leq_{\mathcal{L}}$ $, \leq_{\mathcal{J}}$ is a closed set of* $\mathrm{Mor}(C) \times \mathrm{Mor}(C)$. *Moreover, the $\mathcal{K}$-classes of morphisms are closed sets, for each $\mathcal{K} \in \{\mathcal{R}, \mathcal{L}, \mathcal{D}, \mathcal{H}, \mathcal{J}\}$.*

Proof: Consider a net $(\varphi_i, \psi_i)_{i\in I}$ of morphisms of C, converging to (φ, ψ), such that $\varphi_i \leq_{\mathcal{J}} \psi_i$ for all $i \in I$. Then we have factorizations $\varphi_i = \alpha_i \circ \psi_i \circ \beta_i$ for some nets $(\alpha_i)_{i\in I}$ and $(\beta_i)_{i\in I}$ of morphisms of C. As C is compact, we may take a cluster point (α, β) of $(\alpha_i, \beta_i)_{i\in I}$. By continuity of the composition, we get $\varphi = \alpha \circ \psi \circ \beta$, thus $\varphi \leq_{\mathcal{J}} \psi$. This proves that $\leq_{\mathcal{J}}$ is closed. The proofs for the other relations are similar. Since each class of a closed equivalence relation in a compact space is a closed set (cf. [RS09, Exercise 3.17]), we are done. ∎

As another expression of the link between C and C_{cd}, a morphism of C is said to be regular when it is a regular element of C_{cd}, and a $\mathcal{J}$-class of C is regular when it is a regular $\mathcal{J}$-class of C_{cd}.

Let H be an $\mathcal{H}$-class of morphisms of the category C. Note that $H \subseteq C(c, d)$ for some objects c, d. The *Schützenberger group of H in C*, denoted $\Gamma(H)$, is the quotient of the monoid

$$T(H) = \{\alpha \in C(c, c) \mid H \circ \alpha \subseteq H\}$$

by the monoid congruence $\approx_H$ on $T(H)$ given, whenever $\alpha, \beta \in T(H)$, by

$$\alpha \approx_H \beta \Leftrightarrow [\forall \varphi \in H : \varphi \circ \alpha = \varphi \circ \alpha],$$

or, in what is easily seen to be an equivalent formulation,

$$\alpha \approx_H \beta \Leftrightarrow [\exists \varphi \in H : \varphi \circ \alpha = \varphi \circ \alpha].$$

The useful equality $T(H) = \{\alpha \in C(c, c) \mid H \circ \alpha \cap H \neq \emptyset\}$ is also easy to check. The monoid quotient $T(H)/{\approx_H}$ is indeed a group; clearly, it coincides with the classical Schützenberger group in C_{cd} of H, if we view H as an $\mathcal{H}$-class of C_{cd}. Moreover, if C is a compact category, then $T(H)$ is a closed submonoid of $C(c,c)$ and $\approx_H$ is a closed relation, by the same arguments used in the proof of Lemma A.5, and then the quotient $\Gamma(H) = T(H)/{\approx_H}$ becomes a compact group (cf. [CHK83, Theorem 1.54]). It is this compact group that will be for us the *Schützenberger group of H in C*, when C is a compact category.

Remark A.6. If H contains some idempotent (which implies $c = d$), then $\Gamma(H)$ is isomorphic to H, via the mapping $\varphi \in H \mapsto [\varphi]_{\approx_H} \in \Gamma(H)$, and this mapping is continuous if C is a compact category.

Lemma A.7. *If C is a profinite category, then $\Gamma(H)$ is a profinite group.*

Proof: Let $C = \varprojlim_{i\in I} C_i$ be an inverse limit of finite categories. Let φ, α, β be morphisms of C with $\varphi \circ \alpha \mathrel{\mathcal{H}} \varphi \circ \beta \mathrel{\mathcal{H}} \varphi$ and $\varphi \circ \alpha \neq \varphi \circ \beta$. For $\psi \in C$, denote by ψ_i its projection on C_i, where $i \in I$. Let H be the $\mathcal{H}$-class of φ and H_i be the $\mathcal{H}$-class of φ_i. Take $i_0 \in I$ such that $\varphi_{i_0} \circ \alpha_{i_0} \neq \varphi_{i_0} \circ \beta_{i_0}$. We have a well defined continuous homomorphism $\Gamma(H) \to \Gamma(H_{i_0})$ given by the assignment $[\gamma]_{\approx_H} \mapsto [\gamma_i]_{\approx_{H_i}}$, and we also know that $[\alpha_{i_0}]_{\approx_{H_{i_0}}} \neq [\beta_{i_0}]_{\approx_{H_{i_0}}}$. Therefore $\Gamma(H)$ is profinite: in fact $\Gamma(H)$ embeds as a closed subgroup of the natural inverse limit $\varprojlim_{i\in I} \Gamma(H_i)$. ■

We may use for compact semigroups some of the notation employed for compact categories. For example, $\Gamma(H)$ is the Schützenberger group (as a compact group) of an $\mathcal{H}$-class H of a compact semigroup S. In fact, when S is a compact semigroup, we may view S^I as compact category with a unique object and the elements of S^I as the morphisms.

For any compact category C, the compact groups $\Gamma(H)$ and $\Gamma(K)$ are isomorphic when H and K are $\mathcal{H}$-classes of morphisms of C contained in the same $\mathcal{J}$-class (see the proof of [CHK83, Theorem 3.61]). Hence, when in a compact category C, we may associate to each $\mathcal{J}$-class J its Schützenberger group $\Gamma(J)$, which is (the isomorphism class of) the Schützenberger group of any of the $\mathcal{H}$-classes contained in J. We next consider the labeled topological poset $C^\dagger$ defined by:

(1) the underlying space is the quotient space $\mathrm{Obj}(C)/\mathcal{J}$;
(2) one has $J_1 \leq J_2$ in $C^\dagger$ if and only if $\varphi \leq_{\mathcal{J}} \psi$ for some (equivalently, for all) morphisms $\varphi \in J_1$ and $\psi \in J_2$;
(3) the label of each element J of $C^\dagger$ is the pair $(\varepsilon, \Gamma(J))$ such that $\varepsilon = 1$ if J is regular and $\varepsilon = 0$ if J is not regular, where $\Gamma(J)$ is taken as an isomorphism class of compact groups.

Let $F\colon C \to D$ be a continuous functor between compact categories. We define a map $F^\dagger\colon C^\dagger \mapsto D^\dagger$ by letting $F^\dagger([\varphi]_{\mathcal{J}}) = [F(\varphi)]_{\mathcal{J}}$. This map is well defined, indeed it is immediate that $\varphi \leq_{\mathcal{J}} \psi$ in C implies $F(\varphi) \leq_{\mathcal{J}} F(\psi)$ in D. Note also that $F^\dagger$ is continuous, because it is the map $\mathrm{Obj}(C)/\mathcal{J} \to \mathrm{Obj}(D)/\mathcal{J}$ naturally induced by the continuous map $F\colon \mathrm{Obj}(C) \to \mathrm{Obj}(D)$, and we are dealing with compact quotients of compact spaces (cf. [Wil70, Theorems 9.2 and 9.4]).

Proposition A.8. *Let C, D, E be compact categories. The following hold:*

(1) if the continuous functors $F, G\colon C \to D$ are isomorphic, then $F^\dagger = G^\dagger$;
(2) if F is the identity functor $C \to C$, then $F^\dagger$ is the identity $C^\dagger \to C^\dagger$;
(3) for any functors $F\colon C \to D$ and $G\colon D \to E$, we have $G^\dagger \circ F^\dagger = (G \circ F)^\dagger$;
(4) if $F : C \to D$ is a continuous equivalence, then $F^\dagger$ preserves labels.

Proof: If $\eta\colon F \Rightarrow G$ is a natural isomorphism, then, for every $\varphi \in C(c,d)$, one has $G(\varphi) = \eta_d \circ F(\varphi) \circ \eta_c^{-1}$ and $F(\varphi) = \eta_d^{-1} \circ G(\varphi) \circ \eta_c$, thus $F(\varphi)\ \mathcal{J}\ G(\varphi)$. This establishes the first item. Items 2 and 3 are immediate.

Concerning the last item, it is immediate that if φ is regular then $F(\varphi)$ is regular, where $F\colon C \to D$ is a functor. If F is an equivalence with pseudo-inverse G, then $G(F(\varphi))\ \mathcal{J}\ \varphi$, and so if $F(\varphi)$ is regular then so is φ. Finally, for every morphism φ of C, if H_φ and $H_{F(\varphi)}$ are respectively the $\mathcal{H}$-classes of φ and $F(\varphi)$, then one clearly has $F(T(H_\varphi)) \subseteq T(H_{F(\varphi)})$, with equality if F is an equivalence. This induces a well defined map $\Gamma(H_\varphi) \to \Gamma(H_{F(\varphi)})$ assigning each class $[\alpha]_{\approx_{H_\varphi}}$ to $[F(\alpha)]_{\approx_{H_{F(\varphi)}}}$, such map being a bijection if F is an equivalence. This map is continuous, as we are dealing with compact quotients of compact spaces. Therefore, the Schützenberger groups of φ and $F(\varphi)$ are indeed isomorphic compact groups. ■

Corollary A.9. *If C and D are equivalent compact categories, then $C^\dagger$ and $D^\dagger$ are isomorphic labeled topological posets.*

Next we show the last piece needed for the proof of Proposition A.1.

Proposition A.10. *The mapping $P_{\mathcal{X}}\colon \mathbb{K}(\mathcal{M}_{\mathsf{V}}(\mathcal{X}))^\dagger \to LU(\mathcal{M}_{\mathsf{V}}(\mathcal{X}))^\dagger$ defined by $P_{\mathcal{X}}([(e,u,f)]_{\mathcal{J}}) = [u]_{\mathcal{J}}$ is an isomorphism of labeled topological posets, for every subshift $\mathcal{X}$ of $A^{\mathbb{Z}}$.*

Proof: It is trivial that $P_{\mathcal{X}}$ is surjective. Let us check that it satisfies the remaining conditions for being an isomorphism of posets. Take morphisms

(e,u,f) and (e',v,f') of $\mathcal{M}_{\mathsf{V}}(\mathcal{X})$. Suppose $u=xvy$ for some $x,y\in\widehat{F}_{\mathsf{V}}(A)$. As $u=euf$ and $v=e'vf'$, we may assume that $x=exe'$ and $y=f'yf$, yielding $(e,u,f)=(e,x,e')(e',v,f')(f',y,f)$ and $(e,u,f)\leq_{\mathcal{J}}(e',v,f')$. Conversely, if $(e,u,f)=(e,x,e')(e',v,f')(f',y,f)$ then $u=xvy$. This shows that $P_{\mathcal{X}}$ is a well defined isomorphism of posets.

Because the map $(e,u,f)\mapsto u$ is continuous, and since we are dealing with compact spaces and their compact quotients, the map $P_{\mathcal{X}}$ is continuous.

Fix a morphism (e,u,f) of $\mathbb{K}(\mathcal{M}_{\mathsf{V}}(\mathcal{X}))$. Suppose that u is regular. Then $u=uxu$ for some $x\in\widehat{F}_{\mathsf{V}}(A)$. Since $u=uf=eu$, we may suppose that $x=fxe$, thus $(e,u,f)=(e,u,f)(f,x,e)(e,u,f)$ and so (e,u,f) is regular. Conversely, if (e,u,f) is regular then it is immediate that u is regular.

It remains to show that the Schützenberger groups of (e,u,f) and u are isomorphic compact groups. Let H be the $\mathcal{H}$-class of (e,u,f) in $\mathbb{K}(\mathcal{M}_{\mathsf{V}}(\mathcal{X}))$, and let K be the $\mathcal{H}$-class of u in $\widehat{F}_{\mathsf{V}}(A)$. Suppose that (f,x,f) and (f,y,f) are elements of $T(H)$ such that $(f,x,f)\approx_H(f,y,f)$. This means that we have $(e,u,f)(f,x,f)=(e,u,f)(f,y,f)\in H$. Hence $ux=uy=zu$ for some z, and, according to what we already saw in the first paragraph of the proof, we also know that ux and u are $\mathcal{J}$-equivalent. Since $\widehat{F}_{\mathsf{V}}(A)$ is a stable semigroup, we conclude that $u\mathrel{\mathcal{H}}ux$, and, similarly, $u\mathrel{\mathcal{H}}uy$, thus $x\approx_K y$. Therefore, we have a well defined map $\varphi\colon\Gamma(H)\to\Gamma(K)$ given by $\varphi([(f,x,f)]_{\approx_H})=[x]_{\approx_K}$. This map is continuous, again because we are dealing with compact spaces and their compact quotients. Moreover, φ is clearly a homomorphism.

Suppose that $x\in T(K)$. Then $u=uf$ yields $f\in T(K)$ and $fxf\in T(K)$, thus $[x]_{\approx_K}=[fxf]_{\approx_K}$ as $f=f^2$ and $\Gamma(K)$ is a group. Consider the equality $(e,u,f)(f,fxf,f)=(e,ux,f)$, entailing $(e,u,f)\leq_{\mathcal{R}}(e,ux,f)$ in $\mathbb{K}(\mathcal{M}_{\mathsf{V}}(\mathcal{X}))$. Since x is an arbitrary element of $T(K)$, we know that ux may be any element of K, and so, by the symmetry of the $\mathcal{H}$-relation, we conclude that $(e,u,f)\mathrel{\mathcal{H}}(e,ux,f)$. Therefore, $(e,u,f)(f,fxf,f)=(e,ux,f)$ implies that $(f,fxf,f)\in T(H)$. We deduce that $[x]_{\approx_K}=\varphi([(f,fxf,f)]_{\approx_H})$, and so φ is onto.

If (f,x,f) is an element of $T(H)$ such that $[x]_{\approx_K}$ is the identity of $\Gamma(H)$, then we have $ux=u$, implying $(e,u,f)(f,x,f)=(e,u,f)$. The latter equality entails that $[(f,x,f)]_{\approx_H}$ is the identity of $\Gamma(H)$. This shows that φ is a continuous isomorphism of compact groups, concluding the proof. ∎

Proof of Proposition A.1: According to Propositions A.10 and A.8, the mapping $P_{\mathcal{Y}} \circ F^{\dagger} \circ P_{\mathcal{X}}^{-1}$ is an isomorphism between the labeled topological posets $LU(\mathcal{M}_{\mathsf{V}}(\mathcal{X}))^{\dagger}$ and $LU(\mathcal{M}_{\mathsf{V}}(\mathcal{Y}))^{\dagger}$, which restricts to an isomorphism between $LU(\mathrm{Sha}_{\mathsf{V}}(\mathcal{X}))^{\dagger}$ and $LU(\mathrm{Sha}_{\mathsf{V}}(\mathcal{Y}))^{\dagger}$. ■

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