Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 20–21

INFINITELY MANY MERIDIONAL ESSENTIAL SURFACES OF BOUNDED GENUS IN HYPERBOLIC KNOT EXTERIORS

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KEYWORDS: Meridional essential surface; arbitrarily many boundary components; bounded genus; hyperbolic knot.
AMS SUBJECT CLASSIFICATION (2010): 57M25, 57N10.

ABSTRACT: We show the existence of an infinite collection of hyperbolic knots where each of which has in its exterior meridional essential planar surfaces of arbitrarily large number of boundary components, or, equivalently, that each of these knots has essential tangle decompositions of arbitrarily large number of strings. Moreover, each of these knots has in its exterior meridional essential surfaces of any positive genus and (even) number of boundary components. That is, the compact surfaces that have a meridional essential embedding into a hyperbolic knot exterior have meridional essential embeddings into each of these hyperbolic knots exteriors.

1. Introduction

In geometric topology it is a common approach to use codimension one objects, and the resulting decompositions, to study manifold topology. On the study of 3-manifolds, and knot exteriors in particular, since the work of Haken and Waldhausen, it is common to study their topology through decompositions along embedded surfaces. A very important class of surfaces used in these decompositions are embedded essential surfaces, which has motivated research on the properties and existence of these embeddings. A particularly interesting phenomena is the existence of knots with the property that their exteriors have properly embedded essential surfaces of arbitrarily high Euler characteristics. The first examples of knots with this property were given by Lyon [14], where he proves the existence of fibered knot exteriors each of which with closed essential surfaces of arbitrarily high genus. Later, other contributions by Oertel [23], Gustafson [7], Ozawa and Tsutsumi [26], Eudave-Muñoz and Neumann-Soto [2], Li [12] and more recently

Received July 2, 2020.
This work was partially supported by the Centre for Mathematics of the University of Coimbra - UIDB/00324/2020, funded by the Portuguese Government through FCT/MCTES. This work was also partially suported by the UTAustin|Portugal program CoLab.

Lopez-Garcia [13], for instance, also gave examples of knots having closed essential surfaces of arbitrarily high genus in their exteriors. These results contrast with the number of non-isotopic closed essential surfaces that are acylindrical, that is with no essential annulus in its exterior, in a 3-manifold being finite, as proved by Hass in [8].

In the mentioned examples the Euler characteristics of the collection is unbounded because of the increasing genus. Along these lines, in [19] we have shown that such a collection of surfaces can be of arbitrarily high genus and two boundary components on a prime knot exterior. We have also proved that a collection of compact surfaces embedded in a particular knot exterior can have arbitrarily large Euler characteristics due to the number of boundary components and not the genus. In fact, in [20] it was shown instead that a particular knot exterior can have essential tangle decompositions of any number of strings, that is, they have meridional planar essential surfaces of any even number of boundary components. This is somewhat surprising as in Proposition 2.1 of [17], Mizuma and Tsutsumi proved that for a given knot the number of strings in essential tangle decompositions without parallel strings is bounded. Moreover, in [21] we use the results from [19] and ideas from [20] to actually show the existence of knot exteriors where each of which has meridional essential surfaces of any (even) number of boundary components and genus. Hence, all compact surfaces that have a meridional essential embedding into a knot exterior have one into each knot exterior of [21]. However, these collections of knot exteriors with meridional essential surfaces of arbitrarily high number of boundary components are not of hyperbolic knot exteriors.

For a hyperbolic knot exterior in particular, it is a result attributed to Haken that it cannot have infinitely many surfaces of uniformly bounded Euler characteristics. (See also [10] by Jaco and Oertel, or [8] by Hass.) Without a bound on the Euler characteristics, among the collections of knots with meridional essential surfaces in their exterior with arbitrarily large Euler characteristics, a result in [21] includes hyperbolic knots exteriors in some 3-manifold where the unbounded Euler characteristics of surfaces in the collection comes independently from both the number of boundary components and the genus, but the genus is always higher than zero and the base 3-manifold not necessarily S^3. In this paper we consider the question whether

a hyperbolic knot in the 3-sphere can also have a set of meridional planar essential surfaces of unbounded Euler characteristics properly embedded in its exterior, that is, with arbitrarily high number of boundary components. In other words, we study if a single hyperbolic knot can have essential tangle decompositions of any number (≥ 2) of strings as in [19]. We also consider the more general question if the arbitrarily large Euler characteristics in the collection of meridional essential surfaces in a single hyperbolic knot exterior can be independently due from both the genus and the number of boundary components. We prove the existence of a collection of hyperbolic knot exteriors where each of which has in its exterior all possible meridional essential surfaces. That is, the compact surfaces that have a meridional essential embedding into a hyperbolic knot exterior have meridional essential embeddings into each of these hyperbolic knots exteriors. The main results of this paper are summarized in the following theorem.

Theorem 1. *There are infinitely many hyperbolic knots in the 3-sphere each of which having in its exterior*

(a) *a meridional essential planar surface with* $2n$ *boundary components for any integer* $n \geq 2$*;*
(b) *a meridional essential surface of any positive genus with* $2n$ *boundary components for any positive integer* n*.*

The knots of Theorem 1 also have in their exteriors closed essential surfaces of any genus greater than or equal to two. This follows from Theorem 2.0.3 of [1], or the handle addition lemma [4], applied to the surfaces with two boundary components in Theorem 1. Note that the knot exterior of 8_{16} [11], which is an alternating and hyperbolic knot, also has meridional essential surfaces with two boundary components and any positive genus, together with the corresponding swallow-follow closed surfaces. This derives from 8_{16}'s 2-string essential tangle decomposition and [13], but it does not have meridional essential planar surfaces of arbitrarily high number of boundary components as its essential tangle decomposition is unique [25].

The statement of Theorem 1 also complements and contrasts with the result of Oertel [24], extending work of Jaco and Sedgwick, saying in particular that a knot exterior without essential genus one surface (with or without boundary) has at most finitely many isotopy classes of compact essential surfaces of uniformly bounded genus. In Theorem 1 we show that each those

hyperbolic knot exteriors, that is without an essential torus, has infinitely many essential surfaces of any genus, hence infinitely many isotopy classes of surfaces of bounded genus. Therefore, from the above mentioned Oertel's result, these knot exteriors have necessarily some essential genus one surface (with boundary for being hyperbolic). Such essential genus one surface is already observed for the knots in Theorem 1(b). The result of this paper also complement the results of Menasco in [16] stating that non-split prime alternating knots (which are hyperbolic excluding the alternating torus knots, as proved in the cited paper) have at most finitely many meridional essential meridionally-incompressible surfaces of uniformly bounded genus. It is relevant to note that all surfaces but the 4-punctured sphere as in the statement of Theorem 1 are meridionally-compressible. In the opposite direction of Theorem 1 we know that: small knot exteriors [1] have no meridional essential surfaces; tunnel number one knot exteriors [6] and free genus one knot exteriors [15] have no meridional essential planar surfaces. There are also knots with an unique essential tangle decomposition [25].

The paper is organized as follows: In Section 2 we show the existence of atoroidal 2-string essential tangles with meridional essential surfaces of any positive genus and two boundary components. These tangles are a base to construct in Section 3 a collection of hyperbolic knots which we use to prove Theorem 1 in Section 4 recurring to branched surface theory. Throughout this paper all manifolds are orientable, all submanifolds are assumed to be in general position and we work in the smooth category. We use $N(X)$ to denote a regular neighborhood of X, and $|X|$ to denote the number of components of X.

2. Twice punctured meridional essential surfaces in 2-string tangle exteriors

In this section we will show the existence of a 2-string essential tangle with meridional essential surfaces of any genus and two boundary components. Let J be an hyperbolic knot with a 2-string essential tangle decomposition with tangles $(Q_+; u_+ \cup v_+)$ and $(Q_-; u_- \cup v_-)$, defined by the sphere Z. From Theorem 1.2 of [13], and its proof, there are meridional essential surfaces of any positive genus and two boundary components in its exterior. The surfaces are as follows: there are two twice-punctured swallow-follow essential tori with meridional boundary, obtained from Z, one in each tangle of the

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Acknowledgement

The author would like to thank John Luecke for discussions on this paper, and Tao Li for some clarifications on branched surface theory.

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decomposition, denoted F^+ and F^-. Suppose $F^\pm$ is in $Q_\pm$ (where $\pm$ denotes $+$ or $-$, respectively). The boundary components of $F^\pm$ are on the same string, say $u_\pm$. Each surface $F^\pm$ has accidental peripheral with meridional slope, that is, it has an essential non-boundary parallel simple closed curve co-bounding an annulus with a meridian of $\partial N(J)$ in $E(J) - F^\pm$, which is referred as *meridional accidental annulus*. A meridional accidental annulus A of F^+ associated with the string v_+ is *not centered*. That is, by numbering the boundary components of $F^+ \cup F^-$ from $\partial A \cap N(J)$ in one direction by $\{1, 2\}$ and the other direction as $\{-1, -2\}$, the boundary components with symmetric numbering, *e. g.* 1 and -1, don't belong to the same surface $F^\pm$. (See [13].)

Continuing as in [13], we take n parallel copies of F^+, F_i^+ for $i = 1, \dots, n$, and one copy of F^- and glue them through nested tubing operations, connecting their boundary components following the numeration from $\partial A \cap N(J)$ in both directions of $\partial N(J)$: $\{1, \dots, k\}$ and $\{-1, \dots, -k\}$, respectively. So, the boundary component i is connected to the boundary component $-i$, for $i = 1, \dots, k-1$ in this order. The resulting surface F has genus $n + 1$ and two meridional boundary components, the boundary components k and $-k$, and is essential in the exterior of J. (See [13] for more details.)

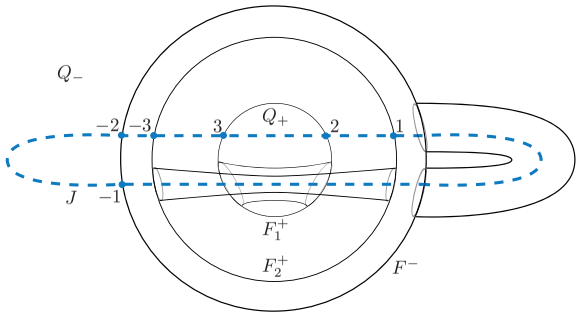

FIGURE 1: A schematic representation of the surfaces F_1^+ and F_2^+ in Q_+, the surface F^- in Q_-, and the numbering of the boundary components of $F_1^+ \cup F_2^+ \cup F^-$ from a meridional accidental annulus A of F^+ in Q_+.

If we order the copies of F^+ on this construction from the inside of Q_+ to the outside, we have that k belongs to F_1^+ and $-k$ to F_2^+. (See Figure 1.) Let us consider the surface S bounding a ball B' intersecting u_+ at a trivial arc

between the points associated with k and $-k$, and the other ball bounded by S let us denote it by B. Denote the string $J - B' \cap u_+$ in B by s. Let Z_i, for $i = 1, 2$, be the copy of Z corresponding to F_i^+. Let us consider the space G between Z_1 and Z_2, and by G_B the intersection $B \cap G$. Denote the arc component of $u_+ \cap G_B$ connecting S and Z_1 by a_+ and the arc component of $u_+ \cap G_B$ disjoint from S by a'_+. Let t be a properly embedded arc in B that lies in the interior of G_B. We choose t so that it is trivial in the exterior of a_+ in G_B and co-bounds a disk in G_B with an arc in S that intersects a'_+ once, being disjoint from s and F_i^+, for $i = 1, 2$, otherwise. Consider the tangle $(B; s \cup t)$. (See Figure 2.)

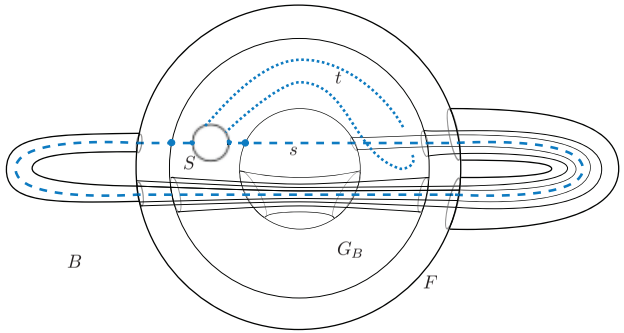

FIGURE 2: A schematic illustration of the tangle $(B; s \cup t)$ and the surface F in the exterior of of $s \cup t$ in B, and the space G_B.

Lemma 1. *The tangle $(B; s \cup t)$ is essential, atoroidal, and each surface F is essential in the exterior of $s \cup t$ in B.*

Proof: First we observe that each surface F is essential in the exterior of $s \cup t$ in B, denoted E_B; otherwise, as the arc $B' \cap J$ is a trivial single arc in B', it would be inessential in the exterior of J, which is a contradiction.
Now we show that the tangle $(B; s \cup t)$ is essential. Suppose, by contradiction, that it is not, and D is a compressing disk for $S - s \cup t$ in $B - s \cup t$. As Z_1 and Z_2 are essential in E_B we have that D is disjoint from Z_1 and Z_2: If D intersects Z_i it intersects it in simple closed curves each of which bounds a disk in D. By choosing an innermost one in D, this curve also has to bound a disk in Z_i, which we can use to cut and paste D and eliminate this intersection with Z_i. Proceeding in this way we can assume that D is

Hence, $\partial_h N(R)$ is incompressible and boundary incompressible in $E(K)$, and there are no monogons of R or disks of contact in R. Therefore, R is an incompressible branched surface. ∎

Let us define F_2 as the surface carried by R with invariant measure $W_F = 1$, $W_A = 1$, $W_{A'} = 0$, $W_{A_D} = 2$ and $W_{P'_D} = 2$. Observe that F_2 is a meridional surface in the exterior of K with $2(n+2)$ boundary components: $2n-2$ from F, 2 from A_D and 4 from P'_D. Denote by F_j, $j \geq 3$, the surface carried with positive weights by R with invariant measure $W_F = 1$, $W_A = j-1$, $W_{A'} = j-2$, $W_{A_D} = 2$ and $W_{P'_D} = j$. Note that F_j, $j \geq 3$, is a meridional surface in the exterior of K with $2(n+j)$ boundary components: $2n-2$ from F, 2 from A_D and $2j$ from P'_D.

The surfaces F_0 and F_1 have genus g and the number of boundary components is $2n$ and $2(n+1)$, respectively. The Euler characteristic of F_j, $j \geq 2$, is $W_F \cdot (2-2g-2n) + W_A \cdot 0 + W_{A'} \cdot 0 + W_{A_D} \cdot 0 + W_{P'_D} \cdot (-2j)$. That is, $2-2g-2(n+j)$. Hence, as F_j has $2(n+j)$ boundary components, its genus is g. We finish the proof of Lemma 7 with the following proposition.

Proposition 3. *The surfaces F_j, $j \geq 2$, are essential in $E(K)$.*

Proof: From Proposition 2 we have that R is an incompressible branched surface without Reeb components. We also have that the surfaces F_j, $j \geq 3$, are carried with positive weights by R. Hence, we are under the conditions of Theorem 2. It then follows that all surfaces carried by R are essential. Therefore, as F_j, $j \geq 2$, is carried by R, it is essential in $E(K)$. ∎

∎

Proof of Theorem 1. Let us consider a knot K in $\mathcal{K}$. Applying Lemma 7 with F the 4-punctured sphere $S - N(s \cup t)$ we have the statement of Theorem 1(a).
Let F be a surface of any positive genus and two boundary components as in Lemma 5. As F is separating in B and has exactly two boundary components, the boundary components have to be on the regular neighborhood of the same string, which we assume to be s. Hence, applying Lemma 7 with F, we have the statement of Theorem 1(b). From Lemma 6, we have that $\mathcal{K}$ is an infinite collection of hyperbolic knots. ∎

neighborhoods of sections A and A', which are connected only through the branched locus component corresponding to b_2. Hence, $A \cup A'$ cannot carry a closed surface. If R would carry a properly embedded annulus O, this annulus had to be transverse to the regular neighborhood of sections of R with boundary components, that is F, A_D or P'_D. In case O is transverse to the regular neighborhood of A_D or F, as it is properly embedded, it has to be transverse to P'_D as well, but in this case it wouldn't be an annulus, which is a contradiction. If O is transverse to the regular neighborhood of P'_D, but not to the one of A_D or F, then it is also transverse to the one of A' and A, each with weight 1 as O has only two boundary components. But in this case O would be a twice punctured torus, which is a contradiction. Therefore, R doesn't carry an annulus and a torus, and, consequently, it has no Reeb component.

Now we prove that R is incompressible in $E(K)$. First observe that there are no (half) disks of contact as no circle on the branched locus of R bounds a disk in $\partial_h N(R)$ and there are no properly embedded arcs on the branched locus of R. The space $N(R)$ decomposes $E(K)$ into three components: a component cut from $E(K)$ by F and A, denoted X; a component cut from $E(K)$ by A_D, P'_D, A and A', denoted X_1; a component cut from $E(K)$ by F, A' and P'_D, denoted X_2.

As b_1 corresponds to $\partial_v N(R)$ in ∂X, we have that $\partial X \cap \partial_h N(R)$ corresponds to the component cut by F in E_B. Therefore, as F is essential in E_B, $\partial X \cap \partial_h N(R)$ is incompressible and boundary incompressible in X; and there are no monogons in X, otherwise, as s is parallel to A, s would be parallel to F, contradicting F being essential in E_B.

Similarly, b_2 corresponds to $\partial_v N(R)$ in ∂X_2, and we have that $\partial X_2 \cap \partial_h N(R)$ corresponds to F in the exterior of X in $E(K)$. As F is essential in $E(K)$, following as in the argument for X, we have that $\partial X_2 \cap \partial_h N(R)$ is incompressible and boundary incompressible in X_2, and there are no monogons in X_2.

In the case of X_1, $\partial A' \cap A_D$ corresponds to $\partial_v N(R)$ in ∂X_1, and X_1 corresponds to the exterior of $s'_1 \cup t'_1$ in B'_1, with $\partial B'_1$ corresponding to two disks singularly glued along b_1 corresponding to $\partial_v N(R)$ in ∂X_1. Hence, there are no monogons in X_1, because any disk intersecting b_1 in ∂X_1 would have to intersect it at least twice. Also, as $\mathcal{T}'_1$ is essential, $\partial X_1 \cap \partial_h N(R)$ is incompressible and boundary incompressible in X_1.

disjoint from Z_i, $i = 1, 2$. Hence, D is in G_B. Let E in G_B be a disk co-bounded by t and an arc in S, and intersecting a'_+ once at a single point p. If E is disjoint from D, as D is disjoint from $Z_1 \cup Z_2$, it cuts a ball from B containing E which is in G_B. But then D separates E from a'_+, which is a contradiction as a'_+ intersects E. Otherwise, assume the intersection of E with D to be non-empty and with minimal number of components $|E \cap D|$. In case there are simple closed curves in $E \cap D$ consider an innermost one in E, say c. Let E_c be the disk cut by c in E. If E_c is disjoint from p, by cutting and pasting D along this disk we reduce $|E \cap D|$, contradicting its minimality. If E_c contains p, together with the disk bounded by c in D we have a sphere in S^3 intersecting J once, which contradicts, for instance, the 3-dimensional Schoenflies theorem. Then, $E \cap D$ is a collection of arcs. Let b be an innermost arc of $E \cap D$ in E and E' the corresponding innermost disk. Following as for the simple closed curve case we get a contradiction. Hence, the tangle $(B; s \cup t)$ is essential.

At last we prove that $(B; s \cup t)$ is atoroidal. Let T be a torus in $B - s \cup t$. As J is hyperbolic the torus T is inessential in $B - s$. As the 2-string tangles $(Q_\pm; u_\pm \cup v_\pm)$ are atoroidal, for J being hyperbolic, the torus T is in G_B, disjoint from Z_1 and Z_2. As t is trivial in the exterior of a_+ in G_B there is a disk O co-bounded by t and an arc in $\partial(G_B - N(a_+))$. If T intersects O by considering an innermost curve c of this intersection in O we obtain a compressing disk for T in $B - s \cup t$ or we eliminate that intersection in case c bounds a disk in T by cutting and pasting. If D_T is a compressing disk of T in $B - s$. Suppose D_T intersects O and assume that $|D_T \cap O|$ is minimal. If there is a simple closed curve in $D_T \cap O$, consider an innermost one in O with innermost disk O'. By cutting and pasting D_T along O' we eliminate this intersection, reducing $|D_T \cap O|$ and contradicting its minimality. If $D_T \cap O$ has no simple closed curves, let b be an outermost arc of the intersection in O with corresponding outermost disk O'. The ends of b are in t, as D_T is disjoint from ∂G_B, and we can isotope t along O' through D_T eliminating this intersection and contradicting the minimality of $|D_T \cap O|$. Then D_T is disjoint from O, and, therefore, T is inessential in $B - s \cup t$. So, $(B; s \cup t)$ is atoroidal. ∎

3. A collection of hyperbolic knots

Let us consider a knot K decomposed into two 2-string tangles by the sphere S: a 2-string essential atoroidal tangle $(B; s \cup t)$ with meridional

essential surfaces of any positive genus and exactly two boundary components in $\partial N(s)$, which from Lemma 1 we know exist; and a 2-string tangle $(B'; s' \cup t')$. Denote these tangles by $\mathcal{T}$ and $\mathcal{T}'$ respectively, as illustrated schematically in Figure 3(a). We assume $\mathcal{T}'$ has a decomposition into a 2-string essential atoroidal tangle $(B'_1; s'_1 \cup t'_1)$ and a 3-string essential atoroidal tangle $(B'_2; s'_2 \cup t'_2)$, denoted $\mathcal{T}'_1$ and $\mathcal{T}'_2$ respectively, by a disk D, with ∂D separating in S the end of s that meets t' from the other ends of $s \cup t$, as illustrated schematically in Figure 3(b). Denote the set of equivalence classes, up to ambient isotopy, of these knots by $\mathcal{K}$.

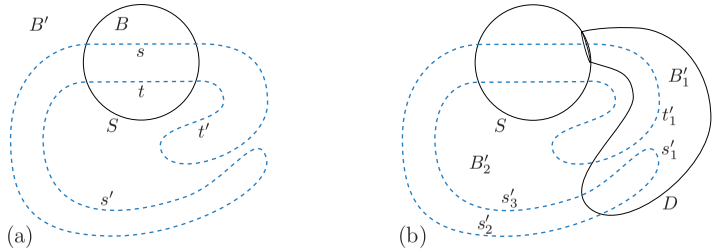

FIGURE 3: Schematic representation of the tangles $\mathcal{T}$, $\mathcal{T}'$, $\mathcal{T}'_1$ and $\mathcal{T}'_2$.

We denote the exterior of the strings of $\mathcal{T}$, $\mathcal{T}'$, $\mathcal{T}'_1$ and $\mathcal{T}'_2$ in B, B', B'_1 and B'_2 by E_B, $E_{B'}$, $E_{B'_1}$ and $E_{B'_2}$, respectively. Let P_D denote the punctured disk $D \cap E_{B'}$.

Lemma 2. *The thrice punctured disk P_D is essential in $E_{B'}$.*

Proof: This lemma follows from P_D being essential in $E_{B'_i} \cap \partial B'_i$ and from $E_{B'_i} \cap \partial B'_i$ being essential in B'_i (for $\mathcal{T}'_i$ being an essential tangle), for $i = 1, 2$. ∎

Lemma 3. *The tangle $\mathcal{T}'$ is essential and atoroidal.*

Proof: Suppose, by contradiction, that $\mathcal{T}'$ is inessential. Let E be a disk in B' separating the strings $s' \cup t'$. Let us consider the intersection of E with P_D and assume that $|E \cap P_D|$ is minimal among all choices of E. If $E \cap P_D$ has circle components, let α be an innermost one in E, otherwise let α be an outermost arc of $E \cap P_D$ in E, and denote by O_E the corresponding outermost disk. Hence, as P_D is essential in $E_{B'}$, from Lemma 2, the disk O_E cannot be a (boundary) compressing disk for P_D. Then, α cuts a disk O_D from P_D. By cutting and pasting E replacing O_E with O_D we reduce $|E \cap P_D|$,

component of $N(K) - S$ which contains s. The boundary components of A bound two disjoint annuli in F with e_i, $i = 1, 2$, respectively, and denote by b_i, $i = 1, 2$, the corresponding boundary components of A. Denote by A' the annulus component of $\partial N(K) - F \cup D$ in the boundary of the component of $N(K) - F \cup D$ which contains s'_2. We assume that ∂D is b_1, and also that $\partial A' \cap F$ is b_2, through an isotopy if necessary. Let A_D and P'_D be the annulus and punctured disk component of $P_D - P_D \cap A'$, respectively. Denote also by F the surface obtained from F by removing the two disjoint annuli bounded by $b_i \cup e_i$, for $i = 1, 2$, in F.

Consider the union of F, the annuli A, A' and A_D, and the punctured disk P'_D, and denote the resulting space by R. We smooth the space R on the intersections of the surfaces F, A, A', A_D and P'_D as explained next. The annuli A' and A_D are smoothed on $P_D \cap A'$ towards P'_D. The punctured disk P'_D is smoothed along b_1 towards F. The annulus A is smoothed along b_1 towards P'_D, and is smoothed along b_2 towards F. From the construction, the space R is a branched surface with sections denoted naturally by F, A, A', A_D and P'_D, as illustrated in Figure 6.

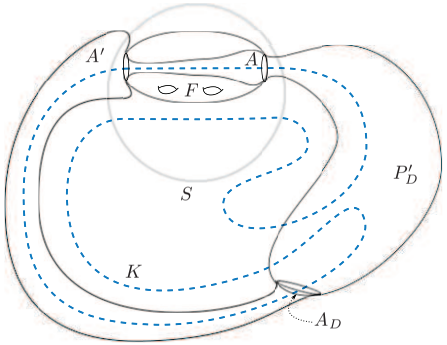

FIGURE 6: A schematic representation of the branched surface R.

Proposition 2. *The branched surface R is incompressible and without Reeb components.*

Proof: First we observe that R doesn't carry a Reeb component. In fact, if R would carry a torus T, this torus couldn't be transverse to the regular neighborhood of sections of R with boundary components, or of sections of R of positive genus. In this case, it could be transverse only to regular

A *monogon* in the closure of $M - N$ is a disc O with $O \cap N = \partial O$ which intersects $\partial_v N$ in a single fiber. (See Figure 5.) We say a branched surface R in M contains a *Reeb component* if R carries a compressible torus or properly embedded annulus, transverse to the fibers of N, bounding a solid torus in M. (This is a weaker version of the definition of Reeb component in [22] by Oertel.)

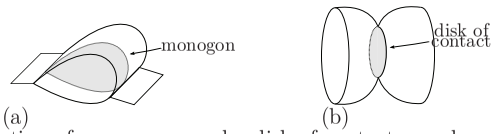

FIGURE 5: Illustration of a monogon and a disk of contact on a branched surface.

A branched surface embedded R in M is said *incompressible* if it satisfies the following three properties:

(i) R has no disk of contact or half-disks of contact;
(ii) $\partial_h N$ is incompressible and boundary incompressible in the closure of $M - N$, where a boundary compressing disk is assumed to have boundary defined by an arc in ∂M and an arc in $\partial_h N$;
(iii) There are no monogons in the closure of $M - N$;

and *without Reeb components* if it satisfies the following property:

(iv) B doesn't carry a Reeb component.

The following theorem proved by Oertel in [22] let us infer if a surface carried by a branched surface is essential. Note that condition (iv), R not carrying a torus or an annulus cutting a solid torus from M, implies the non-existence of Reeb components in the sense of Oertel [22].

Theorem 2 (Oertel, [22])**.** *If R is an incompressible branched surface without Reeb components (i.e. satisfies (i)-(iv)) and R carries some surface with positive weights then any surface carried by R is essential.*

We now proceed to define the surfaces F_j, $j \geq 2$, and prove that they are essential, through a Branched surface under the conditions of Theorem 2 that we proceed to construct.
Let s'_2 be the string of $\mathcal{T}'_2$ connecting an end of s to an end of s'_1 in $\mathcal{T}'_1$. (See Figure 3.) Denote by e_i, $i = 1, 2$, the boundary components of F in $N(s)$ closer to s'_i, $i = 1, 2$, respectively.
Denote by A the annulus component of $\partial N(K) - F$ in the boundary of the

contradicting its minimality. Hence $\mathcal{T}'$ is essential.
Suppose now, also by contradiction, that $B' - s' \cup t'$ has an embedded essential torus P. As $\mathcal{T}'_1$ and $\mathcal{T}'_2$ are atoroidal, P intersects D. Let us consider the intersection of P with D and assume that $|P \cap D|$ is minimal among all choices of P. Suppose a component of $P \cap D$ bounds a disk in P and consider an innermost disk O_P among these in P. As P_D is essential in $E_{B'}$, ∂O_P bounds a disk O_D in P_D. By cutting and pasting P replacing O_P with O_D we reduce $|P \cap D|$, contradicting its minimality. Hence, all components of $P \cap D$ in P are parallel essential curves in P and the components of $P - P \cap D$ are all annuli. As P is a torus and D is separating, there are at least two annuli and one of these, say A, is in B'_2 and has both boundary components in D. As each string of $\mathcal{T}'_2$ has only one end in D, the annulus A' that ∂A bounds in D is disjoint from $s' \cup t'$. Let P' be the torus $A \cup A'$. In case the component P' bounds in B'_2 is not solid torus, as A is incompressible in $B'_2 - s'_2 \cup t'_2$, this implies that P' is incompressible in $B'_2 - s'_2 \cup t'_2$, contradicting $\mathcal{T}'_2$ being atoroidal. Otherwise, the component P' bounds in B'_2 is a solid torus, and, therefore, we have that A is boundary compressible, and hence is parallel to A'. Then, we can isotope P along A through A' and reduce $|P \cap D|$, contradicting its minimality. That is, $\mathcal{T}'$ is atoroidal. ■

Lemma 4. *The knot K is hyperbolic.*

Proof: The knot K has a 2-string atoroidal essential tangle decomposition, as $\mathcal{T}$ is essential and atoroidal, by definition, and $\mathcal{T}'$ is essential and atoroidal, from Lemma 3. The statement then follows from Theorem 1 in [27] by Soma. ■

Lemma 5. *The exterior of the knot K has meridional essential surfaces in B of any positive genus and exactly two boundary components in $\partial N(s)$.*

Proof: Let F be a surface in B as in Lemma 1, which can be chosen of any positive genus and two boundary components in $\partial N(s)$.
As F is essential in E_B and ∂B is essential in $E(K)$, it follows that F is essential in $E(K)$. ■

Lemma 6. *The set $\mathcal{K}$ is an infinite collection of hyperbolic knots.*

Proof: From Lemma 4, each element of $\mathcal{K}$ is an hyperbolic knot.
Consider a 2-string tangle $\mathcal{T}$ decomposed by a collection of disjoint disks $\{X_1, \ldots, X_n\}$, $n \in \mathbb{N}$, in B, each intersecting $s \cup t$ at two points, one point

from each string, such that the 2-string tangle $\mathcal{T}_i$ cut by $X_i \cup X_{i+1}$ from $\mathcal{T}$ is essential and atoroidal. For $\mathcal{T}_i$ being essential, also such a $\mathcal{T}$ is essential. From Theorem 2 of [27], as $\mathcal{T}_i$, $i = 1, \ldots, n$, is atoroidal, we have that $\mathcal{T}$ is atoroidal. If we consider $\mathcal{T}_1$ as in Lemma 1, we have that the surfaces in the exterior $\mathcal{T}$ remain essential in the exterior of $\mathcal{T}$. Hence, the tangle $\mathcal{T}$ has meridional essential surfaces of any positive genus and exactly two boundary components in the boundary of a regular neighborhood of one string. Therefore, there are knots in $\mathcal{K}$ with $(B; s \cup t)$ as $\mathcal{T}$. Denote by K_n a knot of the collection $\mathcal{K}$ with the tangle $\mathcal{T}$ as just defined, for each $n \in \mathbb{N}$. Note that K_n has at least n disjoint non-parallel meridional planar essential surfaces in its exterior. Therefore, from Kneser-Haken finiteness theorem for essential surfaces with boundary properly embedded in 3-manifolds, as for instance in Jaco's book [9], and from knots being determined by their complements [5], we have that the set of equivalence classes of knots $\{K_n : n \in \mathbb{N}\}$ contains an infinite subset of non-equivalent knots. As $\mathcal{K}$ contains this set, it also has infinite cardinality. ∎

4. Meridional essential surfaces from a branched surface

In this section we prove the Theorem 1 by showing that each knot K in $\mathcal{K}$ is as in the statement of the theorem.

Lemma 7. *Suppose that the exterior of $K \in \mathcal{K}$ has a meridional essential surface F in B of genus g and $2n$ boundary components with exactly two boundary components in the boundary of $N(s)$. Then the exterior of K has meridional essential surfaces of genus g and $2b$ boundary components for every $b \geq n$.*

Proof: We define the surfaces in the exterior of K as in the statement of the lemma, denoted by $F_0, \ldots F_j, \ldots$, where F_j has $2(n + j)$ boundary components, and afterwards we prove they are essential in $E(K)$. We will use branched surface theory for this process.
The surface F_0 is defined as F in $E(K)$. We isotope the closest boundary component of F in $\partial N(s)$ to ∂D (which we can isotope to be in $\partial N(s)$) and identify it with ∂D. We define F_1 as the resulting surface from the union of F and P_D along ∂D.

Proposition 1. *The surfaces F_0 and F_1 are meridional essential surfaces in $E(K)$.*

Proof: The surface F_0 is essential in $E(K)$ from Lemma 5. Suppose, by contradiction, that F_1 is inessential in $E(K)$. Considering a (boundary) compressing disk for F_1 in $E(K)$, and its intersection with S, through an innermost curve\outermost arc argument, we obtain a (boundary) compressing disk for F in E_B or for P_D in $E_{B'}$, which is a contradiction to F being essential in E_B or P_D being essential in $E_{B'}$. ∎

For the remaining surfaces, F_j for $j \geq 2$, we will call on branched surfaces. First, we start by revising the definitions and result relevant to this paper from Oertel's work in [22], and also Floyd and Oertel's work in [3].
A *branched surface* R with generic branched locus is a compact space locally modeled on Figure 4(a). Hence, a union of finitely many compact smooth surfaces in a 3-manifold M, glued together to form a compact subspace of M respecting the local model, is a branched surface. We denote by $N = N(R)$ a fibered regular neighborhood of R (embedded) in M, locally modelled on Figure 4(b). The boundary of N is the union of three compact surfaces $\partial_h N$, $\partial_v N$ and $\partial M \cap \partial N$, where a fiber of N meets $\partial_h N$ transversely at its endpoints and either is disjoint from $\partial_v N$ or meets $\partial_v N$ in a closed interval in its interior. We say that a surface S is *carried* by R if it can be isotoped into N so that it is transverse to the fibers. Furthermore, S is carried by R with *positive weights* if S intersects every fiber of N. If we associate a weight $w_i \geq 0$ to each component on the complement of the branch locus in R we say that we have an *invariant measure* provided that the weights satisfy *branch equations* as in Figure 4(c). Given an invariant measure on R we can define a surface carried by R, with respect to the number of intersections between the fibers and the surface. We also note that if all weights are positive then the surface carried can be isotoped to be transverse to all fibers of N, and hence is carried with positive weights by R.

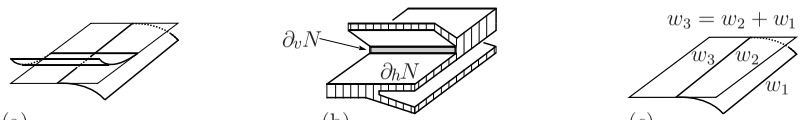

FIGURE 4: Local model for a branched surface, in (a), and its regular neighborhood, in (b).

A *disc of contact* is a disc O embedded in N transverse to fibers and with $\partial O \subset \partial_v N$. A *half-disc of contact* is a disc O embedded in N transverse to fibers with ∂O being the union of an arc in $\partial M \cap \partial N$ and an arc in $\partial_v N$.