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# ON JOINS OF COMPLEMENTED SUBLOCALES

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Abstract: The system $\mathcal{S}_c(L)$ consisting of joins of closed sublocales of a locale $L$ is known to be a frame, and for $L$ subfit it coincides with the Booleanization $\mathcal{S}_b(L)$ of the coframe of sublocales of $L$. In this paper, we study $\mathcal{S}_b(L)$ for a general locale $L$. We show that $\mathcal{S}_c(L)$ is always a subframe of $\mathcal{S}_b(L)$. Moreover, if $X$ is a $T_D$-space, we prove that $\mathcal{S}_b(\Omega(X))$ is precisely the set of classical subspaces of $X$, and that a locale $L$ is $T_D$-spatial iff the Boolean algebra $\mathcal{S}_b(L)$ is atomic. Some functoriality properties of $\mathcal{S}_b(L)$ are also studied.

Keywords: Locale, Frame, Sublocale, Booleanization, Induced sublocale, Complemented sublocale, Subfit locale, $T_D$-axiom.
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## 1. Introduction

This paper deals with several aspects concerning sublocale lattices, i.e. the natural subobject lattices in the category $\mathsf{Loc}$ of locales. Sublocales of a given locale $L$ constitute under the natural inclusion ordering a coframe $\mathcal{S}(L)$, that is, its order-theoretic dual $\mathcal{S}(L)^{op}$ is again a locale.

Even if $\mathcal{S}(L)^{op}$ is quite disconnected (for example, it is zero-dimensional as a locale, and also ultraparacompact [12]), it is typically non-Boolean. Nevertheless, in a recent work [11] Picado, Pultr and Tozzi show that under a weak point-free separation axiom (namely, subfitness) the subset $\mathcal{S}_c(L)$ of $\mathcal{S}(L)$ consisting of joins of closed sublocales is a Boolean algebra, and moreover it coincides precisely with the Booleanization $\mathcal{S}_b(L)$ of $\mathcal{S}(L)$. Subsequently, $\mathcal{S}_c(L)$ has been further investigated by several authors: its (non-)functoriality properties [1], the naturality of the construction as an envelope [2], its role as a discretization of a locale for modeling non-continuous localic maps [9], as well as for studying several point-free counterparts of topological properties (see e.g. [5]).

Somewhat surprisingly, it was also shown in [11] that $\mathcal{S}_c(L)$ is *always* a frame, even in the non-subfit case. In this paper we continue the study of

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the general case. We provide several equivalent descriptions of $\mathcal{S}_b(L)$ for a general locale $L$. Among others we discuss the connection between $\mathcal{S}_b(L)$ and $\mathcal{S}_c(L)$ and generalize some of the results in [11]. For example, it was shown there that if $X$ is a $T_1$-space, there is an isomorphism $\mathcal{S}_c(\Omega(X)) \cong \mathcal{P}(X)$. In this paper we extend this result by proving that $\mathcal{S}_b(\Omega(X)) \cong \mathcal{P}(X)$ whenever $X$ is a $T_D$-space, therefore showing that the Booleanization of the coframe of sublocales can still be seen as a natural discretization for $T_D$-spatial locales. Moreover, we prove that the Booleanization $\mathcal{S}_b(L)$ can be used as a tool for detecting $T_D$-spatiality of a locale, by showing that a locale $L$ is $T_D$-spatial if and only if the Boolean algebra $\mathcal{S}_b(L)$ is atomic. Functoriality properties of $\mathcal{S}_b$ are also studied, providing suitable examples and non-examples.

The organization of the paper is as follows. Section 2 concerns the Booleanization of $\mathcal{S}(L)$ when $L$ is not necessarily subfit. Among others, we establish the basic facts and study the connection between the frames $\mathcal{S}_b(L)$ and $\mathcal{S}_c(L)$. Section 3 is devoted to study the relation between $\mathcal{S}_b(L)$ and $T_D$-spatiality. Finally, Section 4 deals with functoriality properties of $\mathcal{S}_b(L)$.

**Background and notation.** We shall assume that the reader is familiar with the categories of frames and locales, for a comprehensive account on the topic we refer to Johnstone [7] or the more recent Picado-Pultr [8]. Our notation and terminology will be that of [8]. The Heyting operator in a locale $L$, right adjoint to the meet operator, will be denoted by $\to$; for each $a \in L$, $a^* = a \to 0$ is the pseudocomplement of $a$. The open (resp. closed) sublocale associated to an $a \in L$ will be denoted by $\mathfrak{o}(a)$ (resp. $\mathfrak{c}(a)$). We will write $S^{\#}$ for the supplement (i.e. co-pseudocomplement) of a sublocale $S \subseteq L$.

## 2. Joins of complemented sublocales

**2.1. Joins of closed sublocales.** Let $\mathcal{S}_c(L)$ denote the subset of $\mathcal{S}(L)$ consisting of joins of closed sublocales i.e.

$$\mathcal{S}_c(L) = \{ \bigvee_{a \in A} \mathfrak{c}(a) \mid A \subseteq L \},$$

endowed with the inclusion order inherited from $\mathcal{S}(L)$. In the recent paper [11], Picado, Pultr and Tozzi show (a.o.) that $\mathcal{S}_c(L)$ is *always* a frame which is embedded as a join-sublattice in the coframe $\mathcal{S}(L)$. One of the main results from [11] is that

> *if $L$ is subfit, and only in that case, $\mathcal{S}_c(L)^{op}$ is a Boolean algebra and coincides precisely with the Booleanization of $\mathcal{S}(L)^{op}$.*

$\mathcal{S}_c(L)$ has subsequently attracted attention in point-free topology (see for example [5, 9, 2, 1] for applications). We shall shortly see that actually no separation axiom is needed for obtaining a nice description of the Booleanization of $\mathcal{S}(L)^{op}$.

**2.2. Joins of complemented sublocales.** A sublocale $S$ is said to be *smooth* if it is a join of complemented sublocales in $L$. Moreover, a sublocale $S$ is *locally closed* if it is of the form $S = \mathfrak{o}(a) \cap \mathfrak{c}(b)$ for some $a, b \in L$. Any complemented sublocale is a join of locally closed sublocales. Indeed, if $S$ is a complemented sublocale, then $S^\# = \bigcap_i \mathfrak{c}(a_i) \vee \mathfrak{o}(b_i)$ for some $a_i, b_i \in L$. Thus

$$S = S^{\#\#} = \bigvee_i (\mathfrak{c}(a_i) \vee \mathfrak{o}(b_i))^\# = \bigvee_i \mathfrak{o}(a_i) \cap \mathfrak{c}(b_i).$$

Hence,

*a sublocale is smooth iff it is a join of locally closed sublocales.*

We denote by $\mathcal{S}_b(L)$ the subset of $\mathcal{S}(L)$ consisting of smooth sublocales, i.e.

$$\mathcal{S}_b(L) = \Big\{ \bigvee_{a \in A, b \in B} \mathfrak{c}(a) \cap \mathfrak{o}(b) \mid A, B \subseteq L \Big\}.$$

The system $\mathcal{S}_b(L)$ will be the main object of study of this paper, and we shall always consider it endowed with the ordering inherited from $\mathcal{S}(L)$, that is, inclusion between sublocales.

**2.3. Booleanization of $\mathcal{S}(L)^{op}$.** Observe that complemented elements of a locale are always regular, and meets of regular elements are regular, consequently any meet of complemented elements is regular. Moreover:

**Lemma 2.1.** *Any regular element of a zero-dimensional locale is a meet of complemented elements.*

*Proof*: Let $L$ be a zero-dimensional locale and $a = b^*$ for some $b \in L$. Then there exists a family $\{c_i\}_{i \in I}$ of complemented elements such that $b = \bigvee_i c_i$. By the first De Morgan law: $a = b^* = (\bigvee_i c_i)^* = \bigwedge_i c_i^* = \bigwedge_i c_i^c$. ■

**Corollary 2.2.** *Let $L$ be a zero-dimensional locale. Then the Booleanization $B_L$ of $L$ is precisely the set of all meets of complemented elements.*

A fundamental fact in locale theory is that $\mathcal{S}(L)^{op}$ is a zero-dimensional locale. From the above we immediatelly have the following:

**Corollary 2.3.** *The Booleanization of $\mathcal{S}(L)^{op}$ is precisely $\mathcal{S}_b(L)^{op}$ — i.e. one has $\mathcal{S}_b(L) = \{ S \in \mathcal{S}(L) \mid S = S^{\#\#} \}$.*

Recall from 2.1 that $L$ is subfit if and only if the Booleanization of $\mathcal{S}(L)^{op}$ is $\mathcal{S}_c(L)^{op}$. Combining this with the previous corollary yields the following:

**Corollary 2.4.** *$L$ is subfit if and only if $\mathcal{S}_c(L) = \mathcal{S}_b(L)$.*

**2.4. A few properties of the embedding of $L$ into $\mathcal{S}_b(L)$.** Since open sublocales are complemented, we note that there is a map

$$\mathfrak{o}_L \colon L \longrightarrow \mathcal{S}_b(L)$$

sending $a$ to $\mathfrak{o}(a)$. This map is clearly injective, and moreover, it is a frame homomorphism (because $\mathfrak{o}\left(\bigvee a_i\right) = \bigvee \mathfrak{o}(a_i)$ and joins in $\mathcal{S}(L)$ are joins in $\mathcal{S}_b(L)$, and $\mathfrak{o}(a \wedge b) = \mathfrak{o}(a) \cap \mathfrak{o}(b)$ and finite intersections in $\mathcal{S}(L)$ of complemented sublocales are complemented, and thus they are meets in $\mathcal{S}_b(L)$). Actually, we have that

*$\mathfrak{o}_L$ is an injective epimorphism in* $\mathsf{Frm}$*,*

(the fact that it is an epimorphism is an easy consequence of the fact that frame homomorphisms commute with complements). The map $\mathfrak{o}_L$ has in fact a quite canonical nature: it is the maximal essential extension of $L$ in the category of frames,cby an application of [2, Proposition 4.3]. For a more detailed account on this notion we refer to [2].

If instead we look at the localic side, the right adjoint $(\mathfrak{o}_L)_* \colon \mathcal{S}_b(L)^{op} \twoheadrightarrow L$ is a surjection of locales; and can thus be regarded as a "discrete cover" of the locale $L$; similar to the canonical surjection $(\mathfrak{c}_L)_* \colon \mathcal{S}(L)^{op} \twoheadrightarrow L$, but this time more discrete.

**2.5. The connection between $\mathcal{S}_b(L)$ and $\mathcal{S}_c(L)$.** Recall that $\mathcal{S}_c(L)$ denotes the set of joins of closed sublocales. Since closed sublocales are locally closed, it is clear that we have the inclusion $\mathcal{S}_c(L) \subseteq \mathcal{S}_b(L)$. Moreover this inclusion preserves arbitrary joins, since joins in both of them are just joins is $\mathcal{S}(L)$, i.e. we have a chain

$$\mathcal{S}_c(L) \subseteq \mathcal{S}_b(L) \subseteq \mathcal{S}(L)$$

of sup-semilattice embeddings.

But can we say something (other than closedness under joins) about how $\mathcal{S}_c(L)$ sits inside $\mathcal{S}_b(L)$? In fact we have the following:

**Theorem 2.5.** *$\mathcal{S}_c(L)$ is a subframe of $\mathcal{S}_b(L)$.*

*Proof*: The only point remaining is to show that $\mathcal{S}_c(L)$ is closed under binary meets in $\mathcal{S}_b(L)$. Let $S = \bigvee_i \mathfrak{c}(a_i)$ and $T = \bigvee_j \mathfrak{c}(b_j)$ in $\mathcal{S}_b(L)$. Then $S = (\bigcap_i \mathfrak{o}(a_i))^\#$ and $T = (\bigcap_j \mathfrak{o}(b_j))^\#$ and their meet in $\mathcal{S}_b(L) = B_{\mathcal{S}(L)^{op}}$ is given by the formula $(S \cap T)^{\#\#} = (S^\# \vee T^\#)^\#$. Thus

$$(S \cap T)^{\#\#} = (S^\# \vee T^\#)^\# = \Big(\big(\bigcap_i \mathfrak{o}(a_i)\big)^{\#\#} \vee \big(\bigcap_j \mathfrak{o}(b_j)\big)^{\#\#}\Big)^\#$$
$$= \big(\bigcap_{i,j} \mathfrak{o}(a_i \vee b_j)\big)^{\#\#\#} = \bigvee_{i,j} \mathfrak{c}(a_i \vee b_j),$$

where the third equality follows from the finite De Morgan law in $\mathcal{S}(L)^{op}$, which *always* holds, and from the coframe distributivity of $\mathcal{S}(L)$. Hence the meet of $S$ and $T$ in $\mathcal{S}_b(L)$ lies in $\mathcal{S}_c(L)$ and so it is also their meet in $\mathcal{S}_c(L)$. ■

(In particular, this yields an alternative, more transparent, proof of the fact that $\mathcal{S}_c(L)$ is always a frame, which was presented in [11, Section 2.3]).

**2.6. A different description of smooth sublocales.** Since smooth sublocales play an important role in this paper, we now give a different, somewhat more categorical, characterization of this class of sublocales. For the convenience of the reader, let us first recall the following Frobenius-type formula in the category of locales.

**Proposition 2.6** (Vermeulen)**.** *Let $f\colon L \to M$ a localic map, $C \subseteq M$ a complemented sublocale and $S \subseteq L$ an arbitrary one. Then $f[S \cap f_{-1}[C]] = f[S] \cap C$. In particular, if $f$ is a surjection, $f[f_{-1}[C]] = C$ for every complemented $C \subseteq M$ — that is, arbitrary surjections are pullback-stable along complemented inclusions.*

We can now characterize smooth sublocales exactly as those sublocales such that pulling back along their inclusion preserves arbitrary surjections:

**Proposition 2.7.** *Let $L$ be a locale and $S \in \mathcal{S}(L)$. Then the following are equivalent:*

(i) *$S$ is smooth*;
(ii) *For every surjection $f\colon M \twoheadrightarrow L$ in* $\mathsf{Loc}$*, $f[f_{-1}[S]] = S$.*

*Proof*: (1) $\Longrightarrow$ (2): The inclusion $\subseteq$ follows by the adjunction $f[-] \dashv f_{-1}[-]$. Let us now show the inclusion $S \subseteq f[f_{-1}[S]]$. By zero-dimensionality of

$\mathcal{S}(L)^{op}$ write $f[f_{-1}[S]] = \bigcap_i C_i$ for a suitable family $\{C_i\}_{i\in I}$ of complemented sublocales, and by smoothness write $S = \bigvee_j D_j$ for a suitable family $\{D_j\}_{j\in J}$ of complemented sublocales. Let $i \in I$ and $j \in J$. One has $f[f_{-1}[D_j]] \subseteq f[f_{-1}[S]] \subseteq C_i$ and by Proposition 2.6 the left-hand side equals $D_j$. Hence we have $D_j \subseteq C_i$ for each $i \in I$ and $j \in J$, that is, $S = \bigvee_j D_j \subseteq \bigcap_j C_i = f[f_{-1}[S]]$.

(2) $\Longrightarrow$ (1): Let $S$ be a sublocale satisfying the property. Consider the frame monomorphism $\mathfrak{o}_L \colon L \to \mathcal{S}_b(L)$ from 2.4 and its right adjoint localic surjection $p \colon \mathcal{S}_b(L) \twoheadrightarrow L$. By looking at the left adjoint frame homomorphisms, we see that the composite $\mathcal{S}_b(L) \xrightarrow{(-)^\#} \mathcal{S}_b(L)^{op} \xhookrightarrow{\iota} \mathcal{S}(L)^{op} \xrightarrow{\mathfrak{c}_*} L$ equals $p$. Now, we have $(c_*\iota)_{-1}[S] = \iota_{-1}[\mathfrak{c}(S)] = \mathcal{S}_b(L)^{op} \cap \mathfrak{c}(S) = \mathcal{S}_b(L)^{op} \cap \mathfrak{c}(S^{\#\#}) = (c_*\iota)_{-1}[S^{\#\#}]$. Hence $p_{-1}[S] = p_{-1}[S^{\#\#}]$. By adjunction the latter yields $p[p_{-1}[S]] \subseteq S^{\#\#}$ and by assumption the left-hand side equals $S$. Therefore $S \subseteq S^{\#\#}$ and $S$ is smooth. ■

## 3. Booleanization and the $T_D$-axiom

The aim of this section is to explore a strong connection between the construction $L \mapsto \mathcal{S}_b(L)$ and the $T_D$-axiom. It was proved in [11] that whenever $X$ is a $T_1$ topological space, then the system $\mathcal{S}_c(\Omega(X))$ of joins of closed sublocales of $\Omega(X)$ is isomorphic to the power set $\mathcal{P}(X)$ of $X$. We shall start by observing that an analogous result holds for $T_D$-spaces in our more general setting.

**3.1. Primes and covered primes.** Recall that an element $p \neq 1$ of a locale $L$ is said to be *prime* if whenever $a \wedge b \leq p$, one has $a \leq p$ or $b \leq p$. It is easy to show that a subset of $L$ of the form $\{1, a\}$ is a sublocale if and only if $a$ is prime. An element $p \in L$ is a *covered prime* if whenever $p = \bigwedge_i a_i$, there is some $i \in I$ with $p = a_i$. For further discussion on the terminology cf. the Introduction in [4]. We shall need the following important property concerning covered primes:

**Lemma 3.1** ([6, Proposition 10.2]). *Let $p$ be a prime in $L$. Then $\{1, p\}$ is complemented in $\mathcal{S}(L)$ if and only if $p$ is a covered prime.*

If $X$ is a space, every element of the form $X - \overline{\{x\}}$ is prime in $\Omega(X)$. Moreover, if $X$ is a $T_D$-space, then every prime $X - \overline{\{x\}}$ is covered (see for example [3, 2.3.2]).

**3.2. Computing $\mathcal{S}_b(\Omega(X))$ for a $T_D$-space $X$.** If $A$ is a subspace of a topological space $X$, let us denote by $\widetilde{A}$ the corresponding induced sublocale in $\mathcal{S}(\Omega(X))$ (cf. [8, VI 1.1]).

**Proposition 3.2.** *Let $X$ be a $T_D$-space. The map $\pi\colon \mathcal{P}(X) \longrightarrow \mathcal{S}(\Omega(X))$ given by $\pi(A) = \widetilde{A}$ restricts to an isomorphism $\pi\colon \mathcal{P}(X) \longrightarrow \mathcal{S}_b(\Omega(X))$, i.e. classical subspaces correspond precisely to smooth sublocales.*

*Proof*: It is well-known that $\pi$ is injective if and only if $X$ is $T_D$ (see [8, VI 1.2]) and that $\pi$ always preserves joins (see e.g. [10, p. 66]). The only task remaining is to show that the image of $\pi$ coincides with $\mathcal{S}_b(\Omega(X))$.

Let $A \subseteq X$. Then we have $\pi(A) = \widetilde{A} = \widetilde{\bigcup_{x\in A}\{x\}} = \bigvee_{x\in A}\widetilde{\{x\}} = \bigvee_{x\in A}\{\, X, X - \overline{\{x\}}\,\}$, so by Lemma 3.1 and the comment thereafter, one has $\pi(A) \in \mathcal{S}_b(\Omega(X))$. Finally, if $S \in \mathcal{S}_b(\Omega(X))$, we have $S = \bigvee_i C_i$ with $C_i$ complemented. But it is well-known that complemented sublocales are always induced, i.e. $C_i = \pi(A_i)$ for suitable $A_i \subseteq X$. Hence $S = \pi(\bigcup_i A_i)$. ■

**3.3. More on $T_D$-spatiality.** Actually, a converse of the previous proposition also holds. A locale is said to be *$T_D$-spatial* if it is isomorphic to one of the form $\Omega(X)$ for a $T_D$-space $X$. We first need to recall a few facts about the $T_D$-duality.

**3.3.1.** *The Banaschewski-Pultr $T_D$-duality.* Let $\mathsf{pt}_D(L)$ denote the set of covered prime elements of $L$. For each $a \in L$, we set $\Sigma'_a = \{\, p \in \mathsf{pt}_D(L) \mid a \not\leq p \,\}$. Then, $\Sigma'(L) := (\mathsf{pt}_D(L), \{\, \Sigma'_a \mid a \in L \,\})$ is a $T_D$ topological space by [3, Proposition 3.3.2]. Moreover, there is a *$T_D$-spatialization* frame surjection $\varphi_L\colon L \twoheadrightarrow \Omega(\Sigma'(L))$ which sends $a$ to $\Sigma'_a$ (cf. [3, 3.4]). We shall need the following lemma:

**Lemma 3.3.** *A locale $L$ is $T_D$-spatial if and only if every element is a meet of covered primes.*

*Proof*: If $X$ is a $T_D$-space, $U = \bigwedge_{x\notin U} X - \overline{\{x\}}$ with each $X - \overline{\{x\}}$ covered. For the converse, assume that every element is a meet of covered primes. Then obviously $\Sigma'_a = \Sigma'_b$ implies $a = b$, whence the map $\varphi_L$ defined above is also injective and thus an isomorphism. Hence $L \cong \Omega(\Sigma'(L))$ with $\Sigma'(L)$ a $T_D$-space. ■

**3.3.2.** *The Boolean algebra $\mathcal{S}_b(L)$ and $T_D$-spatiality.* We are now in position the main result of the section:

**Theorem 3.4.** *A locale $L$ is $T_D$-spatial if and only if the Boolean algebra $\mathcal{S}_b(L)$ is atomic—i.e. if and only if $\mathcal{S}_b(L)$ is spatial.*

*Proof*: The "only if" part follows from Proposition 3.2. Let us show the converse. It is known that for any locale $L$ a prime of the locale $\mathcal{S}(L)^{op}$ is of the form $\{1, p\}$ for a prime $p$ of $L$, i.e. that there is a bijection between the spectrum of $L$ and that of $\mathcal{S}(L)^{op}$ — in fact this follows readily from the universality of $\mathcal{S}(L)^{op}$. Hence, the set of prime elements of $\mathcal{S}_b(L)^{op}$ is precisely the set of sublocales $\{1, p\}$ (with $p$ prime in $L$) contained in $\mathcal{S}_b(L)$. But a two-element sublocale is complemented as soon as it belongs to $\mathcal{S}_b(L)$ (indeed, if $\{1, p\} = \bigvee_i C_i$ with each $C_i$ complemented, then there is a $C_i \neq \{1\}$ and hence $C_i = \{1, p\}$). Accordingly, by virtue of Lemma 3.1 we have that prime elements of $\mathcal{S}_b(L)^{op}$ are precisely the $\{1, p\}$ with $p$ covered in $L$.

Since $\mathcal{S}_b(L)^{op}$ is spatial, each smooth sublocale is a meet in $\mathcal{S}_b(L)^{op}$ of primes in $\mathcal{S}_b(L)^{op}$. Let $a \in L$. In particular, $\mathfrak{c}(a) = \bigwedge_i^{\mathcal{S}_b(L)^{op}} \{1, p_i\} = \bigvee_i \{1, p_i\}$ for a suitable family $\{p_i\}_{i \in I}$ of covered primes in $L$. Hence $a \in \mathfrak{c}(a)$ is of the form $\bigwedge_i x_i$ with $x_i \in \{1, p_i\}$ for each $i \in I$. We have therefore shown that every element of $L$ is a meet of covered primes. The result now follows from the previous lemma. ■

## 4. Remarks on functoriality

Given that the Booleanization $B_L$ of a locale $L$ is not a functorial construction, it should of course not come as a surprise that neither is the assignment $L \mapsto \mathcal{S}_b(L)$. If $f\colon L \to M$ is a frame homomorphism, the function $\mathcal{S}_b(f)\colon \mathcal{S}_b(L) \to \mathcal{S}_b(M)$ given by

$$\mathcal{S}_b(f)(S) = \bigvee \{ \mathfrak{o}(f(a)) \cap \mathfrak{c}(f(b)) \mid \mathfrak{o}(a) \cap \mathfrak{c}(b) \subseteq S \}$$

is the only possible candidate for obtaining a commutative square

$$\begin{tikzcd} \mathcal{S}_b(L) \arrow[r, "\mathcal{S}_b(f)"] & S_b(M) \\ L \arrow[u, hook, "\mathfrak{o}_L"] \arrow[r, "f"] & M \arrow[u, hook, "\mathfrak{o}_M"'] \end{tikzcd}$$

in $\mathsf{Frm}$. We shall say that $f$ *lifts* if the $\mathcal{S}_b(f)$ defined above is indeed a frame homomorphism (clearly, the square always commutes).

The following result indicates that the problem of studying lifts of frame homomorphisms amounts to studying lifts of surjections and subframes.

**Proposition 4.1.** *A frame homomorphism lifts if and only if both halves of its* (*Regular Epi, Mono*) *factorization lift.*

*Proof*: The "if" part is clear, so let us show the "only if". Assume that a frame homomorphism $f\colon L \to M$ lifts to a frame homomorphism $h\colon \mathcal{S}_b(L) \to \mathcal{S}_b(M)$ which fits in the diagram

$$\begin{tikzcd} \mathcal{S}_b(L) \arrow[r, "h"] & S_b(M) \\ L \arrow[u, tail, "\mathfrak{o}_L"] \arrow[r, "f"] & M \arrow[u, tail, "\mathfrak{o}_M"'] \end{tikzcd} \tag{4.1}$$

We factor $h$ through its image, $\mathcal{S}_b(L) \xrightarrow{e} h(S_b(L)) \xrightarrow{m} \mathcal{S}_b(M)$ —i.e. $h = m \circ e$ where $e$ is surjective and $m$ is injective. The surjection $e\colon \mathcal{S}_b(L) \twoheadrightarrow h(\mathcal{S}_b(L))$ corresponds to a sublocale of the Boolean locale $\mathcal{S}_b(L)$, and every sublocale of a Boolean locale is open, hence $e$ corresponds to an open sublocale, i.e. there is a $T \in \mathcal{S}_b(L)$ and an isomorphism $k\colon {\downarrow}T \to h(\mathcal{S}_b(L))$ such that $e = k \circ ((-) \wedge T)$. Therefore, we have that $M = h(L) = e(L) = e(T) = h(T)$. Since $T \in \mathcal{S}_b(L)$, write $T = \bigvee_{\mathfrak{o}(a)\cap\mathfrak{c}(b)\subseteq T} \mathfrak{o}(a) \cap \mathfrak{c}(b)$, and so the definition of $h$ yields

$$\begin{aligned} M = h(T) &= \bigvee_{\mathfrak{o}(a)\cap\mathfrak{c}(b)\subseteq T} h(\mathfrak{o}(a)) \cap h(\mathfrak{c}(b)) = \bigvee_{\mathfrak{o}(a)\cap\mathfrak{c}(b)\subseteq T} \mathfrak{o}(f(a)) \cap \mathfrak{c}(f(b)) \\ &= \bigvee_{\mathfrak{o}(a)\cap\mathfrak{c}(b)\subseteq T} (f_*)_{-1}[\mathfrak{o}(a) \cap \mathfrak{c}(b)]. \end{aligned}$$

The colocalic map $f_*[-]\colon \mathcal{S}(M) \longrightarrow \mathcal{S}(L)$ preserves joins, and hence we obtain $f_*[M] = \bigvee_{\mathfrak{o}(a)\cap\mathfrak{c}(b)\subseteq T} f_*[(f_*)_{-1}[\mathfrak{o}(a) \cap \mathfrak{c}(b)]] \subseteq \bigvee_{\mathfrak{o}(a)\cap\mathfrak{c}(b)\subseteq T} \mathfrak{o}(a) \cap \mathfrak{c}(b) = T$ where the inclusion follows because of the adjunction $f_*[-] \dashv (f_*)_{-1}[-]$. Hence we have inclusions of sublocales $f_*[M] \subseteq T \subseteq L$. Now, let $i\colon L \twoheadrightarrow T$ and $j\colon T \twoheadrightarrow f_*[M]$ be the corresponding frame surjections. Then $f$ factors as

$$L \overset{i}{\twoheadrightarrow} T \overset{j}{\twoheadrightarrow} f_*[M] \overset{n}{\rightarrowtail} M$$

(note that $(j \circ i, n)$ is the (Regular Epi, Mono) factorization of $f$). We also observe that since $T \in \mathcal{S}_b(L)$, then $\mathcal{S}_b(T) = {\downarrow}T$. For the remainder of the proof it is convenient to consider the diagram

$$\begin{tikzcd}
\mathcal{S}_b(L) \arrow[r, two heads, "(-)\wedge T"] & \mathcal{S}_b(T) \arrow[rr, tail, "m\circ k"] & & \mathcal{S}_b(M) \\
L \arrow[u, tail, "\mathfrak{o}_L"] \arrow[r, two heads, "i"] & T \arrow[u, tail, "\mathfrak{o}_T"] \arrow[r, two heads, "j"] & f_*[M] \arrow[r, tail, "n"] & M \arrow[u, tail, "\mathfrak{o}_M"]
\end{tikzcd} \tag{4.2}$$

The left hand side square commutes because for all $a \in L$ one has $\mathfrak{o}_T(i(a)) = \mathfrak{o}_L(a) \cap T$ (cf. [8, III 6.2.1]). Commutativity of the right hand side square follows from the commutativity of the left hand side square, commutativity of the outer square (i.e. (4.1) above) and the fact that $i$ is an epimorphism. Now, since the composite $\mathfrak{o}_M \circ n \circ j$ is monic, so is $j$, thus it is an isomorphism, i.e. $f_*[M] = T$. Diagram (4.2) therefore displays the desired lifts. ■

**Corollary 4.2.** *Let $f\colon L \twoheadrightarrow S$ be a frame surjection onto a sublocale $S$ of $L$. Then $f$ lifts if and only if $S$ is smooth.*

*Proof*: $\Longrightarrow$: On the way of proving the previous proposition we showed that the image of the map which is lifted always corresponds to a join of complemented sublocales of its domain, hence this implication follows.

$\Longleftarrow$: If $S \in \mathcal{S}_b(L)$, it is immediate to check that $\mathcal{S}_b(S) = {\downarrow} S \subseteq \mathcal{S}_b(L)$. Hence there is a surjection $\mathcal{S}_b(L) \to \mathcal{S}_b(S)$ which maps $T$ to $S \wedge T$. The fact that $\mathfrak{o}_S(f(a)) = S \cap \mathfrak{o}(a)$ ensures that the relevant diagram commutes. ■

In particular, we can strengthen [1, Theorem 4.5] as follows:

**Corollary 4.3.** *Let $p$ be a prime in a locale $L$. Then the frame surjection associated to the sublocale $\{1, p\}$ lifts if and only if $p$ is a covered prime.*

*Proof*: Recall Lemma 3.1 and that $\{1, p\}$ belongs to $\mathcal{S}_b(L)$ iff it is complemented. The result therefore follows from the previous corollary. ■

A frame homomorphism is a *D-homomorphism* [3] if its right adjoint preserves coveredness of primes. The following is a necessary condition:

**Corollary 4.4.** *If a frame homomorphism lifts, it is a D-homomorphism.*

*Proof*: Let $f\colon L \to M$ lift and let $p$ be a covered prime in $M$. Since $p$ is a covered prime, by virtue of the previous corollary the left adjoint of the inclusion $\{1, p\} \hookrightarrow M$ lifts, and so does $f$, hence the left adjoint of the upper-left composite in

$$\begin{tikzcd} M \arrow[r, "f_*"] & L \\ \{1, p\} \arrow[u, hook] \arrow[r, two heads, "f_*"] & \{1, f_*(p)\} \arrow[u, hook] \end{tikzcd}$$

lifts. Now, the bottom-right composite corresponds to its image factorization so by Proposition 4.1 both components lift. In particular, the left adjoint of $\{1, f_*(p)\} \hookrightarrow L$ lifts, which by Corollary 4.3 implies that $f_*(p)$ is covered. ∎

As we have seen, plenty of surjections lift. Nevertheless, for monic frame homomorphisms the behaviour seems to be worse, for example the canonical embedding into the assembly seldom lifts:

**Proposition 4.5.** *Let $L$ be fit. The canonical monomorphism $\mathfrak{c}\colon L \rightarrowtail \mathcal{S}(L)^{op}$ lifts if and only if $\mathcal{S}(L)$ is Boolean.*

*Proof*: The "if" implication is trivial, so let us show the other one. Consider the onto coframe homomorphism $f\colon S(L) \to \mathcal{S}_b(L)^{op}$ sending $S$ to $S^\#$. Let us show that it is also injective. Let $S, T \in \mathcal{S}(L)$ with $S^\# = T^\#$. Since $L$ is fit, we can write $S = \bigcap_i \mathfrak{o}(a_i)$ and $T = \bigcap_j \mathfrak{o}(b_j)$. Now, from $S^\# = T^\#$, we obtain $\bigvee_i \mathfrak{c}(a_i) = \bigvee_j \mathfrak{c}(b_j)$. Since the lift preserves arbitrary joins, it follows that $\bigvee_i \mathfrak{c}(\mathfrak{c}(a_i)) = \bigvee_j \mathfrak{c}(\mathfrak{c}(b_j))$. But the $\mathfrak{c}(a_i)$ and $\mathfrak{c}(b_j)$ are complemented in $\mathcal{S}(L)^{op}$, so we can write the last equality as $\bigvee_i \mathfrak{o}(\mathfrak{o}(a_i)) = \bigvee_j \mathfrak{o}(\mathfrak{o}(b_j))$, and therefore $\mathfrak{o}(\bigvee_i \mathfrak{o}(a_i)) = \mathfrak{o}(\bigvee_j \mathfrak{o}(b_j))$. Accordingly, $\bigvee_i \mathfrak{o}(a_i) = \bigvee_j \mathfrak{o}(b_j)$ in $\mathcal{S}(L)^{op}$, i.e. $\bigcap_i \mathfrak{o}(a_i) = \bigcap_j \mathfrak{o}(b_j)$. Hence $S = T$. Therefore $\mathcal{S}(L)$ is Boolean. ∎

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